CONSIDERATION OF COMPACT MINIMAL SURFACES IN 4-DIMENSIONAL FLAT TORI IN TERMS OF DEGENERATE GAUSS MAP

TOSHIHIRO SHODA

ABSTRACT. In this paper, we study a compact minimal surface in a 4-dimensional flat torus via degenerate Gauss map. Then, we give a refinement of Nagano-Smyth's argument [9] and an alternative proof of Arezzo-Pirola's proposition [2] about the Moduli space of compact minimal surfaces in flat 4-tori. Moreover, we show that a compact minimal surface of genus 4 in a 4-dimensional flat torus is either a holomorphic minimal surface, or a non-holomorphic hyperelliptic minimal surface, or a non-holomorphic trigonal minimal surface, where a trigonal Riemann surface is a Riemann surface which can be represented as a 3-sheeted branched cover of the sphere.

1. INTRODUCTION

Our study is about a compact minimal surface M_g of genus g in a flat torus $\mathbf{R}^n/\Lambda$, so an n-periodic minimal surface in $\mathbf{R}^n$. By the isothermal coordinates, we can consider M_g as Riemann surface. The main subject is to consider a compact minimal surface in a 4-dimensional flat torus from viewpoints of degenerate Gauss map. Then we refine the Nagano-Smyth's argument and give an alternative proof of Arezzo-Pirola's proposition about the Moduli of compact minimal surfaces in 4-dimensional flat tori. Moreover, we classify compact minimal surfaces of genus 4 in 4-dimensional flat tori. In the next paragraph, we will explain about the backgrounds.

An example of periodic minimal surfaces was found by H. A. Schwarz [12] first and, after that, it has been known many examples [6], [11]. On the other hands, Nagano-Smyth [9] and W. H. Meeks [5] established general theory of periodic minimal surfaces. The basic ingredient is the following (p.884 [5]):

Theorem 1.1 (Generalized Weierstrass Representation)**.** *If $f : M_g \longrightarrow \boldsymbol{R}^n/\Lambda$ is a conformal minimal immersion then, after a translation, f can be represented by*

$$(1) \qquad f(p) = \Re \int_{p_0}^{p} (\omega_1, \omega_2, \dots, \omega_n)^T \quad Mod\ \Lambda,$$

2000 MATHEMATICS SUBJECT CLASSIFICATION. 49Q05, 53A10.
KEY WORDS AND PHRASES. MINIMAL SURFACES, FLAT TORI.

where $p_0 \in M_g$, the superscript T means the transposed matrix and ω_1, ω_2, $\dots$, ω_n are holomorphic differentials on M_g satisfying

$$\{\omega_1, \omega_2, \dots, \omega_n\} \text{ has no common zeros,} \tag{2}$$

$$\sum_{i=1}^{n} \omega_i^2 = 0\,, \tag{3}$$

$$\left\{ \Re \int_\gamma (\omega_1, \omega_2, \dots, \omega_n)^T \mid \gamma \in H_1(M_g, \boldsymbol{Z}) \right\} \text{ is a sublattice of } \Lambda. \tag{4}$$

Conversely, every minimal surface in $\boldsymbol{R}^n/\Lambda$ is obtained by the above construction.

Condition (4) is called the periodic condition and guarantees that path integral (1) is well-defined. Now we define the associate immersion f_θ of f:

$$f_\theta(p) := \Re \int_{p_0}^{p} e^{i\,\theta}\,(\omega_1, \omega_2, \dots, \omega_n)^T$$

If f_θ is well-defined, then we call it the associate surface of f. The conjugate surface of f is the associate surface $f_{\pi/2}$. Recall that the Gauss map of the minimal surface defined by (1) is given as a holomorphic map from M_g to the quadric $Q_{n-2} := \{[w] \in \mathbf{C}P^{n-1} | w \cdot w = \sum_i (w^i)^2 = 0\}$, where " $\cdot$ " is the complex bilinear inner product:

$$\begin{aligned} M_g &\longrightarrow Q_{n-2} \subset \mathbf{C}P^{n-1} \\ p &\longmapsto (\omega_1, \omega_2, \cdots, \omega_n) \end{aligned}$$

In $\mathbf{C}P^{n-1}$, $(\omega_1, \omega_2, \cdots, \omega_n)$ agrees with $e^{i\,\theta}\,(\omega_1, \omega_2, \cdots, \omega_n)$. But they are different objects as minimal surface because we take real parts of the path integrals. Thus, it is an important subject to study gap between complex category and real category for minimal surfaces. Existence of associate surfaces is a part of this problem, in fact, if a minimal surface in a flat 3-torus has some associate surfaces, then the Jacobi variety can be represented by three elliptic curves and $(g-3)$-dimensional complex torus (Corollary 5.1 in [5]). So existence of associate surfaces has an effect on complex category since the Jacobi variety is a complex object. Nagano-Smyth [8], [9] considered a criterion for existence of associate surfaces. We now review the Nagano-Smyth's argument. Given $f : M_g \longrightarrow \mathbf{R}^n/\Lambda$ a conformal minimal immersion of a compact Riemann surface M_g into a flat torus $\mathbf{R}^n/\Lambda$, we may assume $f(p_0) = id_{\mathbf{R}^n/\Lambda}$. Then it is known that the Jacobi variety $J(M_g)$ of M_g has the universal property that $f = h \circ j$, where $h : J(M_g) \longrightarrow \mathbf{R}^n/\Lambda$ is a real homomorphism of tori and $j : M_g \longrightarrow J(M_g)$ is the Abel-Jacobi map. Hence, Jacobi variety contains a good information for the minimal surface. Nagano-Smyth considered complex kernel V defined below to investigate the Jacobi variety for study of the minimal surface: The kernel of h determines a real subspace U of the tangent space to $J(M_g)$ at the identity called the real kernel of f and its maximal complex subspace V is called the complex kernel of f. In this paper, we consider a full minimal surface, that is, $f(M_g)$ does not lie in any subtorus of $\mathbf{R}^n/\Lambda$. It follows that $\dim_{\mathbf{R}} U = 2g - n$. The complex kernel V plays an important role to study existence of associate surfaces. Nagano-Smyth [9] calculated the dimension of V under the assumption of irreducible symmetry for f ($\dim_{\mathbf{C}} V = g - n$). On the other hands, M. Micallef [7] dealt with the gap by considering the converse of Wirtinger's result. Micallef proved that a stable minimal surface in a 4-dimensional flat torus is holomorphic with respect to some complex structure of the torus. Correspondingly, we considered whether the stability in Micallef's Theorem can be removed or not. In case $g = 2$, every minimal surface in 4-dimensional flat torus is holomorphic, and hence, it is removable condition. But, in case $g = 3$, we can construct a family of minimal surfaces which are not holomorphic with respect to any complex structure of the tori [14]. This family of minimal surfaces motivates us to study the Moduli of compact minimal surfaces in flat 4-tori. The Moduli space of compact minimal surfaces in flat tori has been studied by C. Arezzo and G. P. Pirola [2], [10]. The key space is $\mathcal{M}_g^n$ which is defined as a subvariety of a bundle over the Teichmüller space (we refer to $\mathcal{M}_g^n$ later). In particular, for $n = 4$, it is possible to give precise description of $\mathcal{M}_g^4$. Arezzo-Pirola [2] gave four components of $\mathcal{M}_g^4$ and calculated their dimensions. In a previous paper, we found other components of $\mathcal{M}_g^4$ and calculated their dimensions [13]. Therefore, our interest is the case $n = 4$. Now we consider the dimension of the complex kernel V from the point of view of Gauss map instead of irreducible symmetry in case $n = 4$. Before we state our first result, we review degenerate Gauss map. If the Gauss image lies in a hyperplane of $\mathbf{C}P^{n-1}$, that is, if there exists a non-zero vector $A \in \mathbf{C}^n$ such that $A \cdot (\omega_1, \omega_2, \cdots, \omega_n) \equiv 0$, then we call the Gauss map is degenerate. Moreover, the Gauss map is said to be k-degenerate if and only if k is the maximal number of independent vectors for A. Note that a 0-degenerate in this terminology is non-degenerate. We observe a good nature of minimal surface in flat 4-torus with degenerate Gauss map in the next Remark.

Remark 1.1. *It is known that, in case $n = 4$, (1) the Gauss map is 2-degenerate if and only if f is full holomorphic with respect to some complex structure of the torus, (2) the Gauss map is 3-degenerate if and only if the image of f lies on a totally geodesic 2-dimensional subtorus of the torus (Proposition 4.6 in [4]).*

By (2) in Remark 1.1, we deal with the case $0 \leq k \leq 2$. Our first result is the following:

Proposition 1.1. *Let $M_g \longrightarrow \boldsymbol{R}^4/\Lambda$ be a conformal minimal immersion. Then, the Gauss map is k-degenerate ($0 \leq k \leq 2$) if and only if $\dim_{\boldsymbol{C}} V = g + k - 4$. As a result, Nagano-Smyth's situation is the case $k = 0$, that is, non-degenerate Gauss map.*

By the way, in case $n = 3$, Nagano-Smyth [8] showed that $\dim_{\mathbf{C}} V = g-3$ without irreducible symmetry for f. They also proved that any two minimal surfaces of genus 3 (if and only if $V = 0$) in flat 3-tori must be associates. Moreover, we can also investigate structure of minimal surface of genus 3 in flat 3-torus, in fact, such a minimal surface is hyperelliptic. This suggests that the condition $V = 0$ may give a good property. In case $n = 4$, the condition $V = 0$ implies $g = 4 - k$, that is, $g = 2$, 3, 4. Minimal surface of genus 2 is holomorphic curve as we mention above. By Arezzo-Micallef's method [1], minimal surface of genus 3 is either holomorphic or hyperelliptic (we show this fact in Remark 4.1). These results motivate us to study general theory of minimal surfaces of genus 4 in 4-dimensional flat tori. Now we state our next result:

Theorem 1.2. *Let $f : M_4 \longrightarrow \boldsymbol{R}^4/\Lambda$ be a conformal minimal immersion. Then either f is holomorphic with respect to some complex structure of the torus, or f is a hyperelliptic minimal immersion with non-degenerate Gauss map, or f is a trigonal minimal immersion with non-degenerate or* 1*-degenerate Gauss map.*

By (1) in Remark 1.1, the minimal surface is not holomorphic with respect to any complex structure of the torus in the latter two case in Theorem 1.2.

Finally, we consider the Moduli of minimal surfaces in 4-dimensional flat tori. First, we review Arezzo-Pirola scheme $\mathcal{M}_g^n$ [2]. We then consider a fixed symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^g$ of $H_1(M_g, \mathbf{Z})$, and define

$$\mathcal{T}_g := \{(M_g, \{\alpha_i, \beta_i\}_{i=1}^g)\}/\sim,$$

where $(M_g, \{\alpha_i, \beta_i\}_{i=1}^g) \sim (M_g', \{\alpha_i, \beta_i\}_{i=1}^g)$ if and only if there exists a biholomorphism $\phi : M_g \longrightarrow M_g'$ such that $\phi_*([\alpha_i]) = [\alpha_i]$ and $\phi_*([\beta_i]) = [\beta_i]$ ($[\ \ ]$ denotes the class in homotopy). We also define

$$\begin{aligned}\mathcal{H}_g^n := \{(M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \cdots, \omega_n \,|\, (M_g, \{\alpha_i, \beta_i\}_{i=1}^g) \in \mathcal{T}_g,\\ \omega_1, \omega_2, \cdots, \omega_n \in H^0(M_g, K) \text{ without common zeros}\},\end{aligned}$$

and

$$\mathcal{M}_g^n := \{(M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \cdots, \omega_n \in \mathcal{H}_g^n \,|\, \omega_1^2 + \omega_2^2 + \cdots + \omega_n^2 = 0\}.$$

Clearly $\mathcal{H}_g^n$ is a smooth manifold of complex dimension $3g-3+ng$ and $\mathcal{M}_g^n$ is a subvariety of $\mathcal{H}_g^n$. Arezzo-Pirola defined the period map on $\mathcal{M}_g^n$ and considered the infinitesimal deformation of the period map for study of the Moduli. Since we will consider the case $n = 4$, we next review the structure of $\mathcal{M}_g^4$.

If $n = 4$, Q_2 is isomorphic to the product of two lines, and isomorphism is given by the Veronese map V (p.19 [4]). We have the following commutative diagram:

$$\begin{tikzcd}
M_g \arrow[rr, "G"] \arrow[dr, "\varphi"'] & & Q_2 \subset \mathbf{C}P^3 \\
& \mathbf{C}P^1 \times \mathbf{C}P^1 \arrow[ur, "V"'] &
\end{tikzcd}$$

V can be given by

$$V((s_1, s_2), (t_1, t_2)) = (s_1\, t_1 - s_2\, t_2,\ i\,(s_1\, t_1 + s_2\, t_2),\ s_1\, t_2 + s_2\, t_1,\ i\,(s_1\, t_2 - s_2\, t_1)).$$

Let us call φ_1 and φ_2 the projection of φ on each of the two factors. Because the Gauss map is evaluation of holomorphic differentials, we know that $G^*(\mathcal{O}(1)) = [(\omega_i)] = K$. If we put $\varphi_i^*(\mathcal{O}(1)) = L_i$ $(i = 1, 2)$, the commutativity of the diagram implies that

$$L_1 + L_2 = K. \tag{5}$$

This pair of line bundles still does not identify the minimal immersion, because we are not able to reconstruct the holomorphic differentials. This can be done precisely if we know not just the line bundles but also a pair of holomorphic sections for each of them. In fact, $\varphi_1 = [s_1, s_2]$, and $\varphi_2 = [t_1, t_2]$, where $s_i \in H^0(M_g, L_1)$ and $t_i \in H^0(M_g, L_2)$ $(i = 1, 2)$. Knowing these sections, we can recover the holomorphic differentials just using the expression of the Veronese map.

We define

$$\widetilde{\mathcal{M}}_g^4 := \{(M_g, \{\alpha_i, \beta_i\}_{i=1}^g), L, s_1, s_2, t_1, t_2 \,|\, s_i \in H^0(M_g, L),\ t_i \in H^0(M_g, K - L)\},$$

and construct a surjection $\Psi : \widetilde{\mathcal{M}}_g^4 \longrightarrow \mathcal{M}_g^4$ given by

$$\begin{aligned}
\omega_1 &= s_1\, t_1 - s_2\, t_2,\\
\omega_2 &= i\,(s_1\, t_1 + s_2\, t_2),\\
\omega_3 &= s_1\, t_2 + s_2\, t_1,\\
\omega_4 &= i\,(s_1\, t_2 - s_2\, t_1).
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\Psi^{-1}((M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \omega_3, \omega_4)\\
&\quad = \{(M_g, \{\alpha_i, \beta_i\}_{i=1}^g), L, \lambda\, s_1, \lambda\, s_2, \lambda^{-1}\, t_1, \lambda^{-1}\, t_2 \,|\, \lambda \in \mathbf{C}^*\}
\end{aligned}$$

and therefore

$$\mathcal{M}_g^4 \cong \widetilde{\mathcal{M}}_g^4 / \mathbf{C}^*.$$

In this paper we restrict ourselves to the study of minimal immersions free of branch points. A direct calculation shows that this is equivalent to $s_1 = s_2 = 0$ or $t_1 = t_2 = 0$. L and $K - L$ have two holomorphic sections without common zeros. This means $|D|$ and $|K - D|$ are base point free, where D is the divisor coressponding to L and K means the canonical divisor. Therefore we can assume $h^0(L) > 1$ and $h^0(K - L) > 1$, unless one of these bundles is trivial. It is easy to see that this can happen if and only if the immersion is holomorphic with respect to a complex structure of the torus.

Arezzo-Pirola considered degenerative components of $\mathcal{M}_g^4$ for study of the Moduli. Recall that an irreducible component $\mathcal{C}_g^4$ of $\mathcal{M}_g^4$ is called degenerative if there exists $((M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \omega_3, 0) \in \mathcal{C}_g^4$ such that $\dim(span_{\mathbf{C}}\{\omega_1, \omega_2, \omega_3\}) = 3$. Then, they proved that for $g \geq 4$, the generic point $((M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \omega_3, \omega_4)$ in each degenerative component is complex full, that is, $\dim(span_{\mathbf{C}}\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 4$ (Proposition 1 in [2]). We give an alternative proof of this proposition by the theory of degenerate Gauss map and spin bundles.

2. Application of degenerate Gauss map to Nagano-Smyth's argument

In this section, we prove Proposition 1.1.

Proof. First, we observed $\dim_{\mathbf{R}} U = 2g - 4$. Then,

$$2g \geq \dim_{\mathbf{R}}(U + JU) = \dim_{\mathbf{R}} U + \dim_{\mathbf{R}} JU - \dim_{\mathbf{R}}(U \cap JU),$$

where J is the complex structure of $J(M_g)$. Hence, we obtain $\dim_{\mathbf{R}} V = \dim_{\mathbf{R}}(U \cap JU) \geq 2g - 8$. On the other hands,

$$\dim_{\mathbf{R}} V = \dim_{\mathbf{R}}(U \cap JU) \leq \dim_{\mathbf{R}} U = 2g - 4.$$

These inequalities imply that $g - 4 \leq \dim_{\mathbf{C}} V \leq g - 2$. Next, we consider the case $\dim_{\mathbf{C}} V = g - 2, g - 3$, and $g - 4$, respectively.

Recall that $T_{id}J(M_g)/U$ is isomorphic to $T_{f(p_0)}\mathbf{R}^4/\Lambda$ via h_* and the Abel-Jacobi map $j : M_g \longrightarrow J(M_g)$ is given by

$$j(p) = \Re \int_{p_0}^{p} (\eta_1, \cdots, \eta_g, i\,\eta_1, \cdots, i\,\eta_g)^T,$$

where $\{\eta_i\}_i$ is a basis of the space of holomorphic differential $H^0(M_g, K)$. Then $T_{id}J(M_g)$ can be identified with $H^0(M_g, K) = span_{\mathbf{C}}\{\eta_1, \cdots, \eta_g\}$.

case $\dim_{\mathbf{C}} V = g - 2$

In this case, $\dim_{\mathbf{R}} V = \dim_{\mathbf{R}} U = 2g - 4$. It follows that $U = V$. Then, since $T_{id}J(M_g)/U$ is a 2-dimensional complex vector space with respect to the original complex structure J of $J(M_g)$, it can be spanned by some holomorphic 1-forms ω_1 and ω_2 (i.e., it can be spanned by ω_1, $i\,\omega_1$, ω_2, and $i\,\omega_2$ in the real category). Hence, it is equivalent to 2-degeneracy of the Gauss map.

case $\dim_{\mathbf{C}} V = g - 3$

$T_{id}J(M_g)/(U + JU)$ is a 1-dimensional complex vector space (i.e., it can be spanned by some holomorphic 1-form ω) with respect to the original complex structure J of $J(M_g)$. Thus, $T_{id}J(M_g)/U$ is isomorphic to $W \oplus T_{id}J(M_g)/(U + JU)$, where W is a real 2-dimensional vector space (not complex space). So it is equivalent to 1-degeneracy of the Gauss map.

case $\dim_{\mathbf{C}} V = g - 4$

$T_{id}J(M_g)/U$ is a real 4-dimensional vector space (not complex vector space with respect to the original complex structure J of $J(M_g)$). Hence, it is equivalent to non-degeneracy of the Gauss map because it can be spanned by linearly independent complex 1-forms ω_1, ω_2, ω_3, and ω_4.

The above arguments complete the proof. □

3. The Moduli space of minimal surfaces in flat tori

In this section, we give an alternative proof of Arezzo-Pirola's Proposition. First, we review surveys of linear series (or system) over Riemann surfaces in [3]. The geometry of $\mathcal{M}_g^4$ is closely related to the classical theory of linear series over Riemann surfaces and we give notation about linear series. Let V be a vector subspace of $H^0(M_g, \mathcal{O}(D))$ and $\mathcal{P}V$ is the projective space. A linear series $\mathcal{P}V$ is said to be a g_d^r if $\deg D = d$, $\dim V = r + 1$.

We introduce the variety $W_d^r(M_g)$ (p.153, p.176 [3]):

$$\begin{aligned} supp(W_d^r(M_g)) =& \{L \in Pic^d(M_g) \,|\, h^0(L) \geq r+1\} \\ =& \{|D| \,|\, \deg\, D = d\,,\ \ h^0(D) \geq r+1\}, \end{aligned}$$

where $r \geq d - g$. The following Lemma explain about structure of $W_d^r(M_g)$:

Lemma 3.1. (p.182 [3]) *Suppose $r \geq d-g$. Then no component of $W_d^r(M_g)$ is entirely contained in $W_d^{r+1}(M_g)$. In particular, if $W_d^r(M_g) \neq \phi$, then $W_d^r(M_g) - W_d^{r+1}(M_g) \neq \phi$.*

Next, we show the following Lemma for our proof of Arezzo-Pirola's result.

Lemma 3.2. *Let $f : M_g \longrightarrow \boldsymbol{R}^4/\Lambda$ be a conformal minimal immersion and we use notation given in Introduction. If the Gauss map is 1-degenerate, then $L_1 = L_2$, that is, L_i is the spin bundle by* (5)*.*

Proof. We first note that we may assume that $s_1\, t_1 + s_2\, t_2 = \alpha(s_1\, t_1 - s_2\, t_2)$ for a constant $\alpha \in (0, 1)$ by Lemma 4.5 in [4]. It follows that $s_2\, t_2 = -\beta s_1\, t_1$ $(\beta \in (0, 1))$. Hence, the image of the Gauss map is given by

$$(s_1\, t_1, s_1\, t_2, s_2\, t_1, s_2\, t_2) \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ -1 & i & 0 & 0 \end{pmatrix} \in Q_2$$

By the following projective transformation, the Gauss image can be reduced to:

$$
\begin{aligned}
&(s_1\,t_1, s_1\,t_2, s_2\,t_1, s_2\,t_2)\begin{pmatrix} -2\sqrt{\beta} & 0 & 0 & \beta \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\qquad = (-2\sqrt{\beta} s_1\,t_1, s_1\,t_2 + s_2\,t_1, i\,(s_1\,t_2 - s_2\,t_1), 0) \in Q_1 \subset Q_2
\end{aligned}
$$

Then, by direct calculation, we obtain the following diagram:

$$
\begin{tikzcd}
M_g \arrow[rr, "G"] \arrow[dr] \arrow[ddr, "\varphi = (\varphi_1, \varphi_2)"'] & & Q_1 \subset Q_2 \\
& \mathbf{C}P^1 \arrow[ur, "\Phi"] \arrow[d, "\psi"'] & \\
& \mathbf{C}P^1 \times \mathbf{C}P^1 \arrow[uur, "V"'] &
\end{tikzcd}
$$

where ψ, Φ, and V are given by

$$
\begin{aligned}
&\psi(\sqrt{\beta}\,s_1, s_2) = ((s_1, s_2), (t_1, t_2)) = ((s_1, s_2), (-\frac{1}{\sqrt{\beta}}\,s_2, \sqrt{\beta}\,s_1)) \\
&\Phi(\sqrt{\beta}\,s_1, s_2) = (2\sqrt{\beta}\,s_1\,s_2, \beta\,s_1^2 - s_2^2, i\,(\beta\,s_1^2 + s_2^2), 0) \\
&V((s_1, s_2), (t_1, t_2)) = (-2\sqrt{\beta}\,s_1\,t_1, s_1\,t_2 + s_2\,t_1, i\,(s_1\,t_2 - s_2\,t_1), 0)
\end{aligned}
$$

Note that the image of Φ agrees with V in Q_1. It can be rewritten as follows:

$$
\begin{tikzcd}
M_g \arrow[rr, "G"] \arrow[dr] \arrow[ddr, "\varphi = (\varphi_1, \varphi_2)"'] & & Q_1 \subset Q_2 \\
& \mathbf{C}P^1 \arrow[ur, "\Phi"] \arrow[d, "\psi"'] & \\
& \mathbf{C}P^1 \times \mathbf{C}P^1 \arrow[uur, "V"'] &
\end{tikzcd}
$$

where ψ and Φ are given by

$$\psi(s,t) = ((\frac{s}{\sqrt{\beta}}, t), (-\frac{t}{\sqrt{\beta}}, s)),$$
$$\Phi(s,t) = (2st, s^2 - t^2, i\,(s^2 + t^2), 0)$$

Therefore, we conclude $L_1 = L_2 = [(s)]$ by the diagram, where $[(s)]$ means the line bundle defined by the divisor (s). The proof is complete. $\square$

By Lemma 3.2, we can show Arezzo-Pirola's proposition as follows: Let $p := ((M_g, \{\alpha_i, \beta_i\}_{i=1}^g), \omega_1, \omega_2, \omega_3, 0)$ be an element of degenerative component of $\mathcal{M}_g^4$. Thus the minimal surface defined by p has 1-degenerate Gauss map. It follows that the corresponding line bundle L_i is the spin bundle with $h^0(L_i) > 1$ by Lemma 3.2. We observe that the set of spin bundle L satisfying $h^0(L) > 1$ has codimension 1 in the Riemannian Moduli space (p.401 [10]). Hence, by a suitable deformation of p in $\mathcal{M}_g^4$, we can obtain elements of $\mathcal{M}_g^4$ such that the line bundles are not spin. Therefore, the corresponding minimal surfaces have non-degenerate Gauss map by Lemma 3.2 and give complex full elements of the degenerative component. Obviously, these points are generic in $\mathcal{M}_g^4$. This completes the proof.

Finally, we refer to the area of the Gauss map.

Remark 3.1. *It is known that the metric $Q_2 \cong \boldsymbol{C}P^1 \times \boldsymbol{C}P^1$ is just the product metric, where each factor $\boldsymbol{C}P^1$ has the metric on a sphere with area half that of the standard unit sphere. Thus each factor is a sphere of radius $1/\sqrt{2}$ (p.22 [4]).*

φ_1, φ_2 are finite-sheeted coverings of each of the copies of $\mathbf{C}P^1$ since Gauss map of minimal surface is holomorphic. Because each of the latter has area 2π in the given metric, it follows that the area of $G(M_g)$ is $2N\pi$ for some positive integer N. We state the following theorem (Theorem 2.1 p.23 [4]):

Theorem 3.1. *Let G be the Gauss map of a minimal surface $f : M_g \longrightarrow \boldsymbol{R}^4/\Lambda$. Then the area of $G(M_g)$ is of the form $2N\pi$ for some integer N.*

4. PROOF OF THEOREM 1.2

In this section, we give a proof of Theorem 1.2. Let $f : M_4 \longrightarrow \mathbf{R}^4/\Lambda$ be a compact minimal surface and G the Gauss map. It is well-known that the total curvature is just the negative of the area of the Gauss map (p.37 [4]):

$$Area(G(M_g)) = -\int_{M_g} K dv.$$

In case $g = 4$, we can conclude $N = 6$ in Theorem 3.1 by the above equation and Gauss-Bonnet theorem. We now consider the case $N = 0+6$, $1+5$, $2+4$, and $3 + 3$, respectively.

case $N = 0 + 6$:
one of the projections φ_i is constant, and thus, f is holomorphic with respect to some complex sturcture of the torus.

case $N = 1 + 5$:
one of the projections φ_i is a biholomorphism from M_4 to $\mathbf{C}P^1$ since a holomorphic map between Riemann surfaces with degree 1 is a biholomorphism. It contradicts the fact that the genus of M_4 is 4.

case $N = 2 + 4$:
we may assume that φ_1 is 2-sheeted branched covering and φ_2 is 4-sheeted branched covering of $\mathbf{C}P^1$, respectively. It follows that M_4 become hyperelliptic Riemann surface via $L_1 = g^1_2$. We observe that there exist no spin bundles with base point free over hyperelliptic curves in even genus case (32 p.288 [3]). Thus, we conclude the Gauss map is non-degenerate by Lemma 3.2. We remark that it is known that the Gauss map of hyperelliptic minimal surface factors through the covered sphere $\mathbf{C}P^1$:

$$
\begin{tikzcd}
M_4 \arrow[rr, "G"] \arrow[dr, "\varphi_1"] \arrow[ddr, "\varphi = (\varphi_1, \varphi_2)"'] & & Q_2 \\
& \mathbf{C}P^1 \arrow[ur, "\Phi"] \arrow[d, "\psi"] & \\
& \mathbf{C}P^1 \times \mathbf{C}P^1 \arrow[uur, "V"'] &
\end{tikzcd}
$$

ψ and Φ can be given by

$$
\begin{aligned}
&\psi(s,t) := ((s,t),(s^2,t^2)),\\
&\Phi(s,t) := (s^3 - t^3, i\,(s^3 + t^3), st^2 + s^2t, i\,(st^2 - s^2t))
\end{aligned}
$$

case $N = 3 + 3$:
we first note that a Riemann surface M_g is trigonal if and only if it has a base point free linear series g^1_3. We observe that $W^1_3(M_4)$ consists of a single point g^1_3 or two distinct points g^1_3 (p.206 in [3]). By Lemma 3.1, these g^1_3 are complete. Recall that each L_i is a base point free line bundle satisfying $h^0(L_i) > 1$ and $\deg L_i = 3$. These imply $L_i \in W^1_3(M_4)$. So we conclude that each L_i gives a base point free complete g^1_3, and therefore, M_4 has a trigonal structure. Next, we consider the following two cases, $L_1 = L_2$ or $L_1 \neq L_2$:
(i) in case $L_1 = L_2$. Then we obtain $2L_1 = K$ by (5). Hence, L_1 is the spin

bundle. Moreover, the following diagram holds :

$$\begin{tikzcd}
M_4 \arrow[rr, "G"] \arrow[dr, "\varphi_1"] \arrow[ddr, "\varphi = (\varphi_1, \varphi_1)"'] & & Q_2 \\
& \mathbf{C}P^1 \arrow[ur, "\Phi"'] \arrow[d, "\psi"'] & \\
& \mathbf{C}P^1 \times \mathbf{C}P^1 \arrow[uur, "V"'] &
\end{tikzcd}$$

ψ and Φ can be given by

$$\psi(s,t) := ((s,t),(s,t)),$$
$$\Phi(s,t) := (s^2 - t^2, i\,(s^2 + t^2), 2st, 0)$$

Note that the Gauss map is 1-degenerate.
(ii) in case $L_1 \neq L_2$. Then, by Lemma 3.2, the Gauss map is non-degenerate since L_i is not spin bundle via (5).

The proof is now complete.

Remark 4.1. *By the same arguments, we can prove that minimal surface of genus* 3 *is either holomorphic curve or hyperelliptic. In fact,* $g = 3$ *implies* $N = 4$ *in Theorem 3.1. We consider the case* $N = 0+4$, $1+3$, *and* $2+2$, *respectively. In case* $N = 0+4$, *the minimal surface is holomorphic curve. Next, in case* $N = 1+3$, *it cannot happen because the genus is* 3. *Finally,* $N = 2+2$ *implies the minimal surface is hyperelliptic.*

References

1. C. Arezzo and M. Micallef, *Minimal surfaces in flat tori*, Geom. Funct. Anal 10 (2000), 679–701.
2. C. Arezzo and G. P. Pirola, *On the existence of periodic minimal surfaces*, J. Alg. Geom 8 (1999), 765–785.
3. P. A. Griffiths E. Arbarello, M.Cornalba and J. Harris, *Geometry of Algebraic Curves. I*, Springer-Verlag, 1985.
4. D. A. Hoffman and R. Osserman, *The geometry of the generalized Gauss map*, Mem. Amer. Math. Soc. vol.28, No.236 (1980).
5. W. H. Meeks III, *The theory of triply periodic minimal surfaces*, Indiana Univ Math Journal 39 No3 (1990), 877–935.
6. H. Karcher, *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions*, manuscripta Math 64 (1989), 291–357.
7. M. Micallef, *Stable minimal surfaces in Euclidean space*, J. Diff. Geom. 19 (1984), 57–84.
8. T. Nagano and B. Smyth, *Periodic minimal surfaces*, Comment. Math. Helv 35 (1978), 29–55.
9. ———, *Periodic minimal surfaces and Weyl groups*, Acta. Math 145 (1980), 1–27.

10. G. P. Pirola, *The infinitesimal variation of the spin abelian differentials and periodic minimal surfaces*, Communications in Analysis and Geometry 6 (1998), 393–426.
11. A. Schoen, *Infinite periodic minimal surfaces without self-intersections*, Technical Note D-5541, N. A. S. A, Cambridge, Mass., May 1970.
12. H. A. Schwarz, *Gesammelte Mathematische Abhandlungen, vol 1*, Berlin: Springer-Verlag, 1890.
13. T. Shoda, *New components of the moduli space of minimal surfaces in* 4*-dimensional flat tori*, to appear in J. London. Math. Soc.
14. ______, *A construction of a dense family of compact minimal surfaces in* 4*-dimensional flat tori*, J. Math. Soc. Japan 56 (2004), 627–633.

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551 Japan
E-mail address: tshoda@math.titech.ac.jp