CLASSIFYING SPACES OF DEGENERATING MIXED HODGE STRUCTURES, III: SPACES OF NILPOTENT ORBITS

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ABSTRACT. We construct toroidal partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients. They are moduli spaces of log mixed Hodge structures with polarized graded quotients. We construct them as the spaces of nilpotent orbits.

Contents

Introduction

This is Part III of our series of papers to study degeneration of mixed Hodge structures, and an expanded, full detailed version of our previous three announcements [KNU10a], [KNU10b], and [KNU10c]. In this part, we construct toroidal partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients, study their properties, and prove that they are moduli spaces of log mixed Hodge structures with polarized graded quotients. We also apply them to investigate Néron models including degenerations of intermediate Jacobians.

We explain the above more precisely. For a complex analytic space S with an fs log structure, a log mixed Hodge structure (LMH, for short) on S is an analytic family parametrized by S of "mixed Hodge structures which may have logarithmic degeneration".

The main subject of this paper is to construct a moduli space of LMH. A morphism from S to this moduli space is in one-to-one correspondence with an isomorphism class of LMH on S. This space gives a toroidal partial compactification of the moduli of

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mixed Hodge structure. This moduli space is not necessarily a complex analytic space, but it belongs to a category $\mathcal{B}(\log)$ which contains the category of fs log analytic spaces as a full subcategory. In the category $\mathcal{B}(\log)$, the space is a fine moduli. See Theorems 2.6.6 and 3.3.3.

This paper is the mixed Hodge theoretic version of the main part of [KU09] (a summary of which is [KU99]), where the degeneration of Hodge structures was studied. We generalize results in [KU09] to the mixed Hodge theoretic versions. In particular, we prove that the above moduli space has nice properties, especially, that it is a log manifold. See Theorems 2.5.1–2.5.6 and 3.2.8.

Every proof goes in a parallel way to the pure case [KU09], and we further enhance the story by replacing "fan" in [KU09] with "weak fan", which is introduced in [KNU10c], for the reason of applications including the study of intermediate Jacobians. In fact, with this relaxed concept of fan, the theory can be applied more vastly. In this paper, we illustrate a few of such applications, but this important subject of the degeneration of intermediate Jacobians should be investigated more in a forthcoming paper. Thus, this paper is more general than [KU09] even in the pure case, though the proofs are still parallel.

We organize this paper as follows. First, we present the main results in §1–§3. Next, we prove them in §4 and §5. Lastly, we gather applications, examples, and complements in §6–§7.

More closely, each section is as follows.

In §1, we review the notion of log mixed Hodge structures, explaining the concept admissibility of the local monodromy, which appears in the mixed Hodge case.

In §2, we consider the moduli space of log mixed Hodge structures with polarized graded quotients for the weight filtration. Let D be a classifying space of mixed Hodge structures with polarized graded quotients ([U84]). It is a complex analytic manifold, and is a mixed Hodge theoretic generalization of a classifying space of polarized Hodge structures defined by Griffiths ([G68a]). In §2, we consider toroidal partial compactifications $\Gamma \backslash D_\Sigma$ of $\Gamma \backslash D$ for discrete subgroups Γ of $\mathrm{Aut}(D)$, and state a theorem (2.6.6) that it is a moduli space of LMH with polarized graded quotients. Here D_Σ is a space of nilpotent orbits in the directions in Σ. This space sits in the following fundamental diagram.

$$
\begin{array}{ccc}
D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
\downarrow & & \downarrow
\end{array}
$$

$$
\begin{array}{ccccccc}
D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & &
\end{array}
$$

We constructed D_{BS} in Part I of this series of papers ([KNU09]), and $D_{\mathrm{SL}(2)}$ in Part II ([KNU.p]).

For the degeneration of polarized (pure) Hodge structures, this diagram was constructed in [KU09].

The results of this section was announced in [KNU10b], but only in the case of fan, not weak fan. Hence, this section is essentially the same as [KNU10b] rewritten by using weak fan.

In §3, we consider a relative theory fixing a base S. We assume that we are given $(H_{(w)})_{w \in \mathbf{Z}}$, where $H_{(w)}$ is a pure log Hodge structure of weight w on an object S of $\mathcal{B}(\log)$. In this section, we consider the moduli of LMH with the given graded quotients $(H_{(w)})_{w \in \mathbf{Z}}$. In the case where all $H_{(w)}$ are polarizable, we state that there exists the moduli space of LMH whose w-th graded quotient for the weight filtration is $H_{(w)}$ for each w (3.3.3). We expect that the construction works without assuming $H_{(w)}$ are polarizable. In some cases, we can actually show that this is true (see 7.1.4).

For the relation of results of §2 and those of §3, see 3.4.

The proofs of the main results in §2 and §3 are given in §4 and §5, by using the theory of SL(2)-orbits in Part II.

In §6, we give the theory of Néron models which appear for example when an intermediate Jacobian degenerates.

In §7, we give examples and explain the theories of log abelian varieties and log complex tori in [KKN08] from the point of view of this paper. We also explain the necessity of weak fan and discuss completeness of weak fans.

§1. Log mixed Hodge structures

§1.1. The category $\mathcal{B}(\log)$ and relative log manifolds

We review the category $\mathcal{B}(\log)$ introduced in [KU09], which plays a central role in this paper. We introduce a notion "relative log manifold" (1.1.6), which is a relative version of the notion "log manifold" introduced in [KU09].

1.1.1. Strong topology.

Let Z be a local ringed space over $\mathbf{C}$, and S a subset of Z. The *strong topology* of S in Z is defined as follows. A subset U of S is open for this topology if and only if for any analytic space A (analytic space means a complex analytic space) and any morphism $f : A \to Z$ of local ringed spaces over $\mathbf{C}$ such that $f(A) \subset Z$, $f^{-1}(U)$ is open in A. This topology is stronger than or equal to the topology as a subspace of Z.

1.1.2. Let Z be a local ringed space (resp. an fs log local ringed space, i.e., a local ringed space endowed with an fs log structure ([KU09] 2.1.5)) over $\mathbf{C}$, and S a subset of Z. The *strong subspace of Z defined by S* is the subset S with the strong topology in Z (1.1.1) and with the inverse image of $\mathcal{O}_Z$ (resp. the inverse images of $\mathcal{O}_Z$ and M_Z, where M_Z denotes the log structure of Z). A morphism $\iota : S \to Z$ of local ringed spaces (resp. fs log local ringed spaces) over $\mathbf{C}$ is a *strong immersion* if S is isomorphic to a strong subspace of Z via ι. If this is the case, it is easy to see by definition that any morphism $A \to Z$ from an analytic space (resp. an fs log analytic space) uniquely factors through S if and only if it factors through S set theoretically.

Lemma 1.1.3. *A composite of two strong immersions is a strong immersion.*

Proof. It is enough to show the following: Let Z be a local ringed space over $\mathbf{C}$ and $T \subset S \subset Z$ are two subsets of Z. Endow S with the strong topology in Z. Then, the strong topology of T in Z coincides with the strong topology of T in S.

First, let U be an open subset of T with respect to the strong topology of T in Z. Let $A \to S$ be a morphism whose image is contained in T. Then, the image of the composite $A \to S \to Z$ is also contained in T and the inverse image of U in A is open. Hence, U is open with respect to the strong topology of T in S. Conversely, let U be an open subset of T with respect to the strong topology of T in S. Let $A \to Z$ be a morphism. If its image is contained in T, then it is contained in S, and $A \to Z$ factors through $A \to S$ (1.1.2). Since the image of the last morphism is contained in T, the inverse image of U in A is open. Therefore, U is open with respect to the strong topology of T in Z. $\square$

1.1.4. The category $\mathcal{B}(\log)$.

As in [KU09], in the theory of moduli spaces of log Hodge structures, we have to enlarge the category of analytic spaces, for the moduli spaces are often not analytic spaces.

Let $\mathcal{A}$ be the category of analytic spaces and let $\mathcal{A}(\log)$ be the category of fs log analytic spaces (i.e., analytic spaces endowed with an fs log structure). We enlarge $\mathcal{A}$ and $\mathcal{A}(\log)$ to $\mathcal{B}$ and $\mathcal{B}(\log)$, respectively, as follows. In this paper, we will work mainly in the category $\mathcal{B}(\log)$, and, as we did in [KU09], we will find the moduli spaces in $\mathcal{B}(\log)$.

$\mathcal{B}$ (resp. $\mathcal{B}(\log)$) is the category of all local ringed spaces S over $\mathbf{C}$ (resp. local ringed spaces over $\mathbf{C}$ endowed with a log structure) having the following property:

S is locally isomorphic to a strong subspace of an analytic space (resp. fs log analytic space).

The category $\mathcal{B}(\log)$ has fiber products ([KU09] 3.5.1).

1.1.5. Log manifold ([KU09] 3.5.7).

In [KU09], we saw that moduli spaces of polarized log Hodge structures were logarithmic manifolds (abbreviated as log manifolds in this paper).

An object S of $\mathcal{B}(\log)$ is a *log manifold* if, locally on S, there are an fs log analytic space Z which is log smooth over $\mathbf{C}$ ([KU09] 2.1.11), elements $\omega_1, \ldots, \omega_n$ of $\Gamma(Z, \omega_Z^1)$, and an open immersion from S to the strong subspace of Z defined by the subset $\{z \in Z \mid \omega_1, \ldots, \omega_n \text{ are zero in } \omega_z^1\}$. Here ω_Z^1 denotes the sheaf of log differential 1-forms on Z ([KU09] 2.1.7), and ω_z^1 denotes the space of log differential 1-forms on the point $z = \mathrm{Spec}(\mathbf{C})$ which is endowed with the inverse image of the log structure of Z.

Example. $S = (\mathbf{C} \times \mathbf{C} \smallsetminus (\{0\} \times \mathbf{C})) \cup \{(0,0)\}$ has a natural structure of a log manifold. See [KU09] 0.4.17.

1.1.6. *Relative log manifold* over an object S of $\mathcal{B}(\log)$.

It is an object T of $\mathcal{B}(\log)$ over S such that, locally on S and on T, there exist a log smooth morphism $Y \to X$ of fs log analytic spaces over $\mathbf{C}$, a morphism $S \to X$, an open subspace U of $S \times_X Y$, elements $\omega_1, \ldots, \omega_n \in \Gamma(U, \omega_U^1)$, and an open immersion from T to the strong subspace of U defined by the subset $\{u \in U \mid \omega_1, \ldots, \omega_n \text{ are zero in } \omega_u^1\}$.

§1.2. Relative monodromy filtrations

1.2.1. Let V be an abelian group, let $W = (W_w)_{w \in \mathbf{Z}}$ be an increasing filtration on V, and let $N \in \mathrm{End}(V, W)$. Assume that N is nilpotent.

Then an increasing filtration $M = (M_w)_{w \in \mathbf{Z}}$ on V is called the *relative monodromy filtration of N with respect to W* if it satisfies the following conditions (1) and (2).

(1) $NM_w \subset M_{w-2}$ for any $w \in \mathbf{Z}$.

(2) For any $w \in \mathbf{Z}$ and $m \geq 0$, we have an isomorphism

$$ N^m : \mathrm{gr}^M_{w+m} \mathrm{gr}^W_w \xrightarrow{\sim} \mathrm{gr}^M_{w-m} \mathrm{gr}^W_w . $$

The relative monodromy filtration of N with respect to W need not exist. If it exists, it is unique ([D80] 1.6.13). If it exists, we denote it by $M(N, W)$ as in [SZ85] (2.5). It exists in the case $W_w = V$ and $W_{w-1} = 0$ for some w ([D80] 1.6.1).

Relative monodromy filtration is also called relative weight filtration, or relative monodromy weight filtration.

If V is a vector space over a field K and the W_w are K-vector subspaces of V, and if N is a K-linear map, then the relative monodromy filtration, if it exists, is formed by K-subspaces of V. If K is of characteristic 0, for an increasing filtration M on V consisting of K-vector subspaces, M is the relative monodromy filtration $M(N, W)$ if and only if for any $w \in \mathbf{Z}$, there is an action of the Lie algebra $\mathfrak{sl}(2, K)$ on gr^W_w satisfying the following conditions (3) and (4). Let $Y := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}(2, K)$.

(3) For $k \in \mathbf{Z}$, $M_k \mathrm{gr}^W_w$ is the sum of $\{v \in \mathrm{gr}^W_w \mid Yv = \ell v\}$ for all integers $\ell \leq k - w$.

(4) $N : \mathrm{gr}^W_w \to \mathrm{gr}^W_w$ coincides with the action of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, K)$.

We have the following compatibility of relative monodromy filtrations with various operators (direct sum, tensor product, etc.). For $j = 1, 2$, let V_j be a vector space over a field K endowed with an increasing filtration $W_\bullet V_j$ consisting of K-vector subspaces, and let $N_j \in \mathrm{End}_K(V_j, W_\bullet V_j)$ be nilpotent.

1.2.1.1. Assume that the relative monodromy filtration $M_\bullet V_j = M(N_j, W_\bullet V_j)$ exists. Then, the filtration $M_\bullet V_1 \oplus M_\bullet V_2$ on $V_1 \oplus V_2$ is the relative monodromy filtration of $N_1 \oplus N_2 \in \mathrm{End}(V_1 \oplus V_2, W_\bullet V_1 \oplus W_\bullet V_2)$. Assume furthermore that K is of characteristic 0. Then, the filtration $M_\bullet V_1 \otimes M_\bullet V_2$ on $V_1 \otimes V_2$ is the relative monodromy filtration of $N_1 \otimes 1 + 1 \otimes N_2 \in \mathrm{End}(V_1 \otimes V_2, W_\bullet V_1 \otimes W_\bullet V_2)$. (Here the w-th filter of the filtration $M_\bullet V_1 \otimes M_\bullet V_2$ is $\sum_{j+k=w} M_j V_1 \otimes M_k V_2$, and the filtration $W_\bullet V_1 \otimes W_\bullet V_2$ is defined similarly.) The filtration M on $H := \mathrm{Hom}_K(V_1, V_2)$ defined by $M_w = \{f \in \mathrm{Hom}_K(V_1, V_2) \mid f(M_k V_1) \subset M_{k+w} V_2 \text{ for all } k\}$ is the relative monodromy filtration of $N \in \mathrm{End}(H, W)$, where W is the filtration on H defined by $W_w H = \{f \in \mathrm{Hom}_K(V_1, V_2) \mid f(W_k V_1) \subset W_{k+w} V_2 \text{ for all } k\}$ and $N : H \to H$ is the map $f \mapsto N_2 f - f N_1$.

1.2.1.2. Assume that the relative monodromy filtration M of $N_1 \oplus N_2 \in \mathrm{End}(V_1 \oplus V_2, W_\bullet V_1 \oplus W_\bullet V_2)$ exists. Then the relative monodromy filtration $M_\bullet V_j = M(N_j, W_\bullet V_j)$ exists for $j = 1, 2$, and $M = M_\bullet V_1 \oplus M_\bullet V_2$.

By 1.2.1.1 and 1.2.1.2, we have the compatibility with symmetric powers and exterior powers because these are direct summands of tensor products.

These 1.2.1.1 and 1.2.1.2 are certainly well known. They are proved as follows.

Proof of 1.2.1.1. We give the proof for the tensor product. The proof for Hom is similar and the proof for the direct sum is easy. For $j = 1, 2$ and $w \in \mathbf{Z}$, let $\rho_{j,w}$ be the action of $\mathfrak{sl}(2, K)$ on $\operatorname{gr}_w^W V_j$ satisfying the above conditions (3) and (4) (with M and W in (3) and (4) replaced by $M_\bullet V_j$ and $W_\bullet V_j$, respectively, and with N in (4) replaced by N_j). For $w \in \mathbf{Z}$, let ρ_w be the representation of $\mathfrak{sl}(2, K)$ on $\bigoplus_{k+\ell=w} \operatorname{gr}_k^W V_1 \otimes \operatorname{gr}_\ell^W V_2$ defined to be $\bigoplus_{k+\ell=w} \rho_{1,k} \otimes \rho_{2,\ell}$. Then ρ_w satisfies the above conditions (3) and (4) (with M and W in (3) and (4) replaced by $M_\bullet V_1 \otimes M_\bullet V_2$ and $W_\bullet V_1 \otimes W_\bullet V_2$, respectively, and with N in (4) replaced by $N_1 \otimes 1 + 1 \otimes N_2$). Hence $M_\bullet V_1 \otimes M_\bullet V_2$ is the relative monodromy filtration of $N_1 \otimes 1 + 1 \otimes N_2$ with respect to $W_\bullet V_1 \otimes W_\bullet V_2$. $\square$

Proof of 1.2.1.2. By the uniqueness of the relative monodromy filtration, M is invariant under the maps $V_1 \oplus V_2 \xrightarrow{\sim} V_1 \oplus V_2$, $(x, y) \mapsto (ax, by)$ for all $a, b \in K^\times$. If K is not $\mathbf{F}_2$, this shows that M is the direct sum of a filtration M' on V_1 and a filtration M'' on V_2. Also, in the case $K = \mathbf{F}_2$, too, the last property can be proved by using a finite extension of K to get a, b such that $a \neq b$. It is easy to see that M' (resp. M'') is the relative monodromy filtration of N_1 (resp. N_2) with respect to $W_\bullet V_1$ (resp. $W_\bullet V_2$). $\square$

Remark. If the characteristic of K is $p > 0$, 1.2.1.1 need not hold for tensor products and Hom. For example, let V_1 be of dimension p and with basis $e_1, \ldots, e_p$ such that $N_1(e_1) = 0$ and $N_1(e_j) = e_{j-1}$ for $2 \leq j \leq p$, let V_2 be of dimension 2 with basis e_1', e_2' such that $N_2(e_1') = 0$ and $N_2(e_2') = e_1'$, and let $W_0 V_j = V_j$ and $W_{-1} V_j = 0$ for $j = 1, 2$. Then for $M = M_\bullet V_1 \otimes M_\bullet V_2$, gr_p^M (resp. gr_{-p}^M) is of dimension 1 and generated by the class of $e_p \otimes e_2'$ (resp. $e_1 \otimes e_1'$). But $(N_1 \otimes 1 + 1 \otimes N_2)^p$ induces the zero map $\operatorname{gr}_p^M \to \operatorname{gr}_{-p}^M$ because $(N_1 \otimes 1 + 1 \otimes N_2)^p = N_1^p \otimes 1 + 1 \otimes N_2^p$ and $N_1^p(e_p) = 0$, $N_2^p(e_2') = 0$.

In §6, in the proof of Theorem 6.2.1, we will use the following fact concerning the relative monodromy filtration.

1.2.1.3. Let (V, W) and N be at the beginning of 1.2.1, and assume that the relative monodromy filtration $M = M(N, W)$ exists. Assume that $W_w = 0$ for $w \ll 0$. Then we have

$$\operatorname{Ker}(N) \cap W_w \subset M_w \quad \text{for all } w \in \mathbf{Z}.$$

Proof of 1.2.1.3. We prove $\operatorname{Ker}(N) \cap W_w \subset M_w + W_k$ ($w, k \in \mathbf{Z}$) by downward induction on k starting from the trivial case $k = w$. Assume that we have already shown $\operatorname{Ker}(N) \cap W_w \subset M_w + W_k$ with some $k \leq w$. Let $x \in \operatorname{Ker}(N) \cap W_w$, and write $x = y + z$ with $y \in M_w$ and $z \in W_k$. Since $N : \operatorname{gr}_j^M(\operatorname{gr}_k^W) \to \operatorname{gr}_{j-2}^M(\operatorname{gr}_k^W)$ are injective for all $j > k$, we have that the map $N : \operatorname{gr}_k^W / M_j \operatorname{gr}_k^W \to \operatorname{gr}_k^W / M_{j-2} \operatorname{gr}_k^W$ is injective if $j \geq k$. Since $N(z) = -N(y) \in M_{w-2}$ and $w \geq k$, this shows that the class of z in gr_k^W belongs to $M_w \operatorname{gr}_k^W$. Hence we can write $z = y' + z'$ with $y' \in M_w$ and $z' \in W_{k-1}$. Hence $x = (y + y') + z' \in M_w + W_{k-1}$. $\square$

1.2.2. Let V be an $\mathbf{R}$-vector space endowed with an increasing filtration W (by $\mathbf{R}$-linear subspaces). Let σ be a cone in some $\mathbf{R}$-vector space (this means that $0 \in \sigma$, σ is stable under the addition, and stable under the multiplication by elements of $\mathbf{R}_{\geq 0}$),

and let $\sigma_{\mathbf{R}}$ be the $\mathbf{R}$-linear span of σ. Assume that the cone σ is finitely generated. Assume that we are given an $\mathbf{R}$-linear map

$$h : \sigma_{\mathbf{R}} \to \operatorname{End}_{\mathbf{R}}(V, W)$$

whose image consists of mutually commuting nilpotent operators.

We say that the action of σ on V via h is *admissible (with respect to W)* if there exists a family $(M(\tau, W))_{\tau}$ of increasing filtrations $M(\tau, W)$ on V given for each face τ of σ satisfying the following conditions (1)–(4).

(1) $M(\sigma \cap (-\sigma), W) = W$.

(2) For any face τ of σ, any $N \in \sigma$ and any $w \in \mathbf{Z}$, we have $h(N)M(\tau, W)_w \subset M(\tau, W)_w$.

(3) For any face τ of σ, any $N \in \tau$ and any $w \in \mathbf{Z}$, we have

$$h(N)M(\tau, W)_w \subset M(\tau, W)_{w-2}.$$

(4) For any faces τ, τ' of σ such that τ' is the smallest face of σ containing τ and N, $M(\tau', W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(\tau, W)$.

(We remark that in [KKN08] 2.1.4, the above condition (2) was missed.)

We call the above filtration $M(\tau, W)$ the *relative monodromy filtration of τ with respect to W*. The condition (4) (by considering the case where τ and $\sigma \cap (-\sigma)$ are taken as τ' and τ in (4), respectively) implies that $M(\tau, W) = M(N, W)$ for any N in the interior of τ.

By 1.2.1.1 and 1.2.1.2, we have the following compatibility of this admissibility with various operators. For $j = 1, 2$, Let V_j be a vector space over $\mathbf{R}$ endowed with an increasing filtration $W_{\bullet}V_j$, and assume that we are given an $\mathbf{R}$-linear map $\sigma_{\mathbf{R}} \to \operatorname{End}_{\mathbf{R}}(V_j, W_{\bullet}V_j)$ whose image consists of mutually commuting nilpotent operators.

1.2.2.1. Assume that for $j = 1, 2$, the action of σ on V_j is admissible with respect to $W_{\bullet}V_j$. Then, the diagonal action of σ on $V_1 \oplus V_2$ is admissible with respect to the filtration $W_{\bullet}V_1 \oplus W_{\bullet}V_2$, and $M(\tau, W_{\bullet}V_1 \oplus W_{\bullet}V_2) = M(\tau, W_{\bullet}V_1) \oplus M(\tau, W_{\bullet}V_2)$ for any face τ of σ. The action of σ on $V_1 \otimes V_2$ ($N \in \sigma$ acts by sending $x \otimes y$ to $Nx \otimes y + x \otimes Ny$) is admissible with respect to the filtration $W_{\bullet}V_1 \otimes W_{\bullet}V_2$, and $M(\tau, W_{\bullet}V_1 \otimes W_{\bullet}V_2) = M(\tau, W_{\bullet}V_1) \otimes M(\tau, W_{\bullet}V_2)$. The action of σ on $H := \operatorname{Hom}_{\mathbf{R}}(V_1, V_2)$ ($N \in \sigma$ acts by $f \mapsto Nf - fN$) is admissible with respect to the filtration $W_{\bullet}H$ defined by $W_w H = \{f \in \operatorname{Hom}_{\mathbf{R}}(V_1, V_2) \mid f(W_k V_1) \subset W_{k+w}V_2 \text{ for all } k\}$, and $M(\tau, W_{\bullet}H)$ on H is given by $M(\tau, W_{\bullet}H)_w = \{f \in \operatorname{Hom}_{\mathbf{R}}(V_1, V_2) \mid f(M(\tau, W)_k V_1) \subset M(\tau, W)_{k+w}V_2 \text{ for all } k\}$.

1.2.2.2. Assume that the diagonal action of σ on $V_1 \oplus V_2$ is admissible with respect to $W_{\bullet}V_1 \oplus W_{\bullet}V_2$. Then the action of σ on V_j is admissible with respect to $W_{\bullet}V_j$, and $M(\tau, W_{\bullet}V_1 \oplus W_{\bullet}V_2) = M(\tau, W_{\bullet}V_1) \oplus M(\tau, W_{\bullet}V_2)$.

Lemma 1.2.3. *Let the notation be as in 1.2.2, let σ' be a finitely generated cone, and assume that we are given an $\mathbf{R}$-linear map $\sigma'_{\mathbf{R}} \to \sigma_{\mathbf{R}}$ which sends σ' in σ.*

(i) *Assume that the action of σ on V is admissible with respect to W. Then the action of σ' on V is also admissible with respect to W. For a face τ' of σ', $M(\tau', W) = M(\tau, W)$, where τ denotes the smallest face of σ which contains the image of τ'.*

(ii) *Assume that the action of σ' on V is admissible with respect to W and that the map $\sigma' \to \sigma$ is surjective. Then the action of σ on V is admissible with respect to W. For a face τ of σ, $M(\tau, W) = M(\tau', W)$, where τ' denotes the inverse image of τ in σ'.*

Proof. (i) This is essentially included in [KKN08] 2.1.5. The proof is easy.

(ii) Denote the given map $\sigma' \to \sigma$ by f. Let α and β be faces of σ, let $N \in \beta$, and assume that β is the smallest face of σ which contains α and N. It is sufficient to prove that $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(f^{-1}(\alpha), W)$. Let x be an element of the interior of $f^{-1}(\beta)$. Since β is the smallest face of σ which contains α and N, there are $y \in \beta$, $z \in \alpha$, $c \in \mathbf{R}_{>0}$ such that $f(x) + y = z + cN$. Take $y' \in f^{-1}(\beta)$ and $z' \in f^{-1}(\alpha)$ such that $f(y') = y$ and $f(z') = z$, and let $u = x + y'$. Since u is in the interior of $f^{-1}(\beta)$, $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(f(u))$ with respect to $M(f^{-1}(\alpha), W)$. Since $f(u) = f(z') + cN$, this shows that $M(f^{-1}(\beta), W)$ is the relative monodromy filtration of $h(N)$ with respect to $M(f^{-1}(\alpha), W)$. $\square$

1.2.4. Let S be an object of $\mathcal{B}(\log)$, and let the topological space $S^{\log}$ and the continuous proper map $\tau : S^{\log} \to S$ be as in [KN99], [KU09] 2.2. Let $L = (L, W)$ be a locally constant sheaf L of finite dimensional $\mathbf{R}$-vector spaces over $S^{\log}$ endowed with an increasing filtration W consisting of locally constant $\mathbf{R}$-subsheaves. We say that the local monodromy of L is *admissible* if the following two conditions (1) and (2) are satisfied for any $s \in S$ and any $t \in s^{\log} = \tau^{-1}(s) \subset S^{\log}$.

(1) For any $\gamma \in \pi_1(s^{\log})$, the action of γ on the stalk L_t is unipotent.

(2) Let $C(s) := \mathrm{Hom}((M_S/\mathcal{O}_S^{\times})_s, \mathbf{R}_{\geq 0}^{\mathrm{add}})$, where $\mathbf{R}_{\geq 0}^{\mathrm{add}}$ denotes $\mathbf{R}_{\geq 0}$ regarded as an additive monoid. Then the action of the cone $C(s)$ on L_t via $C(s)_{\mathbf{R}} = \mathbf{R} \otimes \pi_1(s^{\log}) \to \mathrm{End}_{\mathbf{R}}(L_t, W)$, defined by $\gamma \mapsto \log(\gamma)$ $(\gamma \in \pi_1(s^{\log}))$, is admissible with respect to W.

Lemma 1.2.5. *Admissibility of local monodromy is preserved by pulling back by a morphism $S' \to S$ in $\mathcal{B}(\log)$.*

Proof. This follows from 1.2.3 (i). $\square$

Remark 1.2.6. There are weaker versions of the notion of admissibility of a cone with respect to W.

For example, one version is

(1) The relative monodrmy filtration $M(N, W)$ exists for any $N \in \sigma$.

Another version is

(2) The relative monodromy filtration $M(N, W)$ exists for any $N \in \sigma$, and it depends only on the smallest face of σ which contains N.

In the situation with polarized graded quotients explained in 1.3.3 below, it is shown that these weaker admissibilities coincide with the admissibility considered in this subsection §1.2. See 1.3.2–1.3.4.

In the papers [KNU10a], [KNU10b], and [KNU10c], we adopted the admissibility in (2) above. But the difference with the admissibility in this subsection is not essential for this paper. See 2.2.5.

§1.3. Log mixed Hodge structures

1.3.1. Log mixed Hodge structure (LMH, for short) on S ([KKN08] 2.3, [KU09] 2.6).

Let S be an object in $\mathcal{B}(\log)$. Recall that we have a sheaf of commutative rings $\mathcal{O}_S^{\log}$ on $S^{\log}$ which is an algebra over the inverse image $\tau^{-1}(\mathcal{O}_S)$ of $\mathcal{O}_S$ ([KN99], [KU09] 2.2).

A *pre-log mixed Hodge structure (pre-LMH) on S* is a triple $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$, where

$H_{\mathbf{Z}}$ is a locally constant sheaf of finitely generated free $\mathbf{Z}$-modules on $S^{\log}$,

W is an increasing filtration on $H_{\mathbf{R}}$ by locally constant rational subsheaves, and

$H_{\mathcal{O}}$ is an $\mathcal{O}_S$-module on S which is locally free of finite rank, endowed with a decreasing filtration $(F^p H_{\mathcal{O}})_p$ and endowed with an isomorphism of $\mathcal{O}_S^{\log}$-modules $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ on $S^{\log}$,

satisfying the condition that $F^p H_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^p H_{\mathcal{O}}$ are locally free of finite rank for any $p \in \mathbf{Z}$.

Denote $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^p H_{\mathcal{O}}$ ($p \in \mathbf{Z}$) the filtration on $\mathcal{O}_S^{\log}$-module $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. The filtrations $F^p H_{\mathcal{O}}$ and F^p determine each other so that we can define a pre-LMH as a triple $(H_{\mathbf{Z}}, W, F)$, where F is a decreasing filtration on $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, satisfying the corresponding conditions. In the rest of this paper, we freely use both formulations.

Let $s \in S$ and let $t \in \tau^{-1}(s) \subset S^{\log}$. By a *specialization* at t, we mean a ring homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ such that the composition $\mathcal{O}_{S,s} \to \mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ is the evaluation at s. Since the evaluation at s is the unique $\mathbf{C}$-algebra homomorphism $\mathcal{O}_{S,s} \to \mathbf{C}$, any $\mathbf{C}$-algebra homomorphism $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ is a specialization at t.

Let s be an fs log point (i.e., an fs log analytic space whose underlying analytic space is a one-point set endowed with the ring $\mathbf{C}$).

A pre-LMH $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ on s is a *log mixed Hodge structure (LMH) on s*, if it satisfies the following conditions (1)–(3).

(1) The local monodromy of $(H_{\mathbf{R}}, W)$ is admissible in the sense of 1.2.4.

(2) Let $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$. Then

$$\nabla F^p \subset \omega_s^{1,\log} \otimes_{\mathcal{O}_s^{\log}} F^{p-1} \quad \text{for all } p.$$

(3) Let $t \in s^{\log}$. For a specialization $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$ at t, let $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ be the filtration on $\mathbf{C} \otimes_{\mathbf{Z}} H_{\mathbf{Z},t}$. If a is sufficiently twisted in the sense explained below, then, for any face τ of the cone $C(s)$ in 1.2.4, $(H_{\mathbf{Z},t}, M(\tau, W), F(a))$ is a mixed Hodge structure in the usual sense.

Here, fixing a finite family $(q_j)_{1 \le j \le n}$ of elements of $M_s \setminus \mathbf{C}^{\times}$ such that $(q_j \bmod \mathbf{C}^{\times})_j$ generates $M_s/\mathbf{C}^{\times}$ (note that $\mathbf{C}^{\times} = \mathcal{O}_s^{\times} \subset M_s$), we say a is sufficiently twisted if

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$\exp(a(\log(q_j)))$ are sufficiently near to 0. This notion "sufficiently twisted" is in fact independent of the choice of $(q_j)_j$.

Let S be an object in $\mathcal{B}(\log)$ again.

A pre-LMH on S is a *log mixed Hodge structure (LMH)* on S if its pull back to each fs log point $s \in S$ is an LMH on s ([KKN08] 2.3, [KU09] 2.6).

Direct sums, tensor products, symmetric powers, exterior powers, and inner-Hom of pre-LMH are defined in the evident ways. By using 1.2.2.1 and 1.2.2.2, we see that these operations preserve LMH.

For $w \in \mathbf{Z}$, we call a pre-LMH (resp. LMH) such that $W_w = H_{\mathbf{R}}$ and $W_{w-1} = 0$ simply a *pre-log Hodge structure (pre-LH)* (resp. *log Hodge structure (LH))* of weight w. In this case, we often omit the W.

1.3.2. Polarized log Hodge structure (PLH) on S ([KU99] §5, [KKN08] 2.5, [KU09] 2.4).

Let w be an integer.

A *pre-polarized log Hodge structure (pre-PLH) of weight w* is a triple $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$, where

$(H_{\mathbf{Z}}, H_{\mathcal{O}})$ is a pre-LH of weight w, and

$\langle\ ,\ \rangle : H_{\mathbf{R}} \times H_{\mathbf{R}} \to \mathbf{R}$ is a rational non-degenerate $(-1)^w$-symmetric bilinear form, satisfying $\langle F^p, F^q \rangle = 0$ when $p + q > w$.

A pre-PLH $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$ of weight w is a *polarized log Hodge structure (PLH) of weight w* if for any $s \in S$, the pull back of $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \langle\ ,\ \rangle)$ to the fs log point s has the property 1.3.1 (2) and the following property: In the notation t, a and $F(a)$ in 1.3.1 (3), $(H_{\mathbf{Z},t}, \langle\ ,\ \rangle, F(a))$ is a polarized Hodge structure of weight w in the usual sense for any sufficiently twisted specialization a (1.3.1).

A PLH of weight w is an LH of weight w. This is by results of [CK82] and [Scm73].

1.3.3. By results of Cattani-Kaplan, Schmid, and Kashiwara, under the existence of polarizations on graded quotients, the definition of LMH becomes simpler as follows.

Let s be an fs log point and let $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ be a pre-LMH on s. Assume that $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ satisfies the condition 1.3.1 (2). Assume moreover that, for each $w \in \mathbf{Z}$, a rational non-degenerate $(-1)^w$-symmetric bilinear form $\langle\ ,\ \rangle_w : H(\mathrm{gr}_w^W)_{\mathbf{R}} \times H(\mathrm{gr}_w^W)_{\mathbf{R}} \to \mathbf{R}$ is given and that the triple $(H(\mathrm{gr}_w^W)_{\mathbf{Z}}, H(\mathrm{gr}_w^W)_{\mathcal{O}}, \langle\ ,\ \rangle_w)$ is a PLH. Let $C(s)$ be the cone in 1.2.4. Fix a set of generators $\gamma_1, \dots, \gamma_n$ of $C(s)$. By [KU09] 2.3.3, (1) in 1.2.4 is satisfied for any $t \in s^{\log}$ and we can define $N_j := \log(\gamma_j) \in \mathrm{End}(H_{\mathbf{R},t}, W)$ for all j.

Proposition 1.3.4. *Let the assumption be as in 1.3.3. Then, the following are equivalent.*

(1) $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ *is an LMH.*

(2) *The local monodromy of $(H_{\mathbf{R}}, W)$ is admissible.*

(3) *The relative monodromy filtrations $M(N_j, W)$ exist for all j.*

Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are evident.

We prove $(2) \Rightarrow (1)$. Assume (2). Since each gr is a PLH, it is also an LH. Hence 1.3.1 (3) is satisfied for each gr. By [K86] 5.2.1, 1.3.1 (3) is satisfied, that is, (1) holds.

We prove (3) $\Rightarrow$ (2). Assume (3). Let F be the specialization (1.3.1) of the Hodge filtration with respect to some $a : \mathcal{O}^{\log}_{s,t} \to \mathbf{C}$ at t. Then, $(H_{\mathbf{C},t}; W_{\mathbf{C},t}; F, \bar{F}; N_1, \dots, N_n)$ is a pre-IMHM in the sense of [K86] 4.2. By 4.4.1 of [K86], it is an IMHM. In particular, for any subset $J \subset \{1, \dots, n\}$, $M(\sum_{j \in J} N_j, W)$ exists. Let $M(\tau, W) := M(\sum_{j \in J} N_j, W)$, where τ is the face of $C(s)$ generated by γ_j for j running in J. Then, this family satisfies the condition 1.2.2 (4) by 5.2.5 of [K86]. Hence, the local monodromy of $(H_{\mathbf{R}}, W)$ is admissible. $\square$

1.3.5. Let S be an object of $\mathcal{B}(\log)$. An *LMH with polarized graded quotients on S* is a quadruple $H = (H_{\mathbf{Z}}, W, H_{\mathcal{O}}, (\langle \, , \, \rangle_w)_w)$ such that $(H_{\mathbf{Z}}, W, H_{\mathcal{O}})$ is an LMH on S and $(H(\mathrm{gr}^W_w)_{\mathbf{Z}}, H(\mathrm{gr}^W_w)_{\mathcal{O}}, \langle \, , \, \rangle_w)$ is a PLH of weight w for any w.

Example 1.3.6. Let $S = \Delta^{n+r}$, where Δ is the unit disk. Let $S^* = (\Delta^*)^n \times \Delta^r$, where $\Delta^* = \Delta - \{0\}$. Let X be an analytic space and $X \to S$ be a projective morphism which is smooth over S^*. Let $X^* = X \times_S S^*$. Let E be a divisor on X such that $E \cap X^*$ is relatively normal crossing over S^*. Let $m \in \mathbf{Z}$. Let $h : X^* \smallsetminus E \to S^*$. Assume that the local monodromy of $H_{\mathbf{R}} := R^m h_*(\mathbf{R})$ along E is unipotent. Then, the natural variation of mixed Hodge structure (VMHS) with polarized graded quotients on S^* underlain by $H_{\mathbf{R}}$ canonically extends to an LMH with polarized graded quotients. This was proved by Steenbrink-Zucker [SZ85], Kashiwara [K86], Saito [Sa90], and Fujino [F04]. See the explanation of the last part of [KNU08] 12.10.

§2. MODULI SPACES OF LOG MIXED HODGE STRUCTURES WITH POLARIZED GRADED QUOTIENTS

§2.1. CLASSIFYING SPACE D

2.1.1. In this §2, we fix $\Lambda = (H_0, W, (\langle \, , \, \rangle_w)_w, (h^{p,q})_{p,q})$, where

H_0 is a finitely generated free $\mathbf{Z}$-module,

W is an increasing rational filtration on $H_{0,\mathbf{R}} = \mathbf{R} \otimes H_0$,

$\langle \, , \, \rangle_w$ for each $w \in \mathbf{Z}$ is a rational non-degenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ which is symmetric if w is even and is anti-symmetric if w is odd,

$h^{p,q}$ are non-negative integers given for each $(p,q) \in \mathbf{Z}^2$ satisfying the following conditions (1)–(3).

(1) $\sum_{p,q} h^{p,q} = \mathrm{rank}_{\mathbf{Z}}(H_0)$,

(2) $\sum_{p+q=w} h^{p,q} = \dim_{\mathbf{R}}(\mathrm{gr}^W_w)$ for any $w \in \mathbf{Z}$,

(3) $h^{p,q} = h^{q,p}$ for any (p,q).

2.1.2. Let D be the classifying space of gradely polarized mixed Hodge structures in [U84] associated with the data fixed in 2.1.1. As a set, D consists of all increasing filtrations F on $H_{0,\mathbf{C}} = \mathbf{C} \otimes H_0$ such that $(H_0, W, (\langle \, , \, \rangle_w)_w, F)$ is a gradely polarized mixed Hodge structures with $\dim_{\mathbf{C}} F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q}) = h^{p,q}$ for all p, q. Let $\check{D}$ be its compact dual. Then D and $\check{D}$ are complex analytic manifolds, and D is open in $\check{D}$. Cf. [KNU09] 1.5.

2.1.3. We recall the notation used in this series of papers.

As in [KNU09] 1.6, for $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let G_A be the group of the A-automorphisms of $(H_{0,A}, W)$ whose gr_w^W are compatible with $\langle \ , \ \rangle_w$ for all w. Then $G_{\mathbf{C}}$ acts on $\check{D}$ and $G_{\mathbf{R}}$ preserves D.

As in [KNU09] 1.7, for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let $\mathfrak{g}_A$ be the associated Lie algebra of G_A, which is identified with the set of all A-endomorphisms N of $(H_{0,A}, W)$ whose gr_w^W satisfies $\langle \mathrm{gr}_w^W(N)(x), y \rangle_w + \langle x, \mathrm{gr}_w^W(N)(y) \rangle_w = 0$ for all w, x, y.

As in [KNU09] 1.6 and 1.7, let $G_{A,u} = \{\gamma \in G_A \mid \mathrm{gr}^W(g) = 1\}$, $\mathfrak{g}_{A,u} = \{N \in \mathfrak{g}_A \mid \mathrm{gr}^W(N) = 0\}$. Then $G_A/G_{A,u} = G_A(\mathrm{gr}^W) := \prod_w G_A(\mathrm{gr}_w^W)$ and $\mathfrak{g}_A/\mathfrak{g}_{A,u} = \mathfrak{g}_A(\mathrm{gr}^W) := \prod_w \mathfrak{g}_A(\mathrm{gr}_w^W)$, where $G_A(\mathrm{gr}^W)$ (resp. $\mathfrak{g}_A(\mathrm{gr}_w^W)$) is "the G_A (resp. $\mathfrak{g}_A$) for gr_w^W".

§2.2. THE SETS D_Σ AND $D_\Sigma^\sharp$

Here we define the sets D_Σ and $D_\Sigma^\sharp$ in our mixed Hodge case. See [KU09] for the pure case. Everything is parallel to and generalize the pure case, except that we use weak fans instead of fans.

2.2.1. *Nilpotent cone.* A subset σ of $\mathfrak{g}_{\mathbf{R}}$ is called a *nilpotent cone* if the following conditions (1) and (2) are satisfied.

(1) All elements of σ are nilpotent and $NN' = N'N$ for any $N, N' \in \sigma$, as linear maps $H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$.

(2) σ is finitely generated. That is, there is a finite family $(N_j)_j$ of elements of σ such that $\sigma = \sum_j \mathbf{R}_{\geq 0} N_j$.

A nilpotent cone is said to be *rational* if we can take $N_j \in \mathfrak{g}_{\mathbf{Q}}$ in the above condition (2).

For a nilpotent cone σ, let $\sigma_{\mathbf{R}}$ be the $\mathbf{R}$-linear span of σ in $\mathfrak{g}_{\mathbf{R}}$, and let $\sigma_{\mathbf{C}}$ be the $\mathbf{C}$-linear span of σ in $\mathfrak{g}_{\mathbf{C}}$.

We say that σ is *admissible* if the action of σ on $H_{0,\mathbf{R}}$ is admissible with respect to W in the sense of 1.2.2.

2.2.2. Let $N_1, \ldots, N_n \in \mathfrak{g}_{\mathbf{R}}$ be mutually commuting nilpotent elements and let $F \in \check{D}$. We say $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit* if the following (1)–(3) are satisfied.

(1) The action of the cone $\sum_{j=1}^n \mathbf{R}_{\geq 0} N_j$ on $H_{0,\mathbf{R}}$ is admissible (1.2.2) with respect to W.

(2) $N_j F^p \subset F^{p-1}$ for $1 \leq j \leq n$ and for any $p \in \mathbf{Z}$.

(3) If $y_j \in \mathbf{R}_{>0}$ and y_j are sufficiently large, we have $\exp(\sum_{j=1}^n i y_j N_j) F \in D$.

Note that these conditions depend only on the cone $\sigma := \sum_{j=1}^n \mathbf{R}_{\geq 0} N_j$ and F. We say also that the pair (σ, F) generates a nilpotent orbit if these conditions are satisfied.

Let σ be a nilpotent cone. A subset Z of $\check{D}$ is called a σ-*nilpotent orbit* (resp. σ-*nilpotent i-orbit*) if the following conditions (4) and (5) are satisfied for some $F \in Z$.

(4) $Z = \exp(\sigma_{\mathbf{C}}) F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}}) F$).

(5) (σ, F) generates a nilpotent orbit.

Such a pair (σ, Z) is called a *nilpotent orbit* (resp. *nilpotent i-orbit*).
Note that if (4) and (5) are satisfied for one $F \in Z$, they are satisfied for all $F \in Z$.

Remark 1. The notion of nilpotent orbit is essentially the same as the notion of LMH on an fs log point. Cf. [KU09] 2.5 for the pure case.

Remark 2. The notion of nilpotent orbit is closely related to the notion infinitesimal mixed Hodge module (IMHM) by Kashiwara ([K86]). A precise relation is as follows. (The statement in [KNU10b] 2.1.3 was not accurate.)

Let H_0 and W be as in 2.1.1. Let F be a decreasing filtration on $H_{0,\mathbf{C}}$. Let $\bar{F}$ be its complex conjugation. Let $N_1, \ldots, N_n$ be a mutually commuting set of nilpotent endomorphisms of $H_{0,\mathbf{Q}}$. Then, $(H_{0,\mathbf{C}}; W_{\mathbf{C}}; F, \bar{F}; N_1, \ldots, N_n)$ is an IMHM if and only if there exists a Λ extending (H_0, W) such that $N_1, \ldots, N_n \in \mathfrak{g}_{\mathbf{R}}$ and $F \in \check{D}$, and that $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit. This was seen in the course of the proof of 1.3.4.

2.2.3. Weak fan in $\mathfrak{g}_{\mathbf{Q}}$ ([KNU10c]).

A *weak fan* Σ in $\mathfrak{g}_{\mathbf{Q}}$ is a non-empty set of sharp rational nilpotent cones in $\mathfrak{g}_{\mathbf{R}}$ satisfying the following conditions (1) and (2). Here a nilpotent cone σ is said to be *sharp* if $\sigma \cap (-\sigma) = \{0\}$.

(1) If $\sigma \in \Sigma$, all faces of σ belong to Σ.

(2) Let $\sigma, \sigma' \in \Sigma$, and assume that σ and σ' have a common interior point. Assume that there is an $F \in \check{D}$ such that (σ, F) and (σ', F) generate nilpotent orbits. Then $\sigma = \sigma'$.

A *fan* Σ in $\mathfrak{g}_{\mathbf{Q}}$ is defined similarly by replacing the condition (2) in the above with the condition "$\sigma \cap \sigma'$ is a face of σ for any $\sigma, \sigma' \in \Sigma$".

For simplicity, we call a weak fan (resp. a fan) in $\mathfrak{g}_{\mathbf{Q}}$ a weak fan (resp. a fan).

An example of a fan is

$$\Xi = \{\mathbf{R}_{\geq 0} N \mid N \in \mathfrak{g}_{\mathbf{Q}}, \ N \text{ is nilpotent}\}.$$

The next lemma, which was announced in [KNU10c] 1.7, implies that a fan in $\mathfrak{g}_{\mathbf{Q}}$ is a weak fan.

Lemma 2.2.4. *In the definition 2.2.3, under the condition* (1), *the condition* (2) *is equivalent to* (2)$'$ *below and also is equivalent to* (2)$''$ *below.*

(2)$'$ *If $\sigma, \sigma' \in \Sigma$ and if there is an $F \in \check{D}$ such that (σ, F), (σ', F), and $(\sigma \cap \sigma', F)$ generate nilpotent orbits, then $\sigma \cap \sigma'$ is a face of σ.*

(2)$''$ *Let τ be a nilpotent cone. Let (τ, F) $(F \in \check{D})$ generate a nilpotent orbit. If the set $A := \{\sigma \in \Sigma \mid \tau \subset \sigma, (\sigma, F)$ generates a nilpotent orbit$\}$ is not empty, A has a smallest element, and it is a face of any element of A.*

Proof. (2)$'' \Rightarrow$ (2)$'$. Let σ, σ' and F be as in (2)$'$. Let $\tau = \sigma \cap \sigma'$. Then, σ and σ' belong to A. Let σ_0 be the smallest element of A. Then, σ_0 is a face of σ and also is a face

of σ', and hence $\sigma_0 \subset \sigma \cap \sigma' = \tau$. On the other hand, since $\sigma_0 \in A$, we have $\tau \subset \sigma_0$. Thus, $\sigma_0 = \tau$. Hence, $\sigma \cap \sigma' = \tau = \sigma_0$ is a face of σ.

$(2)' \Rightarrow (2)$. Let σ, σ' and F be as in (2). Since $\sigma \cap \sigma'$ contains an interior point of σ, $(\sigma \cap \sigma', F)$ generates a nilpotent orbit. Hence, $(2)'$ implies that $\sigma \cap \sigma'$ is a face of σ, but again, since $\sigma \cap \sigma'$ contains an interior point of σ, we have $\sigma \cap \sigma' = \sigma$. By symmetry, $\sigma \cap \sigma' = \sigma'$. Therefore, $\sigma = \sigma'$.

$(2) \Rightarrow (2)''$. Let (τ, F) be as in $(2)''$. For an element $\sigma \in A$, we write as σ_0 the face of σ spanned by τ. Then, (σ_0, F) generates a nilpotent orbit because both (τ, F) and (σ, F) generate nilpotent orbits. Thus, $\sigma_0 \in A$. It is enough to show $\sigma_0 = \sigma_0'$ for any $\sigma, \sigma' \in A$, because, then, this common σ_0 is the smallest element of A as soon as A is not empty, and it is indeed a face of any $\sigma \in A$. But, since an interior point of τ is a common interior point of σ_0 and σ_0', we have certainly $\sigma_0 = \sigma_0'$ by (2). $\square$

2.2.5. Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Let D_Σ (resp. $D_\Sigma^\sharp$) be the set of all nilpotent orbits (resp. nilpotent i-orbits) (σ, Z) with $\sigma \in \Sigma$.

We have embeddings
$$D \subset D_\Sigma, \quad F \mapsto (\{0\}, \{F\}),$$
$$D \subset D_\Sigma^\sharp, \quad F \mapsto (\{0\}, \{F\}).$$

We have a canonical surjection

$$D_\Sigma^\sharp \to D_\Sigma, \quad (\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z),$$

which is compatible with the above embeddings of D.

Remark 1. If $'\Sigma \subset \Sigma$ denotes the subset of Σ consisting of all admissible $\sigma \in \Sigma$, and if $''\Sigma \subset {}'\Sigma$ denotes the subset of Σ consisting of all $\sigma \in \Sigma$ such that a σ-nilpotent orbit exists, then $'\Sigma$ and $''\Sigma$ are weak fans, and we have $D_\Sigma = D_{'\Sigma} = D_{''\Sigma}$, $D_\Sigma^\sharp = D_{'\Sigma}^\sharp = D_{''\Sigma}^\sharp$.

Remark 2. In our previous papers [KNU10b] and [KNU10c], we adopted the following formulations (1)–(3) which are slightly different from those in the present paper. In those papers:

(1) Nilpotent cone was assumed to be sharp.

(2) In the definition of nilpotent orbit, we adopted the admissibility condition 1.2.6 (2) on the cone instead of the condition 2.2.2 (1).

For sharp cones, the definition of nilpotent orbit in [KNU10b] and [KNU10c] is equivalent to that in this paper. Cf. 1.2.6, 1.3.4.

(3) In the definition of weak fan (resp. fan), in [KNU10c] (resp. [KNU10b] and [KNU10c]), we put the condition that any cone in a weak fan (resp. fan) should satisfy the admissibility condition 1.2.6 (2). But we do not do it in the present paper, for we prefer a simpler definition here.

These changes of the formulations do not make any difference for the space of nilpotent orbits D_Σ (see Remark 1 above) and for the moduli spaces $\Gamma \backslash D_\Sigma$ of LMH which appear in §2.4 below.

In the pure case, the definitions of nilpotent cone, nilpotent orbit, and fan in this paper coincide with those in [KU09].

2.2.6. Compatibility of a weak fan and a subgroup of $G_{\mathbf{Z}}$.

Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$.
We say that Σ and Γ are *compatible* if the following condition (1) is satisfied.

(1) If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\mathrm{Ad}(\gamma)\sigma \in \Sigma$.

If Σ and Γ are compatible, Γ acts on D_Σ and also on $D_\Sigma^\sharp$ by $(\sigma, Z) \mapsto (\mathrm{Ad}(\gamma)\sigma, \gamma Z)$ $(\gamma \in \Gamma)$.

We say that Σ and Γ are *strongly compatible* if they are compatible and furthermore the following condition (2) is satisfied.

(2) If $\sigma \in \Sigma$, any element of σ is a sum of elements of the form aN, where $a \in \mathbf{R}_{\geq 0}$ and $N \in \sigma$ satisfying $\exp(N) \in \Gamma$.

2.2.7. For a sharp rational nilpotent cone σ, let $\mathrm{face}(\sigma)$ be the set of all faces of σ. It is a fan in $\mathfrak{g}_{\mathbf{Q}}$. Let

$$D_\sigma := D_{\mathrm{face}(\sigma)}, \quad D_\sigma^\sharp := D_{\mathrm{face}(\sigma)}^\sharp.$$

Let

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma) \subset \Gamma(\sigma)^{\mathrm{gp}} = \Gamma \cap \exp(\sigma_{\mathbf{R}}) = \{ab^{-1} \mid a, b \in \Gamma(\sigma)\} \subset G_{\mathbf{Z}}.$$

Then $\Gamma(\sigma)$ is a sharp torsion free fs monoid and $\Gamma(\sigma)^{\mathrm{gp}}$ is a finitely generated free abelian group.

Let Σ and Γ be as in 2.2.6. Assume that they are strongly compatible. Then, for any $\sigma \in \Sigma$, the fan $\mathrm{face}(\sigma)$ and the group $\Gamma(\sigma)^{\mathrm{gp}}$ are strongly compatible.

2.2.8. We will use the following fact in the proof of Theorem 6.2.1.

Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be an admissible nilpotent cone. Then the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}$ is admissible.

Proof. By 1.2.2.1, the adjoint action of σ on $A := \mathrm{Hom}_{\mathbf{R}}(H_{0,\mathbf{R}}, H_{0,\mathbf{R}})$ is admissible. For a face τ of σ, let $M(\tau)$ be the relative monodromy filtration of τ on A for this admissible action. The adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_w^W)$ for $w < 0$ is admissible with the relative monodromy filtrations induced by $M(\tau)$ because $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_w^W) = A(\mathrm{gr}_w^W)$. The adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_0^W)$ is admissible with the relative monodromy filtrations induced by $M(\tau)$ by 1.2.2.2 because $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_0^W)$ is a direct summand of $A(\mathrm{gr}_0^W)$ for the adjoint action of σ. It follows that the adjoint action of σ on $\mathfrak{g}_{\mathbf{R}}$ is admissible with the relative monodromy filtrations induced by $M(\tau)$. $\square$

§2.3. THE SETS E_σ AND $E_\sigma^\sharp$

Here we define the sets E_σ and $E_\sigma^\sharp$.

2.3.1. *Toric varieties.*

Assume that we are given Σ and Γ as in 2.2.6. We assume that they are strongly compatible.

Fix $\sigma \in \Sigma$ in the following. Let $P(\sigma) = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$. Define

$$\mathrm{toric}_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{C}^{\mathrm{mult}}) \supset |\mathrm{toric}|_\sigma := \mathrm{Hom}(P(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0}).$$

Here $\mathbf{C}^{\mathrm{mult}}$ (resp. $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$) denotes the set $\mathbf{C}$ (resp. $\mathbf{R}_{\geq 0}$) regarded as a multiplicative monoid. Let

$$\mathrm{torus}_\sigma := \Gamma(\sigma)^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{C}^\times \supset |\mathrm{torus}|_\sigma := \Gamma(\sigma)^{\mathrm{gp}} \otimes_{\mathbf{Z}} \mathbf{R}_{>0}.$$

We regard torus_σ as an open set of toric_σ and $|\mathrm{torus}|_\sigma$ as an open set of $|\mathrm{toric}|_\sigma$ via the embeddings

$$\mathrm{torus}_\sigma = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{C}^\times) \subset \mathrm{toric}_\sigma, \quad |\mathrm{torus}|_\sigma = \mathrm{Hom}(P(\sigma)^{\mathrm{gp}}, \mathbf{R}_{>0}) \subset |\mathrm{toric}|_\sigma.$$

We have a natural action of torus_σ on toric_σ, and a natural action of $|\mathrm{torus}|_\sigma$ on $|\mathrm{toric}|_\sigma$. We have an exact sequence

$$0 \to \Gamma(\sigma)^{\mathrm{gp}} \xrightarrow{\log} \sigma_{\mathbf{C}} \xrightarrow{\mathbf{e}} \mathrm{torus}_\sigma \to 0,$$

where

$$\mathbf{e}(\log(\gamma) \otimes z) = \gamma \otimes e^{2\pi i z} \text{ for } \gamma \in \Gamma(\sigma), z \in \mathbf{C}.$$

For a face τ of σ, the surjective homomorphism $P(\sigma) \to P(\tau)$ induces an embedding $\mathrm{toric}_\tau \to \mathrm{toric}_\sigma$. Let $0_\tau \in \mathrm{toric}_\tau \subset \mathrm{toric}_\sigma$ be the point corresponding to the homomorphism $P(\tau) \to \mathbf{C}^{\mathrm{mult}}$ which sends $1 \in P(\tau)$ to 1 and all the other elements of $P(\tau)$ to 0. (This point $0_\tau \in \mathrm{toric}_\sigma$ is denoted by 1_τ in [KU09], but we change the notation.)

Any element q of toric_σ (resp. $|\mathrm{toric}|_\sigma$) is written in the form $q = \mathbf{e}(a) \cdot 0_\tau$ (resp. $q = \mathbf{e}(ib) \cdot 0_\tau$) for $a \in \sigma_{\mathbf{C}}$ (resp. $b \in \sigma_{\mathbf{R}}$) and for a face τ of σ. The face τ is determined by q, and a modulo $\tau_{\mathbf{C}} + \log(\Gamma(\sigma)^{\mathrm{gp}})$ (resp. b modulo $\tau_{\mathbf{R}}$) is determined by q.

2.3.2. Let

$$\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}.$$

We endow $\check{E}_\sigma$ with the inverse image of the log structure of toric_σ.

We define a canonical pre-LMH on $\check{E}_\sigma$. See [KU09] for the pure case. Let $(H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$ be the locally constant sheaf of $\mathbf{Z}$-modules on $\mathrm{toric}^{\mathrm{log}}_\sigma$ endowed with a weight filtration and polarizations which is characterized by the property that the stalk at the unit point $1 \in \mathrm{torus}_\sigma \subset \mathrm{toric}^{\mathrm{log}}_\sigma$ is $(H_0, W, (\langle\ ,\ \rangle_w)_w)$ and the action of $\pi_1(\mathrm{toric}^{\mathrm{log}}_\sigma) = \Gamma(\sigma)^{\mathrm{gp}}$ is given by $\Gamma(\sigma)^{\mathrm{gp}} \subset G_{\mathbf{Z}}$. We consider its pull-back to $\check{E}_\sigma$ still denoted by the same symbol $(H_{\sigma,\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w)$. Similarly as in the pure case, [KU09] 2.3.7 gives a canonical isomorphism $\mathcal{O}^{\mathrm{log}}_{\check{E}_\sigma} \otimes H_{\sigma,\mathbf{Z}} = \mathcal{O}^{\mathrm{log}}_{\check{E}_\sigma} \otimes H_0$. The $\mathcal{O}_{\check{D}} \otimes H_0$ has the universal Hodge filtration, which induces by pulling back a filtration on $\mathcal{O}^{\mathrm{log}}_{\check{E}_\sigma} \otimes H_0$, and by τ_* a filtration on $H_{\sigma,\mathcal{O}} := \tau_*(\mathcal{O}^{\mathrm{log}}_{\check{E}_\sigma} \otimes H_0)$. Then, the triple $H_\sigma := (H_{\sigma,\mathbf{Z}}, W, H_{\sigma,\mathcal{O}})$ is a pre-LMH, which we call a canonical pre-LMH on $\check{E}_\sigma$. Note that each graded piece of this pre-LMH together with $\langle\ ,\ \rangle_w$ is regarded as a pre-PLH of weight w.

2.3.3. We define the subsets $\tilde{E}_\sigma$, E_σ, and $E^\sharp_\sigma$ of $\check{E}_\sigma$. Let H_σ be the canonical pre-LMH on $\check{E}_\sigma$ defined in 2.3.2. Then, the subset $\tilde{E}_\sigma$ (resp. E_σ) is defined as the set of

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points $x \in \check{E}_\sigma$ such that the pull-back of H_σ on x satisfies the Griffiths transversality (resp. is an LMH with polarized graded quotients). Thus, E_σ is contained in $\tilde{E}_\sigma$.

Consider the subset $|\mathrm{toric}|_\sigma$ of toric_σ, let $\check{E}_\sigma^\sharp = |\mathrm{toric}|_\sigma \times \check{D}$, and let $E_\sigma^\sharp := E_\sigma \cap \check{E}_\sigma^\sharp$. E_σ and $E_\sigma^\sharp$ are characterized as follows (cf. [KNU10b] 2.2.3).

For $q \in \mathrm{toric}_\sigma$ (resp. $q \in |\mathrm{toric}|_\sigma$), write $q = \mathbf{e}(a) \cdot 0_\tau$ (resp. $q = \mathbf{e}(ib) \cdot 0_\tau$), where $a \in \sigma_{\mathbf{C}}$ ($b \in \sigma_{\mathbf{R}}$), τ is the face of σ and 0_τ is as in 2.3.1. Then, a point (q, F) of $\check{E}_\sigma = \mathrm{toric}_\sigma \times \check{D}$ (resp. $\check{E}^\sharp = |\mathrm{toric}|_\sigma \times \check{D}$) belongs to E_σ (resp. $E_\sigma^\sharp$) if and only if $(\tau, \exp(a)F)$ (resp. $(\tau, \exp(ib)F)$) generates a nilpotent orbit.

The proofs of these facts are similar to the pure case (cf. [KU09] 3.3.7).

2.3.4. In the above notation, we have canonical projections $\varphi : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ and $\varphi^\sharp : E_\sigma^\sharp \to D_\sigma^\sharp$ by

$$\varphi(q, F) = (\tau, \exp(\tau_{\mathbf{C}}) \exp(a)F) \bmod \Gamma(\sigma)^{\mathrm{gp}},$$

$$\varphi^\sharp(q, F) = (\tau, \exp(i\tau_{\mathbf{R}}) \exp(ib)F).$$

§2.4. Topology, complex structures, and log structures

Let Σ and Γ be as in 2.2.6. We assume that they are strongly compatible.

We define a structure of log local ringed space over $\mathbf{C}$ on $\Gamma \backslash D_\Sigma$ and a topology on $D_\Sigma^\sharp$.

2.4.1. As in 2.4.1 of [KNU10b], for each $\sigma \in \Sigma$, we endow the subsets E_σ and $\tilde{E}_\sigma$ of $\check{E}_\sigma$ in 2.3.3 with the following structures of log local ringed spaces over $\mathbf{C}$. The topology is the strong topology in $\check{E}_\sigma$. The sheaf $\mathcal{O}$ of rings and the log structure M are the inverse images of $\mathcal{O}$ and M of $\check{E}_\sigma$, respectively.

We endow a topology on $E_\sigma^\sharp$ as a subspace of E_σ.

2.4.2. We endow $\Gamma \backslash D_\Sigma$ with the strongest topology for which the maps $\pi_\sigma : E_\sigma \xrightarrow{\varphi} \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma \to \Gamma \backslash D_\Sigma$ are continuous for all $\sigma \in \Sigma$. Here φ is as in 2.3.4. We endow $\Gamma \backslash D_\Sigma$ with the following sheaf of rings $\mathcal{O}_{\Gamma \backslash D_\Sigma}$ over $\mathbf{C}$ and the following log structure $M_{\Gamma \backslash D_\Sigma}$. For any open set U of $\Gamma \backslash D_\Sigma$ and for any $\sigma \in \Sigma$, let $U_\sigma := \pi_\sigma^{-1}(U)$ and define $\mathcal{O}_{\Gamma \backslash D_\Sigma}(U)$ (resp. $M_{\Gamma \backslash D_\Sigma}(U)$) := {map $f : U \to \mathbf{C} \mid f \circ \pi_\sigma \in \mathcal{O}_{E_\sigma}(U_\sigma)$ (resp. $\in M_{E_\sigma}(U_\sigma)$) ($\forall \sigma \in \Sigma$)}. Here we regard $M_{E_\sigma}(U_\sigma)$ as a subset of $\mathcal{O}_{E_\sigma}(U_\sigma)$ via the structural map $M_{E_\sigma}(U_\sigma) \to \mathcal{O}_{E_\sigma}(U_\sigma)$, which is injective.

2.4.3. We introduce on $D_\Sigma^\sharp$ the strongest topology for which the maps $E_\sigma^\sharp \xrightarrow{\varphi^\sharp} D_\sigma^\sharp \to D_\Sigma^\sharp$ ($\sigma \in \Sigma$) are continuous. Here $\varphi^\sharp$ is as in 2.3.4. Note that the surjection $D_\Sigma^\sharp \to \Gamma \backslash D_\Sigma$ (cf. 2.2.5) becomes continuous.

§2.5. Properties of $\Gamma \backslash D_\Sigma$

In this §2.5, let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$ and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ (2.2.6).

The following Theorem 2.5.1–Theorem 2.5.6 are the mixed Hodge theoretic versions of 4.1.1 Theorem A (i)–(vi) of [KU09].

Theorem 2.5.1. *For $\sigma \in \Sigma$, E_σ is open in $\tilde{E}_\sigma$ in the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$. Both $\tilde{E}_\sigma$ and E_σ are log manifolds.*

We say Γ is *neat* if for any $\gamma \in \Gamma$, the subgroup of $\mathbf{C}^\times$ generated by all eigen values of the action of γ on $H_{0,\mathbf{C}}$ is torsion free. It is known that there is a neat subgroup of $G_\mathbf{Z}$ of finite index.

Theorem 2.5.2. *If Γ is neat, then $\Gamma \backslash D_\Sigma$ is a log manifold.*

Theorem 2.5.3. *Let $\sigma \in \Sigma$ and define the action of $\sigma_\mathbf{C}$ on E_σ over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by*

$$a \cdot (q, F) := (\mathbf{e}(a) \cdot q, \exp(-a)F) \quad (a \in \sigma_\mathbf{C}, \ (q, F) \in E_\sigma),$$

where $\mathbf{e}(a) \in \mathrm{torus}_\sigma$ is as in 2.3.1 and $\mathbf{e}(a) \cdot q$ is defined by the natural action of torus_σ on toric_σ. Then, $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a $\sigma_\mathbf{C}$-torsor in the category of log manifolds. That is, locally on the base $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, E_σ is isomorphic as a log manifold to the product of $\sigma_\mathbf{C}$ and the base endowed with the evident action of $\sigma_\mathbf{C}$.

Theorem 2.5.4. *If Γ is neat, then, for any $\sigma \in \Sigma$, the map*

$$\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma \to \Gamma \backslash D_\Sigma$$

is locally an isomorphism of log manifolds.

Theorem 2.5.5. *The topological space $\Gamma \backslash D_\Sigma$ is Hausdorff.*

Theorem 2.5.6. *If Γ is neat, then there is a homeomorphism of topological spaces*

$$(\Gamma \backslash D_\Sigma)^{\log} \simeq \Gamma \backslash D_\Sigma^\sharp,$$

which is compatible with $\tau : (\Gamma \backslash D_\Sigma)^{\log} \to \Gamma \backslash D_\Sigma$ and the projection $\Gamma \backslash D_\Sigma^\sharp \to \Gamma \backslash D_\Sigma$ induced by $D_\Sigma^\sharp \to D_\Sigma$ in 2.2.5.

The proofs of these theorems will be given in §4–§5.

§2.6. Moduli

We define the moduli functor of LMH with polarized graded quotients.

2.6.1. Fix $\Phi = (\Lambda, \Sigma, \Gamma)$, where Λ is as in 2.1.1 and Σ and Γ are as in 2.5.
Let S be an object of $\mathcal{B}(\log)$. Recall that an LMH with polarized graded quotients on S is a quadruple $H = (H_\mathbf{Z}, W, H_\mathcal{O}, (\langle\ ,\ \rangle_w)_w)$ such that $(H_\mathbf{Z}, W, H_\mathcal{O})$ is an LMH on S and $(H(\mathrm{gr}_w^W)_\mathbf{Z}, H(\mathrm{gr}_w^W)_\mathcal{O}, \langle\ ,\ \rangle_w)$ is a PLH of weight w for any w.

2.6.2. Let S be an object of $\mathcal{B}(\log)$. By an *LMH with polarized graded quotients of type Φ on S*, we mean an LMH with polarized graded quotients $H = (H_\mathbf{Z}, W, H_\mathcal{O}, (\langle\ ,\ \rangle_k)_k)$ endowed with a global section μ of the sheaf $\Gamma \backslash \mathrm{Isom}((H_\mathbf{Z}, W, (\langle\ ,\ \rangle_k)_k), (H_0, W, (\langle\ ,\ \rangle_k)_k))$ on $S^{\log}$ (called a Γ-*level structure*) which satisfies the following conditions (1) and (2).

(1) $\mathrm{rank}_\mathbf{Z}(H_\mathbf{Z}) = \sum_{p,q} h^{p,q}$, $\quad \mathrm{rank}_{\mathcal{O}_S}(F^p H_\mathcal{O}(\mathrm{gr}_{p+q}^W))/(F^{p+1} H_\mathcal{O}(\mathrm{gr}_{p+q}^W)) = h^{p,q}$ $\quad$ for all p, q.

(2) For any $s \in S$ and $t \in S^{\log}$ lying over s, if $\tilde{\mu}_t : (H_{\mathbf{Z},t}, W, (\langle \ , \ \rangle_k)_k) \xrightarrow{\sim} (H_0, W, (\langle \ , \ \rangle_k)_k)$ is a representative of the germ of μ at t, then there exists $\sigma \in \Sigma$ such that the image of the composite map

$$\mathrm{Hom}(M_{S,s}/\mathcal{O}_{S,s}^{\times}, \mathbf{N}) \hookrightarrow \pi_1(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Z},t}, W, (\langle \ , \ \rangle_k)_k) \xrightarrow{\text{by } \tilde{\mu}_t} \mathrm{Aut}(H_0, W, (\langle \ , \ \rangle_k)_k)$$

is contained in σ and such that the $\exp(\sigma_{\mathbf{C}})$-orbit Z including $\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t)$, which is independent of the choice of a specialization $\mathcal{O}_{S,t}^{\log} \to \mathbf{C}$ at t (1.3.1), is a σ-nilpotent orbit.

We call an LMH with polarized graded quotients of type Φ on S also an *LMH with polarized graded quotients, with global monodromy in Γ, and with local monodromy in Σ.*

Remark. There is an error in the condition (1) in [KNU10b] 3.2.2. The correct condition is the above (1).

2.6.3. Moduli functor LMH_{Φ}.

Let $\mathrm{LMH}_{\Phi} : \mathcal{B}(\log) \to (\text{set})$ be the contravariant functor defined as follows: For an object S of $\mathcal{B}(\log)$, $\mathrm{LMH}_{\Phi}(S)$ is the set of isomorphism classes of LMH with polarized graded quotients of type Φ on S.

2.6.4. Period maps.

We will have some period maps associated to an LMH with polarized graded quotients of type Φ. In the following, assume that Γ is neat.

(1) $\mathrm{LMH}_{\Phi} \to \mathrm{Map}(-, \Gamma \backslash D_{\Sigma})$.

Set theoretically, this period map is described as follows. Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_{\Phi}(S)$. Then, the image of $s \in S$ by the map corresponding to H is $((\sigma, Z) \bmod \Gamma) \in \Gamma \backslash D_{\Sigma}$. Here σ is the smallest cone of Σ satisfying 2.6.2 (2), which exists by 2.2.4 (2)$''$, and Z is the associated $\exp(\sigma_{\mathbf{C}})$-orbit which appeared in 2.6.2 (2).

(2) $\mathrm{LMH}_{\Phi} \to \mathrm{Map}(-^{\log}, \Gamma \backslash D_{\Sigma}^{\sharp})$.

This is the composite of (1) and $(S \xrightarrow{f} \Gamma \backslash D_{\Sigma}) \mapsto (S^{\log} \xrightarrow{f^{\log}} (\Gamma \backslash D_{\Sigma})^{\log} \simeq \Gamma \backslash D_{\Sigma}^{\sharp})$ (cf. 2.5.6). Set theoretically, this is described as follows. Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_{\Phi}(S)$. Then, the image of $t \in S^{\log}$ by the map corresponding to H is $((\sigma, Z) \bmod \Gamma) \in \Gamma \backslash D_{\Sigma}^{\sharp}$. Here σ is the same as in (1), and Z is the $\exp(i\sigma_{\mathbf{R}})$-orbit including $\tilde{\mu}_t(\mathbf{C} \otimes_{\mathcal{O}_{S,t}^{\log}} F_t)$ which appeared in 2.6.2 (2).

(3) A variant of (2). Let $\mathcal{B}(\log)^{\log}$ be the category of pairs (S, U), where S is an object of $\mathcal{B}(\log)$ and U is an open set of $S^{\log}$. Let $\mathrm{LMH}_{\Phi}^{\log}$ be the functor on $\mathcal{B}(\log)^{\log}$ defined as follows. $\mathrm{LMH}_{\Phi}^{\log}((S, U))$ is the set of isomorphism classes of an $H \in \mathrm{LMH}_{\Phi}(S)$ plus a representative $\tilde{\mu} : (H_{\mathbf{Z}}, W, (\langle \ , \ \rangle_k)_k) \simeq (H_0, W, (\langle \ , \ \rangle_k)_k)$ of μ given on U. Then the map $\mathrm{LMH}_{\Phi}^{\log}((S, U)) \to \mathrm{Map}(U, D_{\Sigma}^{\sharp})$ is given, whose set-theoretical description is similar to (2).

2.6.5. The idea of the definition of the period map

$$\mathrm{LMH}_{\Phi} \to \mathrm{Map}(-, \Gamma \backslash D_{\Sigma})$$

of functors on $\mathcal{B}(\log)$ in 2.6.4 (1) is the same as in [KU09]: Let $S \in \mathcal{B}(\log)$. Let $H \in \mathrm{LMH}_\Phi(S)$. Then, locally on S, the corresponding $S \to \Gamma \backslash D_\Sigma$ comes from a morphism $S \to E_\sigma$ for some $\sigma \in \Sigma$.

Theorem 2.6.6. *Let* $\Phi = (\Lambda, \Sigma, \Gamma)$ *be as in* 2.6.1. *Assume that* Γ *is neat. Then* LMH_Φ *is represented by* $\Gamma \backslash D_\Sigma$ *in* $\mathcal{B}(\log)$.

The proof will be given in §4–§5.

§3. Moduli spaces of log mixed Hodge structures with given graded quotients

Let S be an object of $\mathcal{B}(\log)$, and assume that we are given a family $Q = (H_{(w)})_{w \in \mathbf{Z}}$, where $H_{(w)}$ is a pure LMH on S of weight w for each w and $H_{(w)} = 0$ for almost all w. Shortly speaking, the subject of §3 is to construct spaces over S which classify LMH H with $H(\mathrm{gr}_w^W) = H_{(w)}$ for any w.

We assume that all $H_{(w)}$ are polarizable, though we expect that the method of §3 works (with some suitable modifications) without assuming the polarizability of $H_{(w)}$ (it actually works successfully in some non-polarizable case; cf. 7.1.4).

More precisely, let

$$\mathrm{LMH}_Q$$

be the contravariant functor on the category $\mathcal{B}(\log)/S$ defined as follows. For an object S' of $\mathcal{B}(\log)$ over S, $\mathrm{LMH}_Q(S')$ is the set of all isomorphism classes of an LMH H on S' endowed with an isomorphism $H(\mathrm{gr}_w^W) \simeq H_{(w)}|_{S'}$ for each $w \in \mathbf{Z}$. Here $H_{(w)}|_{S'}$ denotes the pull-back of $H_{(w)}$ on S'. The functor LMH_Q is usually not representable. Our subject is to construct relative log manifolds over S which represent big subfunctors of LMH_Q.

We can often construct such a relative log manifold over S by the method of §2: As is explained in [KNU10a], we can construct such a space as the fiber product of $S \to \prod_{w \in \mathbf{Z}} \Gamma_w \backslash D(\mathrm{gr}_w^W)_{\sigma_w} \leftarrow \Gamma \backslash D_\Sigma$, where $S \to \Gamma_w \backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ is the period map of $H_{(w)}$ (if it exists) and D_Σ is a moduli space in §2 (see §3.4 for details). However, this method has a disadvantage that the period map $S \to \Gamma_w \backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ can be defined in general only after blowing up S as is explained in [KU09] §4.3. (Cf. 3.4.3.)

The main purpose of this §3 is to construct more relative log manifolds over S which represent big subfunctors of LMH_Q, by improving the method of §2. The difference of §2 and §3 lies in that though cones considered in §2 were inside $\mathfrak{g}_\mathbf{R}$, the cones considered in §3 exist outside $\mathfrak{g}_\mathbf{R}$. In particular, by the method of this §3, we will obtain in §6 the Néron model and the connected Néron model, an open subspace of the Néron model, both of which represent big subfunctors of LMH_Q (see Theorem 6.1.1). In this method, we do not need blow up S.

§3.1. Relative formulation of cones, the set $D_{S,\Sigma}$ of nilpotent orbits

Here we give a formulation of cones which is suitable for the study of the above functor LMH_Q of log mixed Hodge structures relative to the prescribed Q.

3.1.1. Assume that we are given an object S of $\mathcal{B}(\log)$ and a polarized log Hodge structure $H_{(w)}$ on S of weight w for each $w \in \mathbf{Z}$. Assume that $H_{(w)} = 0$ for almost all w. Write $Q = (H_{(w)})_{w \in \mathbf{Z}}$.

Then, locally on S, we can find the following (1)–(5).

(1) $\Lambda = (H_0, W, (\langle \ , \ \rangle_w)_w, (h^{p,q})_{p,q})$ as in 2.1.1.

(2) A sharp fs monoid P.

(3) A homomorphism

$$\Gamma' := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z}) \to G'_{\mathbf{Z}} := G_{\mathbf{Z}}(\mathrm{gr}^W) = \prod_w G_{\mathbf{Z}}(\mathrm{gr}_w^W)$$

whose image consists of unipotent automorphisms.

((4) and (5) are stated after a preparation.) Let

$$\sigma' := \mathrm{Hom}(P, \mathbf{R}_{\geq 0}^{\mathrm{add}}), \quad \mathrm{toric}_{\sigma'} := \mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}.$$

Let

$$\sigma'_{\mathbf{R}} \to \mathfrak{g}'_{\mathbf{R}} := \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$$

be the $\mathbf{R}$-linear map induced by the logarithm of the homomorphism in (3).

We define the set $E'_{\sigma'}$. The definition is parallel to the definition of E_σ in 2.3.3. Let $E'_{\sigma'}$ be the set of all points $s \in \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$, where $\check{D}(\mathrm{gr}^W) = \prod_w \check{D}(\mathrm{gr}_w^W)$, such that for each w, the pull-back of the canonical pre-PLH of weight w on $\mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}_w^W)$ to the fs log point s is a PLH. Here the canonical pre-PLH of weight w on $\mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}_w^W)$ is defined in the same way as in 2.3.2.

In other words, $E'_{\sigma'}$ is the set of all pairs $(q', F') \in \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$ satisfying the following condition:

If we write $q' = \mathbf{e}(a) \cdot 0_{\tau'}$ with τ' a face of σ' and $a \in \sigma'_{\mathbf{C}}$ (2.3.1), then $(\tau'_{\mathfrak{g}'}, \exp(a_{\mathfrak{g}'})F')$ generates a nilpotent orbit, where $(-)_{\mathfrak{g}'}$ denotes the image under the $\mathbf{C}$-linear map $\sigma'_{\mathbf{C}} \to \mathfrak{g}'_{\mathbf{C}} := \mathfrak{g}_{\mathbf{C}}(\mathrm{gr}^W)$ induced by the above $\mathbf{R}$-linear map $\sigma'_{\mathbf{R}} \to \mathfrak{g}'_{\mathbf{R}}$.

We endow $E'_{\sigma'}$ with the structure of an object of $\mathcal{B}(\log)$ by using the embedding $E'_{\sigma'} \subset \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$.

(4) A strict morphism $S \to E'_{\sigma'}$ in $\mathcal{B}(\log)$.

Here a morphism $X \to Y$ in $\mathcal{B}(\log)$ is said to be *strict* if via this morphism, the log structure of X coincides with the inverse image of the log structure of Y.

(5) An isomorphism of PLH for each $w \in \mathbf{Z}$ between $H_{(w)}$ and the pull-back of the canonical PLH of weight w on $E'_{\sigma'}$ under the morphism in (4).

3.1.2. The local existence of (1)–(5) in 3.1.1 follows from a general theory of PLH in [KU09].

In fact, take $s_0 \in S$, and take a point $t_0 \in S^{\log}$ lying over s_0. Let $H_0 := \bigoplus_w H_{(w),\mathbf{Z},t_0}$ with the evident weight filtration W on $H_{0,\mathbf{R}}$ and with the bilinear form $\langle \ , \ \rangle_w$ given by the polarization of $H_{(w)}$. For each w, let $(h^{p,q})_{p,q}$, $p + q = w$, be the Hodge type of $H_{(w)}$ at s_0.

Let $P := (M_S/\mathcal{O}_S^\times)_{s_0}$. Then $\Gamma' := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z})$ is identified with the fundamental group $\pi_1(s_0^{\log})$, and the actions of $\pi_1(s_0^{\log})$ on $H_{(w),\mathbf{Z},t_0}$ for $w \in \mathbf{Z}$ give the homomorphism $\Gamma' \to G'_{\mathbf{Z}}$.

On an open neighborhood of s_0 in S, there is a homomorphism $P \to M_S$ which induces the identity map $P \to (M_S/\mathcal{O}_S^\times)_{s_0}$ and which is a chart ([KU09] 2.1.5) of the fs log structure M_S. We replace S by this neighborhood of s_0. Then this homomorphism induces a strict morphism $S \to \mathrm{toric}_{\sigma'}$. By [KU09] 8.2.1–8.2.3, we obtain a strict morphism $S \to E'_{\sigma'}$ over $\mathrm{toric}_{\sigma'}$ and the isomorphism (5) in 3.1.1.

3.1.3. Assume that we are given S, $H_{(w)}$, $Q = (H_{(w)})_{w \in \mathbf{Z}}$, and (1)–(5) of 3.1.1 and fix them.

We will fix a splitting $H_0 = \bigoplus_w H_0(\mathrm{gr}_w^W)$ of the filtration W on H_0. We will denote $H_0(\mathrm{gr}_w^W)$ also by $H_{0,(w)}$.

Let μ be the Γ'-level structure on $H_{\mathbf{Z}}(\mathrm{gr}^W) := \bigoplus_w H_{(w),\mathbf{Z}}$, i.e., the class of local isomorphisms $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_0$ on $S^{\log}$ which is global mod Γ', defined by the isomorphism (5) in 3.1.1.

We will denote $E'_{\sigma'}$ often by S_0.

In §3, the notions nilpotent cone, nilpotent orbit, fan, and weak fan are different from those in §2. Recall $\sigma' = \mathrm{Hom}(P, \mathbf{R}_{\geq 0}^{\mathrm{add}})$ together with the linear map $\sigma'_{\mathbf{R}} \to \mathfrak{g}'_{\mathbf{R}} := \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$. In §3, nilpotent cones, fans, and weak fans appear in the fiber product $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, and nilpotent orbits appear in the product $\sigma'_{\mathbf{C}} \times \check{D}$, as explained below.

3.1.4. Nilpotent cone.

In §3, a *nilpotent cone* means a finitely generated cone σ in the fiber product

$$\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$$

satisfying the following (1) and (2).

(1) For any $N \in \sigma$, the $\mathfrak{g}_{\mathbf{R}}$-component $N_{\mathfrak{g}} \in \mathfrak{g}_{\mathbf{R}}$ of N is nilpotent.

(2) For any $N_1, N_2 \in \sigma$, $N_{1,\mathfrak{g}} N_{2,\mathfrak{g}} = N_{2,\mathfrak{g}} N_{1,\mathfrak{g}}$.

We say a nilpotent cone σ is *admissible* if the following condition (3) is satisfied.

(3) The action of σ on $H_{0,\mathbf{R}}$ via $\sigma \to \mathfrak{g}_{\mathbf{R}}$, $N \mapsto N_{\mathfrak{g}}$, is admissible with respect to W in the sense of 1.2.2.

3.1.5. Nilpotent orbit.

Let σ be a nilpotent cone. A σ-*nilpotent orbit* is a subset Z of $\sigma'_{\mathbf{C}} \times \check{D}$ satisfying the following conditions (1) and (2).

(1) If $(a, F) \in Z$ ($a \in \sigma'_{\mathbf{C}}, F \in \check{D}$), then $Z = \{(a + b', \exp(b_{\mathfrak{g}})F) \mid b \in \sigma_{\mathbf{C}}\}$ (here b' denotes the image of b in $\sigma'_{\mathbf{C}}$).

(2) If $(a, F) \in Z$, then $(\sigma_{\mathfrak{g}}, F)$ generates a nilpotent orbit.

Note that for $F \in \check{D}$ and $a \in \sigma'_{\mathbf{C}}$, $\{(a+b', \exp(b_{\mathfrak{g}})F) \mid b \in \sigma_{\mathbf{C}}\}$ is a σ-nilpotent orbit if and only if the above condition (2) is satisfied, which depends on F but not on a. If this condition (2) is satisfied, we say F *generates a σ-nilpotent orbit* (or (σ, F), or further, $(N_1, \ldots, N_n, F)$ *generates a nilpotent orbit* when σ is generated by $N_1, \ldots, N_n$).

3.1.6. Fan and weak fan.

In §3, a *fan* is a non-empty set of sharp rational nilpotent cones in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ which is closed under the operations of taking a face and taking the intersection.

A *weak fan* is a non-empty set Σ of sharp rational nilpotent cones in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ which is closed under the operation of taking a face and satisfies the following condition.

(1) Let $\sigma, \tau \in \Sigma$, and assume that σ and τ have a common interior point. Assume that there is an $F \in \check{D}$ such that (σ, F) and (τ, F) generate nilpotent orbits in the sense of 3.1.5. Then $\sigma = \tau$.

A fan is a weak fan. (This is proved in the same way as 2.2.4.)
An example of a fan is

$$\Xi = \{ \mathbf{R}_{\geq 0} N \mid N \in \sigma' \times_{\mathfrak{g}_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}, N \text{ is rational, } N_{\mathfrak{g}} \text{ is nilpotent} \}.$$

3.1.7. Recall

$$\Gamma' = \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z}).$$

We denote the given homomorphism $\Gamma' \to G'_{\mathbf{Z}}$ (3.1.1 (3)) by $a \mapsto a_{G'}$. We regard $G'_{\mathbf{Z}}$ as a subgroup of $G_{\mathbf{Z}}$ by the fixed splitting of W of H_0.

As is easily seen, there is a one-to-one correspondence between

(1) A subgroup Γ_u of $G_{\mathbf{Z},u}$ such that $\gamma_{G'} \Gamma_u \gamma_{G'}^{-1} = \Gamma_u$ for all $\gamma \in \Gamma'$.

(2) A subgroup Γ of $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ containing $\Gamma' = \{ (a, a_{G'}) \mid a \in \Gamma' \}$.

The correspondence is:
From (1) to (2). $\Gamma := \{ (a, a_{G'} b) \mid a \in \Gamma', b \in \Gamma_u \}$.
From (2) to (1). $\Gamma_u := \mathrm{Ker}(\Gamma \to \Gamma')$.
In what follows, we will denote the projection $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}} \to G_{\mathbf{Z}}$ by $a \mapsto a_G$.

3.1.8. Let Σ be a weak fan, and let Γ be a subgroup of $\Gamma' \times_{G'_{\mathbf{Z}}} G_{\mathbf{Z}}$ satisfying (2) in 3.1.7.

We say that Σ and Γ are *compatible* if $\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$ for any $\gamma \in \Gamma$ and $\sigma \in \Sigma$. Here $\mathrm{Ad}(\gamma)$ is $(x, y) \mapsto (x, \mathrm{Ad}(\gamma_G)y)$ $(x \in \sigma', y \in \mathfrak{g}_{\mathbf{R}})$.

We say Σ and Γ are *strongly compatible* if they are compatible and, for any $\sigma \in \Sigma$, any element of σ is a sum of elements of the form $a \log(\gamma)$ with $a \in \mathbf{R}_{\geq 0}$ and $\gamma \in \Gamma(\sigma)$. Here $\log$ is the map $\Gamma \to \Gamma' \times \mathfrak{g}_{\mathbf{R}}$, where $\Gamma \to \Gamma'$ is the projection and $\Gamma \to \mathfrak{g}_{\mathbf{R}}$ is $\gamma \mapsto \log(\gamma_G)$, and $\Gamma(\sigma)$ is defined by

$$\Gamma(\sigma) := \{ \gamma \in \Gamma \mid \log(\gamma) \in \sigma \}.$$

3.1.9. Γ-level structure on H.
Let Γ be as in 3.1.8. Let Γ_G be the image of Γ in $G_{\mathbf{Z}}$.
Let $S' \in \mathcal{B}(\log)/S$. Let H be an LMH on S'. Assume that an isomorphism between $H(\mathrm{gr}_w^W)$ and $H_{(w)}|_{S'}$ is given for each w.
Then, there is a one-to-one correspondence between the following (1) and (2).

(1) The class (mod Γ_G) of a collection of local isomorphisms $(H_{\mathbf{Z}}, W) \simeq (H_0, W)$ which is compatible with the given Γ'-level structure on $H_{\mathbf{Z}}(\mathrm{gr}^W)$ and which is global modΓ_G.

(2) The class (mod Γ_G) of a collection of local splittings $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_{\mathbf{Z}}$ of the filtration W on the local system $H_{\mathbf{Z}}$ on $S^{\log}$ which is global mod Γ_G.

Such a class is called a Γ-*level structure* on $H_{\mathbf{Z}}$.

3.1.10. Assume that Σ and Γ as in 3.1.8 are given. Assume that Σ is strongly compatible with Γ. Define

$$D_{S,\Sigma} = \{(s,\sigma,Z) \mid s \in S, \sigma \in \Sigma, Z \subset \sigma'_{\mathbf{C}} \times \check{D} \text{ satisfying (1) and (2) below}\}.$$

(1) Z is a σ-nilpotent orbit (3.1.5).

(2) For any $(a,F) \in Z$, the image of s under $S \to E'_{\sigma'}$ coincides with $(\mathbf{e}(a) \cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_{\mathfrak{g}'})F(\mathrm{gr}^W))$. Here $0_\sigma(\mathrm{gr}^W)$ denotes the image of $0_\sigma \in \mathrm{toric}_\sigma$ in $\mathrm{toric}_{\sigma'}$. (That is, if α denotes the smallest face of σ' which contains the image of $\sigma \to \sigma'$, then $0_\sigma(\mathrm{gr}^W) = 0_\alpha$.)

Note that under the condition (1), the element $(\mathbf{e}(a) \cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_{\mathfrak{g}'})F(\mathrm{gr}^W))$ of $E'_{\sigma'}$ in the condition (2) is independent of the choice of $(a,F) \in Z$.

If $'\Sigma \subset \Sigma$ denotes the subset of Σ consisting of all admissible $\sigma \in \Sigma$, and if $''\Sigma \subset {}'\Sigma$ denotes the subset of Σ consisting of all $\sigma \in \Sigma$ such that a σ-nilpotent orbit exists, then $'\Sigma$ and $''\Sigma$ are weak fans, and we have $D_{S,\Sigma} = D_{S,'\Sigma} = D_{S,''\Sigma}$.

§3.2. Toroidal partial compactifications $\Gamma \backslash D_{S,\Sigma}$

Notation is as in §3.1.

Assume that Σ is a weak fan and is strongly compatible with Γ. In this §3.2, we endow the set $\Gamma \backslash D_{S,\Sigma}$ with a structure of an object of $\mathcal{B}(\log)$. Here $\gamma \in \Gamma$ acts on $D_{S,\Sigma}$ as $(s,\sigma,Z) \mapsto (s, \mathrm{Ad}(\gamma)(\sigma), \gamma Z)$, where γZ is defined by the action of γ on $\sigma'_{\mathbf{C}} \times \check{D}$ given by $(a,F) \mapsto (a - \gamma', \gamma_G F)$ with γ' being the image of γ in Γ' (3.1.7).

3.2.1. Let $\sigma \in \Sigma$. Let
$$\mathrm{toric}_\sigma = \mathrm{Spec}(\mathbf{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}},$$
where $\Gamma(\sigma)^\vee = \mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$.

Let S be an object of $\mathcal{B}(\log)$ and let

$$E_{S,\sigma} = \{(s,z,q,F) \mid s \in S, z \in \sigma'_{\mathbf{C}}, q \in \mathrm{toric}_\sigma, F \in \check{D} \text{ satisfying the following (1) and (2)}\}.\blacksquare$$

(1) If we write $q = \mathbf{e}(b) \cdot 0_\tau$ with τ a face of σ and $b \in \sigma_{\mathbf{C}}$ (2.3.1), then $\exp(b_{\mathfrak{g}})F$ generates a τ-nilpotent orbit (3.1.5).

(2) The image of s under $S \to E'_{\sigma'}$ coincides with $(\mathbf{e}(z)q(\mathrm{gr}^W), \exp(-z_{\mathfrak{g}'})F(\mathrm{gr}^W))$, where $q(\mathrm{gr}^W)$ is the image of q in $\mathrm{toric}_{\sigma'}$.

3.2.2. Note that, for $\sigma \in \Sigma$, we have a canonical map

$$E_{S,\sigma} \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S,\sigma}, \quad (s,z,q,F) \mapsto \mathrm{class}(s,\tau,Z).$$

Here (τ, Z) is as follows. Write $q = \mathbf{e}(b) \cdot 0_\tau$ (2.3.1). This determines τ. Further, Z is the unique τ-nilpotent orbit containing $(z + b', \exp(b_{\mathfrak{g}})F)$. $D_{S,\sigma}$ denotes $D_{S,\mathrm{face}(\sigma)}$.

Proposition 3.2.3. *Set-theoretically, $E_{S,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma}$ with respect to the following action of $\sigma_{\mathbf{C}}$ on $E_{S,\sigma}$. For $a \in \sigma_{\mathbf{C}}$, a acts on $E_{S,\sigma}$ as $(s, z, q, F) \mapsto (s, z - a', \mathbf{e}(a)q, \exp(-a_{\mathfrak{g}})F)$, where a' denotes the image of a in $\sigma'_{\mathbf{C}}$.*

This is proved easily.

3.2.4. Let $S_0 = E'_{\sigma'}$ as in 3.1.3. We consider the case that the morphism $S \to S_0$ is an isomorphism (we will express this situation as the case $S = S_0$). We endow $E_{S_0,\sigma}$ with the structure of an object of $\mathcal{B}(\log)$ by using the embedding

$$E_{S_0,\sigma} \subset \sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}, \ \ (s, z, q, F) \mapsto (z, q, F).$$

That is, the topology of $E_{S_0,\sigma}$ is the strong topology in $\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}$, and $\mathcal{O}$ and the log structure M of $E_{S_0,\sigma}$ are the inverse images of $\mathcal{O}$ and M of $\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \times \check{D}$, respectively.

3.2.5. We endow $\Gamma\backslash D_{S_0,\Sigma}$ with the structure of a log local ringed space over $\mathbf{C}$, as follows.

We define the topology of $\Gamma\backslash D_{S_0,\Sigma}$ as follows. A subset U of $\Gamma\backslash D_{S_0,\Sigma}$ is open if and only if, for any $\sigma \in \Sigma$, the inverse image U_σ of U in $E_{S_0,\sigma}$ is open.

We define the sheaf of rings $\mathcal{O}$ on $\Gamma\backslash D_{S_0,\Sigma}$, as a subsheaf of the sheaf of $\mathbf{C}$-valued functions, as follows. For an open set U of $\Gamma\backslash D_{S_0,\Sigma}$ and for a map $f : U \to \mathbf{C}$, f belongs to $\mathcal{O}$ if and only if, for any $\sigma \in \Sigma$, the pull-back of f on the inverse image of U in $E_{S_0,\sigma}$ belongs to the $\mathcal{O}$ of $E_{S_0,\sigma}$.

We define the log structure of $\Gamma\backslash D_{S_0,\Sigma}$ as a subsheaf of $\mathcal{O}$ of $\Gamma\backslash D_{S_0,\Sigma}$, as follows. For an open set U of $\Gamma\backslash D_{S_0,\Sigma}$ and for an $f \in \mathcal{O}(U)$, f belongs to the log structure if and only if, for any $\sigma \in \Sigma$, the pull-back of f on the inverse image of U in $E_{S_0,\sigma}$ belongs to the log structure of $E_{S_0,\sigma}$. Here we regard the log structure of $E_{S_0,\sigma}$ as a subsheaf of $\mathcal{O}$ of $E_{S_0,\sigma}$.

Proposition 3.2.6. (i) *$E_{S_0,\sigma}$ and $\Gamma\backslash D_{S_0,\Sigma}$ are objects of $\mathcal{B}(\log)$.*

(ii) *$E_{S_0,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S_0,\sigma}$ in $\mathcal{B}(\log)$.*

This will be proved in §5.

3.2.7. Note that, as a set, $\Gamma\backslash D_{S,\Sigma}$ is the fiber product of $S \to S_0 \leftarrow \Gamma\backslash D_{S_0,\Sigma}$.

We endow $\Gamma\backslash D_{S,\Sigma}$ with the structure as an object of $\mathcal{B}(\log)$ by regarding it as the fiber product of $S \to S_0 \leftarrow \Gamma\backslash D_{S_0,\Sigma}$ in $\mathcal{B}(\log)$. The following theorem will be proved in §5.

Theorem 3.2.8. (i) *$\Gamma\backslash D_{S,\Sigma}$ is a relative log manifold over S.*

(ii) *For $\sigma \in \Sigma$, the morphism $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_{S,\sigma} \to \Gamma\backslash D_{S,\Sigma}$ is locally an isomorphism in $\mathcal{B}(\log)$.*

(iii) *If S is Hausdorff, $\Gamma\backslash D_{S,\Sigma}$ is Hausdorff.*

3.2.9. We define $D_{S,\Sigma}^\sharp$ in the evident way, replacing nilpotent orbit by nilpotent i-orbit which is defined in the evident way. Then $\Gamma\backslash D_{S,\Sigma}^\sharp$ is identified with $(\Gamma\backslash D_{S,\Sigma})^{\log}$ via the natural map.

§3.3. Moduli

3.3.1. Let Σ be a weak fan in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, where $\sigma' = \mathrm{Hom}(P, \mathbf{R}_{\geq 0}^{\mathrm{add}})$.

Assume that we are given a subgroup Γ_u of $G_{\mathbf{Z},u}$ which satisfies the condition in 3.1.7 (1), and let Γ be the corresponding group as in 3.1.7. Assume that Σ and Γ are strongly compatible.

3.3.2. Let $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$ be the following functor on the category $\mathcal{B}(\log)/S$. For any object S' of $\mathcal{B}(\log)/S$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}(S')$ is the set of all isomorphism classes of the following H.

H is an LMH on S' endowed with a Γ-level structure μ, whose gr^W is identified with $(H_{(w)})_w$ endowed with the given Γ'-level structure (3.1.3), satisfying the following condition.

For any $s \in S'$, any $t \in (S')^{\log}$ lying over s, any representative $\mu_t : H_{\mathbf{Z},t} \simeq H_0$ of μ, and for any specialization a at t (1.3.1), there exists $\sigma \in \Sigma$ such that the image of $\mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^{\times})_s, \mathbf{N}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$ is contained in σ and $\mu_t(F(a))$ generates a σ-nilpotent orbit (3.1.5). Here F denotes the Hodge filtration of H.

We have the canonical morphism $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)} \to \mathrm{LMH}_Q$.

In the case $\Gamma_u = G_{\mathbf{Z},u}$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$ is the following functor.

For any object S' of $\mathcal{B}(\log)/S$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}(S')$ is the set of all isomorphism classes of the following H.

H is an LMH on S', whose gr^W is identified with $(H_{(w)})_w$, satisfying the following condition.

For any $s \in S'$, any $t \in (S')^{\log}$ lying over s, any isomorphism $\mu_t : H_{\mathbf{Z},t} \simeq H_0$ whose gr^W belongs to the given Γ'-level structure (3.1.3), and for any specialization a at t (1.3.1), there exists $\sigma \in \Sigma$ such that the image of $\mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^{\times})_s, \mathbf{N}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$ is contained in σ and $\mu_t(F(a))$ generates a σ-nilpotent orbit (3.1.5). Here F denotes the Hodge filtration of H.

From this, we see that in the case $\Gamma_u = G_{\mathbf{Z},u}$, $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)} \to \mathrm{LMH}_Q$ is injective and hence $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$ is regarded as a subfunctor of LMH_Q. We will denote this subfunctor by $\mathrm{LMH}_Q^{(\Sigma)}$. (This injectivity is explained also from the fact that, in the case $\Gamma_u = G_{\mathbf{Z},u}$, H with the given $\mathrm{gr}^W = (H_{(w)})_w$ is automatically endowed with a Γ-level structure.)

In the case $\Gamma_u = G_{\mathbf{Z},u}$, we denote $\Gamma \backslash D_{S,\Sigma}$ by

$$J_{S,\Sigma}, \quad \text{or simply by} \quad J_\Sigma.$$

Theorem 3.3.3. *As an object of $\mathcal{B}(\log)$ over S, $\Gamma \backslash D_{S,\Sigma}$ represents $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$.*

This will be proved in §5.

3.4.1. Let the situation be as in §2.

Let Σ be a weak fan, and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

Let σ_w be a sharp rational nilpotent cone in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_w^W)$ for each w. Assume that, for any $\sigma \in \Sigma$ and for any w, the image of σ in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_w^W)$ is contained in σ_w.

Let Γ_w be a neat subgroup of $G_{\mathbf{Z}}(\mathrm{gr}_w^W)$ for each w. Assume that for each w, the image of Γ in $G_{\mathbf{Z}}(\mathrm{gr}_w^W)$ is contained in Γ_w.

Assume that the fan $\mathrm{face}(\sigma_w)$ and Γ_w are strongly compatible.

Let S be an object of $\mathcal{B}(\log)$. Assume that we are given a morphism $S \to \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ ∎ for each w. As in the papers [KNU10a] and [KNU10c], we consider the fiber product of

$$S \to \textstyle\prod_w \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w} \leftarrow \Gamma\backslash D_\Sigma.$$

Assume that $S \to \prod_w \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ is strict. Assume also that the homomorphism $\Gamma \to \prod_w \Gamma_w$ is surjective. Then we have the following relation of this fiber product to the space constructed in this section.

Let $H_{(w)}$ be the pull-back on S of the universal polarized log Hodge structure of weight w on $\Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}$, and let $Q = (H_{(w)})_w$.

Locally on S, the morphism $S \to \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ factors as $S \to E(\mathrm{gr}_w^W)_{\sigma_w} \to \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}$. We have the situation of 3.1.1 by taking $H_{0,(w)} = H_0(\mathrm{gr}_w^W)$ and P the dual fs monoid of $\prod_w \Gamma_w(\sigma_w)$. We have $\sigma' = \prod_w \sigma_w$, $\Gamma' = \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}}$, and Σ can be regarded as a weak fan in the sense of 3.1.6 by identifying $\sigma \in \Sigma$ with $\{(N_{\sigma'}, N) \mid N \in \sigma\}$, where $N_{\sigma'}$ is the image of N in σ'.

Let $\tilde{\Gamma} \subset \Gamma$ be the inverse image of $\Gamma' \subset \prod_w \Gamma_w$.

Proposition 3.4.2. *We have a canonical isomorphism*

$$\tilde{\Gamma}\backslash D_{S,\Sigma} \simeq S \times_{\prod_w \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}} \Gamma\backslash D_\Sigma$$

in the category $\mathcal{B}(\log)/S$.

Proof. We have a canonical map $D_{S,\Sigma} \to D_\Sigma$ which sends (s, σ, Z) to $(\sigma, \exp(\sigma_{\mathbf{C}})F)$, where F is any element of $\check{D}$ such that $(a, F) \in Z$ for some $a \in \sigma'_{\mathbf{C}}$.

The induced map $\tilde{\Gamma}\backslash D_{S,\Sigma} \to Y := S \times_{\prod_w \Gamma_w\backslash D(\mathrm{gr}_w^W)_{\sigma_w}} \Gamma\backslash D_\Sigma$ is bijective. In fact, it is easy to see the surjectivity, and the injectivity is reduced to the following Claim 1.

Claim 1. Assume $(s, \sigma, Z_j) \in D_{S,\Sigma}$ and $(a_j, F) \in Z_j$ for $j = 1, 2$, with $s \in S$, $\sigma \in \Sigma$, $Z_j \subset \sigma'_{\mathbf{C}} \times \check{D}$, $a_j \in \sigma'_{\mathbf{C}} = \prod_w \sigma_{w,\mathbf{C}}$, $F \in \check{D}$. Then $a_1 = a_2$ (hence $Z_1 = Z_2$).

We prove Claim 1. By the definition of $D_{S,\Sigma}$, for $j = 1, 2$, the image of s in $S_0 = E'_{\sigma'}$ is $(\mathbf{e}(a_j)\cdot 0_\sigma(\mathrm{gr}^W), \exp(-a_j)F(\mathrm{gr}^W))$. Hence $\exp(-a_1)F(\mathrm{gr}^W) = \exp(-a_2)F(\mathrm{gr}^W)$. Since $(\sigma', F(\mathrm{gr}^W))$ generates a nilpotent orbit, this implies $a_1 = a_2$ by [KU09] Proposition 7.2.9 (i).

We will prove

Claim 2. In the case $S = S_0 = \prod_w E(\mathrm{gr}_w^W)_{\sigma_w}$, the above bijection $\tilde{\Gamma}\backslash D_{S,\Sigma} \to Y$ is in fact an isomorphism $\tilde{\Gamma}\backslash D_{S_0,\Sigma} \xrightarrow{\sim} Y$ in $\mathcal{B}(\log)$.

By the base change by $S \to S_0$, the isomorphism in Claim 2 induces an isomorphism $\tilde{\Gamma} \backslash D_{S,\Sigma} \xrightarrow{\sim} Y$ in $\mathcal{B}(\log)/S$ in general.

We prove Claim 2. By Theorem 2.5.4 and Theorem 3.2.8 (ii), it is sufficient to prove that for each $\sigma \in \Sigma$, the morphism of $\mathcal{B}(\log)$ from $\tilde{\Gamma}(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}$ to the fiber product of

$$ S_0 \to \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}} \backslash D(\mathrm{gr}_w^W)_{\sigma_w} \leftarrow \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma $$

is locally an isomorphism. By Theorem 2.5.3, $S_0 = E'_{\sigma'} \to \prod_w \Gamma_w(\sigma_w)^{\mathrm{gp}} \backslash D(\mathrm{gr}_w^W)_{\sigma_w}$ is a $\sigma'_{\mathbf{C}}$-torsor and E_σ in §2 is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$. Furthermore, $E_{S_0,\sigma}$ is a $\sigma_{\mathbf{C}}$-torsor over $\tilde{\Gamma}(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}$ by Proposition 3.2.6 (ii). Hence Claim 2 is reduced to the evident fact that the canonical morphism $E_{S_0,\sigma} \to \sigma'_{\mathbf{C}} \times E_\sigma$ is an isomorphism. $\square$

3.4.3. Thus, to construct the moduli space of LMH with a given graded quotients of the weight filtration, there are two methods. One is the method in [KNU10a] and [KNU10c] to take the fiber product as in the above 3.4.1, and the other is the method of this §3. As advantages of the method of §3, we have:

(1) As is shown in §6, the Néron model and the connected Néron model can be constructed by the method of §3 even in the case of higher dimensional base S. In the method of §2, to obtain such models, the period map $S \to \Gamma' \backslash D_{\Sigma'}$ of the gr^W is necessary to define the fiber product. But, since such a period map of the gr^W can exist in general only after some blowing-up (see §4.3 of [KU09]), we may have to blow up the base S in the method of §2.

(2) The case with no polarization on gr^W can be sometimes treated. (See 7.1.4 for example.)

§4. ASSOCIATED SL(2)-ORBITS

In the next §5, we will prove main results in §2–§3. This §4 is a preparation for §5. We review some necessary definitions and results on the space $D_{\mathrm{SL}(2)}$ of SL(2)-orbits in [KNU.p], introduce the most important map (called the CKS map) $D_{\Sigma,\mathrm{val}}^{\sharp} \to D_{\mathrm{SL}(2)}$ in the fundamental diagram in Introduction, and prove that it is continuous, which will be crucial in §5.

Let the situation and terminologies (weak fan, nilpotent cone, etc.) be as in §2 (not as in §3).

§4.1. $D_{\mathrm{SL}(2)}$

We review from [KNU.p] the definition of SL(2)-orbit, the space $D_{\mathrm{SL}(2)}$ and the SL(2)-orbit associated to an $(n+1)$-tuple $(N_1, \ldots, N_n, F)$ which generates a nilpotent orbit.

4.1.1. SL(2)-orbit ([KNU.p] 2.3).
Assume first that we are in the pure case (originally considered by [CKS86]), that is, $W_w = H_{0,\mathbf{R}}$ and $W_{w-1} = 0$ for some w. Then an SL(2)-*orbit in n variables* is a pair (ρ, φ), where $\rho : \mathrm{SL}(2, \mathbf{C})^n \to G_{\mathbf{C}}$ is a homomorphism of algebraic groups defined over

$\mathbf{R}$ and $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$ is a holomorphic map, satisfying the following three conditions ([KU02], [KU09]).

(1) $\varphi(gz) = \rho(g)\varphi(z)$ for any $g \in \mathrm{SL}(2, \mathbf{C})^n$, $z \in \mathbf{P}^1(\mathbf{C})^n$.

(2) $\varphi(\mathfrak{h}^n) \subset D$.

(3) $\rho_*(\mathrm{fil}_z^p(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})) \subset \mathrm{fil}_{\varphi(z)}^p(\mathfrak{g}_{\mathbf{C}})$ for any $z \in \mathbf{P}^1(\mathbf{C})^n$ and any $p \in \mathbf{Z}$.

Here in (2), $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$ is the upper-half plane. In (3), ρ_* denotes the Lie algebra homomorphism induced by ρ, and $\mathrm{fil}_z(\mathfrak{sl}(2, \mathbf{C})^{\oplus n})$, $\mathrm{fil}_{\varphi(z)}(\mathfrak{g}_{\mathbf{C}})$ are the filtrations induced by the Hodge filtrations at z, $\varphi(z)$, respectively.

Now we consider the general mixed Hodge situation. A *non-degenerate* $\mathrm{SL}(2)$-*orbit of rank n* is a pair $((\rho_w, \varphi_w)_{w \in \mathbf{Z}}, \mathbf{r})$, where (ρ_w, φ_w) is an $\mathrm{SL}(2)$-orbit in n variables for gr_w^W (that is, an $\mathrm{SL}(2)$-orbit for the pure case) for each $w \in \mathbf{Z}$ and $\mathbf{r}$ is an element of D satisfying the following conditions (4)–(6).

(4) $\mathbf{r}(\mathrm{gr}_w^W) = \varphi_w(\mathbf{i})$ for any $w \in \mathbf{Z}$, where $\mathbf{i} = (i, \ldots, i) \in \mathbf{P}^1(\mathbf{C})^n$.

(5) If $2 \leq j \leq n$, there exists $w \in \mathbf{Z}$ such that the j-th component of ρ_w is a non-trivial homomorphism.

(6) If $\mathbf{r} \in D_{\mathrm{spl}}$ and $n \geq 1$, there exists $w \in \mathbf{Z}$ such that the 1-st component of ρ_w is a non-trivial homomorphism. Here, as in [KNU09], D_{spl} denotes the subset of D consisting of $\mathbf{R}$-split mixed Hodge structures.

Let $((\rho_w, \varphi_w)_w, \mathbf{r})$ be a non-degenerate $\mathrm{SL}(2)$-orbit of rank n.

Then, the associated homomorphism of algebraic groups over $\mathbf{R}$

$$\tau : \mathbf{G}_{m,\mathbf{R}}^n \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$$

and the associated set (or family) of weight filtrations are defined as follows.

Recall that the canonical splitting (over $\mathbf{R}$) of the weight filtration of a mixed Hodge structure is defined as in [KNU08] §1, as is reviewed in [KNU09] §4, [KNU.p] §1.2. It was originally defined in [CKS86]. By associating the canonical splitting $\mathrm{spl}_W(F)$ of W to a mixed Hodge structure $F \in D$, we have a continuous map

$$\mathrm{spl}_W : D \to \mathrm{spl}(W),$$

where $\mathrm{spl}(W)$ denotes the set of all splittings of W.

Let $s_{\mathbf{r}} := \mathrm{spl}_W(\mathbf{r}) : \mathrm{gr}^W \xrightarrow{\sim} H_{0,\mathbf{R}}$ be the canonical splitting of W associated to $\mathbf{r}$. Then

$$\tau(t_1, \ldots, t_n) = s_{\mathbf{r}} \circ \left(\bigoplus_{w \in \mathbf{Z}} \left(\prod_{j=1}^n t_j \right)^w \rho_w(g_1, \ldots, g_n) \text{ on } \mathrm{gr}_w^W \right) \circ s_{\mathbf{r}}^{-1}$$

$$\text{with} \quad g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix}.$$

For $1 \leq j \leq n$, let $\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the j-th component of τ.

For $1 \leq j \leq n$, let $W^{(j)}$ be the increasing filtration on $H_{0,\mathbf{R}}$ defined by

$$W_w^{(j)} = \bigoplus_{k \leq w} \{v \in H_{0,\mathbf{R}} \mid \tau_j(a)v = a^k \ (\forall a \in \mathbf{R}^\times)\}.$$

The associated set (resp. family) of weight filtrations is defined to be the set $\{W^{(1)}, \cdots, W^{(n)}\}$∎ (resp. the family $(W^{(j)})_{1 \leq j \leq n}$).

4.1.2. $D_{\mathrm{SL}(2)}$ ([KNU.p] 2.5).

Two non-degenerate SL(2)-orbits $p = ((\rho_w, \varphi_w)_w, \mathbf{r})$ and $p' = ((\rho'_w, \varphi'_w)_w, \mathbf{r}')$ of rank n are said to be *equivalent* if there is a $t \in \mathbf{R}^n_{>0}$ such that

$$\rho'_w = \mathrm{Int}(\mathrm{gr}^W_w(\tau(t))) \circ \rho_w, \quad \varphi'_w = \mathrm{gr}^W_w(\tau(t)) \circ \varphi_w \quad (\forall\, w \in \mathbf{Z}), \quad \mathbf{r}' = \tau(t)\mathbf{r}.$$

This is actually an equivalence relation.

The associated homomorphism of algebraic groups $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$, and the associated set (resp. family) of weight filtrations depend only on the equivalence class.

Let $D_{\mathrm{SL}(2)}$ be the set of all equivalence classes p of non-degenerate SL(2)-orbits of various ranks satisfying the following condition (1).

(1) If W' is a member of set of weight filtrations associated to p, then the $\mathbf{R}$-spaces $W'_k \mathrm{gr}^W_w$ are rational (i.e. defined over $\mathbf{Q}$) for any k and w.

In [KNU.p], we defined two topologies of $D_{\mathrm{SL}(2)}$, which we call the stronger topology and the weaker topology, respectively, and denoted by $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ the set $D_{\mathrm{SL}(2)}$ endowed with the stronger topology and the weaker topology, respectively. The identity map of sets $D^I_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ is continuous in these topologies. In [KNU.p], we defined some kind of real analytic structures on $D^I_{\mathrm{SL}(2)}$ and on $D^{II}_{\mathrm{SL}(2)}$, but we do not use them in the present paper.

A finite set Ψ of increasing filtrations on $H_{0,\mathbf{R}}$ is called an admissible set of weight filtrations if Ψ is the associated set of weight filtrations of some $p \in D_{\mathrm{SL}(2)}$. For an admissible set Ψ of weight filtrations, let $D^I_{\mathrm{SL}(2)}(\Psi)$ be the subset of $D_{\mathrm{SL}(2)}$ consisting of all points p of $D_{\mathrm{SL}(2)}$ such that the set of weight filtrations associated to p is a subset of Ψ. Then $\{D^I_{\mathrm{SL}(2)}(\Psi)\}_\Psi$ is an open covering of $D^I_{\mathrm{SL}(2)}$.

4.1.3. Associated SL(2)-orbits ([KNU.p] 2.4.2).

Let $(N_1, \ldots, N_n, F)$ generate a nilpotent orbit (2.2.2). Then, we associate to it a non-degenerate SL(2)-orbit $((\rho'_w, \varphi'_w)_w, \mathbf{r}_1)$ as follows.

First, by [CKS86], for each $w \in \mathbf{Z}$, we have the SL(2)-orbit (ρ_w, φ_w) in n variables for gr^W_w associated to $(\mathrm{gr}^W_w(N_1), \ldots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$, which generates a nilpotent orbit for gr^W_w. Let $k = \min(\{j \mid 1 \leq j \leq n,\ N_j \neq 0\} \cup \{n+1\})$. Let

$$J' = \{j \mid 1 \leq j \leq n,\ \text{the } j\text{-th component of } \rho_w \text{ is non-trivial for some } w \in \mathbf{Z}\}.$$

Let $J = J' = \emptyset$ if $k = n+1$, and let $J = J' \cup \{k\}$ if otherwise. Let $J = \{a(1), \ldots, a(r)\}$ with $a(1) < \cdots < a(r)$. Then (ρ'_w, φ'_w) is an SL(2)-orbit on gr^W_w characterized by

$$\rho'_w(g_{a(1)}, \ldots, g_{a(r)}) := \rho_w(g_1, \ldots, g_n), \quad \varphi'_w(z_{a(1)}, \ldots, z_{a(r)}) := \varphi_w(z_1, \ldots, z_n).$$

Next, if $y_j \in \mathbf{R}_{>0}$ and $y_j/y_{j+1} \to \infty$ ($1 \leq j \leq n$, y_{n+1} means 1), the canonical splitting $\mathrm{spl}_W(\exp(\sum_{j=1}^n iy_j N_j)F)$ of W (4.1.1) associated to $\exp(\sum_{j=1}^n iy_j N_j)F$ converges in $\mathrm{spl}(W)$. Let $s \in \mathrm{spl}(W)$ be the limit.

Let $\tau : \mathbf{G}_{m,\mathbf{R}}^n \to \operatorname{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the homomorphism of algebraic groups defined by

$$\tau(t_1, \ldots, t_n) = s \circ \left(\bigoplus_{w \in \mathbf{Z}} \left(\left(\textstyle\prod_{j=1}^n t_j \right)^w \rho_w(g_1, \ldots, g_n) \text{ on } \operatorname{gr}_w^W \right) \right) \circ s^{-1},$$

where g_j is as in 4.1.1. Then, as $y_j > 0$, $y_1 = \cdots = y_k$, $y_j/y_{j+1} \to \infty$ ($k \leq j \leq n$, y_{n+1} means 1),

$$\tau \left(\sqrt{\frac{y_2}{y_1}}, \ldots, \sqrt{\frac{y_{n+1}}{y_n}} \right)^{-1} \exp(\textstyle\sum_{j=1}^n iy_j N_j) F$$

converges in D. Let $\mathbf{r}_1 \in D$ be the limit (cf. [KNU.p] 2.4.2 (ii)).

§4.2. Valuative spaces

4.2.1. D_{val}.

Let D_{val} (resp. $D_{\mathrm{val}}^\sharp$) be the set of all triples (A, V, Z), where

A is a $\mathbf{Q}$-subspace of $\mathfrak{g}_{\mathbf{Q}}$ consisting of mutually commutative nilpotent elements. Let $A_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} A$ and $A_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Q}} A$.

V is an (additive) submonoid of $A^* = \operatorname{Hom}_{\mathbf{Q}}(A, \mathbf{Q})$ satisfying the following conditions: $V \cap (-V) = \{0\}$, $V \cup (-V) = A^*$.

Z is a subset of $\check{D}$ such that $Z = \exp(A_{\mathbf{C}})F$ (resp. $Z = \exp(iA_{\mathbf{R}})F$) for any $F \in Z$ and that there exists a finitely generated rational subcone τ of $A_{\mathbf{R}}$ satisfying the following condition (1).

(1) Z is a τ-nilpotent orbit (resp. i-orbit), and V contains any element h of A^* such that $h : A \to \mathbf{Q}$ sends $A \cap \tau$ to $\mathbf{Q}_{\geq 0}$.

We have a canonical map

$$D_{\mathrm{val}}^\sharp \to D_{\mathrm{val}}, \quad (A, V, Z) \mapsto (A, V, \exp(A_{\mathbf{C}})Z),$$

which is surjective.

4.2.2. $D_{\Sigma,\mathrm{val}}$ and $D_{\Sigma,\mathrm{val}}^\sharp$.

Let Σ be a weak fan (2.2.3) in $\mathfrak{g}_{\mathbf{Q}}$.

Let $D_{\Sigma,\mathrm{val}}$ (resp. $D_{\Sigma,\mathrm{val}}^\sharp$) be the subset of D_{val} (resp. $D_{\mathrm{val}}^\sharp$) consisting of all $(A, V, Z) \in D_{\mathrm{val}}$ (resp. $D_{\mathrm{val}}^\sharp$) such that there exists $\sigma \in \Sigma$ satisfying the following conditions (1) and (2).

(1) $\exp(\sigma_{\mathbf{C}})Z$ is a σ-nilpotent orbit (resp. $\exp(i\sigma_{\mathbf{R}})Z$ is a σ-nilpotent i-orbit).

(2) V contains any element h of A^* such that $h : A \to \mathbf{Q}$ sends $A \cap \sigma$ to $\mathbf{Q}_{\geq 0}$.

Note that (2) implies $A \subset \sigma_{\mathbf{R}}$.

The canonical map $D_{\mathrm{val}}^\sharp \to D_{\mathrm{val}}$ (4.2.1) induces a canonical map $D_{\Sigma,\mathrm{val}}^\sharp \to D_{\Sigma,\mathrm{val}}$ which is surjective.

Our next subject is to define canonical maps $D_{\Sigma,\mathrm{val}} \to D_{\Sigma}$ and $D_{\Sigma,\mathrm{val}}^\sharp \to D_{\Sigma}^\sharp$ for a weak fan Σ.

Lemma 4.2.3. *Let Σ be a weak fan and let $(A, V, Z) \in D_{\Sigma,\mathrm{val}}$. Let S be the set of all $\sigma \in \Sigma$ satisfying the conditions (1) and (2) in 4.2.2. Then S has a smallest element.*

Proof. Let T be the set of all rational nilpotent cones σ which satisfy (1) and (2) in 4.2.2. Hence $S = T \cap \Sigma$. Let T_1 be the set of all finitely generated subcones of $A_{\mathbf{R}}$ satisfying the condition 4.2.1 (1).

Then we have

Claim 1. *If $\sigma_1 \in T_1$ and $\sigma_2 \in T$, then $\sigma_1 \cap \sigma_2 \in T_1$.*

We prove Claim 1. By $(\sigma_1 \cap \sigma_2)^{\vee} = \sigma_1^{\vee} + \sigma_2^{\vee}$ ([O88] A.1 (2)), 4.2.2 (2) is satisfied for $\sigma := \sigma_1 \cap \sigma_2$. Here $(\ )^{\vee}$ denotes the dual of a cone. Then, $\dim \sigma = \dim \sigma_1$. On the other hand, Z generates a σ_1-nilpotent orbit by 4.2.1 (1) for σ_1. Thus Z also generates a σ-nilpotent orbit. Hence $\sigma \in T_1$.

We now prove 4.2.3. Let σ_1 and σ_2 be minimal elements of S. We prove $\sigma_1 = \sigma_2$. Since $(A, V, Z) \in D_{\mathrm{val}}$, the set T_1 is not empty. Let τ be an element of T_1, and let $\tau' = \tau \cap \sigma_1 \cap \sigma_2$. Then $\tau' \in T_1$ by Claim 1. Since Σ is a weak fan, by 2.2.4 (2)$''$, the set $\{\alpha \in S \mid \tau' \subset \alpha\}$ has a smallest element σ. Since σ_1 and σ_2 belong to this set, $\sigma \subset \sigma_1$ and $\sigma \subset \sigma_2$. Since σ_1 and σ_2 are minimal in S, we have $\sigma_1 = \sigma = \sigma_2$. $\square$

4.2.4. Let Σ be a weak fan. We define the maps $D_{\Sigma,\mathrm{val}} \to D_{\Sigma}$ and $D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{\sharp}_{\Sigma}$ as

$$(A, V, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z), \quad (A, V, Z) \mapsto (\sigma, \exp(i\sigma_{\mathbf{R}})Z),$$

respectively, where σ is the smallest element of the set S in 4.2.3.

4.2.5. The sets $E_{\sigma,\mathrm{val}}$ and $E^{\sharp}_{\sigma,\mathrm{val}}$.

Let E_{σ} and $E^{\sharp}_{\sigma}$ be as in 2.3.3, and let $\mathrm{toric}_{\sigma,\mathrm{val}}$ over toric_{σ} and $|\mathrm{toric}|_{\sigma,\mathrm{val}}$ be as in [KU09] 5.3.6. Define

$$E_{\sigma,\mathrm{val}} := \mathrm{toric}_{\sigma,\mathrm{val}} \times_{\mathrm{toric}_{\sigma}} E_{\sigma}, \quad E^{\sharp}_{\sigma,\mathrm{val}} = |\mathrm{toric}|_{\sigma,\mathrm{val}} \times_{|\mathrm{toric}|_{\sigma}} E^{\sharp}_{\sigma}.$$

Let $E_{\sigma,\mathrm{val}} \to E_{\sigma}$ and $E^{\sharp}_{\sigma,\mathrm{val}} \to E^{\sharp}_{\sigma}$ be the canonical projections.

4.2.6. Analogously as in [KU09] 5.3.7, we have the projections $E_{\sigma,\mathrm{val}} \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma,\mathrm{val}}$ ▌ and $E^{\sharp}_{\sigma,\mathrm{val}} \to D^{\sharp}_{\sigma,\mathrm{val}}$.

4.2.7. By 2.4 and [KU09] 3.6.20, we define on $E_{\sigma,\mathrm{val}}$ and $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ structures of log local ringed spaces over $\mathbf{C}$, and on $E^{\sharp}_{\sigma,\mathrm{val}}$ and $D^{\sharp}_{\Sigma,\mathrm{val}}$ topologies, analogously as in [KU09] 5.3.6–5.3.8.

$$\S 4.3. \ \text{The map } D^{\sharp}_{\mathrm{val}} \to D_{\mathrm{SL}(2)}.$$

We prove two theorems 4.3.1 and 4.3.2, which follow from propositions 4.3.3 and 4.3.4.

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Theorem 4.3.1. *Let $p = (A, V, Z) \in D_{\mathrm{val}}^{\sharp}$.*

(i) There exists a family $(N_j)_{1 \leq j \leq n}$ of elements of $A_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} A$ satisfying the following two conditions:

(1) If $F \in Z$, $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit.

(2) Via $(N_j)_{1 \leq j \leq n} : A^* = \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q}) \to \mathbf{R}^n$, V coincides with the set of all elements of A^* whose images in $\mathbf{R}^n$ are ≥ 0 with respect to the lexicographic order.

(ii) Take $(N_j)_{1 \leq j \leq n}$ as in (i), let $F \in Z$ and define $\psi(p) \in D_{\mathrm{SL}(2)}$ to be the class of the SL(2)-orbit associated to $(N_1, \ldots, N_n, F)$. Then $\psi(p)$ is independent of the choices of $(N_j)_{1 \leq j \leq n}$ and F.

By Theorem 4.3.1, we obtain a map

$$(3) \qquad\qquad \psi : D_{\mathrm{val}}^{\sharp} \to D_{\mathrm{SL}(2)}.$$

We call this map the *CKS map*, like in the pure case in [KU09]. (CKS stands for Cattani-Kaplan-Schmid from whose work [CKS86] the map ψ in the pure case was defined in [KU09]).

Theorem 4.3.2. *Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$. Then, $\psi : D_{\Sigma,\mathrm{val}}^{\sharp} \to D_{\mathrm{SL}(2)}^I$ is continuous.*

Thus $\psi : D_{\Sigma,\mathrm{val}}^{\sharp} \to D_{\mathrm{SL}(2)}^I$ is the unique continuous extension of the identity map of D.

The C^{∞}-property of the CKS map will be discussed in a later part of this series.

As in [KU09], these theorems follow from Proposition 4.3.3 below, which is an analog of [KU09] 6.4.1.

Proposition 4.3.3. *Let $(N_s)_{s \in S}$ be a finite family of mutually commuting nilpotent elements of $\mathfrak{g}_{\mathbf{R}}$, let $F \in \check{D}$, and assume that $((N_s)_{s \in S}, F)$ generates a nilpotent orbit. Let $a_s \in \mathbf{R}_{>0}$ for $s \in S$. Assume that S is the disjoint union of nonempty subsets S_j $(1 \leq j \leq n)$. Denote $S_{\leq j} := \bigsqcup_{k \leq j} S_k$ and $S_{\geq j} := \bigsqcup_{k \geq j} S_k$. For $1 \leq j \leq n$, let $\check{D}_j$ be the subset of $\check{D}$ consisting of all $F' \in \check{D}$ such that $((N_s)_{s \in S_{\leq j}}, F')$ generates a nilpotent orbit. Let L be a directed ordered set, let $F_{\lambda} \in \check{D}$ $(\lambda \in L)$, $y_{\lambda,s} \in \mathbf{R}_{>0}$ $(\lambda \in L, s \in S)$, and assume that the following five conditions are satisfied.*

(1) F_{λ} converges to F.

(2) $y_{\lambda,s} \to \infty$ for any $s \in S$.

(3) If $1 \leq j < n$, $s \in S_{\leq j}$ and $t \in S_{\geq j+1}$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \to \infty$.

(4) If $1 \leq j \leq n$ and $s, t \in S_j$, then $\frac{y_{\lambda,s}}{y_{\lambda,t}} \to \frac{a_s}{a_t}$.

(5) For $1 \leq j \leq n$ and $e \geq 0$, there exist $F_{\lambda}^* \in \check{D}$ $(\lambda \in L)$ and $y_{\lambda,t}^* \in \mathbf{R}_{>0}$ $(\lambda \in L, t \in S_{\geq j+1})$ such that

$$
\begin{aligned}
&\exp\left(\sum_{t \in S_{\geq j+1}} i y_{\lambda,t}^* N_t\right) F_{\lambda}^* \in \check{D}_j \quad (\lambda : \textit{sufficiently large}), \\
&y_{\lambda,s}^e d(F_{\lambda}, F_{\lambda}^*) \to 0 \quad (\forall s \in S_j), \\
&y_{\lambda,s}^e |y_{\lambda,t} - y_{\lambda,t}^*| \to 0 \quad (\forall s \in S_j, \forall t \in S_{\geq j+1}).
\end{aligned}
$$

Here d is a metric on a neighborhood of F in $\check{D}$ which is compatible with the analytic structure of $\check{D}$.

For each $1 \leq j \leq n$, take $c_j \in S_j$ and denote $N_j := \sum_{s \in S_j} \frac{a_s}{a_{c_j}} N_s$.

Then, $(N_1, \cdots, N_n, F)$ generates a nilpotent orbit, and we have the convergence

$$\exp(\textstyle\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \to \mathrm{class}(\psi(N_1, \cdots, N_n, F)) \quad in \ \ D_{\mathrm{SL}(2)}^I.$$

By [KNU.p] 3.2.12 (i) which characterizes the topology of $D_{\mathrm{SL}(2)}^I$, for the proof of Proposition 4.3.3, it is sufficient to prove the following

Proposition 4.3.4. *Let the assumption be as in 4.3.3. Let $\mathbf{r}_1 \in D$ be the point and τ be the homomorphism which are associated to $(N_1, \ldots, N_n, F)$ as in 4.1.3.*

Then we have the following convergences (1) and (2).

(1) $\tau\big(\sqrt{\frac{y_{\lambda,c_1}}{y_{\lambda,c_2}}}, \ldots, \sqrt{\frac{y_{\lambda,c_n}}{y_{\lambda,c_{n+1}}}}\big) \exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda \to \mathbf{r}_1$ *in D, where $y_{\lambda,c_{n+1}} = 1$.*

(2) $\mathrm{spl}_W(\exp(\sum_{s \in S} iy_{\lambda,s} N_s) F_\lambda) \to \mathrm{spl}_W(\mathbf{r}_1)$ *in $\mathrm{spl}(W)$.*

Here, for $F' \in D$, $\mathrm{spl}_W(F') \in \mathrm{spl}(W)$ denotes the canonical splitting of W associated to F' as in 4.1.1.

(1) of Proposition 4.3.4 is the case $j = k$ (k is as in 4.1.3) of Lemma 4.3.6 below by $\mathbf{r}_1 = \exp(iN_k)\hat{F}_{(k)}$ ([KNU.p] 2.4.8, Claim).

4.3.5. Let the notation be as in 4.3.3. Before stating the lemma, we recall the definition of $\hat{F}_{(j)}$ and τ_j associated to $(N_1, \ldots, N_n, F)$ (cf. [KNU08] 10.1.1, [KNU.p] 2.4.6).

Let $W_0 = W$ and, for $1 \leq j \leq n$, let $W^{(j)} = M(N_1 + \cdots + N_j, W)$.

For $0 \leq j \leq n$, we define an **R**-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$ and the associated splitting $s^{(j)}$ of $W^{(j)}$ inductively starting from $j = n$ and ending at $j = 0$. First, $(W^{(n)}, F)$ is a mixed Hodge structure as is proved by Deligne (see [K86] 5.2.1). Let $(W^{(n)}, \hat{F}_{(n)})$ be the **R**-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(n)}, F)$. Then $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the **R**-split mixed Hodge structure associated to $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$. Then $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure. This process continues. In this way we define $\hat{F}_{(j)}$ downward-inductively as the **R**-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(j)}, \exp(iN_{j+1})\hat{F}_{(j+1)})$, and define $s^{(j)} : \mathrm{gr}^{W^{(j)}} \simeq H_{0,\mathbf{R}}$ to be the splitting of $W^{(j)}$ associated to $\hat{F}_{(j)}$. The splitting s in 4.1.3 is nothing but $s^{(0)}$ ([KNU08] 10.1.2). The j-th component

$$\tau_j : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W) \quad (0 \leq j \leq n)$$

of the homomorphism τ associated to $(N_1, \ldots, N_n, F)$ (4.1.3) is characterized as follows. For $a \in \mathbf{R}^\times$ and $w \in \mathbf{Z}$, $\tau_j(a)$ acts on $s^{(j)}(\mathrm{gr}_w^{W^{(j)}})$ as the multiplication by a^w.

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Lemma 4.3.6. *Let the notation be as in 4.3.3. Let $1 \le j \le n$. Then*

$$\left(\prod_{j \le k \le n} \tau_k \left(\sqrt{\tfrac{y_{\lambda, c_k}}{y_{\lambda, c_{k+1}}}} \right) \right) \exp(\sum_{s \in S_{\ge j}} i y_{\lambda, s} N_s) F_\lambda \to \exp(i N_j) \hat{F}_{(j)} \quad \text{in } \check{D},$$

where $y_{\lambda, c_{n+1}} := 1$. (Recall $N_j := \sum_{s \in S_j} \frac{a_s}{a_{c_j}} N_s$.)

Proof. This is a mixed Hodge version of [KU09] Lemma 6.4.2, and proved by the same method in the proof of that lemma. At the place in the proof of that lemma where we used [KU09] Lemma 6.1.10, we use its mixed Hodge version [KNU08] 10.3. □

4.3.7. We prove Proposition 4.3.4 (2).

We prove the following assertion (A_j) by downward induction on j. (Note that (A_0) is what we want to prove.)

(A_j) : Proposition 4.3.4 (2) is true in the case $\exp\left(\sum_{t \in S_{\ge j+1}} i y_{\lambda, t} N_t \right) F_\lambda \in \check{D}_j$ for all λ. Here $\check{D}_0 = D$.

Proof. Let $0 \le j \le n$.

Let $p_\lambda = \exp(\sum_{s \in S} i y_{\lambda, s} N_s) F_\lambda$, $t_{\lambda, j} = \prod_{j < k \le n} \tau_k \left(\sqrt{\tfrac{y_{\lambda, c_k}}{y_{\lambda, c_{k+1}}}} \right)$, and $p_{\lambda, j} = t_{\lambda, j} p_\lambda$. Then, by [KNU08] 10.3,

$$p_{\lambda, j} = t_{\lambda, j} p_\lambda = \exp\left(\sum_{s \in S_{\le j}} i \tfrac{y_{\lambda, s}}{y_{\lambda, c_{j+1}}} N_s \right) U_{\lambda, j},$$

$$\text{where} \quad U_{\lambda, j} := t_{\lambda, j} \exp(\sum_{t \in S_{\ge j+1}} i y_{\lambda, t} N_t) F_\lambda.$$

Assume $\exp\left(\sum_{t \in S_{\ge j+1}} i y_{\lambda, t} N_t \right) F_\lambda \in \check{D}_j$. Then $(N_1', \ldots, N_j', U_{\lambda, j})$ generates a nilpotent orbit, where $N_k' = \sum_{s \in S_k} \tfrac{y_{\lambda, s}}{y_{\lambda, c_k}} N_s$ $(1 \le k \le j)$. Let s_λ be the associated limit splitting. By [KNU08] 0.5 (2) and 10.8 (1), there is a convergent power series u_λ whose constant term is 1 and whose coefficients depend on $U_{\lambda, j}$ and $y_{\lambda, s}/y_{\lambda, c_k}$ $(1 \le k \le j, s \in S_k)$ real analytically such that $\text{spl}_W(p_{\lambda, j}) = u_\lambda \left(\tfrac{y_{\lambda, c_2}}{y_{\lambda, c_1}}, \ldots, \tfrac{y_{\lambda, c_{j+1}}}{y_{\lambda, c_j}} \right) s_\lambda$. Since s_λ also depends real analytically on $U_{\lambda, j}$, we have

(1) $\text{spl}_W(p_{\lambda, j})$ converges to $\text{spl}_W(\mathbf{r}_1)$.

This already showed (A_n).

Next, assume $j < n$ and that (A_{j+1}) is true. We prove (A_j) is true.

Choose a sufficiently big $e > 0$ depending on $\tau_{j+1}, \ldots, \tau_n$.

Take F_λ^* and $y_{\lambda, t}^*$ $(t \in S_{\ge j+2})$ by the assumption 4.3.3 (5). Let $y_{\lambda, t}^* = y_{\lambda, t}$ $(t \in S_{\le j+1})$. Define p_λ^*, $t_{\lambda, j}^*$ and $p_{\lambda, j}^*$ similarly as p_λ, $t_{\lambda, j}$ and $p_{\lambda, j}$, respectively.

Then we have

(2) $\text{spl}_W(p_{\lambda, j}^*)$ converges to $\text{spl}_W(\mathbf{r}_1)$, and $y_{\lambda, c_{j+1}}^e d(\text{spl}_W(p_{\lambda, j}), \text{spl}_W(p_{\lambda, j}^*)) \to 0$.

By inductive hypothesis on j, we have

(3) $\text{spl}_W(p_\lambda^*) = t_{\lambda, j}^{*-1} \text{spl}_W(p_{\lambda, j}^*) \text{gr}^W(t_{\lambda, j}^*)$ converges to $\text{spl}_W(\mathbf{r}_1)$.

By (1)–(3), we have $\text{spl}_W(p_\lambda) = t_{\lambda, j}^{-1} \text{spl}_W(p_{\lambda, j}) \text{gr}^W(t_{\lambda, j})$ also converges to $\text{spl}_W(\mathbf{r}_1)$. □∎

§5. Proofs of the main results

In this section, we prove the results in §2 and §3.

The results in §3 are reduced to the ones in §2. We explain this reduction in the last subsection 5.6 in this section.

The proofs 5.1–5.5 for the results in §2 are the mixed Hodge theoretic and weak fan versions of [KU09] §7, and arguments are parallel to those in [KU09]. We replace the results and lemmas used there with the corresponding ones here (e.g., the Hausdorffness of $D_{\mathrm{SL}(2)}$ in the pure case ([KU02] 3.14 (ii)) there is replaced by the corresponding result in the mixed Hodge situation ([KNU.p])), and make obvious modifications (e.g., fans are replaced by weak fans, the associated filtration $W(-)$ is replaced by $M(-,W)$, $\tilde{\rho}$ is replaced by τ,...). For the readers' conveniences, we include several important rewritten arguments from those of [KU09] §7, even in the case they are more or less repetitions of loc. cit.

§5.1. Proof of Theorem 2.5.1

We prove Theorem 2.5.1.

5.1.1. The fact that $\tilde{E}_\sigma$ is a log manifold is proved by the similar arguments in the proof of the pure case [KU09] 3.5.10.

5.1.2. We prove that E_σ is open in $\tilde{E}_\sigma$ for the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$.

For a face τ of σ, let $U(\tau)$ be the open set of toric_σ defined to be $\mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}$ where $P \subset \mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z})$ is the inverse image of $\Gamma(\tau)^\vee \subset \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})$ under the canonical map $\mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})$. Let $V(\tau)$ be the open set of $\tilde{E}_\sigma$ defined to be $\tilde{E}_\sigma \cap (U(\tau) \times \check{D})$. Let V be the union of $V(\tau)$ where τ ranges over all faces of σ which are admissible. Then V is open in $\tilde{E}_\sigma$, and $E_\sigma \subset V$. Concerning the canonical projection $V \to \prod_w \tilde{E}_{\sigma(\mathrm{gr}_w^W)}$, E_σ is the inverse image of $\prod_w E_{\sigma(\mathrm{gr}_w^W)}$ in V. By [KU09] 4.1.1 Theorem A (i), each $E_{\sigma(\mathrm{gr}_w^W)}$ is open in $\tilde{E}_{\sigma(\mathrm{gr}_w^W)}$ in the strong topology of $\tilde{E}_{\sigma(\mathrm{gr}_w^W)}$ in $\check{E}_{\sigma(\mathrm{gr}_w^W)}$. Hence E_σ is open in $\tilde{E}_\sigma$ in the strong topology of $\tilde{E}_\sigma$ in $\check{E}_\sigma$.

This shows that E_σ is a log manifold.

§5.2. Study of $D_\sigma^\sharp$

Let Σ be a weak fan, Γ a subgroup of $G_{\mathbf{Z}}$, and assume that they are strongly compatible. Let $\sigma \in \Sigma$.

We prove some results on $D_\sigma^\sharp$.

5.2.1. The actions of $\sigma_{\mathbf{C}}$

$$\sigma_{\mathbf{C}} \times E_\sigma \to E_\sigma, \quad (a, (q, F)) \mapsto (\mathbf{e}(a)q, \exp(-a)F),$$

where $a \in \sigma_{\mathbf{C}}$, $q \in \mathrm{toric}_\sigma$, $F \in \check{D}$ and $(q, F) \in E_\sigma$, and

$$\sigma_{\mathbf{C}} \times E_{\sigma,\mathrm{val}} \to E_{\sigma,\mathrm{val}}, \quad (a, (q, F)) \mapsto (\mathbf{e}(a)q, \exp(-a)F),$$

where $a \in \sigma_{\mathbf{C}}$, $q \in \mathrm{toric}_{\sigma,\mathrm{val}}$, $F \in \check{D}$ and $(q, F) \in E_{\sigma,\mathrm{val}}$, are proved to be continuous by the same argument in the pure case [KU09] 7.2.1. They induce the actions of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ and $E_{\sigma,\mathrm{val}}^\sharp$.

The quotient spaces for these actions are identified as

$$\sigma_{\mathbf{C}} \backslash E_\sigma = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma, \quad \sigma_{\mathbf{C}} \backslash E_{\sigma,\mathrm{val}} = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma,\mathrm{val}},$$
$$i\sigma_{\mathbf{R}} \backslash E_\sigma^\sharp = D_\sigma^\sharp, \quad i\sigma_{\mathbf{R}} \backslash E_{\sigma,\mathrm{val}}^\sharp = D_{\sigma,\mathrm{val}}^\sharp.$$

Proposition 5.2.2. (i) *The action of $\sigma_{\mathbf{C}}$ on E_σ (resp. $E_{\sigma,\mathrm{val}}$) is free.*

(ii) *The action of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ (resp. $E_{\sigma,\mathrm{val}}^\sharp$) is free.*

Here and in the following, free action always means set-theoretically free action. That is, an action of a group H on a set S is free if and only if $hs \neq s$ for any $h \in H - \{1\}$ and $s \in S$.

Before proving 5.2.2, we state a lemma, whose variant in the pure case is [KU09] 6.1.7 (2).

Lemma 5.2.2.1. *Let $(H_{\mathbf{R}}, W, F)$ be an $\mathbf{R}$-mixed Hodge structure, let*

$$\delta(F) \in L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W)) \subset \mathrm{End}_{\mathbf{R}}(H(\mathrm{gr}^W)_{\mathbf{R}})$$

be as in [KU09] §4, [KNU.p] 1.2.2, and let $s' \in \mathrm{spl}(W)$ be the splitting of W characterized by $F = s'(\exp(i\delta(F))F(\mathrm{gr}^W))$ (loc. cit.).
Let $N \in \mathrm{End}_{\mathbf{R}}(H_{\mathbf{R}})$ and assume that $(s')^{-1}Ns' \in L_{\mathbf{R}}^{-1,-1}(F(\mathrm{gr}^W))$ and that $(s')^{-1}Ns'$ commutes with $\delta(F)$. Let $z \in \mathbf{C}$. Then, we have $\delta(\exp(zN)F) = \delta(F) + \mathrm{Im}(z)(s')^{-1}Ns'$.

Proof. Write $z = x + iy$ with x and y real. Let $\delta = \delta(F)$. Apply $\exp(zN)$ to $F = s'(\exp(i\delta)F(\mathrm{gr}^W))$. Then, we have $\exp(zN)F = (\exp(xN)s')\exp(iy(s')^{-1}Ns')\exp(i\delta)F(\mathrm{gr}^W)$. By the commutativity assumption, $\exp(iy(s')^{-1}Ns')\exp(i\delta) = \exp(i(y(s')^{-1}Ns' + \delta))$, and the conclusion follows. $\square$

Proof of 5.2.2. We show here that the action of $\sigma_{\mathbf{C}}$ on E_σ is free. The rest is proved in the similar way.

Assume $a \cdot (q, F) = (q, F)$ ($a \in \sigma_{\mathbf{C}}$, $(q, F) \in E_\sigma$), that is, $\mathbf{e}(a) \cdot q = q$ and $\exp(-a)F = F$.

By the pure case considered in [KU09] 7.2.9, the image of a in $\mathfrak{g}_{\mathbf{C}}(\mathrm{gr}^W)$ is zero. The rest of the proof of 5.2.2 is parallel to the argument in [KU09] 7.2.9 as follows.

By $\mathbf{e}(a) \cdot q = q$, we have $a = b + c$ ($b \in \sigma(q)_{\mathbf{C}} \cap \mathfrak{g}_{\mathbf{C},u}$, $c \in \log(\Gamma(\sigma)^{\mathrm{gp}} \cap G_{\mathbf{Z},u})$), where $\sigma(q)$ is the unique face of σ such that $q \in \mathbf{e}(\sigma_{\mathbf{C}}) \cdot 0_{\sigma(q)}$. On the other hand, taking $\delta(\ ) = \delta(M(\sigma(q), W), \ )$ of $\exp(-a)F = F$, we have $\delta(F) = \delta(\exp(-a)F) = -\mathrm{Im}(b) + \delta(F)$ by 5.2.2.1. It follows $\mathrm{Im}(b) = 0$ and hence $\exp(\mathrm{Re}(b) + c)F = F$. Take an element y of the interior of $\sigma(q)$ such that $F' := \exp(iy)F \in D$. Then $\exp(\mathrm{Re}(b) + c)F' = F'$. Since $\mathrm{Re}(b) + c \in \mathfrak{g}_{\mathbf{R},u}$, we have $\exp(\mathrm{Re}(b) + c)\mathrm{spl}_W(F') = \mathrm{spl}_W(F')$. This proves $\mathrm{Re}(b) + c = 0$ and hence $a = 0$. $\square$

Proposition 5.2.3. *Let σ and σ' be sharp rational nilpotent cones. Let $\alpha \in E_{\sigma,\mathrm{val}}^\sharp$ and $\alpha' \in E_{\sigma',\mathrm{val}}^\sharp$. Assume that $(\mathbf{e}(iy_\lambda), F_\lambda) \in E_{\sigma,\mathrm{val,triv}}^\sharp$ ($y_\lambda \in \sigma_{\mathbf{R}}$, $F_\lambda \in \check{D}$) (resp. $(\mathbf{e}(iy'_\lambda), F'_\lambda) \in E_{\sigma',\mathrm{val,triv}}^\sharp$ ($y'_\lambda \in \sigma'_{\mathbf{R}}$, $F'_\lambda \in \check{D}$)) converges to α (resp. α') in the strong*

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topology (here $(-)_{\mathrm{triv}}$ means the part where the log structure of $E_{\sigma,\mathrm{val}}$ or of $E_{\sigma',\mathrm{val}}$ becomes trivial, respectively), and that

$$\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda \ \ in \ D.$$

Then we have:

  (i) *The images of α and α' in $D^\sharp_{\mathrm{val}}$ coincide.*

  (ii) *$y_\lambda - y'_\lambda$ converges in $\mathfrak{g}_{\mathbf{R}}$.*

Proof. First of all, we remark that the proof below is just a revision of that of [KU09] 7.2.10. Notice that in the course of the proof of Claim A in the mixed Hodge case below, the map β is undefined on D_{spl} when $W^{(1)} = W$. But, in this case, $\mathbf{r}_1 \notin D_{\mathrm{spl}}$, so β can be still applied.

Since the composite map $E^\sharp_{\sigma,\mathrm{val}}$ (resp. $E^\sharp_{\sigma',\mathrm{val}}$) $\to D^\sharp_{\mathrm{val}} \xrightarrow{\psi} D^I_{\mathrm{SL}(2)}$ is continuous by 4.2.7 and Theorem 4.3.2, the image of α (resp. α') under this composite map is the limit of $\exp(iy_\lambda)F_\lambda$ (resp. $\exp(iy'_\lambda)F'_\lambda$) in $D^I_{\mathrm{SL}(2)}$. Since $\exp(iy_\lambda)F_\lambda = \exp(iy'_\lambda)F'_\lambda$ and since $D^I_{\mathrm{SL}(2)}$ is Hausdorff ([KNU.p] 3.5.17 (i)), these images coincide, say $p \in D^I_{\mathrm{SL}(2)}$. Let m be the rank of p and let $\Psi = (W^{(k)})_{1 \le k \le m}$ be the associated family of weight filtrations (4.1.1, 4.1.2). Let (A, V, Z) (resp. (A', V', Z')) be the image of α (resp. α') in $D^\sharp_{\mathrm{val}}$. Take an excellent basis $(N_s)_{s \in S}$ (resp. $(N'_s)_{s \in S'}$) for (A, V, Z) (resp. (A', V', Z')) in the mixed Hodge case (the definition of an excellent basis [KU09] 6.3.8 obviously extends to the mixed Hodge case, and its existence is seen similarly to [KU09] 6.3.9), and let $(a_s)_{s \in S}$ and $(S_j)_{1 \le j \le n}$ (resp. $(a'_s)_{s \in S'}$ and $(S'_j)_{1 \le j \le n'}$) be as in (the mixed Hodge version of) [KU09] 6.3.3. For each l with $1 \le l \le m$, let $f(l)$ (resp. $f'(l)$) be the smallest integer such that $1 \le f(l) \le n$ (resp. $1 \le f'(l) \le n'$), and that $W^{(l)} = M(\sum_{s \in S_{\le f(l)}} \mathbf{R}_{\ge 0} N_s, W)$ (resp. $= M(\sum_{s \in S'_{\le f'(l)}} \mathbf{R}_{\ge 0} N'_s, W)$). Let $g(l)$ (resp. $g'(l)$) be any element of $S_{f(l)}$ (resp. $S'_{f'(l)}$).

Take an $\mathbf{R}$-subspace B of $\sigma_{\mathbf{R}}$ (resp. B' of $\sigma'_{\mathbf{R}}$) such that $\sigma_{\mathbf{R}} = A_{\mathbf{R}} \oplus B$ (resp. $\sigma'_{\mathbf{R}} = A'_{\mathbf{R}} \oplus B'$). Write

$$y_\lambda = \sum_{s \in S} y_{\lambda,s} N_s + b_\lambda, \quad y'_\lambda = \sum_{s \in S'} y'_{\lambda,s} N'_s + b'_\lambda$$

with $y_{\lambda,s}, y'_{\lambda,s} \in \mathbf{R}$, $b_\lambda \in B$, $b'_\lambda \in B'$. We may assume $y_{\lambda,s}, y'_{\lambda,s} > 0$. Then, by the mixed Hodge version of [KU09] 6.4.11, proved similarly, $y_{\lambda,s}$ ($s \in S$), $y'_{\lambda,s}$ ($s \in S'$), $\frac{y_{\lambda,s}}{y_{\lambda,t}}$ ($s \in S_{\le j}$, $t \in S_{\ge j+1}$) and $\frac{y'_{\lambda,s}}{y'_{\lambda,t}}$ ($s \in S'_{\le j}$, $t \in S'_{\ge j+1}$) tend to ∞, $\frac{y_{\lambda,s}}{y_{\lambda,t}}$ ($s, t \in S_j$) tends to $\frac{a_s}{a_t}$, $\frac{y'_{\lambda,s}}{y'_{\lambda,t}}$ ($s, t \in S'_j$) tends to $\frac{a'_s}{a'_t}$, and b_λ and b'_λ converge.

Claim A. $\frac{y'_{\lambda,g'(l)}}{y_{\lambda,g(l)}}$ *converges to an element of $\mathbf{R}_{>0}$ for $1 \le l \le m$.*

We prove this claim. The assumption of Proposition 4.3.3 is satisfied if we take $(N_s)_{s \in S}$, a_s, S_j, $y_{\lambda,s}$ as above the same ones in 4.3.3, the above $\exp(b_\lambda)F_\lambda$ as F_λ in 4.3.3, the limit of the above $\exp(b_\lambda)F_\lambda$ as F in 4.3.3. Let $\tau : (\mathbf{R}^\times)^m \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}}, W)$ be the homomorphism associated to p (4.1.1, 4.1.2). Then, by 4.3.4 (1), we have

$$\tau\left(\sqrt{\frac{y_{\lambda,g(1)}}{y_{\lambda,g(2)}}}, \ldots, \sqrt{\frac{y_{\lambda,g(m)}}{y_{\lambda,g(m+1)}}}\right) \exp(iy_\lambda)F_\lambda \to \mathbf{r}_1 \quad \text{in } D.$$

See 4.1.3 for $\mathbf{r}_1$. Similarly, for some $t = (t_1, \ldots, t_m) \in \mathbf{R}^m_{>0}$, we have

$$\tau\left(\sqrt{\frac{y'_{\lambda, g'(1)}}{y'_{\lambda, g'(2)}}}, \ldots, \sqrt{\frac{y'_{\lambda, g'(m)}}{y'_{\lambda, g'(m+1)}}}\right) \exp(iy'_\lambda) F'_\lambda \to \tau(t) \cdot \mathbf{r}_1 \quad \text{in } D$$

(4.1.2). Here $y_{\lambda, g(m+1)} = y'_{\lambda, g'(m+1)} = 1$. Recall that $\Psi = \{W^{(1)}, \ldots, W^{(m)}\}$ is the set of weight filtrations associated to p. In the case $W \notin \Psi$ (resp. $W \in \Psi$), take a distance to Ψ-boundary

$$\beta : D \to \mathbf{R}^m_{>0} \quad (\text{resp. } D_{\text{nspl}} := D \smallsetminus D_{\text{spl}} \to \mathbf{R}^m_{>0})$$

in [KNU.p] 3.2.5. So $\beta(\tau(t)x) = t\beta(x)$ ($x \in D$ (resp. D_{nspl}), $t \in \mathbf{R}^m_{>0}$). Note that, in the case $W \in \Psi$, $\mathbf{r}_1 \in D_{\text{nspl}}$. Applying β to the above convergences, taking their ratios, and using $\exp(iy_\lambda) F_\lambda = \exp(iy'_\lambda) F'_\lambda$, we have

$$\frac{y'_{\lambda, g'(l)}}{y_{\lambda, g(l)}} \to t_l^2 \quad (1 \leq l \leq m).$$

Claim A is proved.

Next, we prove the following Claims B_l and C_l ($1 \leq l \leq m+1$) by induction on l.

Claim B_l. $y^{-1}_{\lambda, g(l)}\left(\sum_{s \in S_{\leq f(l)-1}} y_{\lambda, s} N_s - \sum_{s \in S'_{\leq f'(l)-1}} y'_{\lambda, s} N'_s\right)$ converges.

Claim C_l. $\sum_{s \in S_{\leq f(l)-1}} \mathbf{Q} N_s = \sum_{s \in S'_{\leq f'(l)-1}} \mathbf{Q} N'_s$.

Here $y_{\lambda, g(m+1)} := 1$, $S_{\leq f(m+1)-1} := S$, $S'_{\leq f'(m+1)-1} := S'$.

Note that Proposition 5.2.3 follows from Claims B_{m+1} and C_{m+1}. In fact, $A = A'$ follows from Claim C_{m+1}, $V = V'$ follows from Claim B_{m+1}, and $Z = Z'$ follows from the facts that the limit of $\exp(b_\lambda) F_\lambda$ (resp. $\exp(b'_\lambda) F'_\lambda$) is an element of Z (resp. Z') and that these limits coincide by Claim B_{m+1}, Claim C_{m+1} and the assumption $\exp(iy_\lambda) F_\lambda = \exp(iy'_\lambda) F'_\lambda$.

We prove these claims. First, Claims B_1 and C_1 are trivial by definition. Assume $l > 1$. By the hypothesis Claim C_{l-1} of induction, N_s ($s \in S_{\leq f(l-1)-1}$) and elements of σ' are commutative. Hence, by the formula $\exp(x_1 + x_2) = \exp(x_1) \exp(x_2)$ if $x_1 x_2 = x_2 x_1$ and by the assumption $\exp(iy_\lambda) F_\lambda = \exp(iy'_\lambda) F'_\lambda$, we have

$$\exp\left(iy_\lambda - \sum_{s \in S_{\leq f(l-1)-1}} iy_{\lambda, s} N_s\right) F_\lambda = \exp\left(iy'_\lambda - \sum_{s \in S_{\leq f(l-1)-1}} iy_{\lambda, s} N_s\right) F'_\lambda.$$

Applying $\prod_{k=l}^m \tau_k\left(\sqrt{\frac{y_{\lambda, g(k)}}{y_{\lambda, g(k+1)}}}\right)$ to this and using [KNU08] 10.3, we obtain

$$(1) \qquad \exp\left(\sum_{f(l-1) \leq j < f(l)} \sum_{s \in S_j} i\frac{y_{\lambda, s}}{y_{\lambda, g(l)}} N_s\right)$$
$$\cdot \prod_{k \geq l} \tau_k\left(\sqrt{\frac{y_{\lambda, g(k)}}{y_{\lambda, g(k+1)}}}\right) \exp\left(\sum_{s \in S_{\geq f(l)}} iy_{\lambda, s} N_s + ib_\lambda\right) F_\lambda$$
$$= \exp\left(\sum_{s \in S'_{\leq f'(l)-1}} i\frac{y'_{\lambda, s}}{y_{\lambda, g(l)}} N'_s - \sum_{s \in S_{\leq f(l-1)-1}} i\frac{y_{\lambda, s}}{y_{\lambda, g(l)}} N_s\right)$$
$$\cdot \prod_{k \geq l} \tau_k\left(\sqrt{\frac{y_{\lambda, g(k)}}{y_{\lambda, g(k+1)}}}\right) \exp\left(\sum_{s \in S'_{\geq f'(l)}} iy'_{\lambda, s} N'_s + ib'_\lambda\right) F'_\lambda.$$

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By Lemma 4.3.6,

$$(2) \qquad \prod_{k \geq l} \tau_k \left(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}} \right) \exp \left(\sum_{s \in S_{\geq f(l)}} iy_{\lambda,s} N_s + ib_\lambda \right) F_\lambda \text{ and}$$

$$\prod_{k \geq l} \tau_k \left(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}} \right) \exp \left(\sum_{s \in S'_{\geq f'(l)}} iy'_{\lambda,s} N'_s + ib'_\lambda \right) F'_\lambda \text{ converge in } \check{D}.$$

Let d (resp. d') be a metric on a neighborhood in $\check{D}$ of the limit of F_λ (resp. F'_λ) which is compatible with the analytic structure. Let $e \geq 1$ be an integer. Then, since $(\mathbf{e}(iy_\lambda), F_\lambda)$ (resp. $(\mathbf{e}(iy'_\lambda), F'_\lambda)$) converges to α (resp. α') in the strong topology, there exist, by [KU09] 3.1.6, $y^*_\lambda = \sum_{s \in S} y^*_{\lambda,s} N_s + b^*_\lambda \in \sigma_{\mathbf{R}}$ ($b^*_\lambda \in B_{\mathbf{R}}$), $y'^*_\lambda = \sum_{s \in S'} y'^*_{\lambda,s} N'_s + b'^*_\lambda \in \sigma'_{\mathbf{R}}$ ($b'^*_\lambda \in B'_{\mathbf{R}}$) and $F^*_\lambda, F'^*_\lambda \in \check{D}$ having the following three properties:

(3) $y^e_{\lambda,g(l-1)}(y_\lambda - y^*_\lambda)$, $y^e_{\lambda,g(l-1)}(y'_\lambda - y'^*_\lambda)$, $y^e_{\lambda,g(l-1)}d(F_\lambda, F^*_\lambda)$, and $y^e_{\lambda,g(l-1)}d(F'_\lambda, F'^*_\lambda)$ converge to 0.

(4) $y_{\lambda,s} = y^*_{\lambda,s}$ ($s \in S_{\leq f(l-1)}$) and $y'_{\lambda,s} = y'^*_{\lambda,s}$ ($s \in S'_{\leq f'(l-1)}$).

(5) $((N_s)_{s \in S_{\leq f(l)-1}}, \exp \left(\sum_{s \in S_{\geq f(l)}} iy^*_{\lambda,s} N_s + b^*_\lambda \right) F^*_\lambda)$ and $((N'_s)_{s \in S'_{\leq f'(l)-1}}, \exp \left(\sum_{s \in S'_{\geq f'(l)}} iy'^*_{\lambda,s} N'_s + b'^*_\lambda \right) F'^*_\lambda)$ generate nilpotent orbits.

Take e sufficiently large. Then, by the hypothesis Claim B_{l-1} of induction and by (1) and (2), the difference between $\delta(W^{(l-1)}, \ )$ of

$$\exp \left(\sum_{f(l-1) \leq j < f(l)} \sum_{s \in S_j} i \frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s \right)$$
$$\cdot \prod_{k \geq l} \tau_k \left(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}} \right) \exp \left(\sum_{s \in S_{\geq f(l)}} iy^*_{\lambda,s} N_s + ib^*_\lambda \right) F^*_\lambda$$

and $\delta(W^{(l-1)}, \ )$ of

$$\exp \left(\sum_{s \in S'_{\leq f'(l)-1}} i \frac{y'_{\lambda,s}}{y_{\lambda,g(l)}} N'_s - \sum_{s \in S_{\leq f(l-1)-1}} i \frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s \right)$$
$$\cdot \prod_{k \geq l} \tau_k \left(\sqrt{\frac{y_{\lambda,g(k)}}{y_{\lambda,g(k+1)}}} \right) \exp \left(\sum_{s \in S'_{\geq f'(l)}} iy'^*_{\lambda,s} N'_s + ib'^*_\lambda \right) F'^*_\lambda$$

converges to 0. By 5.2.2.1, the former is equal to

$$\sum_{f(l-1) \leq j < f(l)} \sum_{s \in S_j} i \frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s + (\text{a term which converges}),$$

and the latter is equal to

$$\sum_{s \in S'_{\leq f'(l)-1}} i \frac{y'_{\lambda,s}}{y_{\lambda,g(l)}} N'_s - \sum_{s \in S_{\leq f(l-1)-1}} i \frac{y_{\lambda,s}}{y_{\lambda,g(l)}} N_s + (\text{a term which converges}).$$

This proves Claim B_l. Claim C_l follows from Claim B_l easily. $\square$

We recall the notion of proper action. In this series of papers, a continuous map of topological spaces is said to be proper if it is proper in the sense of [B66] and is separated.

Definition 5.2.4 ([B66] Ch.3 §4 no.1 Definition 1). Let H be a Hausdorff topological group acting continuously on a topological space X. H is said to act properly on X if the map

$$H \times X \to X \times X, \quad (h, x) \mapsto (x, hx),$$

is proper.

As is explained above, the meaning of properness of a continuous map is slightly different from that in [B66]. However, for a continuous action of a Hausdorff topological group H on a topological space X, the map $H \times X \to X \times X, (h, x) \mapsto (x, hx)$ is always separated, and hence this map is proper in our sense if and only if it is proper in the sense of [B66].

The following 5.2.4.1–5.2.4.7 hold. Recall (5.2.2) that when we say an action is free, it means free set-theoretically.

5.2.4.1. If a Hausdorff topological group H acts properly on a topological space X, then the quotient space $H\backslash X$ is Hausdorff.

5.2.4.2. If a discrete group H acts properly and freely on a Hausdorff space X, then the projection $X \to H\backslash X$ is a local homeomorphism.

5.2.4.3. Let H be a Hausdorff topological group acting continuously on topological spaces X and X'. Let $\psi : X \to X'$ be an equivariant continuous map.

(i) If ψ is proper and surjective and if H acts properly on X, then H acts properly on X'.

(ii) If H acts properly on X' and if X is Hausdorff, then H acts properly on X.

5.2.4.4. Assume that a Hausdorff topological group H acts on a topological space X continuously and freely. Let X' be a dense subset of X. Then, the following two conditions (1) and (2) are equivalent.

(1) The action of H on X is proper.

(2) Let $x, y \in X$, L be a directed set, $(x_\lambda)_{\lambda \in L}$ be a family of elements of X' and $(h_\lambda)_{\lambda \in L}$ be a family of elements of H, such that $(x_\lambda)_\lambda$ (resp. $(h_\lambda x_\lambda)_\lambda$) converges to x (resp. y). Then $(h_\lambda)_\lambda$ converges to an element h of H and $y = hx$.

If X is Hausdorff, these equivalent conditions are also equivalent to the following condition (3).

(3) Let L be a directed set, $(x_\lambda)_{\lambda \in L}$ be a family of elements of X' and $(h_\lambda)_{\lambda \in L}$ be a family of elements of H, such that $(x_\lambda)_\lambda$ and $(h_\lambda x_\lambda)_\lambda$ converge in X. Then $(h_\lambda)_\lambda$ converges in H.

5.2.4.5. Let H be a Hausdorff topological group acting continuously on Hausdorff topological spaces X_1 and X_2. Let H_1 be a closed normal subgroup of H, and assume that the action of H on X_2 factors through $H_2 := H/H_1$. Assume that for $j = 1, 2$, the action of H_j on X_j is proper and free. Assume further that there are a neighborhood U of 1 in H_2 and a continuous map $s : U \to H$ such that the composition $U \xrightarrow{s} H \to H_2$ is the inclusion map. Then the diagonal action of H on $X_1 \times X_2$ is proper and free.

5.2.4.6. Let H be a topological group acting continuously on a Hausdorff topological space X. Assume that $Y := H \backslash X$ is Hausdorff and assume that X is an H-torsor over Y in the category of topological spaces. Then the action of H on X is proper.

The following 5.2.4.7 is not related to a group action, but we put it here for 5.2.4.1 and 5.2.4.3 (i) imply it in the case $H = \{1\}$.

5.2.4.7. Let $f : X \to Y$ be a continuous map between topological spaces and assume that f is proper and surjective. Assume X is Hausdorff. Then Y is Hausdorff.

For 5.2.4.1, 5.2.4.2, 5.2.4.3, see [B66] Ch.3 §4 no.2 Proposition 3, ibid. Ch.3 §4 no.4 Corollary, ibid. Ch.3 §2 no.2 Proposition 5, respectively. 5.2.4.4 is [KU09] Lemma 7.2.7. See [KU09] for the proof of it.

5.2.4.5 is proved as follows. The freeness is clear. We prove the properness. Let L be a directed set, let $(x_\lambda, y_\lambda)_{\lambda \in L}$ be a family of elements of $X_1 \times X_2$, let $(h_\lambda)_{\lambda \in L}$ be a family of elements of H. Assume that x_λ, y_λ, $h_\lambda x_\lambda$, $h_\lambda y_\lambda$ converge. By 5.2.4.4, it is sufficient to prove that h_λ converges. Let $\bar{h}_\lambda$ be the image of h_λ in H_2. Since the action of H_2 on X_2 is proper, $\bar{h}_\lambda$ converges by 5.2.4.4. Let $h \mod H_1$ ($h \in H$) be the limit. By replacing h_λ by $h^{-1}h_\lambda$, we may assume $\bar{h}_\lambda \to 1$. By replacing h_λ by $s(\bar{h}_\lambda)^{-1}h_\lambda$, we may assume $h_\lambda \in H_1$. Since the action of H_1 on X_1 is proper, h_λ converges by 5.2.4.4.

5.2.4.6 is proved as follows. Let L be a directed set, let $(x_\lambda)_{\lambda \in L}$ be a family of elements of X, and let $(h_\lambda)_{\lambda \in L}$ be a family of elements of H. Assume that x_λ converges to $x \in X$ and $h_\lambda x_\lambda$ converges to $y \in X$. By 5.2.4.4, it is sufficient to prove that h_λ converges. Since $Y = H \backslash X$ is Hausdorff, the images of x and y in Y coincide. Let $z \in Y$ be their image. By replacing Y by a sufficiently small neighborhood of z, we may assume that $X = H \times Y$ with the evident action of H. Then the convergence of $(h_\lambda)_\lambda$ is clear.

Proposition 5.2.5. *The action of $i\sigma_{\mathbf{R}}$ on $E_\sigma^\sharp$ (resp. $E_{\sigma,\mathrm{val}}^\sharp$) is proper.*

Proof. The proof is exactly the same as in the pure case [KU09] 7.2.11, that is, it reduces to 5.2.3 (ii) by 5.2.4.3 (i) and 5.2.4.4. $\square$

By 5.2.5 and 5.2.4.1, we have the following result.

Corollary 5.2.6. *The spaces $D_\sigma^\sharp$ and $D_{\sigma,\mathrm{val}}^\sharp$ are Hausdorff.*

This corollary will be generalized in §5.3 by replacing σ with Σ (5.3.3).

The case (b) of the following Lemma will be used in §5.4 later.

Lemma 5.2.7. *Let $\mathcal{C}$ be either one of the following two categories*:

   (a) *the category of topological spaces.*

   (b) *the category of log manifolds* (1.1.5).

In the case (a) (resp. (b)), let H be a topological group (resp. a complex analytic group), X an object of $\mathcal{C}$ and assume that we have an action $H \times X \to X$ in $\mathcal{C}$. (In the case (b), we regard H as having the trivial log structure.) Assume this action is proper (5.2.4) topologically and is free set-theoretically. Assume moreover the following condition (1) is satisfied.

(1) *For any point $x \in X$, there exist an object S of $\mathcal{C}$, a morphism $\iota : S \to X$ of $\mathcal{C}$ whose image contains x and an open neighborhood U of 1 in H such that $U \times S \to X$, $(h, s) \mapsto h\iota(s)$, induces an isomorphism in $\mathcal{C}$ from $U \times S$ onto an open set of X.*

Then:

(i) *In the case* (b), *the quotient topological space $H \backslash X$ has a unique structure of an object of $\mathcal{C}$ such that, for an open set V of $H \backslash X$, $\mathcal{O}_{H \backslash X}(V)$ (resp. $M_{H \backslash X}(V)$) is the set of all functions on V whose pull-backs to the inverse image V' of V in X belong to $\mathcal{O}_X(V')$ (resp. $M_X(V')$).*

(ii) $X \to H \backslash X$ *is an H-torsor in the category $\mathcal{C}$.*

Proof. This is [KU09] Lemma 7.3.3. See [KU09] for the proof.

Proposition 5.2.8. $E_\sigma^\sharp \to i\sigma_{\mathbf{R}} \backslash E_\sigma^\sharp = D_\sigma^\sharp$ *and* $E_{\sigma,\mathrm{val}}^\sharp \to i\sigma_{\mathbf{R}} \backslash E_{\sigma,\mathrm{val}}^\sharp = D_{\sigma,\mathrm{val}}^\sharp$ *are $i\sigma_{\mathbf{R}}$-torsors in the category of topological spaces.*

Proof. This is proved by using the arguments in [KU09] 7.3.5 for the pure case. We apply Lemma 5.2.7 by taking $H = i\sigma_{\mathbf{R}}$, $X = E_\sigma^\sharp$, and $\mathcal{C}$ to be the category of topological spaces as in 5.2.7 (a). By Proposition 5.2.5, Proposition 5.2.2 and Lemma 5.2.7, it is sufficient to prove that the condition 5.2.7 (1) is satisfied. Let $x = (q, F) \in E_\sigma^\sharp$. Let $S_1 \subset E_\sigma$ be the log manifold containing x constructed in the same way as the log manifold denoted by S in the pure case [KU09] 7.3.5. (In the argument to construct this log manifold, we replace $W(\sigma(q))[-w]$ there by $M(\sigma(q), W)$ here.) Let $S_2 = E_\sigma^\sharp \cap S_1$, let U be a sufficiently small neighborhood of 0 in $\sigma_{\mathbf{R}}$, and let $S = \{(q', \exp(a)F') \,|\, (q', F') \in S_2, a \in U\}$. Then S has the desired property. $\square$

§5.3. STUDY OF $D_\Sigma^\sharp$

Let Σ be a weak fan. We prove some results on $D_\Sigma^\sharp$.

Theorem 5.3.1. *For $\sigma \in \Sigma$, the inclusion maps $D_\sigma^\sharp \to D_\Sigma^\sharp$ and $D_{\sigma,\mathrm{val}}^\sharp \to D_{\Sigma,\mathrm{val}}^\sharp$ are open maps.*

Proof. We prove that $D_\sigma^\sharp \to D_\Sigma^\sharp$ is an open map. For $\sigma, \tau \in \Sigma$, $D_\sigma^\sharp \cap D_\tau^\sharp$ is the union of $D_\alpha^\sharp$, where α ranges over all elements of Σ having the following property: α is a face of σ and a face of τ at the same time. Hence, by the definition of the topology of $D_\Sigma^\sharp$, it is sufficient to prove the following (1).

(1) If $\sigma \in \Sigma$ and τ is a face of σ, the inclusion $D_\tau^\sharp \to D_\sigma^\sharp$ is a continuous open map.

We prove (1). Let the open set $U(\tau)$ of toric_σ be as in 5.1.2, and let $|U|(\tau)$ be the open set $U(\tau) \cap |\mathrm{toric}|_\sigma$ of $|\mathrm{toric}|_\sigma$. Then $|\mathrm{toric}|_\tau \subset |U|(\tau)$ as subsets of $|\mathrm{toric}|_\sigma$. Let $|\tilde{U}|(\tau) \subset E_\sigma^\sharp$ be the inverse image of $|U|(\tau)$ under $E_\sigma^\sharp \to |\mathrm{toric}|_\sigma$. We have commutative diagrams of topological spaces

$$
\begin{array}{ccccc}
E_\tau^\sharp & \overset{\subset}{\longrightarrow} & |\tilde{U}|(\tau) & \overset{\subset}{\longrightarrow} & E_\sigma^\sharp \\
\downarrow & & \downarrow & & \downarrow \\
|\mathrm{toric}|_\tau & \overset{\subset}{\longrightarrow} & |U|(\tau) & \overset{\subset}{\longrightarrow} & |\mathrm{toric}|_\sigma,
\end{array}
\qquad
\begin{array}{ccc}
E_\tau^\sharp & \overset{\subset}{\longrightarrow} & E_\sigma^\sharp \\
\downarrow & & \downarrow \\
D_\tau^\sharp & \overset{\subset}{\longrightarrow} & D_\sigma^\sharp.
\end{array}
$$

Furthermore, the inverse image of $D_\tau^\sharp$ under $E_\sigma^\sharp \to D_\sigma^\sharp$ coincides with $|\tilde{U}|(\tau)$. The second diagram shows that $D_\tau^\sharp \to D_\sigma^\sharp$ is continuous. Let B be an **R**-vector subspace of $\sigma_{\mathbf{R}}$ such that $\tau_{\mathbf{R}} \oplus B = \sigma_{\mathbf{R}}$. Then we have a homeomorphism

$$|\text{toric}|_\tau \times B \overset{\sim}{\to} |U|(\tau), \quad (a, b) \mapsto \mathbf{e}(ib)a.$$

From this, we see that the projection $|\tilde{U}|(\tau) \to D_\tau^\sharp$ factors through $|\tilde{U}|(\tau) \to E_\tau^\sharp$ which sends $(\mathbf{e}(ib)a, F)$ to $(a, \exp(-ib)F)$. This shows that a subset U of $D_\tau^\sharp$ is open in $D_\sigma^\sharp$, that is, its inverse image in $E_\sigma^\sharp$ is open, if and only if it is open in $D_\tau^\sharp$, that is, its inverse image in $E_\tau^\sharp$ is open. This completes the proof of (1).

The val version is proved similarly. $\square$

Proposition 5.3.2. *The map $D_{\Sigma,\text{val}}^\sharp \to D_\Sigma^\sharp$ is proper.*

Proof. By Theorem 5.3.1, we are reduced to the case $\Sigma = \{\text{face of } \sigma\}$. In this case, by Proposition 5.2.8, we are reduced to the fact that $E_{\sigma,\text{val}}^\sharp \to E_\sigma^\sharp$ is proper, and hence to the fact that $|\text{toric}|_{\sigma,\text{val}} \to |\text{toric}|_\sigma$ is proper (cf. 4.2.5). $\square$

Proposition 5.3.3. *$D_\Sigma^\sharp$ and $D_{\Sigma,\text{val}}^\sharp$ are Hausdorff spaces.*

Proof. The statement for $D_\Sigma^\sharp$ follows from that for $D_{\Sigma,\text{val}}^\sharp$, since $D_{\Sigma,\text{val}}^\sharp \to D_\Sigma^\sharp$ is proper by Proposition 5.3.2 and is surjective.

We prove that $D_{\Sigma,\text{val}}^\sharp$ is Hausdorff. By Theorem 5.3.1, it is sufficient to prove the following (1).

(1) Let σ and σ' be rational nilpotent cones and let $\beta \in D_{\sigma,\text{val}}^\sharp$ and $\beta' \in D_{\sigma',\text{val}}^\sharp$. Assume $x_\lambda \in D$ converges to β in $D_{\sigma,\text{val}}^\sharp$ and to β' in $D_{\sigma',\text{val}}^\sharp$. Then $\beta = \beta'$ in $D_{\text{val}}^\sharp$.

We prove this. By Proposition 5.2.8, there exist an open neighborhood U of β in $D_{\sigma,\text{val}}^\sharp$ (resp. U' of β' in $D_{\sigma',\text{val}}^\sharp$) and a continuous section $s_\sigma : U \to E_{\sigma,\text{val}}^\sharp$ (resp. $s_{\sigma'} : U' \to E_{\sigma',\text{val}}^\sharp$) of the projection $E_{\sigma,\text{val}}^\sharp \to D_{\sigma,\text{val}}^\sharp$ (resp. $E_{\sigma',\text{val}}^\sharp \to D_{\sigma',\text{val}}^\sharp$). Write

$$s_\sigma(x_\lambda) = (\mathbf{e}(iy_\lambda), F_\lambda), \quad s_{\sigma'}(x_\lambda) = (\mathbf{e}(iy_\lambda'), F_\lambda').$$

Write $\alpha = s_\sigma(\beta)$, $\alpha' = s_{\sigma'}(\beta')$. Then the assumption of Proposition 5.2.3 is satisfied. Hence we have $\beta = \beta'$ by Proposition 5.2.3 (i). $\square$

Lemma 5.3.4. *Let V be a vector space over a field K and let $N, h : V \to V$ be K-linear maps such that $Nh = hN$. Let I be an increasing filtration on V with $I_k = 0$ for $k \ll 0$ and $I_k = V$ for $k \gg 0$. Assume that $NI_k \subset I_k$ and $hI_k \subset I_{k-1}$ for all k, and that the relative monodromy filtration M of N relative to I exists. Then $\text{gr}^M(h) = 0$.*

Proof. We prove the following statement (A_j) for $j \geq 1$, by induction on j.

(A_j) $hI_k \, \text{gr}_l^M \subset I_{k-j} \, \text{gr}_l^M$ for any $k, l \in \mathbf{Z}$.

First, (A_1) holds by the assumption $hI_k \subset I_{k-1}$. Let $j \geq 1$, and assume that (A_j) holds. We prove that (A_{j+1}) holds. It is sufficient to prove that the map $\text{gr}_k^I \, \text{gr}_l^M \to \text{gr}_{k-j}^I \, \text{gr}_l^M$ induced by h is the zero map for any $k, l \in \mathbf{Z}$. Note $\text{gr}_k^I \, \text{gr}_l^M = (I_k \cap$

$M_l)/(I_{k-1} \cap M_l + I_k \cap M_{l-1}) = \mathrm{gr}_l^M \, \mathrm{gr}_k^I$. We prove that the map $\mathrm{gr}_l^M \, \mathrm{gr}_k^I \to \mathrm{gr}_l^M \, \mathrm{gr}_{k-j}^I$ induced by h is the zero map. If $l \geq k$, we denote the kernel of $N^{(l-k)+1} : \mathrm{gr}_l^M \, \mathrm{gr}_k^I \to \mathrm{gr}_{2k-l-2}^M \, \mathrm{gr}_k^I$ by $P(\mathrm{gr}_l^M \, \mathrm{gr}_k^I)$. By [D80] 1.6, $\mathrm{gr}_l^M \, \mathrm{gr}_k^I$ is the sum of the images of $N^s : P(\mathrm{gr}_{l'}^M \, \mathrm{gr}_k^I) \to \mathrm{gr}_l^M \, \mathrm{gr}_k^I$, where (s, l') ranges over all pairs of integers such that $l' \geq k$, $s \geq 0$, and $l = l' - 2s$. Hence it is sufficient to prove that the map $P(\mathrm{gr}_l^M \, \mathrm{gr}_k^I) \to \mathrm{gr}_l^M \, \mathrm{gr}_{k-j}^I$ induced by h is the zero map for any $k, l \in \mathbf{Z}$ such that $l \geq k$. Let $x \in \mathrm{gr}_l^M \, \mathrm{gr}_{k-j}^I$ be an element of the image of $P(\mathrm{gr}_l^M \, \mathrm{gr}_k^I)$ under this map. By the definition of $P(\mathrm{gr}_l^M \, \mathrm{gr}_k^I)$ and by $Nh = hN$, the map $N^{l-k+1} : \mathrm{gr}_l^M \, \mathrm{gr}_{k-j}^I \to \mathrm{gr}_{l-2(l-k+1)}^M \, \mathrm{gr}_{k-j}^I$ kills x. On the other hand, since M is the monodromy filtration of N relative to I, the map $N^{l-(k-j)} : \mathrm{gr}_l^M \, \mathrm{gr}_{k-j}^I \to \mathrm{gr}_{l-2(l-(k-j))}^M \, \mathrm{gr}_{k-j}^I$ is an isomorphism. Since $l - (k-j) \geq l - k + 1$, we have $x = 0$. $\square$

Let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

Theorem 5.3.5. *Assume that Γ is neat.*

    (i) *Let $p \in D_\Sigma^\sharp$, $\gamma \in \Gamma$, and assume $\gamma p = p$. Then $\gamma = 1$.*

    (ii) *Let $p \in D_\Sigma$, $\gamma \in \Gamma$, and assume $\gamma p = p$. Then $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$.*

Proof. Let $(\sigma, Z) \in D_\Sigma$ and $\gamma \in \Gamma$. Assume $\gamma(\sigma, Z) = (\sigma, Z)$, that is, $\mathrm{Ad}(\gamma)(\sigma) = \sigma$, $\gamma Z = Z$. We prove first

(1) $\gamma N = N\gamma$ for any $N \in \sigma_{\mathbf{C}}$.

Since $\gamma \Gamma(\sigma) \gamma^{-1} = \Gamma(\sigma)$, we have an automorphism $\mathrm{Int}(\gamma) : y \mapsto \gamma y \gamma^{-1}$ $(y \in \Gamma(\sigma))$ of the sharp fs monoid $\Gamma(\sigma)$. Since the automorphism group of a sharp fs monoid is finite (see for example, [KU09] 7.4.4; an alternative proof: see the generators of the 1-faces), this automorphism of $\Gamma(\sigma)$ is of finite order. Since $\sigma_{\mathbf{C}}$ is generated over $\mathbf{C}$ by $\log(\Gamma(\sigma))$, the $\mathbf{C}$-linear map $\mathrm{Ad}(\gamma) : \sigma_{\mathbf{C}} \to \sigma_{\mathbf{C}}$, $y \mapsto \gamma y \gamma^{-1}$, is of finite order. On the other hand, any eigenvalue a of this $\mathbf{C}$-linear map is equal to bc^{-1} for some eigenvalues b, c of the $\mathbf{C}$-linear map $\gamma : H_{0,\mathbf{C}} \to H_{0,\mathbf{C}}$, and hence the neat property of Γ shows $a = 1$. Thus we have $\gamma y \gamma^{-1} = y$ for any $y \in \sigma_{\mathbf{C}}$.

Let $M := M(\sigma, W)$. By (1), we have $\gamma M = M$.

Take $F \in Z$. Since $\gamma Z = Z$, there exists $N \in \sigma_{\mathbf{C}}$ such that

(2) $$\gamma F = \exp(N)F.$$

Recall that the pure Hodge theoretic version of 5.3.5 is proved in [KU09] 7.4.5, and its proof shows $\log(\mathrm{gr}^W(\gamma)) = \mathrm{gr}^W(N)$. Hence $\mathrm{gr}^W(\gamma)$ is in the image of $\Gamma(\sigma)^{\mathrm{gp}}$, and we may assume $\mathrm{gr}^W(\gamma) = 1$.

Since $\exp(N)$ acts on $\mathrm{gr}^M := (\mathrm{gr}^M)_{\mathbf{C}}$ trivially, $\mathrm{gr}^M(\gamma)F(\mathrm{gr}^M) = F(\mathrm{gr}^M)$ follows.

Claim 1. $\mathrm{gr}^M(\gamma) = 1$.

This follows from (1) and Lemma 5.3.4 applied to $h = \gamma - 1$.

Claim 2. $\log(\gamma)M_k \subset M_{k-2} \quad (\forall k \in \mathbf{Z})$.

In fact, by (2), we have

(3) $$(\log(\gamma) - N)F^p \subset F^p \quad (\forall p \in \mathbf{Z}).$$

Since $N(M_k) \subset M_{k-2}$, (3) shows that the map $\mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$ induced by $\log(\gamma)$, which we denote as $\mathrm{gr}_{-1}^M(\log(\gamma)) : \mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$, satisfies

(4) $$\mathrm{gr}_{-1}^M(\log(\gamma))F^p(\mathrm{gr}_k^M) \subset F^p(\mathrm{gr}_{k-1}^M) \quad (\forall k, \forall p).$$

By taking the complex conjugation of (4), we have

(5) $$\mathrm{gr}_{-1}^M(\log(\gamma))\overline{F}^p(\mathrm{gr}_k^M) \subset \overline{F}^p(\mathrm{gr}_{k-1}^M) \quad (\forall k, \forall p).$$

Since

$$\mathrm{gr}_k^M = \bigoplus_{p+q=k} F^p(\mathrm{gr}_k^M) \cap \overline{F}^q(\mathrm{gr}_k^M) \quad \text{and}$$
$$F^p(\mathrm{gr}_{k-1}^M) \cap \overline{F}^q(\mathrm{gr}_{k-1}^M) = 0 \quad \text{for } p+q = k > k-1,$$

(4) and (5) show that the map $\mathrm{gr}_{-1}^M(\log(\gamma)) : \mathrm{gr}_k^M \to \mathrm{gr}_{k-1}^M$ is the zero map. This proves Claim 2.

By (3), $\log(\gamma)F^p \subset F^{p-1}$, $\log(\gamma)\overline{F}^p \subset \overline{F}^{p-1}$ $(\forall p \in \mathbf{Z})$. These and Claim 2 show, by [KU09] Lemma 6.1.8 (iv),

$$\log(\gamma), N \in L^{-1,-1}(M, F).$$

Hence, by [KU09] Lemma 6.1.8 (iii), $\gamma F = \exp(N)F$ proves $\log(\gamma) = N$. This proves $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$.

If $\gamma(\sigma, Z') = (\sigma, Z')$ for some $(\sigma, Z') \in D_\Sigma^\sharp$ then, for $Z := \exp(\sigma_{\mathbf{R}})Z'$, we have $(\sigma, Z) \in D_\Sigma$ and $\gamma(\sigma, Z) = (\sigma, Z)$. In the above argument, we take $N \in i\sigma_{\mathbf{R}}$, and have $\log(\gamma) = N$. Since $\log(\gamma)$ is real and N is purely imaginary, this shows that $\log(\gamma) = 0$. Hence $\gamma = 1$. $\square$

Theorem 5.3.6. (i) *The actions of Γ on $D_\Sigma^\sharp$ and $D_{\Sigma,\mathrm{val}}^\sharp$ are proper. In particular, the quotient spaces $\Gamma \backslash D_\Sigma^\sharp$ and $\Gamma \backslash D_{\Sigma,\mathrm{val}}^\sharp$ are Hausdorff.*

(ii) *Assume Γ is neat. Then the canonical maps $D_\Sigma^\sharp \to \Gamma \backslash D_\Sigma^\sharp$ and $D_{\Sigma,\mathrm{val}}^\sharp \to \Gamma \backslash D_{\Sigma,\mathrm{val}}^\sharp$ are local homeomorphisms.*

Proof. We first prove (i). By [KNU.p] 3.5.17, the action of Γ on $D_{\mathrm{SL}(2)}^I$ is proper. By 5.2.4.3 (ii), the statement for $D_{\Sigma,\mathrm{val}}^\sharp$ follows from this, from the continuity of $\psi : D_{\Sigma,\mathrm{val}}^\sharp \to D_{\mathrm{SL}(2)}^I$, and from Proposition 5.3.3. By 5.2.4.3 (i), the statement for $D_\Sigma^\sharp$ follows from the above result, since $D_{\Sigma,\mathrm{val}}^\sharp \to D_\Sigma^\sharp$ is proper by Proposition 5.3.2 and is surjective.

Next, by 5.2.4.2, (ii) follows from (i) and Theorem 5.3.5. $\square$

Let Σ be a weak fan, let $\sigma \in \Sigma$, and let Γ be a subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ.

In this subsection, we prove Theorem 2.5.3 and the local cases (the cases where $\Sigma = \mathrm{face}(\sigma)$) of Theorems 2.5.2, 2.5.5, 2.5.6.

Lemma 5.4.1. *Let $\sigma \in \Sigma$ and assume that a σ-nilpotent orbit exists. Let $\mathcal{I}$ be the set of all admissible sets Ψ of weight filtrations on $H_{0,\mathbf{R}}$ (4.1.2) such that $M(\sigma, W) \in \Psi$. Then, the image of the CKS map $D_{\sigma,\mathrm{val}}^\sharp \to D_{\mathrm{SL}(2)}^I$ is contained in the open set $\bigcup_{\Psi \in \mathcal{I}} D_{\mathrm{SL}(2)}^I(\Psi)$ (4.1.2) of $D_{\mathrm{SL}(2)}^I$.*

Proof. Let $\alpha = (A, V, Z) \in D_{\sigma,\mathrm{val}}^\sharp$. Then there exist $N_1, \ldots, N_n \in A \cap \sigma$ which generate A over $\mathbf{Q}$ such that, for the injective homomorphism $(N_j)_{1 \le j \le n} : \mathrm{Hom}_{\mathbf{Q}}(A, \mathbf{Q}) \to \mathbf{R}^n$, V coincides with the inverse image of the subset of $\mathbf{R}^n$ consisting of all elements which are ≥ 0 for the lexicographic order. Take elements $N_{n+1}, \ldots, N_m$ of $\mathfrak{g}_{\mathbf{Q}} \cap \sigma$ such that $N_1, \ldots, N_m$ generate the cone σ. By assumption, there is an $F \in \check{D}$ which generates a σ-nilpotent orbit. Let $\Psi := \{M(N_1 + \cdots + N_j, W) \mid 1 \le j \le m\} \supset \Psi' := \{M(N_1 + \cdots + N_j, W) \mid 1 \le j \le n\}$. Then Ψ coincides with the set of weight filtrations associated to the class of the SL(2)-orbit associated to $((N_j)_{1 \le j \le m}, F)$, and Ψ' coincides with the set of weight filtrations associated to the image p of α in $D_{\mathrm{SL}(2)}$. We have $p \in D_{\mathrm{SL}(2)}^I(\Psi)$ and $M(\sigma, W) = M(N_1 + \cdots + N_m, W) \in \Psi$. $\square$

The following lemma will be improved as Proposition 5.4.6 later.

Lemma 5.4.2. *Let $\sigma \in \Sigma$, let τ be a face of σ such that a τ-nilpotent orbit exists, let $U(\tau)$ be the open set of toric_σ defined in 5.1.2, and let $\tilde{U}(\tau)$ (resp. $\tilde{U}(\tau)_{\mathrm{val}}$) be the inverse image of $U(\tau)$ under $E_\sigma \to \mathrm{toric}_\sigma$ (resp. $E_{\sigma,\mathrm{val}} \to \mathrm{toric}_\sigma$). Then the action (5.2.1) of $\sigma_{\mathbf{C}}$ on $\tilde{U}(\tau)$ (resp. $\tilde{U}(\tau)_{\mathrm{val}}$) is proper.*

Proof. Since $\tilde{U}(\tau)_{\mathrm{val}} \to \tilde{U}(\tau)$ is proper and surjective, it is sufficient to consider $\tilde{U}(\tau)_{\mathrm{val}}$ (5.2.4.3 (i)).

Let W' be the filtration $M(\tau, W)(\mathrm{gr}^W)$ on gr^W induced by $M(\tau, W)$. We consider five continuous maps

$$f_1 : \tilde{U}(\tau)_{\mathrm{val}} \to E_{\sigma,\mathrm{val}}^\sharp, \quad f_2 : \tilde{U}(\tau)_{\mathrm{val}} \to D_{\tau,\mathrm{val}}^\sharp,$$

$$f_3 : \tilde{U}(\tau)_{\mathrm{val}} \to \mathrm{spl}(W), \quad f_4 : \tilde{U}(\tau)_{\mathrm{val}} \to \mathrm{spl}(W'),$$

$$f_5 : \tilde{U}(\tau)_{\mathrm{val}} \to \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}})).$$

Here f_1 is induced by $|-| : E_{\sigma,\mathrm{val}} \to E_{\sigma,\mathrm{val}}^\sharp$, $(q, F) \mapsto (|q|, F)$, where for $q : \Gamma(\sigma)^\vee \to \mathbf{C}^{\mathrm{mult}}$, $|q|$ denotes the composition of q and $\mathbf{C} \to \mathbf{R}_{\ge 0}$, $a \mapsto |a|$. The map f_2 is induced from f_1 via the canonical map $E_{\sigma,\mathrm{val}}^\sharp \to D_{\sigma,\mathrm{val}}^\sharp$. The map f_3 is induced from f_2, the CKS map $D_{\tau,\mathrm{val}}^\sharp \to D_{\mathrm{SL}(2)}^I$, and the canonical map $\mathrm{spl}_W : D_{\mathrm{SL}(2)}^I \to \mathrm{spl}(W)$ ([KNU.p] §3.2). The map f_4 is induced from the map f_2, the CKS map $D_{\tau,\mathrm{val}}^\sharp \to \bigcup_{\Psi \in \mathcal{I}} D_{\mathrm{SL}(2)}^I(\Psi)$ where $\mathcal{I}$ denotes the set of all admissible sets of weight filtrations Ψ such that $M(\tau, W) \in \Psi$

(here we apply 5.4.1 replacing σ there by τ), and the Borel-Serre splitting $\mathrm{spl}_{W'}^{\mathrm{BS}}$: $\bigcup_{\Psi \in \mathcal{I}} D_{\mathrm{SL}(2)}^I(\Psi) \to \mathrm{spl}(W')$ ([KNU.p] §3.2). The map f_5 is the composition

$$\tilde{U}(\tau)_{\mathrm{val}} \to U(\tau) \to (\Gamma(\sigma)^{\mathrm{gp}}/\Gamma(\tau)^{\mathrm{gp}}) \otimes \mathbf{C}^{\times} \to \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$$

where the second arrow is defined by the fact

$$\mathrm{Ker}(\mathrm{Hom}(\Gamma(\sigma)^{\mathrm{gp}}, \mathbf{Z}) \to \mathrm{Hom}(\Gamma(\tau)^{\mathrm{gp}}, \mathbf{Z})) \subset P,$$

where P is the fs monoid which appeared in the definition of $U(\tau) = \mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}}$ (5.1.2), and the last arrow is the homomorphism $\gamma \otimes e^{2\pi i z} \mapsto \mathrm{Re}(z) \log(\gamma)$ ($\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$, $z \in \mathbf{C}$).

These maps f_j have the following compatibilities with the action of $\sigma_{\mathbf{C}}$ on $\tilde{U}(\tau)$:

For any $a \in \sigma_{\mathbf{C}}$ and any $x \in \tilde{U}(\tau)_{\mathrm{val}}$, $f_j(a \cdot x) = a \cdot f_j(x)$,

where $\sigma_{\mathbf{C}}$ acts on the target space of f_j as follows. For $x = (q, F) \in E_{\sigma,\mathrm{val}}^{\sharp}$, $a \cdot x = (\mathbf{e}(i \, \mathrm{Im}(a)) \cdot q, \exp(-a)F)$. Note that in the case $a \in i\sigma_{\mathbf{R}}$, this action coincides with the original action of a on $E_{\sigma,\mathrm{val}}^{\sharp}$. For $x = (A, V, Z) \in D_{\tau,\mathrm{val}}^{\sharp}$, $a \cdot x = (A, V, \exp(-\mathrm{Re}(a))Z)$. For $s \in \mathrm{spl}(W)$, $a \cdot s = \exp(-\mathrm{Re}(a)) \circ s \circ \mathrm{gr}^W(\exp(\mathrm{Re}(a)))$. For $s \in \mathrm{spl}(W')$, $a \cdot s = \exp(-\mathrm{Re}(a)) \circ s \circ \mathrm{gr}^{W'}(\exp(\mathrm{Re}(a)))$. For $x \in \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$, $a \cdot x = x + \mathrm{Re}(a)$.

Let $H = \sigma_{\mathbf{C}}$, and define closed subgroups $H(j)$ ($0 \leq j \leq 5$) of H such that $0 = H(0) \subset H(1) \subset \cdots \subset H(5) = H$ as follows. $H(1) = i\sigma_{\mathbf{R}}$, $H(2) = H(1) + \log(\Gamma(\sigma)^{\mathrm{gp}})$, $H(3) = H(2) + \tau_{\mathbf{R},u}$ where $\tau_{\mathbf{R},u} = \tau_{\mathbf{R}} \cap \mathfrak{g}_{\mathbf{R},u}$, $H(4) = H(3) + \tau_{\mathbf{R}}$. Define the spaces X_j ($1 \leq j \leq 5$) with actions of H as follows: $X_1 = E_{\sigma,\mathrm{val}}^{\sharp}$, $X_2 = D_{\tau,\mathrm{val}}^{\sharp}$, X_3 is the quotient space of $\mathrm{spl}(W)$ under the action of $\Gamma(\sigma)^{\mathrm{gp}}$ given by $s \mapsto \gamma \circ s \circ \mathrm{gr}^W(\gamma)^{-1}$ ($s \in \mathrm{spl}(W)$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$), X_4 is the quotient space of $\mathrm{spl}(W')$ under the action of $\Gamma(\sigma)^{\mathrm{gp}}$ given by $s \mapsto \gamma \circ s \circ \mathrm{gr}^{W'}(\gamma)^{-1}$ ($s \in \mathrm{spl}(W')$, $\gamma \in \Gamma(\sigma)^{\mathrm{gp}}$), and $X_5 = \sigma_{\mathbf{R}}/(\tau_{\mathbf{R}} + \log(\Gamma(\sigma)^{\mathrm{gp}}))$. Then for $1 \leq j \leq 5$, the action of $H(j)$ on X_j factors through $H_j := H(j)/H(j-1)$, and the action of H_j on X_j is proper and free. In fact, for $j = 1$, this is by Proposition 5.2.5 and Proposition 5.2.2 (ii). For $j = 2$, this is by Theorem 5.3.6 (i) and Theorem 5.3.5 (i). For $j = 3, 4, 5$, this is easily seen. Hence by 5.2.4.5, the action of H on $X_1 \times X_2 \times X_3 \times X_4 \times X_5$ is proper. By 5.2.4.3 (ii), this proves that the action of $\sigma_{\mathbf{C}}$ on $\tilde{U}(\tau)_{\mathrm{val}}$ is proper. $\square$

Proposition 5.4.3. *The log local ringed space $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma}$ over $\mathbf{C}$ is a log manifold.*

We prove this proposition together with Theorem 2.5.3 (that is, E_{σ} is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\sigma}$).

Proof. Since $E_{\sigma} = \bigcup_{\tau} \tilde{U}(\tau)$ where τ ranges over all faces of σ such that a τ-nilpotent orbit exists, it is sufficient to prove that for such τ, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\tau} = \sigma_{\mathbf{C}} \backslash \tilde{U}(\tau)$ is a log manifold and $\tilde{U}(\tau)$ is a $\sigma_{\mathbf{C}}$-torsor over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{\tau}$.

For such τ, we apply Lemma 5.2.7 by taking $H = \sigma_{\mathbf{C}}$, $X = \tilde{U}(\tau)$ and $\mathcal{C}$ to be the category of log manifolds as in 5.2.7 (b). By Lemma 5.4.2, Proposition 5.2.2 (i) and Lemma 5.2.7, it is sufficient to prove that the condition 5.2.7 (1) is satisfied. Let $x = (q, F) \in \tilde{U}(\tau)$. Let $S_1 \subset E_{\sigma}$ be the log manifold which contains x constructed in the same way as the log manifold S in the pure case [KU09] 7.3.5. (In the argument to

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construct this log manifold, we replace $W(\sigma(q))[-w]$ there by $M(\sigma(q), W)$ here.) Let $S = S_1 \cap \tilde{U}(\tau)$. Then S has the desired property. $\square$

Since $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ belongs to $\mathcal{B}(\log)$ by 5.4.3, the topological space $(\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)^{\log}$ is defined.

Proposition 5.4.4. *We have a canonical homeomorphism*

$$\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp \simeq (\Gamma(\sigma^{\mathrm{gp}}) \backslash D_\sigma)^{\log}$$

over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$.

Proof. We define a canonical homeomorphism $(\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)^{\log} \simeq \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp$ over $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$. We have

$$(\mathrm{toric}_\sigma)^{\log} \simeq \mathrm{Hom}(\Gamma(\sigma)^\vee, \mathbf{S}^1 \times \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \simeq (\mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}) \times |\mathrm{toric}|_\sigma \ .$$

Hence the homeomorphisms

$$(E_\sigma)^{\log} \simeq (\mathrm{toric}_\sigma)^{\log} \times_{\mathrm{toric}_\sigma} E_\sigma, \quad E_\sigma^\sharp \simeq |\mathrm{toric}|_\sigma \times_{\mathrm{toric}_\sigma} E_\sigma$$

induce a homeomorphism

$$(1) \qquad\qquad (E_\sigma)^{\log} \simeq (\mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}) \times E_\sigma^\sharp.$$

This homeomorphism is compatible with the actions of $\sigma_{\mathbf{C}}$. Here $z = x + iy \in \sigma_{\mathbf{C}}$ ($x, y \in \sigma_{\mathbf{R}}$) acts on the left-hand side of (1) as the map $(E_\sigma)^{\log} \to (E_\sigma)^{\log}$ induced by the action of z on E_σ (5.2.1) and on the right-hand side as $(u, q, F) \mapsto (\mathbf{e}(x)u, \mathbf{e}(iy)q, \exp(-z)F)$ ($u \in \mathbf{S}^1 \otimes_{\mathbf{Z}} \Gamma(\sigma)^{\mathrm{gp}}$, $q \in |\mathrm{toric}|_\sigma$, $F \in \check{D}$). The homeomorphism between the quotient spaces of these actions is

$$(\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)^{\log} \simeq \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp. \quad \square$$

Corollary 5.4.5. *The space* $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ *is Hausdorff.*

The space $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp$ is Hausdorff by 5.3.6 (i) and the map $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is proper and surjective by 5.4.4. Hence $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is Hausdorff by 5.2.4.7.

Proposition 5.4.6. *The action of* $\sigma_{\mathbf{C}}$ *on* E_σ *(resp.* $E_{\sigma, \mathrm{val}}$*) is proper.*

The result for E_σ follows from Theorem 2.5.3 and Corollary 5.4.5 by 5.2.4.6. The result for $E_{\sigma, \mathrm{val}}$ follows from it by 5.2.4.3 (ii).

$$\S 5.5. \ \text{STUDY OF } \Gamma \backslash D_\Sigma$$

In this subsection, we prove Theorems 2.5.2, 2.5.4, 2.5.5, 2.5.6, and 2.6.6.

Lemma 5.5.1. *Let X be a topological space with a continuous action of a discrete group Γ and let Y be a set with an action of Γ. Let $f : X \to Y$ be a Γ-equivariant surjective map. Let Γ' be a subgroup of Γ. We introduce the quotient topologies of X on $\Gamma' \backslash Y$ and on $\Gamma \backslash Y$. Let V be an open set of $\Gamma' \backslash Y$ and let U be the inverse image of V in $\Gamma' \backslash X$. We assume moreover the three conditions (1)–(3) below. Then, $V \to \Gamma \backslash Y$ is a local homeomorphism.*

(1) $X \to \Gamma \backslash X$ *is a local homeomorphism and $\Gamma \backslash X$ is Hausdorff.*

(2) $U \to V$ *is proper.*

(3) *If $x \in X$ and $\gamma \in \Gamma$, and if the images of γx and x in $\Gamma' \backslash Y$ are contained in V and they coincide, then $\gamma \in \Gamma'$.*

Proof. This is [KU09] Lemma 7.4.7. See [KU09] for a proof.

5.5.2. *Proof of Theorem 2.5.4.* We prove that $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma \to \Gamma \backslash D_\Sigma$ is locally an isomorphism.

We use Lemma 5.5.1 for $X = D_\Sigma^\sharp$, $Y = D_\Sigma$, $\Gamma = \Gamma$, $\Gamma' = \Gamma(\sigma)^{\mathrm{gp}}$, $V = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ and $U = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp$. Theorem 5.3.6 for $D_\Sigma^\sharp$ shows that the condition (1) in Lemma 5.5.1 is satisfied. Theorem 2.5.6 for $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, proved in §5.4, shows that the condition (2) in 5.5.1 is satisfied. Theorem 5.3.5 shows that the condition (3) in 5.5.1 is satisfied. $\square$

5.5.3. *Proof of Theorem 2.5.2.* We prove that $\Gamma \backslash D_\Sigma$ is a log manifold.
This follows from Theorem 2.5.2 for $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ (5.4.3) by Theorem 2.5.4. $\square$

5.5.4. *Proof of Theorem 2.5.6.* We prove that $(\Gamma \backslash D_\Sigma)^{\log} = \Gamma \backslash D_\Sigma^\sharp$.
By using Theorem 2.5.4, it is easily seen that the canonical homeomorphisms $(\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)^{\log} \simeq \blacksquare$
$\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^\sharp$ (Theorem 2.5.6 for $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, proved in §5.4) glue uniquely to a homeomorphism $(\Gamma \backslash D_\Sigma)^{\log} \simeq \Gamma \backslash D_\Sigma^\sharp$. $\square$

5.5.5. *Proof of Theorem 2.5.5.* We prove that $\Gamma \backslash D_\Sigma$ is a Hausdorff space.
Replacing Γ by a subgroup of finite index, we may assume Γ is neat. Then, by Theorem 2.5.6, the map $\Gamma \backslash D_\Sigma^\sharp \to \Gamma \backslash D_\Sigma$ is proper and surjective. Since $\Gamma \backslash D_\Sigma^\sharp$ is Hausdorff by Theorem 5.3.6 (i), it follows by 5.2.4.7 that $\Gamma \backslash D_\Sigma$ is Hausdorff. $\square$

A local description of the log manifold $\Gamma \backslash D_\Sigma$ is given by the following.

Theorem 5.5.6. *Let Σ be a weak fan and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ. Let $(\sigma, Z) \in D_\Sigma$ and let p be its image in $\Gamma \backslash D_\Sigma$.*

(i) *Let $F_{(0)} \in Z$. Then there exists a locally closed analytic submanifold Y of $\check{D}$ satisfying the following conditions (1) and (2).*

(1) $F_{(0)} \in Y$.

(2) *The canonical map $T_Y(F_{(0)}) \oplus \sigma_{\mathbf{C}} \to T_{\check{D}}(F_{(0)})$ is an isomorphism. Here $T_Y(F_{(0)})$ denotes the tangent space of Y at $F_{(0)}$ and $T_{\check{D}}(F_{(0)})$ denotes the tangent space of $\check{D}$ at $F_{(0)}$.*

(ii) *Let $F_{(0)}$ and Y be as in* (i). *Let $X := \mathrm{toric}_\sigma \times Y$, and let S be the log manifold defined by*

$$S := \{(q, F) \in X \mid N(F^p) \subset F^{p-1} \ (\forall N \in \sigma(q), \forall p \in \mathbf{Z})\},$$

where $\sigma(q)$ denotes the face of σ corresponding to q. Then, there exist an open neighborhood U_1 of $(0, F_{(0)})$ in S in the strong topology, an open neighborhood U_2 of p in $\Gamma \backslash D_\Sigma$, and an isomorphism $U_1 \xrightarrow{\sim} U_2$ of log manifolds sending $(0, F_{(0)})$ to p.

Proof. This is the mixed Hodge version of [KU09] 7.4.13 and proved similarly. □

5.5.7. We prove Theorem 2.6.6. This is the mixed Hodge theoretic version of Theorem B in [KU09]. Since everything is prepared in §2.5, especially Theorem 2.5.3, the proof of 2.6.6 is exactly parallel to the pure case [KU09] 8.2. That is, we first describe the functor represented by E_σ. Second we describe the functor represented by $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ by taking quotients where we use 2.5.3. Finally, gluing them up, we show that the functor LMH_Φ is represented by $\Gamma \backslash D_\Sigma$ (cf. 2.6.5).

<center>§5.6. Proofs for Section 3</center>

Let the notation be as in §3.

5.6.1. We reduce the results in §3 to the results in §2 in the following way. We define a modification $\Lambda^e = (H_0^e, W^e, (\langle\ ,\ \rangle_w^e)_w, (h^{e,p,q})_{p,q})$ of $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ of 3.1.1 (1). Then, the results in §3 are reduced to the results in §2 for Λ^e.

Take an integer r such that $W_{2r} = 0$. Let $H_{0,(w)}^e$ be Γ' if $w = 2r - 2$, be $\mathbf{Z}$ if $w = 2r$, and be $H_{0,(w)}$ if $w \neq 2r - 2, 2r$. Let $H_0^e = \bigoplus_{w \in \mathbf{Z}} H_{0,(w)}^e = \Gamma' \oplus \mathbf{Z} \oplus H_0$. Let the increasing filtration W^e on $H_{0,\mathbf{R}}^e$ be the evident one.

Define the bilinear form $\langle\ ,\ \rangle_w^e : H_{0,(w),\mathbf{R}}^e \times H_{0,(w),\mathbf{R}}^e \to \mathbf{R}$ as follows. If $w \neq 2r - 2, 2r$, it is just the bilinear form $\langle\ ,\ \rangle_w$ on $H_{0,(w)}$. If $w = 2r - 2$, it is any fixed positive definite symmetric bilinear form $\mathbf{R} \otimes \Gamma' \times \mathbf{R} \otimes \Gamma' \to \mathbf{R}$ which is rational over $\mathbf{Q}$. If $w = 2r$, it is the form $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(x, y) \mapsto xy$.

Let $h^{e,p,q}$ be $\mathrm{rank}(\Gamma')$ if $p = q = r - 1$, be 1 if $p = q = r$, and be $h^{p,q}$ otherwise.

We denote by D^e the classifying space in §2 for $\Lambda^e = (H_0^e, W^e, (\langle\ ,\ \rangle_w^e)_w, (h^{e,p,q})_{p,q})$.

For a nilpotent cone σ in §3, we regard σ as a nilpotent cone in §2 for Λ^e in the following way. For $N \in \sigma$, we regard it as a linear map $H_{0,\mathbf{R}}^e \to H_{0,\mathbf{R}}^e$ whose restriction to $H_{0,\mathbf{R}}$ is $N_{\mathfrak{g}} : H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$, whose restriction to $\mathbf{R} \otimes \Gamma' = H_{0,(2r-2),\mathbf{R}}^e$ is 0, and which sends $1 \in \mathbf{R} = H_{0,(2r)}^e$ to the image of N in $\mathbf{R} \otimes \Gamma' = H_{0,(2r-2),\mathbf{R}}^e$ under $\sigma \to \sigma' \subset \sigma_\mathbf{R}' \simeq \mathbf{R} \otimes \Gamma'$ (3.1.1).

By this, a weak fan Σ in §3 is regarded as a weak fan in §2 for Λ^e.

We denote the objects E_σ, $G_\mathbf{Z}$ etc. of §2 for Λ^e, by E_σ^e, $G_\mathbf{Z}^e$, etc.

Γ in §3 is regarded as a subgroup of $G_\mathbf{Z}^e$ in the following way. We regard $(a, b) \in \Gamma$ ($a \in \Gamma'$, $b \in G_\mathbf{Z}$) as an element of $G_\mathbf{Z}^e$ whose restriction to H_0 is b, whose restriction to $\Gamma' = H_{0,(2r-2)}^e$ is the identity map, and which sends $1 \in \mathbf{Z} = H_{0,(2r)}^e$ to the sum of $1 \in H_{0,(2r)}^e$ and the element a of $\Gamma' = H_{0,(2r-2)}^e$. Σ and Γ are compatible (resp. strongly compatible) in the sense of §3 if and only if they are compatible (resp. strongly compatible) in the sense of §2 for Λ^e. The notation $\Gamma(\sigma)$ in §3 and that in §2 for Λ^e are compatible.

We define a morphism $\sigma_\mathbf{C}' \times \check{D} \to \check{D}^e$ as follows. An element $a \in \sigma_\mathbf{C}'$ corresponds to a decreasing filtration $f(a)$ on $V_\mathbf{C} := \mathbf{C} \otimes \Gamma' \oplus \mathbf{C} = H_{0,(2r-2),\mathbf{C}}^e \oplus H_{0,(2r),\mathbf{C}}^e$ given by

$$f(a)^{r+1} = 0 \subset f(a)^r = \mathbf{C} \cdot (a, 1) \subset f(a)^{r-1} = V_\mathbf{C},$$

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where a in $(a, 1)$ is identified with an element of $\mathbf{C} \otimes \Gamma'$ via the canonical isomorphism $\sigma'_{\mathbf{C}} \simeq \mathbf{C} \otimes \Gamma'$. The map $\sigma'_{\mathbf{C}} \times \check{D} \to \check{D}^e$ sends an element (a, F) of $\sigma'_{\mathbf{C}} \times \check{D}$ to the element $f(a) \oplus F$ of $\check{D}^e$ defined as the direct sum of $f(a)$ and F on $V_{\mathbf{C}} \oplus H_{0,\mathbf{C}} = H^e_{0,\mathbf{C}}$.

Note that, via the above morphism, $\sigma'_{\mathbf{C}} \times \check{D}$ is a closed analytic submanifold of $\check{D}^e$.

Proposition 5.6.2. *Let $S_0 = E'_{\sigma'}$ as in 3.1.1–3.1.3. Then the following diagram is cartesian in $\mathcal{B}(\log)$.*

$$
\begin{array}{ccc}
E_{S_0,\sigma} & \longrightarrow & E^e_\sigma \\
\downarrow & & \downarrow \\
\sigma'_{\mathbf{C}} \times \check{D} & \longrightarrow & \check{D}^e.
\end{array}
$$

Here the lower horizontal arrow sends (z, F) ($z \in \sigma'_{\mathbf{C}}, F \in \check{D}$) to $f(z) \oplus F$ as in 5.6.1, the left vertical arrow sends (s, z, q, F) to (z, F) ($s \in S_0, z \in \sigma'_{\mathbf{C}}$, $q \in \mathrm{toric}_\sigma$, $F \in \check{D}$), the right vertical arrow sends (q, F) to F ($q \in \mathrm{toric}_\sigma$, $F \in \check{D}^e$), and the upper horizontal arrow sends (s, z, q, F) to $(q, f(z) \oplus F)$.

This is seen easily.

5.6.3. We give preliminaries to Propositions 5.6.4, 5.6.5.

Let X, Y be topological spaces, let $r : X \to Y$ be a continuous map, and assume that, locally on Y, there is a continuous map $s : Y \to X$ such that $r \circ s$ is the identity map of Y. Let B be a subspace of Y and let $A = r^{-1}(B)$.

5.6.3.1. Endow A with the topology as a subspace of X. Then as is easily seen, the following two topologies on B coincide. The image of the topology of A under $r : A \to B$, and the inverse image of the topology of Y.

5.6.3.2. Assume furthermore that X (resp. Y) is endowed with a subsheaf $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) of rings over $\mathbf{C}$ of the sheaf of $\mathbf{C}$-valued continuous functions on X (resp. Y). Assume that the stalks of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) are local rings. Assume that r respects these sheaves and assume that, locally on Y, we can take s as above which respects these sheaves.

Let $\mathcal{O}_A$ be the subsheaf of the sheaf of $\mathbf{C}$-valued continuous functions on A defined as follows. For a $\mathbf{C}$-valued continuous function f on an open set U of A, f belongs to $\mathcal{O}_A$ if and only if, for any $x \in U$, there are an open neighborhood U' of x in U, an open neighborhood V of x in X such that $U' \subset V$, and an element g of $\mathcal{O}_X(V)$ such that the restrictions of f and g to U' coincide.

Then it is easy to see the coincidence of the following two subsheaves (1) and (2) of the sheaf of $\mathbf{C}$-valued continuous functions on B: Let f be a $\mathbf{C}$-valued continuous function on an open set U of B. f belongs to the sheaf (1) if and only if $f \circ r$ belongs to $\mathcal{O}_A(r^{-1}(U))$. f belongs to the sheaf (2) if and only if, for any $y \in U$, there are an open neighborhood U' of y in U, an open neighborhood V of y in Y such that $U' \subset V$, and an element g of $\mathcal{O}_Y(V)$ such that the restrictions of f and g to U' coincide.

Let $\mathcal{O}_B$ be this sheaf on B.

5.6.3.3. Assume further that we have elements $f_1, \ldots, f_n$ of $\mathcal{O}_X(X)$ such that $A = \{x \in X \mid f_1(x) = \cdots = f_n(x) = 0\}$ and such that the kernel of $\mathcal{O}_X|_A \to \mathcal{O}_A$ is the ideal generated by $f_1, \ldots, f_n$. Here $\mathcal{O}_X|_A$ is the sheaf-theoretic inverse image of $\mathcal{O}_X$ by

$A \hookrightarrow X$. Then locally on Y, for s as above, $B = \{y \in Y \mid f_1 s(y) = \cdots = f_n s(y) = 0\}$ and the kernel of $\mathcal{O}_Y|_B \to \mathcal{O}_B$ is the ideal generated by $f_1 \circ s, \ldots, f_n \circ s$.

5.6.3.4. Assume further that Y is endowed with a log structure M_Y. Let M_X (resp. M_A) be the inverse image of M_Y on X (resp. A) and assume that the structural map $M_A \to \mathcal{O}_A$ of log structure is injective.

Then the following two log structures (1) and (2) on B coincide. (1) For an open set U of B, $M_B(U)$ is the subset of $\mathcal{O}_B(U)$ consisting of all elements whose pull back to $r^{-1}(U)$ belongs to the image of $M_A \to \mathcal{O}_A$. (2) The inverse image of M_Y on B.

5.6.3.5. Assume furthermore that X and Y belong to $\mathcal{B}(\log)$. Let H be a complex Lie group. Assume that an action of H on X in $\mathcal{B}(\log)$ is given, and assume that X is an H-torsor over Y in $\mathcal{B}(\log)$. Note that A belongs to $\mathcal{B}(\log)$. By the above, B also belongs to $\mathcal{B}(\log)$. Assume that H acts also on A in $\mathcal{B}(\log)$ and assume that this action is compatible with the action of H on X. Then A is an H-torsor over B in $\mathcal{B}(\log)$.

Proof of 5.6.3.5. Considering locally on B, we may assume that we have $s : Y \to X$ as above. Hence we have $H \times Y \to X$, $(h, y) \mapsto hs(y)$. Let $t : X \to H$ be the composition $X \simeq H \times Y \to H$. Then we have a morphism $A \to H \times B$, $a \mapsto (t(a), r(a))$ which is the inverse map of the morphism $H \times B \to A$, $(h, b) \mapsto hs(b)$. These morphisms are inverse to each other in $\mathcal{B}(\log)$. This proves that A is an H-torsor over B in $\mathcal{B}(\log)$. $\square$

It follows that the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

in $\mathcal{B}(\log)$ is cartesian.

Proposition 5.6.4. (i) $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}$ *is a relative log manifold over* S_0.

(ii) $E_{S_0,\sigma}$ *is a relative log manifold over* S_0.

(iii) $E_{S_0,\sigma} \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}$ *is a* $\sigma_{\mathbf{C}}$*-torsor in the category* $\mathcal{B}(\log)$.

Proof. We apply 5.6.3 by taking

$$
X = E_\sigma^e, \quad Y = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^e, \quad A = E_{S_0,\sigma}, \quad B = \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}.
$$

By 5.6.3, $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma} \in \mathcal{B}(\log)$ and we have (iii). By (iii), (i) and (ii) are equivalent. It is sufficient to prove that (ii). This is reduced to the fact that the morphism

$$
\sigma'_{\mathbf{C}} \times \mathrm{toric}_\sigma \to \mathrm{toric}_{\sigma'}, \quad (z, q) \mapsto \mathbf{e}(z) \cdot q(\mathrm{gr}^W)
$$

is log smooth, which can be shown easily. $\square$

In the following, Σ is a weak fan in the sense of §3.

Proposition 5.6.5. (i) $\Gamma \backslash D_{S_0,\Sigma}$ *is a relative log manifold over S_0.*

(ii) *The morphism $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma} \to \Gamma \backslash D_{S_0,\Sigma}$ in $\mathcal{B}(\log)$ is locally an isomorphism.*

(iii) $\Gamma \backslash D_{S_0,\Sigma}$ *is Hausdorff.*

Proof. By the next lemma, we can apply 5.6.3 by taking

$$X = \bigsqcup_{\sigma \in \Sigma} \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^e, \quad Y = \Gamma \backslash D_\Sigma^e, \quad A = \bigsqcup_{\sigma \in \Sigma} \Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S_0,\sigma}, \quad B = \Gamma \backslash D_{S_0,\Sigma}.$$

By 5.6.3.1–5.6.3.4, $\Gamma \backslash D_{S_0,\Sigma}$ belongs to $\mathcal{B}(\log)$ and we have (ii) by 2.5.4. (ii) and 5.6.4 (i) imply (i). (iii) follows from the fact that $\Gamma \backslash D_\Sigma^e$ is Hausdorff, proved in §2. $\square$

Lemma 5.6.6. *For $\sigma \in \Sigma$, the following diagrams of sets are cartesian.*

$$
\begin{array}{ccc}
D_{S,\sigma} & \longrightarrow & D_{S,\Sigma} \\
\downarrow & & \downarrow \\
D_\sigma^e & \longrightarrow & D_\Sigma^e,
\end{array}
\qquad
\begin{array}{ccc}
\Gamma(\sigma)^{\mathrm{gp}} \backslash D_{S,\sigma} & \longrightarrow & \Gamma \backslash D_{S,\Sigma} \\
\downarrow & & \downarrow \\
\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma^e & \longrightarrow & \Gamma \backslash D_\Sigma^e.
\end{array}
$$

Recall that we endowed $\Gamma \backslash D_{S,\Sigma}$ with the structure of an object of $\mathcal{B}(\log)$ by regarding it as the fiber product of $S \to S_0 \leftarrow \Gamma \backslash D_{S_0,\Sigma}$, where $S \to S_0$ is a strict morphism in $\mathcal{B}(\log)$ as in 3.1.1 (4). Hence Theorem 3.2.8 follows from Proposition 5.6.5.

5.6.7. We prove Theorem 3.3.3. By §2 for Λ^e, $\Gamma \backslash D_\Sigma^e$ represents the moduli functor LMH_{Φ^e} for $\Phi^e = (\Lambda^e, \Sigma, \Gamma)$. From 5.6.2, we have that the subspace $\Gamma \backslash D_{S_0,\Sigma}$ of $\Gamma \backslash D_\Sigma^e$ represents the subfunctor of LMH_{Φ^e} consisting of the classes of LMH which are the direct sum of an LMH H of weight $> 2r$ and an LMH h of weight $\leq 2r$. The projection $E_{S_0,\sigma} \to S_0 \subset \mathrm{toric}_{\sigma'} \times \check{D}(\mathrm{gr}^W)$ represents $h \oplus H \mapsto (h, H(\mathrm{gr}^W))$. Hence over S_0, $\Gamma \backslash D_{S_0,\Sigma}$ is the moduli space of this H with given $H(\mathrm{gr}^W)$, and hence represents $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}$. $\square$

§6. Néron models

In this section, we construct the Néron model and the connected Néron model in degeneration of Hodge structures. In the case of Hodge structures corresponding to abelian varieties of semi-stable reductions, these coincide with the classical Néron model and the connected Néron model in the theory of degeneration of abelian varieties. We expect that the constructions of such models are useful in the study of degeneration of intermediate Jacobians. In the case of degeneration of intermediate Jacobians of Griffiths [G68a] and [G68b], our connected Néron model coincides with the model of Zucker [Z76] (which need not be Hausdorff) modified by putting slits (then we have a Hausdorff space), and under a certain assumption, our Néron model coincides with the model constructed by Clemens [C83].

In §6.4, we give the proofs of the results in [KNU10c] which were omitted there.

§6.1. Results on Néron models and connected Néron models

In §6.1–§6.3, as in §3, let S be an object of $\mathcal{B}(\log)$, and assume that for each $w \in \mathbf{Z}$, we are given a polarized log Hodge structure $H_{(w)}$ on S. Assume $H_{(w)} = 0$ for almost all w.

Theorem 6.1.1. *There is a relative log manifold J_1 (resp. J_0) over S which is strict over S and which represents the following functor on $\mathcal{B}/S^\circ$:*

$\mathcal{B}/S^\circ \ni S' \mapsto$ the set of all isomorphism classes of LMH H on S' with $H(\mathrm{gr}_w^W) = H_{(w)}$ ($w \in \mathbf{Z}$) satisfying the following condition: Locally on S', there is a splitting $H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}(\mathrm{gr}^W) \simeq H_{\mathbf{Z}}$) of the weight filtration W on the local system $H_{\mathbf{Q}}$ (resp. $H_{\mathbf{Z}}$) on $(S')^{\log}$.

Here S° denotes S regarded as an object of $\mathcal{B}$ by forgetting the log structure. In the description of the functor in 6.1.1, $S' \in \mathcal{B}/S^\circ$ is endowed with the inverse image of the log structure of S.

6.1.2. We call the relative log manifold J_1 the *Néron model*, and J_0 the *connected Néron model*.

In the case of degeneration of abelian varieties, these terminologies agree with the classical ones (see 7.1.4).

6.1.3. Let $w \in \mathbf{Z}$, $w < 0$. Assume $H_{(k)} = 0$ for $k \neq w, 0$, and that $H_{(0)} = \mathbf{Z}$. Let $H' = H_{(w)}$. Consider the exact sequence on $S^{\log}$

$$0 \to H'_{\mathbf{Z}} \to H'_{\mathcal{O}^{\log}}/F^0 \to H'_{\mathbf{Z}} \backslash H'_{\mathcal{O}^{\log}}/F^0 \to 0.$$

Descending this by $\tau : S^{\log} \to S$, we have an exact sequence of sheaves of abelian groups on $\mathcal{B}(\log)/S$

$$0 \to \tau_* H'_{\mathbf{Z}} \backslash (H'_{\mathcal{O}}/F^0 H'_{\mathcal{O}})^{sG} \to \mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}}.$$

Here $(\ )^{sG}$ is the small Griffiths part, i.e., the part restricted by the small Griffiths transversality (cf. [KU09] 2.4.9), and $\mathcal{E}xt^1(\mathbf{Z}, H') := \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, H')$ is the Ext-sheaf of the category of LMH which is identified with the subgroup of $\tau_*(H'_{\mathbf{Z}} \backslash H'_{\mathcal{O}^{\log}}/F^0)^{sG}$ restricted by the condition of the admissibility of local monodromy.

Proposition 6.1.4. *In the situation of* 6.1.3:
(i) *On $\mathcal{B}/S^\circ$, the Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}} \to R^1\tau_* H'_{\mathbf{Q}}).$$

(ii) *On $\mathcal{B}/S^\circ$, the connected Néron model represents*

$$\mathrm{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}}) = \tau_* H'_{\mathbf{Z}} \backslash (H'_{\mathcal{O}}/F^0 H'_{\mathcal{O}})^{sG}.$$

6.1.5. *Remark 1.* 6.1.4 (ii) shows that in the situation of 6.1.3, the connected Néron model is the Zucker model ([Z76]) modified by putting slits induced by the small Griffiths transversality.

Remark 2. Proposition 6.1.4 shows that in the situation of 6.1.3, the Néron model and the connected Néron model are commutative group objects over S which represent subgroup functors of $\mathcal{E}xt^1(\mathbf{Z}, H')$ on $\mathcal{B}/S^\circ$.

For an object S of $\mathcal{B}(\log)$ and for an integer r, we say the log rank of S is $\leq r$ if $\mathrm{rank}_{\mathbf{Z}}((M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s) \leq r$ for any $s \in S$. As an example which appears in 6.1.7 below, if S is a complex analytic manifold with a smooth divisor which gives the log structure of S, then the log rank of S is ≤ 1.

Corollary 6.1.6. *In the situation of* 6.1.3, *assume furthermore* $w = -1$ *and the log rank of* S *is* ≤ 1. *Then, on* $\mathcal{B}/S^\circ$, *the Néron model represents* $\mathcal{E}xt^1(\mathbf{Z}, H')$.

This is because under the assumption of 6.1.6, W on the local system $H_\mathbf{Q}$ of any LMH H splits locally on S.

Corollary 6.1.7. *In the situation of* 6.1.6, *assume that* S *is a complex analytic manifold with a smooth divisor which gives the log structure of* S. *Then we have*

$$(\textit{a section of the Néron model over } S)$$
$$= (\textit{a normal function on } S^* \textit{ which is admissible with respect to } S).$$

(Cf. [Sa96] for admissible normal functions.)

6.1.8. In what follows, the proof of 6.1.1 is given in §6.2, and the proof of 6.1.4 is given in §6.3. Note that for the proofs of 6.1.1 and 6.1.4, we may work locally on S. Locally on S, we construct in §6.2 the Néron model and the connected Néron model by the method of §3, by constructing suitable weak fans explicitly.

§6.2. MODULI SPACES OF LMH WITH $\mathbf{Q}$-SPLITTING LOCAL MONODROMY

Let S and $(H_{(w)})_w$ be as in 6.1. We assume that we are given the data (1)–(5) in 3.1.1. (Note that in general, as we proved in 3.1.2, we have always such (1)–(5) locally on S.)

Let $\Gamma_u = G_{\mathbf{Z},u}$ and let Γ be the corresponding group (3.1.7).

In this subsection, we construct certain weak fans Σ_1 and Σ_0. We will see that J_1 in Theorem 6.1.1 is J_{Σ_1} and J_0 is J_{Σ_0} (3.3.2).

Theorem 6.2.1. *Let* $\sigma' = \mathrm{Hom}(P, \mathbf{R}_{\geq 0}^{\mathrm{add}}) \to \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W) \to \mathfrak{g}_\mathbf{R}$, $a \mapsto a_\mathfrak{g}$, *be the composite map as in* §3. *Let* Υ *be a subgroup of* $G_{\mathbf{Q},u}$. *Let* $\Sigma(\Upsilon)$ *be the set of cones* $\sigma_{\tau',\upsilon} := \{(x, \mathrm{Ad}(\upsilon)x_\mathfrak{g}) \mid x \in \tau'\} \subset \sigma' \times \mathfrak{g}_\mathbf{R}$, *where* τ' *ranges over all faces of* σ' *and* υ *ranges over all elements of* Υ. *Then* $\Sigma(\Upsilon)$ *is a weak fan.*

Proof. By construction, it is easy to see the rationality and the closedness under the operation of taking a face. We examine the condition (1) in 3.1.6.

Let $\tau', \rho' \in \mathrm{face}(\sigma')$ and let $\tau_\mathfrak{g}, \rho_\mathfrak{g}$ be their images in $\mathfrak{g}_\mathbf{R}$, respectively. Let $\upsilon \in \Upsilon$ and let $\rho_{\mathfrak{g},\upsilon} = \mathrm{Ad}(\upsilon)(\rho_\mathfrak{g})$. Note that $\rho_{\mathfrak{g},\upsilon}$ coincides with the image of $\sigma_{\rho',\upsilon}$ in $\mathfrak{g}_\mathbf{R}$.

Assume that N is in the interior of $\sigma_{\tau',1}$ and of $\sigma_{\rho',\upsilon}$. Let $F \in \check{D}$. Assume that $(\sigma_{\tau',1}, F)$ and $(\sigma_{\rho',\upsilon}, F)$ generate nilpotent orbits.

We first claim that $N_\mathfrak{g}\upsilon = \upsilon N_\mathfrak{g}$. In fact, $\mathrm{gr}^W(N_\mathfrak{g}) = \mathrm{gr}^W(\mathrm{Ad}(\upsilon)^{-1}N_\mathfrak{g})$ because $\upsilon \in \Upsilon \subset G_{\mathbf{Q},u}$. On the other hand, $N_\mathfrak{g} \in \tau_\mathfrak{g}$ and $\mathrm{Ad}(\upsilon)^{-1}N_\mathfrak{g} \in \rho_\mathfrak{g}$ are pure of weight 0 with respect to W under the fixed splitting of W. Hence $N_\mathfrak{g} = \mathrm{Ad}(\upsilon)^{-1}N_\mathfrak{g}$, that is, $N_\mathfrak{g}\upsilon = \upsilon N_\mathfrak{g}$.

Since $\mathrm{face}(\sigma')$ is a fan and the image of N in σ' is in the interior of τ' and of ρ', we see $\tau' = \rho'$.

Since $(\sigma_{\tau',1}, F)$ generates a nilpotent orbit, the action of $\tau_\mathfrak{g}$ on $H_{0,\mathbf{R}}$ is admissible with respect to W. By 2.2.8, the adjoint action of $\tau_\mathfrak{g}$ on $\mathfrak{g}_\mathbf{R}$ is admissible with respect to the filtration $W\mathfrak{g}_\mathbf{R}$ on $\mathfrak{g}_\mathbf{R}$ induced by W.

Let $M = M(\mathrm{Ad}(N_{\mathfrak{g}}), W\mathfrak{g}_{\mathbf{R}})$. Since $v - 1 \in (W\mathfrak{g}_{\mathbf{R}})_{-1}$ and $\mathrm{Ad}(N_{\mathfrak{g}})(v-1) = 0$, we see $v - 1 \in M_{-1}$ by 1.2.1.3.

Let $h \in \tau_{\mathfrak{g}}$. Then $h \in M_{-2}$. Hence, by the above result, $hv - vh = h(v-1) - (v-1)h \in M_{-3}$. Applying v^{-1} from the right, we have $h - vhv^{-1} \in M_{-3}$. Since $(\tau_{\mathfrak{g}}, F)$ and $(\tau_{\mathfrak{g},v}, F)$ generate nilpotent orbits in the sense of §2 (cf. 3.1.5), we have $h \in F^{-1}\mathfrak{g}_{\mathbf{C}}$ and $vhv^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, respectively (Griffiths tranversality). Hence $h - vhv^{-1} \in F^{-1}\mathfrak{g}_{\mathbf{C}}$, and hence $h - vhv^{-1} \in M_{-3} \cap F^{-1}\mathfrak{g}_{\mathbf{C}} \cap \bar{F}^{-1}\mathfrak{g}_{\mathbf{C}} = 0$. Here we use the fact that $(M, F\mathfrak{g}_{\mathbf{C}})$ is an $\mathbf{R}$-mixed Hodge structure. Thus we have $\tau_{\mathfrak{g}} = \tau_{\mathfrak{g},v} = \rho_{\mathfrak{g},v}$. Here the second equality follows from $\tau' = \rho'$. Hence $\sigma_{\tau',1} = \sigma_{\rho',v}$ and the condition (1) in 3.1.6 is verified. $\quad\square$

[KNU10c] §2 contains an analogous result (ibid. Theorem 2.1) and its proof by the method of §2 in this paper. Theorem 6.2.1 above is its generalization by the method of §3.

6.2.2. Recall that $\Gamma_u = G_{\mathbf{Z},u}$. So Γ is a semi-direct product of Γ' in 3.1.1 (3) and $G_{\mathbf{Z},u}$.

Take a subgroup Υ of $G_{\mathbf{Q},u}$. Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$.

(2) $\mathrm{Int}(\gamma)\Upsilon = \Upsilon$ for any $\gamma \in \Gamma_G := \mathrm{Image}(\Gamma \to G_{\mathbf{Z}})$.

$G_{\mathbf{Z},u}$ and $G_{\mathbf{Q},u}$ are examples of Υ which satisfies (1) and (2).

Lemma 6.2.3. *Under the assumption on Υ in 6.2.2, $\Sigma(\Upsilon)$ is strongly compatible with Γ.*

We will deduce this from the following lemma.

Lemma 6.2.4. *Let τ' be a face of σ' and let $v \in G_{\mathbf{Q},u}$. Let $\Gamma'(\tau')$ be the face of $\mathrm{Hom}(P, \mathbf{N})$ corresponding to τ'. Then, the canonical map $\Gamma(\sigma_{\tau',v}) \to \Gamma'(\tau')$ is of Kummer type, that is, it is injective and some power of any element of $\Gamma'(\tau')$ belongs to its image.*

Proof. First, let γ be an element of $\Gamma(\sigma_{\tau',v})$. Let γ_G be the image of γ in $G_{\mathbf{Z}}$. Let γ' be the image of γ in $\Gamma'(\tau')$. Let γ'_G be the image of γ' in $G'_{\mathbf{Z}}$. γ'_G is identified with its trivial extension in $G_{\mathbf{Z}}$. Since $v^{-1}\gamma_G v$ is also the trivial extension of γ'_G, it coincides with γ'_G. Hence $\gamma_G = v\gamma'_G v^{-1}$. It implies that γ_G is recovered from γ'_G, and hence γ is recovered by γ'. This means that the canonical map $\Gamma(\sigma_{\tau',v}) \to \Gamma'(\tau')$ is injective.

Next, let γ' be an element of $\Gamma'(\tau')$. Let γ'_G be the image of γ' in $G'_{\mathbf{Q}} \subset G_{\mathbf{Q}}$. Then, γ' belongs to the image of the above map if and only if $v\gamma'_G v^{-1}$ belongs to $\Gamma_G = \mathrm{Image}(\Gamma \to G_{\mathbf{Z}})$, which is equivalent to that $v\gamma'_G v^{-1}$ belongs to $G_{\mathbf{Z}}$. Hence, to prove that the canonical map is of Kummer type, it suffices to show that there exists $n \geq 1$ such that $v(\gamma'_G)^n v^{-1}$ belongs to $G_{\mathbf{Z}}$ for any $\gamma' \in \Gamma'(\tau')$. But, we can take an n such that $n\,\mathrm{Ad}(v)\log(\gamma'_G)$ sends H_0 into H_0 for any $\gamma' \in \Gamma'(\tau')$. Multiplying n if necessary, we may assume further that $\mathrm{Int}(v)((\gamma'_G)^n)$ sends H_0 into H_0 for any $\gamma' \in \Gamma'(\tau')$. This n satisfies the desired condition. $\quad\square$

Proof of Lemma 6.2.3. First, we show that $\Sigma := \Sigma(\Upsilon)$ is compatible with Γ. Let $\tau' \in \mathrm{face}(\sigma')$ and $v \in \Upsilon$. Let $\tau_1 := \sigma_{\tau',v}$ as in 6.2.1. Fix $\gamma \in \Gamma$. We have to show $\mathrm{Ad}(\gamma)(\tau_1) \in \Sigma$. Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ (6.2.2), and since $G_{\mathbf{Z},u}$

is contained in Υ by the property (1) of Υ, we may assume that $\gamma \in \Gamma'$. By the property (2) of Υ, there exists $v' \in \Upsilon$ such that $\gamma_{G'} v = v' \gamma_{G'}$. Then, $\mathrm{Ad}(\gamma)(\tau_1) = \mathrm{Ad}(v')(\sigma_{\tau',\gamma})$, where $\sigma_{\tau',\gamma} := \{(x, \mathrm{Ad}(\gamma_{G'}) x_{\mathfrak{g}}) \mid x \in \tau'\}$. Hence it is enough to show that $\sigma_{\tau',\gamma}$ belongs to Σ. But, $\sigma_{\tau',\gamma} = \sigma_{\tau',1}$, and this certainly belongs to Σ. The compatibility of Σ and Γ is proved.

Next, we show that Σ and Γ are strongly compatible. Let τ', v, and τ_1 be as in the beginning of this proof. We have to show that any element of τ_1 can be written as a finite $\mathbf{R}_{\geq 0}$-combination of elements $N \in \tau_1$ satisfying $\exp(N) \in \Gamma$. Any element of τ_1 can be written as an $\mathbf{R}_{\geq 0}$-combination of $N_1, \ldots, N_m \in \tau_1$ satisfying $\exp(N_{j,\mathfrak{g}}) \in v \Gamma'_G v^{-1}$ for all $1 \leq j \leq m$, where $\Gamma'_G = \mathrm{Image}(\Gamma' \to G'_{\mathbf{Z}})$. On the other hand, by 6.2.4, for some $n \geq 1$, the image of $\exp(nN_j)$ in $\Gamma'(\tau')$ belongs to the image of the canonical map $\Gamma(\tau_1) \to \Gamma'(\tau')$ for all j. By the proof of 6.2.4, this implies that $\exp(nN_j)$ belongs to $\Gamma(\tau_1)$ for all j. Since any element of τ_1 can be written as an $\mathbf{R}_{\geq 0}$-combination of $nN_1, \ldots, nN_m$, this implies the desired statement. $\square$

As in 3.3.2, for a weak fan Σ which is strongly compatible with Γ, let $J_\Sigma = \Gamma \backslash D_{S,\Sigma}$.

We say a morphism $f : X \to Y$ in $\mathcal{B}(\log)$ is of Kummer type if, for any $x \in X$, the homomorphism $(M_Y/\mathcal{O}_Y^\times)_y \to (M_X/\mathcal{O}_X^\times)_x$ of fs monoids is of Kummer type, where $y = f(x)$.

Proposition 6.2.5. (*Characterization of J_Σ for $\Sigma = \Sigma(\Upsilon)$ with Υ as in 6.2.2.*)

(i) J_Σ *is of Kummer type over S.*

(ii) *For S' of Kummer type over S, a morphism $S' \to J_\Sigma$ over S corresponds in one-to-one manner to an isomorphism class of LMH H over S' with $H(\mathrm{gr}_w^W) = H_{(w)}$ ($w \in \mathbf{Z}$) such that, locally on S', there is a splitting $H_\mathbf{Q}(\mathrm{gr}^W) \simeq H_\mathbf{Q}$ of the filtration W on the local system $H_\mathbf{Q}$ on $(S')^{\log}$ satisfying the following condition: For any local isomorphism $H_\mathbf{Z}(\mathrm{gr}^W) \simeq H_0(\mathrm{gr}^W)$ on $(S')^{\log}$ which belongs to the given Γ'-level structure on $H_\mathbf{Z}(\mathrm{gr}^W)$, the composition $H_\mathbf{Q} \simeq H_\mathbf{Q}(\mathrm{gr}^W) \simeq H_{0,\mathbf{Q}}(\mathrm{gr}^W) = H_{0,\mathbf{Q}}$ sends $H_\mathbf{Z}$ onto vH_0 for some $v \in \Upsilon$.*

Proof. (i) is by 6.2.4. We prove (ii). Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ (6.2.2), any $H \in \mathrm{LMH}_Q(S')$ has automatically a unique Γ-level structure whose gr^W belongs to the given Γ'-level structure. For $t \in (S')^{\log}$, take a representative $\mu_t : H_{\mathbf{Z},t} \xrightarrow{\sim} H_0$ of such Γ-level structure, and put $\mu'_t := \mu_t(\mathrm{gr}^W)$.

The condition that H belongs to $\mathrm{LMH}_{Q,\Gamma}^{(\Sigma)}(S')$ (3.3.2), which J_Σ represents by 3.3.3, is equivalent to that, for any $s \in S'$ and any $t \in \tau^{-1}(s) \subset (S')^{\log}$, there exist a face ρ' of σ' and $v \in \Upsilon$ such that the image of the homomorphism

$$ f : \pi_1^+(s^{\log}) \to \sigma' \times \mathfrak{g}_\mathbf{R} $$

in 3.3.2 is contained in $\sigma_{\rho',v}$. Here $\pi_1^+(s^{\log}) := \mathrm{Hom}((M_{S'}/\mathcal{O}_{S'}^\times)_s, \mathbf{N}) \subset \pi_1(s^{\log})$. If this is the case, $a := \mu_t^{-1} v^{-1} \mu'_t : H_{\mathbf{Q},t}(\mathrm{gr}^W) \xrightarrow{\sim} H_{\mathbf{Q},t}$ extends to a local section over the inverse image by $\tau : (S')^{\log} \to S'$ of some open neighborhood of s in S'. Since the composition $\mu'_t a^{-1} = v\mu_t$ sends $H_{\mathbf{Z},t}$ onto vH_0, this a satisfies the condition in (ii).

Conversely, if there is such a splitting $a : H_{\mathbf{Q}}(\mathrm{gr}^W) \xrightarrow{\sim} H_{\mathbf{Q}}$ satisfying the condition in (ii), the homomorphism $\pi_1^+(s^{\log}) \to \mathrm{Aut}(H_{\mathbf{Q},t})$ factors into

$$\pi_1^+(s^{\log}) \to \prod_w \mathrm{Aut}(H_{\mathbf{Q},t}(\mathrm{gr}_w^W)) \xrightarrow{\mathrm{Int}(a)} \mathrm{Aut}(H_{\mathbf{Q},t}).$$

Define $\upsilon := \mu_t' a^{-1} \mu_t^{-1}$. Then, υ belongs to Υ by the property of a, and the homomorphism $\pi_1^+(s^{\log}) \to \mathrm{Aut}(H_{0,\mathbf{Q}})$ is the composite of $\pi_1^+(s^{\log}) \to \prod_w \mathrm{Aut}(H_{0,(w),\mathbf{Q}}(\mathrm{gr}_w^W))$ and $\mathrm{Int}(\upsilon) : \prod_w \mathrm{Aut}(H_{0,(w),\mathbf{Q}}(\mathrm{gr}_w^W)) \hookrightarrow \mathrm{Aut}(H_{0,\mathbf{Q}}) \to \mathrm{Aut}(H_{0,\mathbf{Q}})$. Thus we see that f is the composite of $f' : \pi_1^+(s^{\log}) \to \sigma' \times \mathfrak{g}_{\mathbf{R}}'$ and $1_{\sigma'} \times \mathrm{Ad}(\upsilon) : \sigma' \times \mathfrak{g}_{\mathbf{R}}' \hookrightarrow \sigma' \times \mathfrak{g}_{\mathbf{R}} \to \sigma' \times \mathfrak{g}_{\mathbf{R}}$. Let ρ' be the smallest cone in $\Sigma' = \mathrm{face}(\sigma')$ which contains the image of f'. ($\sigma' \times_{\mathfrak{g}_{\mathbf{R}}'} \mathfrak{g}_{\mathbf{R}}' = \sigma'$.) Then, the image of f is contained in $\sigma_{\rho',\upsilon} \in \Sigma(\Upsilon)$. $\square$

6.2.6. We now consider the Néron model and the connected Néron model. Let

$$\Sigma_0 := \Sigma(G_{\mathbf{Z},u}),$$

$$\Sigma_1 := \{\sigma_{\tau',\upsilon} \in \Sigma(G_{\mathbf{Q},u}) \mid \Gamma(\sigma_{\tau',\upsilon}) \to \Gamma'(\tau') \text{ is an isomorphism}\}.$$

Then Σ_1 is a weak fan which is strongly compatible with Γ, and J_{Σ_1} coincides with the open set of $J_{\Sigma(G_{\mathbf{Q},u})}$ consisting of all points at which $J_{\Sigma(G_{\mathbf{Q},u})} \to S$ is strict.

Proposition 6.2.7. (i) $\Sigma_0 \subset \Sigma_1$. In particlear, J_{Σ_0} is an open subspace of J_{Σ_1}.

(ii) J_{Σ_1} (resp. J_{Σ_0}) has the property of J_1 (resp. J_0) in Theorem 6.1.1.

Proof. (i) is easy to see. (ii) follows from 3.2.8 (i) and 6.2.5. $\square$

This proposition completes the proof of Theorem 6.1.1.

§6.3. Case of $\mathcal{E}xt^1$

Let the notation be as in §6.2.

We discuss more on the objects in §6.2 in the case where, for some $w < 0$, $H_{(k)} = 0$ for $k \neq w, 0$, and $H_{(0)} = \mathbf{Z}$.

6.3.1. Let $H_0' = \mathrm{gr}_w^W(H_0)$. So $H_0(\mathrm{gr}^W) = H_0' \oplus \mathbf{Z}$.
Hence,

$$G_{\mathbf{Z},u} \simeq H_0', \quad G_{\mathbf{Q},u} \simeq H_{0,\mathbf{Q}}',$$

and Υ satisfying (1) and (2) in 6.2.2 corresponds to a subgroup B of $H_{0,\mathbf{Q}}'$ containing H_0' which is stable under the action of Γ' (3.1.7).

The correspondence is as follows. Let $h \in H_{0,\mathbf{Q}}'$. Then the corresponding element of $G_{\mathbf{Q},u}$ sends $e = 1 \in \mathbf{Z}$ to $e + h$.

6.3.2. In the situation 6.3.1,

(i) $\Upsilon = G_{\mathbf{Z},u}$ corresponds to $B = H_0'$.

(ii) $\Upsilon = G_{\mathbf{Q},u}$ corresponds to $B = H_{0,\mathbf{Q}}'$.

By the construction of the Néron model and the connected Néron model in 6.2.6–6.2.7, Proposition 6.1.4 follows from the following proposition.

Proposition 6.3.3. *On the category of Kummer objects in $\mathcal{B}(\log)$ over S, J_Σ for $\Sigma = \Sigma(\Upsilon)$ represents*

$$\text{Ker}(\mathcal{E}xt^1(\mathbf{Z}, H') \to R^1\tau_* H'_{\mathbf{Z}} \to R^1\tau_* \tilde{B}),$$

where we denote by $\tilde{B}$ the sub local system of $H'_{\mathbf{Q}}$ corresponding to the subgroup $B \subset H'_{0,\mathbf{Q}}$.

Proof. Let $S' \in \mathcal{B}(\log)$ be of Kummer type over S. Then for an $H \in \text{LMH}_Q(S')$, the image of H in $R^1\tau_* \tilde{B}$ vanishes if and only if locally on S', the extension $0 \to H'_{\mathbf{Z}} \to H_{\mathbf{Z}} \to \mathbf{Z} \to 0$ of local systems on $(S')^{\log}$ splits after pushed out by $H'_{\mathbf{Z}} \to \tilde{B}$. The last condition is equivalent to the condition that locally on S', there is a splitting $H_{\mathbf{Q}}(\text{gr}^W) \simeq H_{\mathbf{Q}}$ of W on $H_{\mathbf{Q}}$ such that the local isomorphism $H_{\mathbf{Q}} \simeq H_{0,\mathbf{Q}}$ as in 6.2.5 (ii) sends $H_{\mathbf{Z}}$ to $\mathbf{Z}e + B$, i.e., onto vH_0 for some $v \in \Upsilon$. By 6.2.5 (ii), this is equivalent to $H \in J_\Sigma(S')$. $\square$

Proposition 6.3.4. *Assume that S is of log rank ≤ 1. Assume $\Gamma' \simeq \mathbf{Z}$, and let γ' be a generator of Γ'. Then Σ_1 in 6.2.6 coincides with $\Sigma(\Upsilon_1)$, where Υ_1 is the subgroup of $G_{\mathbf{Q},u}$ corresponding to the subgroup*

$$B_1 := \{b \in H'_{0,\mathbf{Q}} \,|\, \gamma'b - b \in H'_0\}$$

of $H'_{0,\mathbf{Q}}$.

Proof. Replacing γ' by $(\gamma')^{-1}$ if necessary, we may assume $\gamma' \in \text{Hom}(P, \mathbf{N}) \subset \Gamma'$. Let N' be the logarithm of the image of γ' in $\mathfrak{g}'_{\mathbf{Q}}$. By abuse of notation, we denote its trivial extension in $\mathfrak{g}_{\mathbf{Q}}$ also by N'. Let $v \in G_{\mathbf{Q},u}$, and let $b = ve - e \in H'_{0,\mathbf{Q}}$. Then the homomorphism $\Gamma(\mathbf{R}_{\geq 0} \text{Ad}(v)N') \to \Gamma'$ is an isomorphism if and only if $v\gamma'_G v^{-1} \in G_{\mathbf{Z}}$. Since $v\gamma'_G v^{-1} e - e = b - \gamma'b$, the last condition is equivalent to $b - \gamma'b \in H_0$, i.e., to $v \in \Upsilon_1$. $\square$

6.3.5. We review some relationships with other works by summarizing [KNU10c] 4.18. (In [KNU10c] 4.18, "4.4, 4.15" should be replaced by "4.15".)

Assume that we are in the situation of the beginning of this subsection.

If S is a disk and $w = -1$, Green-Griffiths-Kerr [GGK10] constructed a Néron model, which is homeomorphic to ours (6.1.1, 6.3.4, [KNU10a] 8.2) as proved by Hayama [H.p].

If S is of higher dimensional, Brosnan-Pearlstein-Saito [BPS.p] constructed a generalization of the Néron model of Green-Griffiths-Kerr. In [Scn.p], Schnell constructed a connected Néron model. The relationships between theirs and the (connected) Néron model in this paper are not known.

6.3.6. Though we assumed $H_{(0)} = \mathbf{Z}$ in this §6.3, it is not essential. In fact, we can study the case where there are only two non-trivial gr^W_w, say gr^W_a and gr^W_b with $a < b$ by using its special case where $b = 0$ and $H_{(0)} = \mathbf{Z}$, and by the canonical isomorphism

$$\mathcal{E}xt^1(\text{gr}^W_b, \text{gr}^W_a) = \mathcal{E}xt^1(\mathbf{Z}, (\text{gr}^W_b)^* \otimes \text{gr}^W_a).$$

§6.4. THE PROOFS FOR THE PREVIOUS ANNOUNCEMENTS

Here we give the proofs of the results in [KNU10c] which were omitted there.

Fix $\Lambda = (H_0, W, (\langle \ , \ \rangle_w)_w, (h^{p,q})_{p,q})$ as in 2.1.1. In this subsection, we use the method of §2.

6.4.1. In [KNU10c], we constructed some Néron models by the method of §2 in this paper as follows. Let Σ'_w be a weak fan on gr^W_w for each w. Let Γ'_w be a subgroup of $G_{\mathbf{Z}}(\mathrm{gr}^W_w)$ for each w, which is strongly compatible with Σ'_w. Let $\Sigma' := \prod_w \Sigma'_w$ and $\Gamma' = \prod_w \Gamma'_w$. Let Υ be a subgroup of $G_{\mathbf{Q},u}$. Let $\Sigma = \Sigma_\Upsilon$ be the weak fan consisting of all Υ-translations of the trivial extension of Σ' ([KNU10c] 2.1). Let Γ be the inverse image of Γ' in $G_{\mathbf{Z}}$.

Assume

(1) $G_{\mathbf{Z},u} \subset \Upsilon$, and

(2) $\gamma \Upsilon \gamma^{-1} = \Upsilon$ for any $\gamma \in \Gamma$.

Then, Σ is strongly compatible with Γ ([KNU10c] 4.2). This is shown in the same way as 6.2.3. (6.2.3 is shown by using 6.2.4. Similarly, as said in loc. cit., this strong compatibility is shown by using the fact that for any $\sigma' \in \Sigma'$ and any $v \in G_{\mathbf{Q},u}$, the map $\Gamma(\mathrm{Ad}(v)(\sigma)) \to \Gamma'(\sigma')$ is of Kummer type, where σ is the trivial extension of σ' (ibid. 4.3), which is an anlogue of 6.2.4, and shown in the same way as 6.2.4).

6.4.2. Let $D' := D(\mathrm{gr}^W)$. Let S be an object of $\mathcal{B}(\log)$. Assume that a morphism $S \to \Gamma' \backslash D'_{\Sigma'}$ is given. Then, we can obtain a moduli space of LMH with the given gr^W on S as the fiber product J_Σ of $S \to \Gamma' \backslash D'_{\Sigma'} \leftarrow \Gamma \backslash D_\Sigma$.

6.4.3. More precisely, J_Σ represents the functor associating with $T \in \mathcal{B}(\log)/S$ the set of isomorphism classes of an LMH H on T endowed with a Γ-level structure μ with the given gr^W satisfying the following conditions (1) and (2).

(1) Locally on T, there is a splitting of the local system $a : H_{\mathbf{Q}}(\mathrm{gr}^W) \simeq H_{\mathbf{Q}}$ such that for any isomorphism $b : H_{\mathbf{Z}} \to H_0 = H_0(\mathrm{gr}^W)$ belonging to μ, we have $ba\,\mathrm{gr}^W(b)^{-1} \in \Upsilon$.

(2) For any $t \in T^{\log}$, any element of the smallest $\sigma' \in \Sigma'$ whose exp contains the image of the induced map $\pi_1^+(\tau^{-1}\tau(t)) \to \prod_w \mathrm{Aut}(H_0(\mathrm{gr}^W_w))$ satisfies Griffiths transversality with respect to $\mathrm{gr}^W(b_t)a_t^{-1}(\mathbf{C} \otimes_{\mathcal{O}^{\log}_{T,\tau(t)}} F_{\tau(t)})$ on H_0, where F is the Hodge filtration of H.

We do not use this fact in this paper and omit the proof.

6.4.4. In the following, we assume that $\Sigma' = \mathrm{face}(\prod \sigma'_w)$ for some $\sigma'_w \in \Sigma'_w$. Let σ be the trivial extension of $\sigma' := \prod \sigma'_w$. Let Υ_1 be the subset of $G_{\mathbf{Q},u}$ consisting of all elements v such that $\Gamma(\mathrm{Ad}(v)(\sigma)) \to \Gamma'(\sigma')$ is an isomorphism. Assume further that there are only two non-trivial gr^W_k. Then, as [KNU10c] 4.6 (i) said, Υ_1 forms a subgroup and satisfies (1) and (2) in 6.4.1. This is seen as follows.

Let a, b be integers with $a < b$, and assume $\mathrm{gr}^W_w = 0$ unless $w = a, b$. In the following, we use the same symbol for an element of $G_{\mathbf{R}}(\mathrm{gr}^W)$ and its trivial extension to $G_{\mathbf{R}}$ by abuse of notation. Let $v \in G_{\mathbf{Q},u}$. As seen as in the proof of 6.2.4, v belongs Υ_1 if and only if $v\gamma'v^{-1}$ belongs to $G_{\mathbf{Z}}$ for any $\gamma' \in \Gamma'(\sigma')$. Since $\gamma' \in G_{\mathbf{Z}}$, the latter condition is satisfied by any $v \in G_{\mathbf{Z},u}$, that is, $G_{\mathbf{Z},u} \subset \Upsilon_1$.

Next, to see that Υ_1 forms a subgroup of $G_{\mathbf{Q},u}$, we describe the above condition as follows. Let q be the $\mathrm{Hom}(\mathrm{gr}_b^W, \mathrm{gr}_a^W)$-component of v. Let γ_a' and γ_b' be the $G_{\mathbf{Z}}(\mathrm{gr}_a^W)$-component and $G_{\mathbf{Z}}(\mathrm{gr}_b^W)$-component of γ' respectively. Then, the $\mathrm{Hom}(\mathrm{gr}_b^W, \mathrm{gr}_a^W)$-component of $v\gamma'v^{-1}$ is $q\gamma_b' - \gamma_a'q$. Hence, the above condition is equivalent to the condition that the element $q\gamma_b' - \gamma_a'q$ of $\mathrm{Hom}(\mathrm{gr}_b^W, \mathrm{gr}_a^W)$ sends $(H_0 \cap W_b)/(H_0 \cap W_{b-1})$ into $(H_0 \cap W_a)/(H_0 \cap W_{a-1})$. If $q_1, q_2 \in G_{\mathbf{Q},u}$ satisfy this last condition, then, $q_1 - q_2$ also satisfies it. Since $G_{\mathbf{Q},u}$ is naturally isomorphic to the additive group $\mathrm{Hom}(\mathrm{gr}_b^W, \mathrm{gr}_a^W)$, our Υ_1 is a subgroup.

Finally, let $\gamma \in \Gamma$, and we prove $\gamma\Upsilon_1\gamma^{-1} = \Upsilon_1$. Let $v \in \Upsilon_1$. Let $\gamma' \in \Gamma'(\sigma')$. We have to prove $(\gamma v\gamma^{-1})\gamma'(\gamma v^{-1}\gamma^{-1}) \in G_{\mathbf{Z}}$. Since Γ is a semi-direct product of Γ' and $G_{\mathbf{Z},u}$ and since any element of $G_{\mathbf{Z},u}$ commutes with v (here we use the assumption that $\mathrm{gr}_w^W = 0$ unless $w = a, b$ again), we may assume that γ is in Γ'. Then, $\gamma^{-1}\gamma'\gamma \in \Gamma'$ and $\log(\gamma^{-1}\gamma'\gamma) = \mathrm{Ad}(\gamma^{-1})\log\gamma' \in \mathrm{Ad}(\gamma^{-1})(\sigma') = \sigma'$, where the last equality is by the fact that $\Sigma' = \mathrm{face}\,(\sigma')$ and Γ' are compatible. Hence $\gamma^{-1}\gamma'\gamma \in \Gamma'(\sigma')$. Since $v \in \Upsilon_1$, this implies that $v(\gamma^{-1}\gamma'\gamma)v^{-1} \in G_{\mathbf{Z}}$, and hence $\gamma v\gamma^{-1}\gamma'\gamma v^{-1}\gamma^{-1} \in G_{\mathbf{Z}}$, which completes the proof.

6.4.5. Assume that we are in the situation of 6.4.4. Let Υ be as in 6.4.1. If $\Upsilon \subset \Upsilon_1$, then, $\Gamma\backslash D_\Sigma \to \Gamma'\backslash D_{\Sigma'}'$ is strict. As is said in [KNU10c] 4.6 (iii), this projection is a relative manifold with slits in the sense of loc. cit. In fact, by 3.4.2, this is reduced to 3.2.8 (i) and the fact that any strict relative log manifold is a relative manifold with slits.

6.4.6. We continue to assume that we are in the situation of 6.4.4. Assume further that there exists a strict morphism $S \to \Gamma'\backslash D_{\Sigma'}'$. Let Σ_{Υ_1} be the weak fan corresponding to Υ_1 (6.4.1). By 6.4.5, $J_{\Sigma_{\Upsilon_1}} \to S$ is strict and a relative manifold with slits.

As is said in [KNU10c] 4.8, the following (i)–(iii) hold.

(i) For an object T of $\mathcal{B}/S^\circ$, there is a natural functorial injection $\iota(T) : \mathrm{Mor}_{S^\circ}(T, J_{\Sigma_{\Upsilon_1}}) \to$ ∎ $\{H \,|\, \mathrm{LMH}$ on T with the given gr^W such that W on $H_{\mathbf{Q}}$ splits locally on $T\}$.

(ii) If $\dim(\sigma') = 1$, then ι is bijective.

(iii) Assume that $s \in S$ is given such that $\pi_1^+(s^{\log}) \to \Gamma'(\sigma')$ is bijective. Let $T \to S$ be a strict morphism, and let $t \in T$ be a point lying over s. Then, the stalk of ι at t is bijective.

(i) is proved similarly as 6.2.5.

We prove (ii) and (iii). Let $T \to S$ be a strict morphism from an object T in $\mathcal{B}(\log)$. Let (H, b) be an LMH over T with polarized graded quotients endowed with a Γ-level structure. Assume that its gr^W is the given one and that W splits locally on T. To prove (ii) (resp. (iii)), it is enough to show that, under the assumption of (ii) (resp. (iii)), (H, b) satisfies the conditions (1) and (2) in 2.6.2 (resp. after replacing T with an open neighborhood of t).

First, (1) is always satisfied.

We consider the condition (2). Let $t \in T$ lying over $s \in S$ and $t' \in T^{\log}$ lying over t. Let $\tilde{b}_{t'}$ be a representative of the germ of b at t'. Since $T \to S \to \Gamma'\backslash D_{\Sigma'}'$ is strict, the map $f' : \pi_1^+(t^{\log}) \xrightarrow{\simeq} \pi_1^+(s^{\log}) \to \Gamma'(\sigma')$ is injective and its image is a face. On the other hand, since the W on the local system splits over $\mathbf{Q}$, there exists $v \in G_{\mathbf{Q},u}$ such that the image of the map $f : \pi_1^+(t^{\log}) \to \mathrm{Aut}(H_0)$ is contained in

$\exp(\mathrm{Ad}(v)(\sigma'))$. We observe that if f' is surjective, then $v \in \Upsilon_1$. This is because f' factors as $\pi_1^+(t^{\log}) \xrightarrow{f} \Gamma(\mathrm{Ad}(v)(\sigma')) \hookrightarrow \Gamma'(\sigma')$ so that $\Gamma(\mathrm{Ad}(v)(\sigma')) \hookrightarrow \Gamma'(\sigma')$ becomes an isomorphism.

Now we assume that S is of log rank ≤ 1. Then, the map f' is either trivial or surjective. Hence, in both cases, the image of f is contained in $\exp(\mathrm{Ad}(v)(\sigma'))$ for some $v \in \Upsilon_1$. Thus the former half of (2) is satisfied. Further, $\tilde{b}_{t'}(\mathbf{C} \otimes_{\mathcal{O}_{T,t'}^{\log}} F_{t'})$ satisfies the Griffiths transversality with respect to each element of the image of f. This implies that it generates a τ-nilpotent orbit, where τ is the smallest cone of Σ_{Υ_1} such that $\exp(\tau)$ contains the image of f (that is, τ is $\{0\}$ or $\mathrm{Ad}(v)(\sigma')$). Therefore, the latter half of (2) is also satisfied, which completes the proof of (ii).

Finally, suppose that we are in the situation of (iii). Then, at least, the image of f at the prescribed t is contained in $\exp(\mathrm{Ad}(v)(\sigma'))$ for some $v \in \Upsilon_1$ because f' at t is bijective by the assumption. Since, for any point u of T which is sufficiently near t, the map f at u naturally factors as $\pi_1^+(u^{\log}) \to \pi_1^+(t^{\log}) \to \mathrm{Aut}(H_0)$, the image of f at u is contained in $\exp(\mathrm{Ad}(v)(\tau'))$, where τ' is the smallest face of σ' such that $\exp(\tau')$ contains the image of f' at u. Hence, after replacing T, the former part of (2) is satisfied. In fact, $\mathrm{Ad}(v)(\tau')$ is the smallest cone which has the above property that its exp contains the image of f at u. Since the Griffiths transversality is satisfied with respect to each element of this cone, the latter half of (2) is also satisfied, which completes the proof of (iii).

6.4.7. Let the assumption be as in 6.4.6. Let Υ_1 be as in 6.4.4. From now on, as in 6.3, assume further $\mathrm{gr}_0^W = \mathbf{Z}$ with the standard polarization, and there is $w < 0$ such that $\mathrm{gr}_k^W = 0$ for $k \neq 0, w$.

Then, Υ_1 corresponds to the subgroup $B_1 = \{x \in H'_{0,\mathbf{Q}} \mid \gamma x - x \in H'_0 \text{ for all } \gamma \in \Gamma'(\sigma')\}$ of $H'_{0,\mathbf{Q}}$. This is seen similarly as in the proof of 6.3.4.

Let Υ be as in 6.4.1 and assume $\Upsilon \subset \Upsilon_1$. Let B be the subgroup of $H'_{0,\mathbf{Q}}$ corresponding to Υ. Then, on $\mathcal{B}/S^\circ$, J_Σ represents the functor $\{$LMH with the given gr^W whose image in $R^1\tau_* H'_{\mathbf{Z}}$ belongs to the kernel of $R^1\tau_* H'_{\mathbf{Z}} \to R^1\tau_* B\}$ ([KNU10c] 4.11). This is easily seen in the same way as 6.3.3.

6.4.8. Let the assumption be as in 6.4.6. In [KNU10c] 4.4, we called J_Σ the connected Néron model when $\Upsilon = G_{\mathbf{Z},u}$. Also, in [KNU10c] 4.7, we called $J_{\Sigma_{\Upsilon_1}}$ the Néron model.

As seen easily, the connected Néron model in this sense is the connected Néron model in this paper (6.1.1). However, the Néron model in this sense is not necessarily the Néron model in this paper (6.1.1). Their relationship is as follows. Consider the weak fan $\Sigma_1 := \{\mathrm{Ad}(v)(\tau') \mid v \in G_{\mathbf{Q},u}, \tau' \in \Sigma', \text{ such that } \Gamma(\mathrm{Ad}(v)(\tau')) \simeq \Gamma'(\tau')\}$. Then Σ_{Υ_1} is a subfan of Σ_1. Let J_{Σ_1} be the fiber product of $S \to \Gamma'\backslash D'_{\Sigma'} \leftarrow \Gamma\backslash D_\Sigma$. Then, this J_{Σ_1} is the Néron model in this paper, and $J_{\Sigma_{\Upsilon_1}}$ is an open subspace of J_{Σ_1} having a group structure.

6.4.9. Finally, we remark that, though the assumption of the existence of a strict morphism $S \to \Gamma'\backslash D'_{\Sigma'}$ in 6.4.6 is rather restrictive, the following holds. Let S be an object of $\mathcal{B}(\log)$ whose log rank is ≤ 1. We assume that we are given a family $Q = (H_{(w)})_{w\in\mathbf{Z}}$ of PLH's over S. Then, as was stated in [KNU10c] 4.14, locally on S, we can always take Σ', Γ', and a strict morphism $S \to \Gamma'\backslash D'_{\Sigma'}$ such that the pull-back

of the canonical family of PLH's on $\Gamma'\backslash D'_{\Sigma'}$ is isomorphic to Q. (The assumption in [KNU10c] 4.14 was inaccurate.) This is shown in [KU09] 4.3.

We remark also that, for a general S, the conclusion of 6.4.7 holds even if the strictness is weakened into the condition that $S \to \Gamma'\backslash D'_{\Sigma'}$ is log injective. Here a morphism $f : X \to Y$ in $\mathcal{B}(\log)$ is called *log injective* if the induced map $f^{-1}(M_Y/\mathcal{O}_Y) \to M_X/\mathcal{O}_X$ is injective.

§7. EXAMPLES AND COMPLEMENTS

In 7.1, basic examples, including log abelian varieties and log complex tori, are described. In 7.2, we discuss by examples why weak fans are essential. In 7.3, we propose some problems on the existence of complete weak fans. In 7.4, we gather some statements on the existences of period maps.

§7.1. BASIC EXAMPLES

7.1.1. $0 \to \mathbf{Z}(1) \to H \to \mathbf{Z} \to 0$.

We take $\Lambda = (H_0, W, (\langle \ , \ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2,$$

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}e_1 \subset W_0 = H_{0,\mathbf{R}},$$

$$\langle e_1', e_1' \rangle_{-2} = 1, \quad \langle e_2', e_2' \rangle_0 = 1,$$

where for $j = 1$ (resp. $j = 2$), e_j' denotes the image of e_j in gr^W_{-2} (resp. gr^W_0),

$$h^{0,0} = h^{-1,-1} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Then $D = \mathbf{C}$, where $z \in \mathbf{C}$ corresponds to $F = F(z) \in D$ defined by

$$F^1 = 0 \subset F^0 = \mathbf{C} \cdot (ze_1 + e_2) \subset F^{-1} = H_{0,\mathbf{C}}.$$

This D appeared in [KNU09] and [KNU.p] as Example I.

Let

$$\Gamma = G_{\mathbf{Z},u} = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix} \subset \mathrm{Aut}(H_0, W, (\langle \ , \ \rangle_w)_w),$$

$$\Sigma = \{\sigma, -\sigma, \{0\}\} \quad \text{with} \ \pm\sigma = \begin{pmatrix} 0 & \pm\mathbf{R}_{\geq 0} \\ 0 & 0 \end{pmatrix} \subset \mathfrak{g}_{\mathbf{R}}.$$

Then Σ is a fan and is strongly compatible with Γ. We have

$$D = \mathbf{C} \to \Gamma\backslash D = \mathbf{C}^\times \subset \Gamma\backslash D_\Sigma = \mathbf{P}^1(\mathbf{C}),$$

where $\mathbf{P}^1(\mathbf{C})$ is endowed with the log structure corresponding to the divisor $\{0, \infty\}$. Here $\mathbf{C} \to \mathbf{C}^\times$ corresponding to $D \to \Gamma\backslash D$ is $z \mapsto e^{2\pi i z}$, and 0 (resp. ∞) $\in \mathbf{P}^1(\mathbf{C})$

corresponds to the class of the nilpotent orbit (σ, Z) (resp. $(-\sigma, Z)$), where $Z = D$. We have $D_{\Sigma,\mathrm{val}} = D_\Sigma$. We have

$$D_\Sigma^\sharp = D_{\Sigma,\mathrm{val}}^\sharp = D_{\mathrm{SL}(2)}^I = D_{\mathrm{SL}(2)}^{II} = \mathbf{R} \times [-\infty, \infty]$$

as topological spaces, where the second equality is given by the CKS map (§4.3), the last equality is given in [KNU.p] 3.6.1 Example I, $D = \mathbf{C}$ is embedded in $\mathbf{R} \times [-\infty, \infty]$ by $x + iy \mapsto (x, y)$ $(x, y \in \mathbf{R})$, and the projection $D_\Sigma^\sharp \to \Gamma \backslash D_\Sigma$ sends $(x, y) \in \mathbf{R} \times [-\infty, \infty]$ to $e^{2\pi i(x+iy)} \in \mathbf{P}^1(\mathbf{C})$.

Next we treat this example by the formulation in §3. Let S be an object of $\mathcal{B}(\log)$ and let $Q = (H_{(w)})_w$ with

$$H_{(-2)} = \mathbf{Z}(1), \quad H_{(0)} = \mathbf{Z}, \quad H_{(w)} = 0 \ \text{for} \ w \neq -2, 0.$$

Then $\mathrm{LMH}_Q = \mathcal{E}xt^1_{LMH}(\mathbf{Z}, \mathbf{Z}(1))$ is identified with the following functor. For an object T of $\mathcal{B}(\log)$ over S, $\mathrm{LMH}_Q(T) = \Gamma(T, M_T^{\mathrm{gp}})$. Here $a \in \Gamma(T, M_T^{\mathrm{gp}})$ corresponds to the class of the following LMH H on T. Define the local system $H_{\mathbf{Z}}$ of $\mathbf{Z}$-modules on $T^{\log}$ by the commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \mathbf{Z}(1) & \to & H_{\mathbf{Z}} & \to & \mathbf{Z} & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{Z}(1) & \to & \mathcal{L}_T & \xrightarrow{\exp} & \tau^{-1}(M_T^{\mathrm{gp}}) & \to & 0
\end{array}
$$

of sheaves on $T^{\log}$, where τ is the canonical map $T^{\log} \to T$, the bottom row is the exact sequence of the sheaf of logarithms $\mathcal{L}_T$ (cf. [KN99], [KU09] 2.2.4), and the right vertical arrow $\mathbf{Z} \to \tau^{-1}(M_T^{\mathrm{gp}})$ is the homomorphism which sends 1 to a^{-1}. Let H be the LMH $(H_{\mathbf{Z}}, W, F)$, where

$$W_{-3} = 0 \subset W_{-2} = W_{-1} = \mathbf{R}(1) \subset W_0 = H_{\mathbf{R}}$$

and F is the decreasing filtration on $\mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}}$ defined by

$$F^1 = 0 \subset F^0 = \mathrm{Ker}(c : \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}} \to \mathcal{O}_T^{\log}) \subset F^{-1} = \mathcal{O}_T^{\log} \otimes H_{\mathbf{Z}}$$

with c being the $\mathcal{O}_T^{\log}$-homomorphism induced by $H_{\mathbf{Z}} \to \mathcal{L}_T \subset \mathcal{O}_T^{\log}$. Take $\sigma' = \{0\}$ in §3.1. The above Σ is regarded as a fan in §3, and the above Γ is regarded as Γ in §3. $\Gamma \backslash D_{S,\Sigma}$ in §3 is identified with $S \times \Gamma \backslash D_\Sigma \simeq S \times \mathbf{P}^1(\mathbf{C})$ with $\Gamma \backslash D_\Sigma$ as above. The inclusion map $\mathrm{LMH}_Q^{(\Sigma)}(T) = \mathrm{Mor}_{\mathcal{B}(\log)}(T, \mathbf{P}^1(\mathbf{C})) \subset \mathrm{LMH}_Q(T) = \Gamma(T, M_T^{\mathrm{gp}})$ is understood by the fact that $\mathbf{P}^1(\mathbf{C})$ with the above log structure represents the subfunctor $T \mapsto \Gamma(T, M_T \cup M_T^{-1})$ of $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$.

7.1.2. $0 \to H^1(E)(1) \to H \to \mathbf{Z} \to 0$ (E a degenerating elliptic curve).

We take $\Lambda = (H_0, W, (\langle \ , \ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^3 = \bigoplus_{j=1}^3 \mathbf{Z}e_j,$$

$$W_{-2} = 0 \subset W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

$$\langle e_2', e_1' \rangle_{-1} = 1, \quad \langle e_3', e_3' \rangle_0 = 1,$$

where for $j = 1, 2$ (resp. $j = 3$), e_j' denotes the image of e_j in gr_{-1}^W (resp. gr_0^W),

$$h^{0,0} = 1, \quad h^{0,-1} = h^{-1,0} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Then $D = \mathfrak{h} \times \mathbf{C}$, where $\mathfrak{h}$ is the upper half plane, and $(\tau, z) \in \mathfrak{h} \times \mathbf{C}$ corresponds to the following $F(\tau, z) \in D$. For $\tau, z \in \mathbf{C}$, we define $F = F(\tau, z) \in \check{D}$ by

$$F^1 = 0 \subset F^0 = \mathbf{C} \cdot (\tau e_1 + e_2) + \mathbf{C} \cdot (z e_1 + e_3) \subset F^{-1} = H_{0,\mathbf{C}}.$$

This D appeared in [KNU09] and [KNU.p] as Example II.

Consider

$$G_{\mathbf{Z},u} = \begin{pmatrix} 1 & 0 & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset \Gamma := \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset G_{\mathbf{Z}}.$$

The quotients $G_{\mathbf{Z},u}\backslash D$ and $\Gamma\backslash D$ are "universal elliptic curves" over $\mathfrak{h}$ and over $\mathbf{Z}\backslash\mathfrak{h} \simeq \Delta^* = \Delta \smallsetminus \{0\}$ ($\Delta := \{q \in \mathbf{C} \mid |q| < 1\}$), respectively, and described as

$$
\begin{array}{ccc}
G_{\mathbf{Z},u}\backslash D = (\mathfrak{h} \times \mathbf{C})/\sim & \longrightarrow & \Gamma\backslash D = \bigcup_{q \in \Delta^*} \mathbf{C}^\times/q^{\mathbf{Z}} \\
\downarrow & & \downarrow \\
D(\mathrm{gr}_{-1}^W) = \mathfrak{h} & \longrightarrow & \Gamma'\backslash D(\mathrm{gr}_{-1}^W) = \Delta^*.
\end{array}
$$

Here, in the above diagram, the notation is as follows. $\mathbf{Z}\backslash\mathfrak{h} \simeq \Delta^*$ is given by $\tau \mapsto e^{2\pi i \tau}$. The equivalence relation $\sim$ on $\mathfrak{h} \times \mathbf{C}$ is defined as follows. For $\tau, \tau' \in \mathfrak{h}$ and $z, z' \in \mathbf{C}$, $(\tau, z) \sim (\tau', z')$ if and only if $\tau = \tau'$ and $z \equiv z' \mod \mathbf{Z}\tau + \mathbf{Z}$. $D(\mathrm{gr}_{-1}^W)$ is the D for $\Lambda(\mathrm{gr}_{-1}^W)$. Γ' denotes the image of Γ in $\mathrm{Aut}(\mathrm{gr}_{-1}^W)$ and is identified with $\mathbf{Z}$. The fiber $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$ of $(\mathfrak{h} \times \mathbf{C})/\sim \to \mathfrak{h}$ over $\tau \in \mathfrak{h}$ is identified with the fiber $\mathbf{C}^\times/q^{\mathbf{Z}}$ of $\bigcup_q \mathbf{C}^\times/q^{\mathbf{Z}} \to \Delta^*$ over $q = e^{2\pi i \tau} \in \Delta^*$, via the isomorphism $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \simeq \mathbf{C}^\times/q^{\mathbf{Z}}$, $z \mod (\mathbf{Z}\tau + \mathbf{Z}) \mapsto e^{2\pi i z} \mod q^{\mathbf{Z}}$.

For $n \in \mathbf{Z}$, define $N_n \in \mathfrak{g}_{\mathbf{Q}}$ by

$$N_n(e_1) = 0, \ N_n(e_2) = e_1, \ N_n(e_3) = n e_1.$$

Define the fans Σ and Σ_0 by

$$\Sigma = \{\sigma_{n,n+1}, \ \sigma_n \ (n \in \mathbf{Z}), \ \{0\}\} \quad \text{with} \ \sigma_{n,n+1} = \mathbf{R}_{\geq 0}N_n + \mathbf{R}_{\geq 0}N_{n+1}, \ \sigma_n = \mathbf{R}_{\geq 0}N_n,$$

$$\Sigma_0 = \{\sigma_n \ (n \in \mathbf{Z}), \ \{0\}\}.$$

Then (Σ, Γ) is strongly compatible and (Σ_0, Γ) is also strongly compatible. Let $\sigma' \subset \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_{-1}^W)$ be the cone $\mathbf{R}_{\geq 0}N'$, where N' is the linear map $\mathrm{gr}_{-1}^W \to \mathrm{gr}_{-1}^W$ which sends e_1' to 0 and e_2' to e_1'. We have a commutative diagram of fs log analytic spaces

$$
\begin{array}{ccccc}
\Gamma\backslash D = \bigcup_{q \in \Delta^*} \mathbf{C}^\times/q^{\mathbf{Z}} & \subset & \Gamma\backslash D_{\Sigma_0} = (\Gamma\backslash D) \cup \mathbf{C}^\times & \subset & \Gamma\backslash D_{\Sigma} = \\
& & & & (\Gamma\backslash D) \cup (\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma'\backslash D(\mathrm{gr}_{-1}^W) = \Delta^* & \subset & \Gamma'\backslash D(\mathrm{gr}_{-1}^W)_{\sigma'} = \Delta & = & \Gamma'\backslash D(\mathrm{gr}_{-1}^W)_{\sigma'}.
\end{array}
$$

Here the fiber of the middle (resp. right) vertical arrow over $0 \in \Delta$ is $\mathbf{C}^{\times}$ (resp. the quotient $\mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ of $\mathbf{P}^1(\mathbf{C})$ obtained by identifying 0 and ∞). The log structures of $\Gamma \backslash D_{\Sigma_0}$ and of $\Gamma \backslash D_{\Sigma}$ are given by the inverse images of $0 \in \Delta$ which are divisors with normal crossings. The element $t \in \mathbf{C}^{\times}$ in the fiber over $0 \in \Delta$ corresponds to the class of the nilpotent orbit (σ_0, Z) with $Z = F(\mathbf{C}, z)$ for $z \in \mathbf{C}$ such that $t = e^{2\pi i z}$. The element $0 = \infty \in \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$ in the fiber over $0 \in \Delta$ corresponds to the class of the nilpotent orbit $(\sigma_{0,1}, Z)$ with $Z = F(\mathbf{C}, \mathbf{C})$.

The infinity of the topological spaces $D_{\Sigma_0}^{\sharp}$, $D_{\Sigma}^{\sharp}$, $D_{\Sigma_0, \mathrm{val}}^{\sharp}$, $D_{\Sigma, \mathrm{val}}^{\sharp}$, the projection $D_{\Sigma}^{\sharp} \to \Gamma \backslash D_{\Sigma}$, and the CKS map $D_{\Sigma, \mathrm{val}}^{\sharp} \to D_{\mathrm{SL}(2)}$ are described as follows. We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{C} \xrightarrow{\sim} D_{\Sigma_0}^{\sharp} \smallsetminus D$, $(n, x, z) \mapsto p_n(x, z)$, where

$$p_n(x, z) = \lim_{y \to \infty} F(x + iy, z + iny).$$

We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{R} \xrightarrow{\sim} D_{\Sigma}^{\sharp} \smallsetminus D_{\Sigma_0}^{\sharp}$, $(n, x, a) \mapsto p_{n,n+1}(x, a)$, where

$$p_{n,n+1}(x, a) = \lim_{y, \, y' \to \infty} F(x + i(y + y'), a + in(y + y') + iy').$$

We have $D_{\Sigma_0}^{\sharp} = D_{\Sigma_0, \mathrm{val}}^{\sharp}$. The points of $D_{\Sigma, \mathrm{val}}^{\sharp} \smallsetminus D_{\Sigma_0, \mathrm{val}}^{\sharp}$ are written uniquely as either $p_r(x, z)$ with $r \in \mathbf{Q} \smallsetminus \mathbf{Z}$, $x \in \mathbf{R}$ and $z \in \mathbf{C}$, or $p_r(x, a)$ with $r \in \mathbf{R} \smallsetminus \mathbf{Q}$ and $x, a \in \mathbf{R}$, or $p_{r,+}(x, a)$ with $r \in \mathbf{Q}$ and $x, a \in \mathbf{R}$, or $p_{r,-}(x, a)$ with $r \in \mathbf{Q}$ and $x, a \in \mathbf{R}$. Here

$$p_r(x, z) = \lim_{y \to \infty} F(x + iy, z + iry) \quad (r \in \mathbf{R}, x \in \mathbf{R}, z \in \mathbf{C})$$

(in the case $r \notin \mathbf{Q}$, we have $p_r(x, z) = p_r(x, \mathrm{Re}(z))$),

$$p_{r,\pm}(x, a) = \lim_{y', \, y/y' \to \infty} F(x + iy, a + iry \pm iy').$$

For $n \in \mathbf{Z}$, the projection $D_{\Sigma, \mathrm{val}}^{\sharp} \to D_{\Sigma}^{\sharp}$ sends $p_r(x, z)$ ($r \in \mathbf{R}$, $n < r < n+1$, $x \in \mathbf{R}$, $z \in \mathbf{C}$) to $p_{n,n+1}(x, \mathrm{Re}(z))$, and sends $p_{r,+}(x, a)$ ($n \leq r < n+1$, $x, a \in \mathbf{R}$) and $p_{r,-}(x, a)$ ($n < r \leq n+1$, $x, a \in \mathbf{R}$) to $p_{n,n+1}(x, a)$. The projection $D_{\Sigma}^{\sharp} \to \Gamma \backslash D_{\Sigma}$ sends $p_n(x, z)$ ($n \in \mathbf{Z}, x \in \mathbf{R}, z \in \mathbf{C}$) to $e^{2\pi i z} \in \mathbf{C}^{\times}$ in the fiber over $0 \in \Delta$, and sends $p_{n,n+1}(x, a)$ to $0 = \infty \in \mathbf{P}^1(\mathbf{C})/(0 \sim \infty)$.

As in [KNU.p] 3.6.1 Example II, we have $D_{\mathrm{SL}(2)}^{I} = D_{\mathrm{SL}(2)}^{II}$ as topological spaces. For the admissible set Ψ of weight filtrations on $H_{0,\mathbf{R}}$ (4.1.2) consisting of single filtration

$$W_{-3}' = 0 \subset W_{-2}' = W_{-1}' = \mathbf{R}e_1 \subset W_0' = H_{0,\mathbf{R}},$$

the open set $D_{\mathrm{SL}(2)}^{I}(\Psi)$ (4.1.2) of $D_{\mathrm{SL}(2)}^{I}$ has a description as a topological space

$$D_{\mathrm{SL}(2)}^{I}(\Psi) = \mathrm{spl}(W) \times \bar{\mathfrak{h}} = \mathbf{R}^2 \times \bar{\mathfrak{h}},$$

where $\bar{\mathfrak{h}} = \{x + iy \mid x \in \mathbf{R}, 0 < y \leq \infty\}$, and we identify $s \in \mathrm{spl}(W)$ with $(a, b) \in \mathbf{R}^2$ such that $s(e_3') = ae_1 + be_2 + e_3$. The embedding $D \to D_{\mathrm{SL}(2)}$ sends $F(\tau, z)$ ($\tau \in \mathfrak{h}$,

$z \in \mathbf{C}$) to $(\mathrm{Re}(z) - \mathrm{Re}(\tau)\,\mathrm{Im}(z)/\,\mathrm{Im}(\tau), -\,\mathrm{Im}(z)/\,\mathrm{Im}(\tau), \tau) \in \mathbf{R}^2 \times \mathfrak{h}$. The CKS map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ has image in $D^I_{\mathrm{SL}(2)}(\Psi)$ and sends $p_r(x, z)$ ($r \in \mathbf{R}, x \in \mathbf{R}, z \in \mathbf{C}$) to $(\mathrm{Re}(z) - rx, -r, x + i\infty) \in \mathbf{R}^2 \times \bar{\mathfrak{h}}$, and sends $p_{r,+}(x, a)$ and $p_{r,-}(x, a)$ ($r \in \mathbf{Q}, x, a \in \mathbf{R}$) to $(a - rx, -r, x + i\infty) \in \mathbf{R}^2 \times \bar{\mathfrak{h}}$.

Using the formulation in §3, let $S = \Delta = \Gamma' \backslash D(\mathrm{gr}^W_{-1})_{\sigma'}$ with the log structure at $0 \in \Delta$, and consider $Q = (H_{(w)})_w$, where $H_{(-1)}$ is the universal log Hodge structure of weight -1 on S (so $H_{(-1)}$ is of rank 2), $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq -1, 0$. Then the above σ' is regarded as σ' in §3. Since $\sigma' \to \mathfrak{g}_\mathbf{R}(\mathrm{gr}^W_{-1})$ is injective, the above Σ and Σ_0 are regarded as fans in §3, and the above Γ is regarded as Γ in §3. (Σ, Γ) and also (Σ_0, Γ) are strongly compatible. We have $\Gamma \backslash D_{S,\Sigma} = \Gamma \backslash D_{\Sigma}$, $\Gamma \backslash D_{S,\Sigma_0} = \Gamma \backslash D_{\Sigma_0}$. Furthermore, Σ_0 is identified with Σ_j in §6 for $j = 0, 1$ (Σ_0 and Σ_1 in §6 coincide in this case). That is, $\Gamma \backslash D_{\Sigma_0} \to S$ is the connected Néron model and is also the Néron model.

7.1.3. $0 \to H^1(E)(b) \to H \to \mathbf{Z} \to 0$ ($b \geq 2$) (E a degenerating elliptic curve).
Let $b \geq 2$. We take $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ in 2.1.1 as follows. Let

$$H_0 = \mathbf{Z}^3 = \bigoplus_{j=1}^3 \mathbf{Z}e_j,$$

$$W_{-2b} = 0 \subset W_{1-2b} = W_{-1} = \mathbf{R}e_1 + \mathbf{R}e_2 \subset W_0 = H_{0,\mathbf{R}},$$

$$\langle e'_2, e'_1 \rangle_{1-2b} = 1, \quad \langle e'_3, e'_3 \rangle_0 = 1,$$

where for $j = 1, 2$ (resp. $j = 3$), e'_j denotes the image of e_j in gr^W_{1-2b} (resp. gr^W_0),

$$h^{0,0} = 1, \quad h^{1-b,-b} = h^{-b,1-b} = 1, \quad h^{p,q} = 0 \text{ for all the other } (p, q).$$

Then $D = \mathfrak{h} \times \mathbf{C}^2$, where $(\tau, z, w) \in \mathfrak{h} \times \mathbf{C}^2$ corresponds to the following $F(\tau, z, w) \in D$. For $\tau, z, w \in \mathbf{C}$, we define $F = F(\tau, z, w) \in \check{D}$ by

$$F^1 = 0 \subset F^0 = F^{2-b} = \mathbf{C}{\cdot}(ze_1 + we_2 + e_3) \subset F^{1-b} = F^{2-b} + \mathbf{C}{\cdot}(\tau e_1 + e_2) \subset F^{-b} = H_{0,\mathbf{C}}.$$

This D in the case $b = 2$ appeared in [KNU09] and [KNU.p] as Example III.
Like in 7.1.2, consider

$$\Gamma = \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \subset G_\mathbf{Z}, \quad N_n = \begin{pmatrix} 0 & 1 & n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_\mathbf{Q} \quad (n \in \mathbf{Z}),$$

$$\Sigma_0 = \{\sigma_n\ (n \in \mathbf{Z}), \{0\}\} \quad \text{with } \sigma_n = \mathbf{R}_{\geq 0} N_n.$$

Then (Σ_0, Γ) is strongly compatible. We have $\Gamma(\sigma_0) = \exp(\mathbf{Z}N_0) \simeq \mathbf{Z}$. We have a commutative diagram in $\mathcal{B}(\log)$

$$\Gamma(\sigma_0)^{\mathrm{gp}} \backslash D = \Delta^* \times \mathbf{C}^2 \quad \subset \quad \Gamma(\sigma_0)^{\mathrm{gp}} \backslash D_{\sigma_0} = \{(q, z, w) \in \Delta \times \mathbf{C}^2 \mid w = 0 \text{ if } q = 0\}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\Gamma' \backslash D(\mathrm{gr}^W_{1-2b}) = \Delta^* \quad \subset \quad \Gamma' \backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta,$$

where the isomorphism $\Gamma(\sigma_0)^{\mathrm{gp}} \backslash D \simeq \Delta^* \times \mathbf{C}^2$ sends the class of $F(\tau, z, w)$ to $(e^{2\pi i \tau}, z - \tau w, w)$. Here Γ' denotes the image of Γ in $\mathrm{Aut}(\mathrm{gr}^W_{1-2b})$, which is isomorphic to $\mathbf{Z}$, and σ' denotes the image of σ_0 in $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{1-2b})$. The element $(0, z, 0) \in \Delta \times \mathbf{C}^2$ corresponds to the class of the nilpotent orbit (σ_0, Z), where $Z = F(\mathbf{C}, z, 0)$.

Note that $\Gamma(\sigma_0)^{\mathrm{gp}} \backslash D_{\sigma_0} \to \Gamma \backslash D_{\Sigma_0}$ is locally an isomorphism (Theorem 2.5.4), and is surjective. Hence, by the above, we have a local description of $\Gamma \backslash D_{\Sigma_0}$. The fiber of $\Gamma \backslash D_{\Sigma_0} \to \Gamma' \backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta$ over $q \in \Delta$ is described, as a group object, as follows. If $q \neq 0$, the fiber is isomorphic to $(\mathbf{C}^\times)^2/\mathbf{Z}$, where $a \in \mathbf{Z}$ acts on $(\mathbf{C}^\times)^2$ as $(u, v) \mapsto (uv^a, v)$. Here, for any fixed τ with $q = e^{2\pi i \tau}$, the isomorphism is given by sending the class of $F(\tau, z, w)$ in the fiber to the class of $(e^{2\pi i z}, e^{2\pi i w}) \in (\mathbf{C}^\times)^2$. The fiber over $0 \in \Delta$ is isomorphic to $\mathbf{C}^\times$. Here the class of the nilpotent orbit $(\sigma_0, F(\mathbf{C}, z, 0))$ in the fiber is sent to $e^{2\pi i z} \in \mathbf{C}^\times$.

If we define $\Sigma = \{\sigma_{n,n+1}, \sigma_n \ (n \in \mathbf{Z}), \{0\}\}$ with $\sigma_{n,n+1} = \mathbf{R}_{\geq 0} N_n + \mathbf{R}_{\geq 0} N_{n+1}$ just as in 7.1.2, we have $D_\Sigma = D_{\Sigma_0}$ in this case.

We have a homeomorphism $\mathbf{Z} \times \mathbf{R} \times \mathbf{C} \xrightarrow{\sim} D^\sharp_{\Sigma_0} \smallsetminus D$, $(n, x, z) \mapsto p_n(x, z)$, where

$$ p_n(x, z) = \lim_{y \to \infty} F(x + iy, z, -n). $$

The projection $D^\sharp_{\sigma_0} = D \cup \{p_0(x, z) \mid x \in \mathbf{R}, z \in \mathbf{C}\} \to \Gamma(\sigma_0)^{\mathrm{gp}} \backslash D_{\sigma_0} \subset \Delta \times \mathbf{C} \times \mathbf{C}$ sends $p_0(x, z)$ to $(0, z, 0)$. We have $D^\sharp_{\Sigma_0} = D^\sharp_{\Sigma_0, \mathrm{val}}$.

We have $D^I_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}$. As in [KNU.p] 3.6.1 Example III, if $b = 2$, for the admissible set Ψ of weight filtrations on $H_{0,\mathbf{R}}$ (4.1.2) consisting of single filtration

$$ W'_{-5} = 0 \subset W'_{-4} = W'_{-3} = \mathbf{R}e_1 \subset W'_{-2} = H_{0,\mathbf{R}}, $$

the open set $D^I_{\mathrm{SL}(2)}(\Psi)$ (4.1.2) of $D^I_{\mathrm{SL}(2)}$ has a description as a topological space

$$ D^I_{\mathrm{SL}(2)}(\Psi) = \mathrm{spl}(W) \times \bar{\mathfrak{h}} \times \bar{L} = \mathbf{R}^2 \times \bar{\mathfrak{h}} \times \bar{L}, $$

where $L = \mathbf{R}^2$ and $\bar{L} = L \cup \{\infty v \mid v \in L \smallsetminus \{0\}\}$ with $\infty v := \lim_{t \to \infty} tv$. In this $b = 2$ case, the CKS map $D^\sharp_{\Sigma_0, \mathrm{val}} \to D_{\mathrm{SL}(2)}$ has the image in $D^I_{\mathrm{SL}(2)}(\Psi)$ and sends $p_n(x, z)$ to

$$ (\mathrm{Re}(z), \ -n \ , x + i\infty \ , \mathrm{Im}(z) \ , 0) \in \mathbf{R}^2 \times \bar{\mathfrak{h}} \times L. $$

Using the formulation in §3, let $S = \Gamma' \backslash D(\mathrm{gr}^W_{1-2b})_{\sigma'} = \Delta$ with the log structure at $0 \in \Delta$, and consider $Q = (H_{(w)})_w$, where $H_{(1-2b)}$ is the universal log Hodge structure of weight $1 - 2b$ on S (so $H_{(1-2b)}$ is of rank 2), $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq 1 - 2b, 0$. Then the above σ' is regarded as σ' in §3. Since $\sigma' \to \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W_{1-2b})$ is injective, the above Σ_0 is regarded as a fan in §3, and the above Γ is regarded as Γ in §3. We have $\Gamma \backslash D_{S, \Sigma_0} = \Gamma \backslash D_{\Sigma_0}$. Furthermore, Σ_0 is identified with Σ_j in §6 for $j = 0, 1$ (Σ_0 and Σ_1 in §6 coincide also in this case). That is, $\Gamma \backslash D_{\Sigma_0} \to S$ is the connected Néron model and is also the Néron model.

7.1.4. Log abelian varieties and log complex tori.
This is a short review of [KKN08] from the point of view of this paper.

Let S be an object of $\mathcal{B}(\log)$. A log complex torus A over S is a sheaf of abelian groups on the category $\mathcal{B}(\log)/S$ which is written in the form $A = \mathcal{E}xt^1_{\mathrm{LMH}}(\mathbf{Z}, H')$ for some log Hodge structure H' on S of weight -1 whose Hodge filtration F satisfies $F^1 H'_{\mathcal{O}} = 0$ and $F^{-1} H'_{\mathcal{O}} = H'_{\mathcal{O}}$. Thus, $A = \mathrm{LMH}_Q$ for $Q = (H_{(w)})_w$, where $H_{(-1)} = H'$, $H_{(0)} = \mathbf{Z}$, and $H_{(w)} = 0$ for $w \neq -1, 0$. This H' is determined by A (see [KKN08]). Note that we do not assume that H' is polarizable. (The definition of LMH_Q in §3 did not use the polarization.) We call a log complex torus A a log abelian variety over S if the pull back of H' to the log point s is polarizable for any $s \in S$. For example, LMH_Q for the Q in the end of 7.1.2 is a log abelian variety over $S = \Delta$.

In [KKN08], for a cone decomposition Σ in the sense of ibid. 5.1.2, a representable subfunctor $A^{(\Sigma)}$ of a log complex torus A is constructed. If S is an fs log analytic space, $A^{(\Sigma)}$ is also an fs log analytic space which is log smooth over S. We explain that this $A^{(\Sigma)}$ is a special case of $\mathrm{LMH}_Q^{(\Sigma)}$ of §3. In the following, for simplicity, we assume that H' is polarizable and hence A is a log abelian variety. For the general case, we have to extend the formulation in §3 a little.

Let H', Q, $A = \mathcal{E}xt^1(\mathbf{Z}, H') = \mathrm{LMH}_Q$ be as above. Then, locally on S, there are finitely generated free $\mathbf{Z}$-modules X and Y and an exact sequence

(1) $$0 \to \mathrm{Hom}(X, \mathbf{Z}) \to H'_{\mathbf{Z}} \to Y \to 0$$

of sheaves on $S^{\log}$ such that the canonical map

$$F^0 H'_{\mathcal{O}} \oplus (\mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{Hom}(X, \mathbf{Z})) \to H'_{\mathcal{O}}$$

is an isomorphism. Locally on S, we can take $H_{0,(-1)}$ in 3.1.1 and an exact sequence $0 \to \mathrm{Hom}(X, \mathbf{Z}) \to H_{0,(-1)} \to Y \to 0$ which corresponds to the above exact sequence (1). Let σ' be as in §3.1. Then a cone decomposition Σ in the sense of [KNN08] 5.1.2 is naturally regarded as a rational fan Σ in $\sigma' \times \mathrm{Hom}(X, \mathbf{R})$, and then regarded as a fan Σ in §3 by identifying $\sigma' \times \mathrm{Hom}(X, \mathbf{R})$ with a part of $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, by the following embedding. We send $(a, l) \in \sigma' \times \mathrm{Hom}(X, \mathbf{R})$ to an element (a, N) of $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$, where N sends $1 \in \mathbf{Z} = H_{0,(0)}$ to $l \in \mathrm{Hom}(X, \mathbf{R}) \subset H_{0,(-1),\mathbf{R}}$ and N induces the linear map $H_{0,(-1),\mathbf{R}} \to H_{0,(-1),\mathbf{R}}$ given by the action of a. Then, we have $A^{(\Sigma)} = \mathrm{LMH}_Q^{(\Sigma)}$.

We consider the relation with classical Néron models of abelian varieties. Let S be a smooth algebraic curve over $\mathbf{C}$, let I be a finite subset of S, and let B be an abelian variety over the function field of S which has good reductions at any point of $U := S \smallsetminus I$ and semi-stable reduction at any point in I. Let H'_U be the variation of Hodge structure on U of weight -1 associated to B. Since B is polarizable, H'_U is polarizable. Furthermore, by the assumption of semi-stability, H'_U has unipotent local monodromy at any point of I. Hence H'_U extends to a polarizable log Hodge structure H' of weight -1 on S by the nilpotent orbit theorem of Schmid ([Scm73]) (which is interpreted in terms of log Hodge structures as in [KMN] Proposition 2.5, [KU09] 2.5.14). Hence we have the log abelian variety A over S corresponding to H'. The usual (analytic) connected Néron model of B over S is the subgroup functor G of A in [KKN08]. It is clear that this G coincides with the connected Néron model in §6. The usual (analytic)

Néron model of B over S is the fiber product of $A \to A/G \leftarrow \Gamma(S, A/G)$. This coincides with the Néron model in §6 as is seen from §6.2 and §6.3.

7.1.5. We consider the degeneration of an intermediate Jacobian. Let

$$E = \bigcup_{q \in \Delta^*} \mathbf{C}^\times / q^{\mathbf{Z}} \to \Delta^*$$

be the family of elliptic curves which appeared in 7.1.2, and consider the family

$$X := E^3 \to S^* := (\Delta^*)^3.$$

Let $S = \Delta^3$ and endow S with the log structure associated to the divisor $S \smallsetminus S^*$. Consider the connected Néron model J_0 over S (§6.1) of the second intermediate Jacobian $J^2(X/S^*)$ of X over S^*. It is the connected Néron model for the case $Q = (H_{(w)})_w$ with $H_{(0)} = \mathbf{Z}$, $H_{(-1)}$ is the unique extension on S as a log Hodge structure of weight -1 (obtained by the nilpotent orbit theorem of Schmid [Scm73]) of the polarizable variation of Hodge structure $H^3(X/S^*)(2)$ on S^*, and $H_{(w)} = 0$ for $w \neq -1, 0$. Here $H^3(X/S^*)$ is the third higher direct image of the constant Hodge structure $\mathbf{Z}$ on X under $X \to S^*$.

As a group object, the fiber of $J_0 \to S$ over $(q_1, q_2, q_3) \in \Delta^3$ has the following structure. If $q_1 q_2 q_3 \neq 0$, the fiber (which is the second intermediate Jacobian of a product of three elliptic curves) is a quotient of $\mathbf{C}^{10}$ by a discrete group which is isomorphic to $\mathbf{Z}^{20}$. If one of q_j is zero and two of them are non-zero, the fiber is isomorphic to a quotient of $\mathbf{C}^9$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^{14}$. If two of q_j are zero and one of them is non-zero, the fiber is isomorphic to a quotient of $\mathbf{C}^8$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^{10}$. If $q_1 = q_2 = q_3 = 0$, the fiber is isomorphic to a quotient of $\mathbf{C}^7$ by a discrete subgroup which is isomorphic to $\mathbf{Z}^7$.

This can be seen as follows. Since the connected Néron model is the slit Zucker model, the fiber has the form V/L, where V is the small Griffiths part of the fiber of the vector bundle $H_{(-1),\mathcal{O}}/F^0 H_{(-1),\mathcal{O}}$ and L is the stalk of $\tau_*(H_{(-1),\mathbf{Z}})$ ($\tau : S^{\log} \to S$). Since the fiber is Hausdorff (Theorem 3.2.8 (iii)), L must be discrete in V. Let H' be the unique extension on Δ of $H^1(E/\Delta^*)(1)$ on Δ^*, as a log Hodge structure of weight -1. Then we have

$$H_{(-1)} \simeq (\textstyle\bigotimes_{j=1}^3 \mathrm{pr}_j^*(H'))(-1) \oplus \mathrm{pr}_1^*(H')^{\oplus 2} \oplus \mathrm{pr}_2^*(H')^{\oplus 2} \oplus \mathrm{pr}_3^*(H')^{\oplus 2}.$$

From this, we have that if $q_1 = 0$ and $q_2 q_3 \neq 0$ (resp. $q_1 = q_2 = 0$ and $q_3 \neq 0$, resp. $q_1 = q_2 = q_3 = 0$), the dimension of V is $3 + 2 + 2 + 2 = 9$ (resp. $2 + 2 + 2 + 2 = 8$, resp. $1 + 2 + 2 + 2 = 7$), and the rank of L is $4 + 2 + 4 + 4 = 14$ (resp. $2 + 2 + 2 + 4 = 10$, resp. $1 + 2 + 2 + 2 = 7$).

The Néron model coincides with the connected Néron model in this case.

§7.2. The necessity of weak fan

We explain by examples that a weak fan (not a fan) is necessary to construct the connected Néron model.

7.2.1. We give an example in which the weak fan Σ_0 in §6, which is used for the construction of the connected Néron model, is not a fan.

In §6, assume $H_{0,(-2)} = L \otimes L$ with $L = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, $H_{0,(0)} = \mathbf{Z}$, $H_{0,(w)} = 0$ for $w \neq -2, 0$, $\sigma' = \mathbf{R}_{\geq 0}N_1' + \mathbf{R}_{\geq 0}N_2'$, where $N_j' : H_{0,(-2),\mathbf{Q}} \to H_{0,(-2),\mathbf{Q}}$ $(j = 1, 2)$ are as follows: $N_1' = N' \otimes 1$, $N_2' = 1 \otimes N'$ with $N' : L \to L$ being the homomorphism defined by $N'(e_1) = 0$, $N'(e_2) = e_1$. Let e be the base 1 of $H_{0,(0)}$, let $\tau = \mathbf{R}_{\geq 0}N_1 + \mathbf{R}_{\geq 0}N_2$ with $N_j \in \mathfrak{g}_{\mathbf{Q}}$ being the extension of N_j' such that $N_j(e) = 0$. Then the weak fan $\Sigma_0 = \{\mathrm{Ad}(\gamma)(\alpha) \mid \gamma \in G_{\mathbf{Z},u}, \alpha \text{ is a face of } \tau\}$ in §6 is not a fan. To see this, let $\gamma_{m,n} \in G_{\mathbf{Z},u}$ for integers $m, n > 0$ be the element which fixes all elements of $H_{0,(-2)}$ and which sends e to $e + me_1 \otimes e_2 - ne_2 \otimes e_1$. Then $\tau \cap \mathrm{Ad}(\gamma_{m,n})(\tau) = \mathbf{R}_{\geq 0} \cdot (mN_1 + nN_2)$ (note that $\mathrm{Ad}(\gamma_{m,n})(mN_1 + nN_2) = mN_1 + nN_2$) and this intersection is not a face of τ.

For a later use in 7.3.3, we remark that if τ_1 is a subcone of τ of rank 2, there is no fan Σ which is stable under the adjoint action of $G_{\mathbf{Z},u}$ such that $\tau_1 \subset \sigma$ for some $\sigma \in \Sigma$. In fact, the interior of τ_1 contains $mN_1 + nN_2$ and $m'N_1 + n'N_2$ for some integers $m, n, m', n' > 0$ such that $mn' - m'n \neq 0$. If such a fan Σ exists, the smallest $\sigma \in \Sigma$ such that $\tau_1 \subset \sigma$ exists, and any interior point of τ_1 is an interior point of σ. Hence σ and $\mathrm{Ad}(\gamma_{m,n})(\sigma)$ contains a common interior point $mN_1 + nN_2$ and hence we should have $\mathrm{Ad}(\gamma_{m,n})(\sigma) = \sigma$ and hence $\mathrm{Ad}(\gamma_{m,n})^k(\sigma) = \sigma$ for all $k \in \mathbf{Z}$. Let $N_0 \in \mathfrak{g}_{\mathbf{Q}}$ be the element which kills $H_{0,(-2)}$ and which sends e to $e_1 \otimes e_1 \in H_{0,(-2)}$. Then $\mathrm{Ad}(\gamma_{m,n})^k(m'N_1 + n'N_2) = m'N_1 + n'N_2 + k(m'n - mn')N_0$. But since σ is a finitely generated sharp cone, it can not contain the set $\{m'N_1 + n'N_2 + k(m'n - mn')N_0 \mid k \in \mathbf{Z}\}$.

This $((H_{0,(w)})_w, \sigma')$ appears in §3 in the case S and $(H_{(w)})_w$ are as follows: $S = \Delta^2$ with the log structure associated to the divisor $S \smallsetminus S^*$ with normal crossings, where $S^* = (\Delta^*)^2$. $H_{(0)} = \mathbf{Z}$. $H_{(-2)}$ is the extension of $\mathrm{pr}_1^* H^1(E/\Delta^*)(1) \otimes \mathrm{pr}_2^* H^1(E/\Delta^*)(1)$ on S^* to S as a log Hodge structure of weight -2, where $E = \bigcup_{q \in \Delta^*} \mathbf{C}^\times / q^{\mathbf{Z}}$ as in 7.1.5 and $H^1(E/\Delta^*)$ is the VHS on Δ^* obtained as the first higher direct image of the constant Hodge structure $\mathbf{Z}$ on E under $E \to \Delta^*$. $H_{(w)} = 0$ for $w \neq -2, 0$.

7.2.2. We show that in the example 7.1.5, we need a weak fan, not a fan, to construct the connected Néron model of the second intermediate Jacobian. Let $X = E^3 \to S^* = (\Delta^*)^3 \subset S = \Delta^3$ be as in 7.1.5. Let $L = \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$ and $N' : L \to L$ be as in 7.2.1. We take $H_{0,(0)} = \mathbf{Z}$, $H_{0,(-1)} = L \otimes L \otimes L \oplus L^{\oplus 2} \oplus L^{\oplus 2} \oplus L^{\oplus 2}$, $H_{0,(w)} = 0$ for $w \neq -1, 0$, and regard L as a stalk of $H^1(E/S^*)(1)_{\mathbf{Z}}$. Define $N_j' \in \mathfrak{g}_{\mathbf{Q}}'$ $(j = 1, 2, 3)$ as $N_1' = (N' \otimes 1 \otimes 1) \oplus (N')^{\oplus 2} \oplus 0 \oplus 0$, $N_2' = (1 \otimes N' \otimes 1) \oplus 0 \oplus (N')^{\oplus 2} \oplus 0$, $N_3' = (1 \otimes 1 \otimes N') \oplus 0 \oplus 0 \oplus (N')^{\oplus 2}$. Extend N_j' to $N_j \in \mathfrak{g}_{\mathbf{Q}}$ by $N_j(e) = 0$, where e is $1 \in \mathbf{Z} = H_{0,(0)}$. Take $P = \mathbf{N}^3$, take the inclusion morphism $S \to \mathrm{toric}_{\sigma'} = \mathrm{Spec}(\mathbf{C}[P])_{\mathrm{an}} = \mathbf{C}^3$, and take the homomorphism $\Gamma' = \mathrm{Hom}(P^{\mathrm{gp}}, \mathbf{Z}) = \mathbf{Z}^3 \to \mathfrak{g}_{\mathbf{R}}'$ given by (N_1', N_2', N_3'). We show that the weak fan Σ_0 is not a fan. We show more strongly that if τ_1 is a finitely generated subcone of $\tau := \mathbf{R}_{\geq 0}N_1 + \mathbf{R}_{\geq 0}N_2 + \mathbf{R}_{\geq 0}N_3$ of rank 3, there is no fan Σ which is stable under the adjoint action of $G_{\mathbf{Z},u}$ such that $\tau_1 \subset \sigma$ for some $\sigma \in \Sigma$. The interior of τ_1 has elements of the form $mN_1 + nN_2 + \ell N_3$ and $m'N_1 + n'N_2 + \ell N_3$ with $m, n, m', n', \ell \in \mathbf{Z}$ such that $mn' - m'n \neq 0$. Let γ be the element of $G_{\mathbf{Z},u}$ which fixes every element of $H_{0,(-1)}$ and which sends $e \in H_{0,(0)}$ to $e + (me_1 \otimes e_2 \otimes e_1 - ne_2 \otimes e_1 \otimes e_1, 0, 0, 0)$. Then $\mathrm{Ad}(\gamma)(mN_1 + nN_2 + \ell N_3) = mN_1 + nN_2 + \ell N_3$, $\mathrm{Ad}(\gamma)^k(m'N_1 + n'N_2 + \ell N_3) = m'N_1 + n'N_2 + \ell N_3 + k(m'n - mn')N_0$, where $N_0(H_{0,(-1)}) = 0$ and $N_0(e) = (e_1 \otimes e_1 \otimes e_1, 0, 0, 0)$. Hence, by the same argument

as in 7.2.1, we can deduce that there is no such fan Σ.

7.2.3. In [KNU10c] 4.13, in a certain case, we gave a necessary condition for Σ_0 in §6 to be a fan.

<h3 style="text-align:center">§7.3. Completeness of weak fan</h3>

We discuss the completeness of a weak fan Σ. We explain that, in the pure case, the conjecture in [KU09] "Added in the proof" after 12.7.7 is false. Here we reformulate such a conjecture by replacing fan by weak fan (generalizing it to the mixed Hodge case).

7.3.1. In the situation of §2 (resp. §3), we say a weak fan Σ is *complete* (resp. *relatively complete*) if the following condition (1) is satisfied.

(1) For any rational nilpotent cone τ such that a τ-nilpotent orbit exists, there is a finite rational subdivision $\{\tau_j\}_{1 \le j \le n}$ of τ having the following property. For each $1 \le j \le n$, there is an element σ of Σ such that $\tau_j \subset \sigma$.

7.3.2. Assume that we are in the situation of §2. In the pure case, for a fan, this completeness is weaker than the completeness defined in [KU09] 12.6.1.

In [KU09] 12.6.8, the following relation of completeness and compactness was proved for the completeness of a fan in the sense of [KU09] 12.6.1. The same proof works for the present completeness of a weak fan. Note that if Γ is a subgroup of finite index of $G_{\mathbf{Z}}$ and Σ is a weak fan which is compatible with Γ, then Σ is strongly compatible with Γ.

Theorem. *Assume that we are in the pure case and that we are in the classical situation* [KU09] 0.4.14. *Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$, let Γ be a subgroup of $G_{\mathbf{Z}}$ of finite index, and assume that Γ is compatible with Σ. Then, $\Gamma \backslash D_{\Sigma}$ is compact if and only if Σ is complete.*

In the pure case and in the classical situation [KU09] 0.4.14, a complete fan was constructed in [AMRT75].

In general, we make a

Conjecture. There is a complete weak fan which is compatible with $G_{\mathbf{Z}}$ (and hence with any subgroup of $G_{\mathbf{Z}}$).

7.3.3. Assume that we are in the situation of §2. We show that even in the pure case, the conjecture in 7.3.2 becomes not true if we replace weak fan by fan.

Consider the case $\Lambda = (H_0, W, (\langle \ , \ \rangle_w)_w, (h^{p,q})_{p,q})$, where

$$H_0 = (L \otimes L) \oplus \mathrm{Sym}^2(L) \quad \text{with } L = \mathbf{Z}^2 = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2,$$

$$W_1 = 0 \subset W_2 = H_{0,\mathbf{R}},$$

$\langle \ , \ \rangle_2$ is induced from the anti-symmetric $\mathbf{Z}$-bilinear form $L \times L \to \mathbf{Z}$ which sends (e_2, e_1) to 1, and

$$h^{2,0} = h^{0,2} = 2, \quad h^{1,1} = 3, \quad h^{p,q} = 0 \text{ for all the other } (p,q).$$

Let $N : L \to L$ be the homomorphism defined by $N(e_1) = 0$, $N(e_2) = e_1$. Define $N_1, N_2, N_3 \in \mathfrak{g}_{\mathbf{Q}}$ by

$$N_1 = (N \otimes 1) \oplus 0, \quad N_2 = (1 \otimes N) \oplus 0, \quad N_3 = 0 \oplus \mathrm{Sym}^2(N),$$

and let $\tau = \mathbf{R}_{\geq 0} N_1 + \mathbf{R}_{\geq 0} N_2 + \mathbf{R}_{\geq 0} N_3$.

Note that there exists a τ-nilpotent orbit. In fact, for any decreasing filtration F on $L_{\mathbf{C}}$ such that $F^0 = L_{\mathbf{C}}$, $\dim_{\mathbf{C}}(F^1) = 1$, and $F^2 = 0$, $(N_1, N_2, N_3, (F \otimes F) \oplus \mathrm{Sym}^2(F))$ generates a nilpotent orbit. (This nilpotent orbit appears in the degeneration of $H^1(E_1) \otimes H^1(E_2) \oplus \mathrm{Sym}^2(H^1(E_3))$, where E_j are degenerating elliptic curves.) We show that if τ_1 is a finitely generated subcone of τ of rank 3, there is no fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\tau_1 \subset \sigma$ for some $\sigma \in \Sigma$.

Assume that such a τ_1 and such a fan Σ exist. Then the interior of the cone τ_1 contains elements of the form $mN_1 + nN_2 + \ell N_3$ and $m'N_1 + n'N_2 + \ell N_3$ with $m, n, m', n', \ell \in \mathbf{Z}$ such that $mn' - m'n \neq 0$. Let γ be the element of $G_{\mathbf{Z}}$ which does not change $(e_1 \otimes e_1, 0)$, $(e_2 \otimes e_2, 0)$, $(0, e_1 e_2)$, $(0, e_1^2)$ and for which

$$\gamma(0, e_2^2) = (0, e_2^2) + (me_1 \otimes e_2 - ne_2 \otimes e_1, 0),$$
$$\gamma(e_1 \otimes e_2, 0) = (e_1 \otimes e_2, 0) + (0, ne_1^2),$$
$$\gamma(e_2 \otimes e_1, 0) = (e_2 \otimes e_1, 0) + (0, me_1^2).$$

Then $\mathrm{Ad}(\gamma)(mN_1 + nN_2 + \ell N_3) = mN_1 + nN_2 + \ell N_3$, $\mathrm{Ad}(\gamma)^k(m'N_1 + n'N_2 + \ell N_3) = m'N_1 + n'N_2 + \ell N_3 + k(m'n - mn')N_0$, where $N_0(0, e_2^2) = (e_1 \otimes e_1, 0)$ and N_0 kills all $(e_j \otimes e_k, 0)$ and kills $(0, e_1^2)$ and $(0, e_1 e_2)$. By the argument in 7.2.1, we see that Σ as above does not exist.

7.3.4. We explain variants of the conjecture in 7.3.2. Let I be the set of all rational nilpotent cones τ for which a τ-nilpotent orbit exists.

Conjecture a. There is a weak fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\bigcup_{\tau \in I} \tau = \bigcup_{\sigma \in \Sigma} \sigma$.

Conjecture b. There is a weak fan Σ which is compatible with $G_{\mathbf{Z}}$ such that $\bigcup_{\tau \in I} \tau \subset \bigcup_{\sigma \in \Sigma} \sigma$.

Clearly we have the implications

$$(\text{Conjecture in } 7.3.2) \Rightarrow (\text{Conjecture b}) \Leftarrow (\text{Conjecture a}).$$

If weak fan in Conjecture a is replaced by fan, it becomes the conjecture stated after [KU09] 12.7.7 (in the pure case). We show that the last conjecture fails. We show more strongly that Conjecture b becomes not true if weak fan in Conjecture b is replaced by fan (even in the pure case).

Consider the example in 7.3.3. Let a_1, a_2, a_3 be positive real numbers which are linearly independent over $\mathbf{Q}$. If a fan Σ as in Conjecture b exists, there is $\sigma \in \Sigma$ such that $a_1 N_1 + a_2 N_2 + a_3 N_3 \in \sigma$. Let $\tau_1 = \tau \cap \sigma$, where $\tau = \mathbf{R}_{\geq 0} N_1 + \mathbf{R}_{\geq 0} N_2 + \mathbf{R}_{\geq 0} N_3$. Since τ_1 is rational and contains $a_1 N_1 + a_2 N_2 + a_3 N_3$ with a_1, a_2, a_3 linearly independent over $\mathbf{Q}$, τ_1 is of rank 3. But this contradicts what we have seen in 7.3.3.

Question. In case where the conjecture in 7.3.2 is valid, we may ask if we can further require that, for any cone σ in the weak fan in the conjecture, a σ-nilpotent orbit exists. Note that a delicate example constructed by Watanabe ([W08]) is harmless to this question.

7.3.5. Next, assume that we are in the situation of §3.

Note that if Γ_u is of finite index in $G_{\mathbf{Z},u}$ and Σ is compatible with Γ, then Σ is strongly compatible with Γ (here Γ_u and Γ are as in 3.1.7).

If Γ_u is of finite index in $G_{\mathbf{Z},u}$ and Σ is compatible with Γ, then the relative completeness of Σ is related to the properness of $\Gamma \backslash D_{S,\Sigma} \to S$ in certain cases ([KNU10a] §5 Proposition).

Conjecture. There is a relatively complete weak fan which is compatible with any Γ as in 3.1.7.

This conjecture becomes not true if we replace weak fan by fan. In the cases 7.2.1 and 7.2.2, there is no relatively complete fan which is stable under the adjoint action of $G_{\mathbf{Z},u}$, as is seen from the arguments in 7.2.1 and 7.2.2.

7.3.6. We show that if the following conditions (1) and (2) are satisfied, then the conjecture in 7.3.5 is true even if we replace weak fan by fan.

(1) There are only two w such that $H_{(w)} \neq 0$.

(2) $\Gamma' \simeq \mathbf{Z}$ (cf. 3.1.1 (3)).

The following construction is a variant of that in [KNU10a] §3.

Assume (1) and (2), and assume $H_{(a)} \neq 0$, $H_{(b)} \neq 0$, $a < b$. Note that the fs monoids P and $\mathrm{Hom}(P, \mathbf{N})$ are isomorphic to $\mathbf{N}$. Let γ' be a generator of $\mathrm{Hom}(P, \mathbf{N}) \subset \Gamma'$, and let $N'_a : H_{0,(a),\mathbf{Q}} \to H_{0,(a),\mathbf{Q}}$ and $N'_b : H_{0,(b),\mathbf{Q}} \to H_{0,(b),\mathbf{Q}}$ be the actions of $\log(\gamma')$ on $H_{0,(a),\mathbf{Q}}$ and on $H_{0,(b),\mathbf{Q}}$, respectively. Let $V = \mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$, and define $\mathbf{Q}$-subspaces X and Y of V as follows:

$$X := \{h \in V \mid hM(N'_b, W(\mathrm{gr}_b^W))_w \subset M(N'_a, W(\mathrm{gr}_a^W))_{w-2} \text{ for all } w\} + \{hN'_b - N'_a h \mid h \in V\},$$
$$Y := \{h \in X \mid hN'_b = N'_a h\} \subset X. \qquad\blacksquare$$

Let $\gamma_a := \mathrm{gr}_a^W(\gamma')$ and $\gamma_b := \mathrm{gr}_b^W(\gamma')$. Fix a finitely generated $\mathbf{Z}$-submodule L of V which is γ_a-stable and γ_b-stable such that

$$L \supset \mathrm{Hom}_{\mathbf{Z}}(H_{0,(b)}, H_{0,(a)}) + \{hN'_b - N'_a h \mid h \in \mathrm{Hom}_{\mathbf{Z}}(H_{0,(b)}, H_{0,(a)})\}.$$

Fix a $\mathbf{Z}$-homomorphism $s : (X \cap L)/(Y \cap L) \to X \cap L$ which is a section of the projection $X \cap L \to (X \cap L)/(Y \cap L)$. For an element x of X/Y, let $d(x)$ be the order of the image of x in $X/((X \cap L) + Y)$ which is a torsion element. Fix a $\mathbf{Z}$-basis $(e_j)_{1 \leq j \leq m}$ of $Y \cap L$. Let v be the image of γ' under the canonical injection $\mathrm{Hom}(P, \mathbf{N}) \to \sigma'$, which is a generator of $\sigma' \simeq \mathbf{R}_{\geq 0}$.

For $x \in X/Y$ and $n = (n_j)_{1 \leq j \leq m} \in \mathbf{Z}^m$, let $\sigma(x, n)$ be nilpotent cone in $\sigma' \times_{\mathfrak{g}'_{\mathbf{R}}} \mathfrak{g}_{\mathbf{R}}$ generated by all elements of the form (v, N), where N is an element of $\mathfrak{g}_{\mathbf{R}}$ satisfying the following (3)–(5).

(3) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(a),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of N coincides with N'_a.

(4) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(b),\mathbf{Q}})$-component of N coincides with N_b'.

(5) The $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of N has the form

$$s(x) + \frac{1}{d(x)} \sum_{j=1}^{m} c_j e_j \quad \text{with} \ n_j \le c_j \le n_j + 1 \ \text{for all} \ j.$$

Let

$$\Sigma = \{\text{face of } \sigma(x, n) \mid x \in X/Y, n \in \mathbf{Z}^m\}.$$

Then we have:

Σ is a relatively complete fan which is compatible with any Γ as in 3.1.7.

First we prove the compatibility. For simplicity, we assume that either N_a' or N_b' is non-trivial and regard $\sigma' \times_{\mathfrak{g}_{\mathbf{R}}'} \mathfrak{g}_{\mathbf{R}} \subset \mathfrak{g}_{\mathbf{R}}$ via the second projection. The case $N_a' = N_b' = 0$ is similar and easier.

Let γ be any element of Γ. It is sufficient to prove $\mathrm{Ad}(\gamma)\sigma(x, n) \in \Sigma$ for any $x \in X/Y$ and $n \in \mathbf{Z}^m$. Write the $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of γ^{-1} by h.

Let $N \in \sigma(x, n)$ be an element satisfying (3)–(5). Then, since γ_b and N_b' commute, the $\mathrm{Hom}_{\mathbf{Q}}(H_{0,(b),\mathbf{Q}}, H_{0,(a),\mathbf{Q}})$-component of $\gamma N \gamma^{-1}$ is

(6) $$\gamma_a s(x) \gamma_b^{-1} + \gamma_a (N_a' h - h N_b') + d(x)^{-1} \sum_{j=1}^{m} c_j e_j.$$

Write $\gamma_a s(x) \gamma_b^{-1} + \gamma_a (N_a' h - h N_b') = s(y) + z$ with $y \in X/Y$ and $z \in Y$. Since L is γ_a-stable and γ_b-stable, we have $d(y) = d(x)$ and $d(x)z \in Y \cap L$. Write $d(x)z = \sum_{j=1}^{m} m_j e_j$ with $m_j \in \mathbf{Z}$, and let $m = (m_j)_j \in \mathbf{Z}^m$. Then, (6) equals to

$$s(y) + d(y)^{-1} \sum_{j=1}^{m} (c_j + m_j) e_j.$$

This means

$$\mathrm{Ad}(\gamma)\sigma(x, n) = \sigma(y, n + m).$$

Next, the relative completeness can be proved by noticing the following two points (7) and (8). Let R be the set of all elements N of $\mathfrak{g}_{\mathbf{R}}$ such that the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(a),\mathbf{R}}, H_{0,(a),\mathbf{R}})$-component of N coincides with N_a' and the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}}, H_{0,(b),\mathbf{R}})$-component of N coincides with N_b'. For $N \in R$, let N_u be the $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}}, H_{0,(a),\mathbf{R}})$-component of N. Then:

(7) Let $N \in R$. Then $M(N, W)$ exists if and only if N_u belongs to the $\mathbf{R}$-subspace $\mathbf{R} \otimes_{\mathbf{Q}} X$ of $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}}, H_{0,(a),\mathbf{R}})$.

(8) Let $N_1, N_2 \in R$ and assume $(N_1)_u, (N_2)_u \in \mathbf{R} \otimes_{\mathbf{Q}} X$ in $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}}, H_{0,(a),\mathbf{R}})$. Then, $N_1 N_2 = N_2 N_1$ if and only if $(N_1)_u - (N_2)_u$ belongs to the $\mathbf{R}$-subspace $\mathbf{R} \otimes_{\mathbf{Q}} Y$ of $\mathrm{Hom}_{\mathbf{R}}(H_{0,(b),\mathbf{R}}, H_{0,(a),\mathbf{R}})$.

7.3.7. *Remark* 1. 7.3.6 gives a proof of [KNU10a] §3, and hence a proof of the theorem in [KNU10a] §2. (In fact, for the proofs, we have to show that the fan Σ defined above satisfies the admissibility in the sense of [KNU10a], i.e., the condition (2) in 1.2.6. But this can be checked easily.)

Remark 2. Here is a correction to [KNU10a]. The submodule L in ibid. §3 should have been imposed to be γ'-stable as in 7.3.6 above. Without this condition, the resulting Σ was not necessarily compatible with Γ.

7.3.8. In the part of 7.1.1 concerning the situation of §3, the fan Σ is relatively complete. In the part of 7.1.2 concerning the situation of §3, the fan Σ is relatively complete. In the part of 7.1.3 concerning the situation of §3, the fan Ξ (3.1.6) is relatively complete. In 7.1.5, there is no relatively complete fan as is seen by the argument in 7.2.2.

§7.4. Extended period maps

In this subsection, we see extensions of period maps.

Theorem 7.4.1. *Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Let H be a variation of mixed Hodge structure on U with polarized graded quotients for the weight filtration, and with unipotent local monodromy along $S \smallsetminus U$. Assume that H extends to a log mixed Hodge structure on S (that is, H is admissible along $S \smallsetminus U$ as a variation of mixed Hodge structure). Fix a base point $u \in U$ and let $\Lambda = (H_0, W, (\langle\ ,\ \rangle_w)_w, (h^{p,q})_{p,q})$ be $(H_{\mathbf{Z},u}, W, (\langle\ ,\ \rangle_{w,u})_w, (\text{the Hodge numbers of } H))$. Let Γ be a subgroup of $G_{\mathbf{Z}}$ which contains the global monodromy group $\text{Image}(\pi_1(U, u) \to G_{\mathbf{Z}})$ and assume that Γ is neat. Let $\varphi : U \to \Gamma \backslash D$ be the associated period map. Then:*

(i) *Let $S_{\text{val}}^{\log}$ be the topological space over $S^{\log}$ defined as in [KU09] 3.6.26, which contains U as a dense open subspace. Then the map $\varphi : U \to \Gamma \backslash D$ extends uniquely to a continuous map*

$$S_{\text{val}}^{\log} \to \Gamma \backslash D_{\text{SL}(2)}^I.$$

(ii) *For any point $s \in S$, there exist an open neighborhood V of s, a log modification V' of V ([KU09] 3.6.12), a commutative subgroup Γ' of Γ, and a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ which is strongly compatible with Γ' such that the period map $\varphi|_{U \cap V}$ lifts to a morphism $U \cap V \to \Gamma' \backslash D$ which extends uniquely to a morphism $V' \to \Gamma' \backslash D_{\Sigma}$ of log manifolds.*

$$
\begin{array}{ccccc}
U & \supset & U \cap V & \subset & V' \\
\varphi \downarrow & & \downarrow & & \downarrow \\
\Gamma \backslash D & \leftarrow & \Gamma' \backslash D & \subset & \Gamma' \backslash D_{\Sigma}.
\end{array}
$$

Furthermore, we have:

(ii-1) *Assume $S \smallsetminus U$ is a smooth divisor. Then we can take $V = V' = S$ and $\Gamma' = \Gamma$. That is, we have a commutative diagram*

$$
\begin{array}{ccc}
U & \subset & S \\
\varphi \downarrow & & \downarrow \\
\Gamma \backslash D & \subset & \Gamma \backslash D_{\Sigma}.
\end{array}
$$

(ii-2) *Assume Γ is commutative. Then we can take $\Gamma' = \Gamma$.*

(ii-3) *Assume that Γ is commutative and that the following condition (1) is satisfied.*

(1) *There is a finite family $(S_j)_{1 \le j \le n}$ of connected locally closed analytic subspaces of S such that $S = \bigcup_{j=1}^{n} S_j$ as a set and such that, for each j, the inverse image of the sheaf $M_S/\mathcal{O}_S^\times$ on S_j is locally constant.*

Then we can take $\Gamma' = \Gamma$ and $V = S$.

Note that in (ii), we can take a fan Σ (we do not need a weak fan).

Proof. (i) (resp. (ii)) is the mixed Hodge version of Theorem 0.5.29 (resp. Theorem 4.3.1) of [KU09]. By using the results in §2 and §4, the proof goes exactly in the same way as in the pure case treated in [KU09]. $\square$

We deduce the following Theorem 7.4.2 from Theorem 7.4.1 (i).

Theorem 7.4.2. *Let the assumption be as in 7.4.1. Let $\tilde{H}_{\mathbf{R}}$ be the unique extension of $H_{\mathbf{R}}$ as a local system on $S_{\mathrm{val}}^{\log}$. Then the canonical continuous splitting of the weight filtration W of $H_{\mathbf{R}}$ (4.1.1) extends unquely to a continuous splitting of the weight filtration of $\tilde{H}_{\mathbf{R}}$.*

Theorem 7.4.2 was proved also by Brosnan and Pearlstein ([BP.p] Theorem 2.21) by another method.

Proof of Theorem 7.4.2. As in [KNU.p], the map $\mathrm{spl}_W : D \to \mathrm{spl}(W)$ in 4.1.1, where $\mathrm{spl}(W)$ denotes the set of all splittings of W, extends to a continuous map $D_{\mathrm{SL}(2)}^I \to \mathrm{spl}(W)$. Hence the period map $S_{\mathrm{val}}^{\log} \to \Gamma \backslash D_{\mathrm{SL}(2)}^I$ in Theorem 7.4.1 (i) induces a continuous map $u : S_{\mathrm{val}}^{\log} \to \Gamma \backslash \mathrm{spl}(W)$. Since the map $\mathrm{spl}(W) \to \Gamma \backslash \mathrm{spl}(W)$ is a local homeomorphism, u lifts locally on $S_{\mathrm{val}}^{\log}$ to a continuous map to $\mathrm{spl}(W)$. This shows that locally on $S_{\mathrm{val}}^{\log}$, the canonical continuous splitting of W of $H_{\mathbf{R}}$ given on U extends to a continuous splitting of W of $\tilde{H}_{\mathbf{R}}$. Since the local extensions are unique, this proves Theorem 7.4.2. $\square$

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