GEOMETRY OF LINES ON CERTAIN MOISHEZON THREEFOLDS. I. EXPLICIT DESCRIPTION OF FAMILIES OF TWISTOR LINES

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ABSTRACT. We study real lines on certain Moishezon threefolds which are potentially twistor spaces of $3\mathbf{CP}^2$. Here, line means a smooth rational curve whose normal bundle is $O(1)^{\oplus 2}$ and the reality implies the invariance under an anti-holomorphic involution on the threefolds. Our threefolds are birational to the double covering of $\mathbf{CP}^3$ branched along a singular quartic surface. On these threefolds we find families of real lines in explicit form and prove that which families have to be chosen as twistor lines depend on how we take small resolutions of the double covering. This is a first step for determining the moduli space of self-dual metrics on $3\mathbf{CP}^2$ of positive scalar curvature, which admit a non-trivial Killing field.

CONTENTS

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1. Introduction

A Riemannian metric on an oriented four-manifold is called self-dual iff the anti-self-dual part of the Weyl conformal curvature of the metric identically vanishes. Basic examples are provided by the round metric on the four-sphere and the Fubini-Study metric on the complex projective plane. In general, one can expect that if two four-manifolds admit self-dual metrics respectively, then their connected sum will also admit a self-dual metric. In fact, Y.S. Poon [12] constructed explicit examples of self-dual metrics on $2\mathbf{CP}^2$, the connected sum of two complex projective planes. He further showed that on $2\mathbf{CP}^2$ there are no self-dual metrics other than his metrics, under the assumption of the positivity of the scalar curvature. Later, C. LeBrun [10] and D. Joyce [7] respectively constructed self-dual metrics of positive scalar curvature on $n\mathbf{CP}^2$ for any $n \geq 1$. These are called LeBrun metrics and Joyce metrics respectively, and have nice characterizations by the (conformal) isometry group. Namely, A. Fujiki [2] proved that if a self-dual metric on $n\mathbf{CP}^2$ has $U(1) \times U(1)$ as the identity component of the isometry group, then it must be a Joyce metric. LeBrun [11] showed that if a self-dual metric on $n\mathbf{CP}^2$ has a non-trivial semi-free $U(1)$-isometry, then the metric must be a LeBrun metric. Here, a $U(1)$-action on a manifold M is called semi-free if the isotropy group is $U(1)$ or identity only, at every point of M.

Now drop the assumption of the semi-freeness for $U(1)$-isometry. We further suppose that the metric is not a Joyce metric, since it is immediate to get non-semi-free $U(1)$-isometries by choosing subgroups of $U(1) \times U(1)$. Then in [5, 6] the author has shown that, for $n = 3$, there already exist self-dual metrics of positive scalar curvature whose identity components of the isometry groups are $U(1)$ acting non-semi-freely on $3\mathbf{CP}^2$. (Note that Poon's metrics on $2\mathbf{CP}^2$ coincide with Joyce metrics and hence there is no self-dual metric on $2\mathbf{CP}^2$ whose isometry group is $U(1)$.) The purpose of the present paper is to study the twistor spaces of these self-dual metrics on $3\mathbf{CP}^2$. In other words, we are interested in twistor spaces of $3\mathbf{CP}^2$ which has a non-trivial holomorphic $U(1)$-action but which are different from LeBrun twistor spaces. Note that on $3\mathbf{CP}^2$ Joyce metrics coincide with LeBrun metrics with torus action.

By a result of Kreußler-Kurke [9] such a twistor space is always Moishezon; namely is birational to a projective variety.123‘455 ¶ ?

First in Section 2 we determine the defining equations of projective models of the above mentioned twistor spaces of $3\mathbf{CP}^2$ (Propositions 2.1 and 2.6). Here note that by a famous theorem of Hitchin [4] the twistor spaces themselves cannot be projective algebraic, so the projective models are only birational to the real twistor spaces. The projective models have a structure of the double covering of $\mathbf{CP}^3$ branched along a singular quartic surface which is birational to an elliptic ruled surface. The defining equations allow us to prove that *our non-semi-free $U(1)$-action on $3\mathbf{CP}^2$ is uniquely determined up to equivariant diffeomorphims* (Proposition 2.7).

Then it naturally arises a question whether, conversely, the singular projective threefold having this kind of structure always becomes birational to a twistor space. Also we want to know how one can obtain the twistor spaces from the projective threefolds by means of resolution of singularities. These are fascinating but difficult problems, since they are equivalent to determine the moduli space of self-dual metrics on $3\mathbf{CP}^2$ of positive scalar curvature, which admit a non-trivial Killing field.

The most primitive way to prove that given threefold is actually a twistor space is to find a family of twistor lines. Here a twistor line is by definition a fiber of the twistor fibration $Z \to M$, where M is the base four-manifold. In Section 3, we see that the image of a real line in our threefold under the double covering map onto $\mathbf{CP}^3$ is a line iff the real line intersects some real smooth rational curve, and that otherwise the image is a real conic whose intersection number with the branch quartic surface is at least two for any intersection points (Proposition 3.2). Following I. Hadan [3], we will call the latter kind of curve *a touching conic*.

Thus in order to find all the twistor lines, we need to study the space of real touching conics for our branch quartic surface. Note that every conic in $\mathbf{CP}^3$ is contained in a plane. In the rest of Section 3 we explicitly write down the defining equations of real touching conics contained in real $\mathbf{C}^*$-invariant planes, where such planes are parametrized by a circle. Our results show that for each of such a real plane, real touching conics in it form just two real one-dimensional families (Propositions 3.4, 3.6 and 3.7 for precisely.) These families are not pencil in general.

In Section 4 we study the inverse images of real touching conics classified in the previous section. The inverse images are of course candidates of twistor lines. It is immediate to see that the inverse image has just two irreducible components which are mapped biholomorphically onto the conic. We show that these two components form the real parts of real pencils on a real $U(1)$-invariant smooth surface (which is the inverse image of a real $U(1)$-invariant plane).

Next in Section 5 we calculate the normal bundle of these two irreducible components inside a smooth threefold which is a small resolution of the singular projective model (the double cover of $\mathbf{CP}^3$). It is easy to show that the normal bundle is always isomorphic to $O(1)^{\oplus 2}$ or $O \oplus O(2)$. But it is quite subtle which one actually occurs. In fact, we will see that *the result depends on the choice of small resolutions*, and that *which one of the two irreducible components must be chosen for twistor line depends on which resolution we take*: for each small resolution, we define a function defined on the parameter space of real invariant planes, and show that the normal bundle degenerates into $O \oplus O(2)$ precisely when the function has a critical point. We then determine the critical points of the functions. As a result, we can exactly determine the images of twistor lines contained in the real $\mathbf{C}^*$-invariant planes (Theorem 5.20). Furthermore, these calculations enable us to prove that *among twenty-four ways of possible small resolutions, only two resolutions can yield a twistor space* (Theorem 5.21), and also to determine which irreducible component must be chosen, for each of the two resolutions.

As a byproduct of these investigations, we can prove that *in our smooth threefolds there always exist families of real smooth rational curves whose normal bundles are $O(1)^{\oplus 2}$ but which cannot be twistor lines, by showing that they can be deformed into a smooth rational curve whose normal bundle is $O \oplus O(2)$ while keeping the reality* (Corollary 5.7 and Proposition 5.5). Thus the twistor line is not characterized by this property in general. However, it should be noted that since the Penrose correspondence between twistor spaces and self-dual metrics is purely local matter, these 'fake twistor lines' certainly define a self-dual metric on some non-compact four-manifold. It will be interesting to study what the four-manifold is, and to clarify the behavior of the self-dual metric near the ends.

Finally in Section 6 we calculate the normal bundle of an irreducible component of the inverse image of a real line in $\mathbf{CP}^3$, which is a candidate of twistor lines mentioned above, and show that it is always real and has $O(1)^{\oplus 2}$ as the normal bundle. (Proposition 6.1). Thus the situation is quite simple in this case. A connection between the subjects in the previous sections is that these lines in $\mathbf{CP}^3$ can be considered as a limit of touching conics studied in Section 3–5. This fact will play an important role for forthcoming investigation of the compactification of the space of real lines in our threefolds.

Finally I mention the relationship between Hadan's elaborate work [3] and ours. Broadly speaking, the threefolds Hadan studied are candidates of the twistor spaces of *generic* self-dual metrics on $3\mathbf{CP}^2$ so that they have *no symmetry*, whereas ours should be the twistor spaces of self-dual metrics on $3\mathbf{CP}^2$ *with semi-free* $U(1)$*-symmetry*. Although both threefolds are birational to the double covering of $\mathbf{CP}^3$ branched along a quartic surface with isolated singularities, and although (as far as I could understand) he also aimed to determine all of the twistor lines, there are quite a few differences. First of all, the branch quartic surface is birational to a K3 surface in Hadan's case, while it is birational to an elliptic ruled surface in our case. Secondly, since his branch quartic surface has much more singularities than ours, there are much more singular plane sections of the quartic. This yields monodromy problem of families of touching conics, which is the main subject of his investigation. Thirdly, he mainly concerns lines which are not necessarily real. But we only consider real ones, and the reality plays a crucial role. Fourthly, he focused his attention to the behavior of generic lines, and therefore did not aim to get the defining equations of touching conics. In contrast, we are concerning twistor lines which can be considered as a limit of generic twistor lines. Finally, his study is mostly on $\mathbf{CP}^3$, and does not investigate the inverse images of touching conics. In particular, he does not consider their normal bundles, whereas a considerable part of our investigation is devoted to determining them. As a consequences of these differences, there are few overlaps between his results and ours.

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2. Defining equations of the branch quartic surfaces and their singularities

Let g be a self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature, and assume that g is not conformally isometric to LeBrun metrics. Let Z be the twistor space of g, and denote by $(-1/2)K_Z$ the fundamental line bundle which is the canonical square root of the anticanonical line bundle of Z. These non-LeBrun twistor spaces of $3\mathbf{CP}^2$ are extensively studied in Kreußler and Kurke [9] and Poon [13], and it has been proved that the fundamental system (the complete linear system associated to the fundamental line bundle) is free and of three-dimensional, and induces a surjective morphism $\Phi : Z \to \mathbf{CP}^3$ which is generically two-to-one, and that the branch divisor B is a quartic surface with only isolated singularities. Furthermore, there is the following diagram:

$$
\begin{array}{ccc}
 & Z & \\
\mu & \downarrow & \searrow \Phi \\
 & Z_0 & \xrightarrow[\Phi_0]{} \mathbf{CP}^3
\end{array}
$$

where $\Phi_0 : Z_0 \to \mathbf{CP}^3$ denotes the double covering branched along B, and μ is *a small resolution* of the singularities of Z_0 over the singular points of B.

Generically, B has only ordinary double points [9, 13] and hence is birational to a K3 surface. As a consequence, one can deduce that Z does not admit a non-zero holomorphic vector field. However, the author showed in [5] and [6] that if B degenerates to have non-ADE singularities, then Z admits a non-zero holomorphic vector field, and that such a twistor space of $3\mathbf{CP}^2$ actually exists. Concerning a defining equation of the branch quartic B for such twistor spaces, we have the following proposition which is the starting point of our investigation.

Proposition 2.1. *Let g be a non-LeBrun self-dual metric on $\mathbf{CP}^2$ of positive scalar curvature, and assume the existence of a non-trivial Killing field. Let $\Phi : Z \to \mathbf{CP}^3$ and $B \subset \mathbf{CP}^3$ be as above. Then there exists a homogeneous coordinate $(y_0 : y_1 : y_2 : y_3)$ on $\mathbf{CP}^3$ fulfilling (i)-(iii) below:*
(i) a defining equation of B is given by

$$(1) \qquad (y_2y_3 + Q(y_0, y_1))^2 - y_0y_1(y_0 + y_1)(ay_0 - by_1) = 0,$$

where $Q(y_0, y_1)$ is a quadratic form of y_0 and y_1 with real coefficients, and a and b are positive real numbers,
(ii) the naturally induced real structure on $\mathbf{CP}^3$ is given by

$$\sigma(y_0 : y_1 : y_2 : y_3) = (\overline{y}_0 : \overline{y}_1 : \overline{y}_3 : \overline{y}_2)\,,$$

(iii) the naturally induced $U(1)$-action on $\mathbf{CP}^3$ is given by

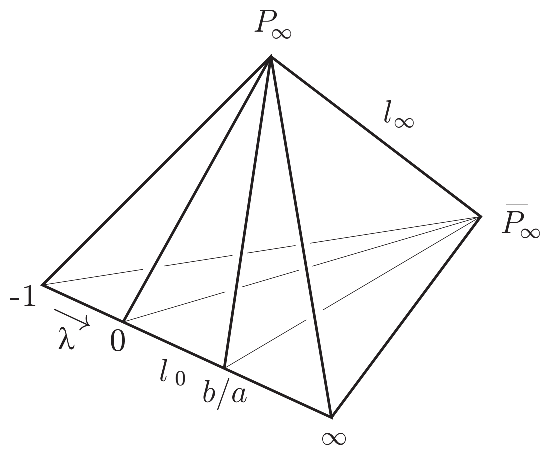
$$(y_0 : y_1 : y_2 : y_3) \mapsto \left(y_0 : y_1 : e^{i\theta}y_2 : e^{-i\theta}y_3\right), \quad e^{i\theta} \in U(1).$$

Proof. If the fundamental system of Z is free, there are just four reducible members, all of which are real [13, 9]. We write $\Phi^{-1}(H_i) = D_i + \overline{D}_i$, $1 \le i \le 4$, where H_i is a real plane in $\mathbf{CP}^3$. The restrictions of Φ onto D_i and $\overline{D}_i$ are obviously birational morphisms onto H_i, so that, together with the fact that $(-1/2)K_Z \cdot L_i = 2$, it can be readily seen that $C_i := \Phi(L_i)$ is a conic contained in B. This implies that the restriction of B onto H_i is a conic of multiplicity two. Namely, C_i, $1 \le i \le 4$, is so called *a trope* of B.

A Killing field naturally gives rise to an isometric $U(1)$-action, which can be canonically lifted to a holomorphic $U(1)$-action on the twistor space. This action naturally goes down to $\mathbf{CP}^3$, and every subvarieties above are clearly preserved by these $U(1)$-actions. In particular, C_i is a $U(1)$-invariant conic on a $U(1)$-invariant plane H_i, where the $U(1)$-action is induced by the vector field. Since the twistor fibration is $U(1)$-equivariant and generically one-to-one on D_i, and since $\Phi|_{D_i} : D_i \to H_i$ is also $U(1)$-equivariant and birational, the $U(1)$-action on any H_i is non-trivial. Hence $U(1)$ acts non-trivially on C_i. For $j \neq i$, put $l_{ij} = H_i \cap H_j$, which is clearly a real $U(1)$-invariant line. Then $C_i \cap l_{ij}$ must be the two $U(1)$-fixed points on C_i, since it is real set and since there is no real point on C_i. This implies that l_{ij} is independent of the choice of $j \neq i$. So we write $l_{ij} = l_\infty$ and let P_∞ and $\overline{P}_\infty$ be the two fixed points of the $U(1)$-action on l_∞. Then H_i, $1 \leq i \leq 4$, must be real members of the real pencil of planes whose base locus is l_∞. Since l_∞ is a real line, we can choose real linear forms y_0 and y_1 such that $l_\infty = \{y_0 = y_1 = 0\}$. Further, since any of H_i is real, by applying a real projective transformation (with respect to (y_0, y_1)), we may assume that $\cup_{i=1}^4 H_i = \{y_0 y_1 (y_0 + y_1)(a y_0 - b y_1) = 0\}$, where $a, b \in \mathbf{R}$ with $a > 0$ and $b > 0$.

As seen above, every C_i goes through P_∞ and $\overline{P}_\infty$. Let l_i, $1 \leq i \leq 4$ be the tangent line of C_i at P_∞. Now we claim that l_i's are lying on the same plane. Let H be the plane containing l_1 and l_2. Then by using $l_1 \cap B = P_\infty = l_2 \cap B$, we can easily deduce that $B \cap H$ is a union of lines, all of which goes through P_∞. Suppose that l_3 is not contained in H. Then the line $H \cap H_3$ is not tangent to C_3, so there is an intersection point of $C_3 \cap H$ other than P_∞. Then the line $H \cap H_3$ is contained in B, because we have already seen that $B \cap H$ is a union of lines all passing through P_∞. This is a contradiction since $B \cap H_3 = 2C_3$. Similarly we have $l_4 \subset H$. Therefore $l_i \subset H$ for any i, as claimed. Because C_i's are real, the plane $\sigma(H)$ contains the tangent lines of C_i's at $\overline{P}_\infty$. Let y_3 be a linear form on $\mathbf{CP}^3$ defining H, and set $y_2 := \overline{\sigma^* y_3}$. Then $(y_0 : y_1 : y_2 : y_3)$ is a homogeneous coordinate on $\mathbf{CP}^3$. By our choice, we have $\sigma(y_0 : y_1 : y_2 : y_3) = (\overline{y}_0 : \overline{y}_1 : \overline{y}_3 : \overline{y}_2)$, $P_\infty = (0 : 0 : 0 : 1)$ and $\overline{P}_\infty = (0 : 0 : 1 : 0)$.

Because the planes $\{y_i = 0\}$ are $U(1)$-invariant, our $U(1)$-action can be linearized with respect to the homogeneous coordinate $(y_0 : y_1 : y_2 : y_3)$. Further, since H_i's are $U(1)$-invariant, the action can be written $(y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : y_1 : e^{i\alpha\theta} y_2 : e^{i\beta\theta} y_3)$ for $e^{i\theta} \in U(1)$, where α and β are relatively prime integers. Moreover, since the conics C_i's are $U(1)$-invariant, we can suppose $\alpha = 1$, $\beta = -1$. Thus the $U(1)$-action can be written in the form (ii) of the proposition.



Let $F = F(y_0, y_1, y_2, y_3)$ be a defining equation of B. Since B is $U(1)$-invariant, monomials appeared in F must be in the ideal $(y_2y_3, y_0^2, y_0y_1, y_1^2)^2$. Moreover, F contains the monomial $y_2^2y_3^2$, since otherwise the restriction onto $\{y_1 = 0\}$ would be the union of two different conics, which contradict to the fact that C_i is a trope. We assume that its coefficient is 1. Then F can be written in the form $(y_2y_3 + Q(y_0, y_1))^2 - q(y_0, y_1)$, where $Q(y_0, y_1) \in (y_0, y_1)^2$ and $q(y_0, y_1) \in (y_0, y_1)^4$ are uniquely determined polynomials with real coefficients. Then it again follows from C_i being a trope that $q(y_0, y_1) = ky_0y_1(y_0 + y_1)(ay_0 - by_1)$ for some constant $k \in \mathbf{R}^\times$. Finally, if k is negative, exchange y_0 and y_1. Then we get the equation of the form (1), and we have proved all of the claims in the proposition. $\square$

Next we study the singular locus of B.

Proposition 2.2. *Let B be a real quartic surface defined by the equation*

$$(y_2y_3 + Q(y_0, y_1))^2 - y_0y_1(y_0 + y_1)(ay_0 - by_1) = 0$$

where $Q(y_0, y_1)$ is a real quadratic form of y_0 and y_1, and a and b are real positive numbers. Let A be the set $\{(y_0 : y_1 : 0 : 0) \,|\, (y_0 : y_1)$ is a multiple root of the quartic equation $Q(y_0, y_1)^2 - y_0y_1(y_0 + y_1)(ay_0 - by_1) = 0\}$. (Here we think of this as an equation on $\mathbf{CP}^1 = \{(y_0 : y_1)\}$.) Then we have: (i) $\mathrm{Sing}(B) = \{P_\infty, \overline{P}_\infty\} \cup A$, where we put $P_\infty = (0 : 0 : 0 : 1)$, (ii) P_∞ and $\overline{P}_\infty$ are elliptic singularities of type $\tilde{E}_7$, and (iii) if $Q(y_0, y_1) \neq 0$, then $(y_0 : y_1 : 0 : 0) \in A$ is an ordinary double point iff its multiplicity is two.

In particular, every singular point of B is isolated.

Proof. (i) First we show that $(\mathrm{Sing}B) \cap \{y_3 \neq 0\} = \{P_\infty\}$, by calculating the Jacobian. Let $x_i = y_i/y_3$ $(0 \leq i \leq 2)$ be affine coordinates on $y_3 \neq 0$. Then the equation of B becomes $(x_2 + Q(x_0, x_1))^2 - x_0x_1(x_0 + x_1)(ax_0 - bx_1) = 0$. Differentiating with respect to x_2, we get $x_2 + Q(x_0, x_1) = 0$ so that we have $x_0x_1(x_0 + x_1)(ax_0 - bx_1) = 0$. Next differentiating with respect to x_0 and x_1 and then substituting $x_2 + Q(x_0, x_1) = 0$, we get $x_1(x_0 + x_1)(ax_0 - bx_1) + x_0x_1(ax_0 - bx_1) + ax_0x_1(x_0 + x_1) = 0$ and $x_0(x_0 + x_1)(ax_0 - bx_1) + x_0x_1(ax_0 - bx_1) - bx_0x_1(x_0 + x_1) = 0$. From the former, we obtain that $x_0 = 0$ implies $x_1 = 0$. Then by $x_2 + Q(x_0, x_1) = 0$ we have $x_2 = 0$. Similar argument shows that if $x_1, x_0 + x_1$ or $ax_0 - bx_1$ is zero, then $x_0 = x_1 = x_2 = 0$. Conversely, it is immediate to see that $(x_0, x_1, x_2) = (0, 0, 0)$ is a double point of B. Thus we get $(\mathrm{Sing}B) \cap \{y_3 \neq 0\} = \{P_\infty\}$. Because the given homogeneous polynomial is symmetric with respect to y_2 and y_3, we have $(\mathrm{Sing}B) \cap \{y_2 \neq 0\} = \{\overline{P}_\infty\}$.

Next we show that $(\mathrm{Sing}B) \cap \{y_2 = y_3 = 0\} = A$. We may suppose $y_1 \neq 0$. Putting $v_i = y_i/y_1$ for $i = 0, 2, 3$, the equation of B becomes $(v_2v_3 + Q(v_0, 1))^2 - f(v_0) = 0$, where we put $f(v_0) = v_0(v_0 + 1)(av_0 - b)$. Substituting $v_2 = v_3 = 0$, we get $Q(v_0, 1)^2 - f(v_0) = 0$. On the other hand, differentiating with respect to v_0 and substituting $v_2 = v_3 = 0$, we get $(Q(v_0, 1)^2 - f(v_0))' = 0$, where the prime denotes differential with respect to v_0. Thus, if $(v_0, v_2, v_3) = (\lambda_0, 0, 0)$ is a singular point of B, λ_0 is a multiple root of $Q(v_0, 1)^2 - f(v_0) = 0$. Conversely, it is easy to see that $(v_0, v_2, v_3) = (\lambda_0, 0, 0)$ is a singular point for such λ_0. Thus we get the claim of (i).

(ii) is obvious if one notes that we can use $(x_0, x_1, x_2 + Q(x_0, x_1))$ instead of (x_0, x_1, x_2) as a local coordinate around P_∞.

Finally we show (iii) by using the coordinate (v_0, v_2, v_3) above. Let λ_0 be a multiple root of $Q(v_0, 1)^2 - f(v_0) = 0$. Then our equation of B can be written $(v_2v_3 + 2Q(v_0, 1))v_2v_3 + g(v_0)(v_0 - \lambda_0)^2 = 0$, where $g(v_0)$ is a polynomial of degree two. Clearly λ_0 is a double root iff $g(\lambda_0) \neq 0$. Suppose $g(\lambda_0) \neq 0$ and define

$$\begin{aligned} w_1 &= \sqrt{g(v_0)} \cdot (v_0 - \lambda_0), \\ w_2 &= \sqrt{2Q(v_0, 1) + v_2v_3} \cdot v_2, \\ w_3 &= \sqrt{2Q(v_0, 1) + v_2v_3} \cdot v_3. \end{aligned} \tag{2}$$

Because $g(\lambda_0) \neq 0$ and $Q(\lambda_0, 1) \neq 0$, (w_1, w_2, w_3) is a local coordinate around $(\lambda_0, 0, 0)$. Then our equation of B becomes $w_2w_3 + w_1^2 = 0$. Thus the singularity is an ordinary double point. Conversely, if $g(\lambda_0) = 0$, it is immediate to see that our equation of B can be written of the form $w_2w_3 + w_1^3 = 0$ or $w_2w_3 + w_1^4 = 0$ depending on whether the multiplicity of λ_0 is three or four. This implies that $(\lambda_0, 0, 0)$ is not an ordinary double point. $\square$

Proposition 2.3. *Let B be as in Proposition 2.2. Put $f(\lambda) = \lambda(\lambda + 1)(a\lambda - b)$. Let $Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. Then if Z_0 admits a small resolution $Z \to Z_0$ such that Z is a twistor space of $3\mathbf{CP}^2$, then $Q(\lambda, 1)^2 - f(\lambda) \geq 0$ for any $\lambda \in \mathbf{R}$ and the equality holds for a unique $\lambda_0 \in \mathbf{R}$. Further, in this case, the multiplicity of λ_0 is two.*

Note that it follows from this proposition that $f(\lambda_0) > 0$ holds, because we have $Q(\lambda_0, 1)^2 = f(\lambda_0)$ and $Q(\lambda_0, 1)$ is a real number which is non-zero because otherwise the restriction $B|_{H_{\lambda_0}}$ would be $y_0 - \lambda_0 y_1 = (y_2 y_3)^2 = 0$ that yields another reducible fundamental divisor.

Proof. By results of Kreußler [8] and Kreußler-Kurke [9], we have $\sum(\mu(x) + c(x)) = 26$ for Z to be a twistor space of $3\mathbf{CP}^2$ for a topological reason, where $\mu(x)$ is the Milnor number of the singularity x of B and $c(x)$ is the number of irreducible components of a small resolution $Z \to Z_0$. Because elliptic singularity of type $\tilde{E}_7$ has $\mu = 9$ and $c = 3$, we get $\sum(\mu(x) + c(x)) = 2$ for other remaining singularities. This implies that there is only one singularity remaining, and that it must be an ordinary double point, which will be denoted by P_0. Therefore, by Proposition 2.2 (i), we have $A = \{P_0\}$. Namely, $Q^2 - f = 0$ has a unique multiple root λ_0. The multiplicity is two by Proposition 2.2 (iii). It is obvious from the uniqueness that this ordinary double point is real. Namely, λ_0 is real.

Next we show that other solutions of $Q(\lambda, 1)^2 - f(\lambda) = 0$ are not real. Assume $\lambda \in \mathbf{R}$, $\lambda \neq \lambda_0$ is a solution. Then by restricting B to the plane $y_0 = \lambda y_1$, we get $(y_2 y_3 + Q(\lambda, 1) y_1^2)^2 - f(\lambda) y_1^4 = (y_2 y_3)^2 + 2Q(\lambda, 1) y_2 y_3 y_1^2 + (Q(\lambda, 1)^2 - f(\lambda)) y_1^4 = y_2 y_3 (y_2 y_3 + 2Q(\lambda, 1) y_1^2)$ $(= 0)$. Therefore, the point $(\lambda : 1 : 0 : 0)$ is a real point of B. Since the multiplicity of the solution λ is one, Proposition 2.2 shows that this is a smooth point of B. This implies that Z has a real point, contradicting to the absence of real points on any twistor spaces. Hence the equation $Q(\lambda, 1)^2 - f(\lambda) = 0$ has no real solution other than λ_0. Because λ_0 is a solution whose multiplicity is two, this implies that the polynomial $Q(\lambda, 1)^2 - f(\lambda)$ has constant sign on $\mathbf{R} \backslash \{\lambda_0\}$. This sign must be clearly positive. $\square$

To investigate the real locus of B, we need the following elementary

Lemma 2.4. *Let $C_\alpha = \{y_2 y_3 = \alpha y_1^2\}$, $\alpha \in \mathbf{R}$ be a real conic in $\mathbf{CP}^2$, where the real structure is given by $(y_1 : y_2 : y_3) \mapsto (\overline{y}_1 : \overline{y}_3 : \overline{y}_2)$. Then C_α has no real point iff $\alpha < 0$.*

Proof. It is immediate to see that the real locus of C_α is

$$\left\{(1:v:\overline{v}) \in \mathbf{CP}^2 \,|\, |v| = \sqrt{\alpha}\right\}.$$

This is empty iff $\alpha < 0$.

Proposition 2.5. *Let B be as in Proposition 2.2 and suppose that the inequality $Q(\lambda,1)^2 - f(\lambda) \geq 0$ holds on $\mathbf{R}$ with the equality holding iff $\lambda = \lambda_0$ as in Proposition 2.3. Put $P_0 := (\lambda_0 : 1 : 0 : 0)$, which is clearly a real point of B. Then we have: (i) there is no real point on B other than P_0 iff the following condition is satisfied: if $f(\lambda) \geq 0$ and $\lambda \neq \lambda_0$, then $Q(\lambda,1) > \sqrt{f(\lambda)} \cdots (*)$, (ii) if $(*)$ is satisfied, then there is no real point on any small resolutions of Z_0.*

Proof. It is immediate to see that any real point of B is contained in some real plane $H_\lambda := \{y_0 = \lambda y_1\}$, $\lambda \in \mathbf{R} \cup \{\infty\}$. An equation of the restriction $B_\lambda := B \cap H_\lambda$ is given by (as in the proof of Proposition 2.3) $\left(y_2y_3 + Q(\lambda,1)y_1^2\right)^2 - f(\lambda)y_1^4 = 0$. This can be rewritten as

$$B_\lambda : \left\{y_2y_3 + \left(Q(\lambda,1) - \sqrt{f(\lambda)}\right)y_1^2\right\}\left\{y_2y_3 + \left(Q(\lambda,1) + \sqrt{f(\lambda)}\right)y_1^2\right\} = 0.$$

Namely, B_λ is a union of two conics. If $\lambda \neq \lambda_0$, we have $Q(\lambda,1)^2 - f(\lambda) > 0$ by our assumption, so both of $Q(\lambda,1) - \sqrt{f(\lambda)}$ and $Q(\lambda,1) + \sqrt{f(\lambda)}$ are non-zero.

Recall that our real structure is given by $\sigma(y_1 : y_2 : y_3) = (\overline{y}_1 : \overline{y}_3 : \overline{y}_2)$ on H_λ (Proposition 2.1, (ii)). Thus each component of B_λ is real iff the coefficients are real; namely $f(\lambda) \geq 0$. Further, the intersection of these two conics is $\{P_\infty, \overline{P}_\infty\}$. Therefore, there is no real point on B_λ if $f(\lambda) < 0$. So suppose $f(\lambda) \geq 0$. In this case, each of the two conics are real, and by Lemma 2.4, both components have no real point iff $Q(\lambda,1) > \sqrt{f(\lambda)}$. On the other hand, we have $B_\infty = \{(y_2y_3 + Q(y_0,0))^2 = 0\}$. Hence again by Lemma 2.4, we have $Q(1,0) > 0$ if B_∞ has no real point. But this follows from the first condition. If $\lambda = \lambda_0$, we have $Q(\lambda_0,1) = \sqrt{f(\lambda_0)}$ and hence one of the components of B_{λ_0} degenerates into a union of two lines whose intersection is P_0. And the other component has no real point since $Q(\lambda_0,1) + \sqrt{f(\lambda_0)} = 2\sqrt{f(\lambda_0)} > 0$. Thus we get (i).

Next we show that Z_0 has no real point other than $\Phi_0^{-1}(P_0)$, under the condition $(*)$. Suppose $y_1 \neq 0$ and use the coordinate (v_0, v_2, v_3) defined in the proof of Proposition 2.2. Then Z_0 is given by the equation $z^2 + (v_2v_3 + Q(v_0,1))^2 - f(v_0) = 0$, where z is a fiber coordinate on the line bundle $O(2)$ over $\mathbf{CP}^3$. Thus to prove $(Z_0)^\sigma = \Phi_0^{-1}(P_0)$ over

$y_1 \neq 0$, it suffices to show that

$$(v_2v_3 + Q(v_0,1))^2 - f(v_0) > 0 \tag{3}$$

for any real $(v_0, v_2, v_3) \neq (\lambda_0, 0, 0)$. Recall that (v_0, v_2, v_3) is real iff $v_0 \in \mathbf{R}$ and $v_2 = \overline{v}_3$. Hence (3) is obvious for real (v_0, v_2, v_3) with $f(v_0) < 0$. Assume $f(v_0) \geq 0$. Using the reality condition, we have $(v_2v_3 + Q(v_0,1))^2 - f(v_0) = |v_2|^4 + 2Q(v_0,1)|v_2|^2 + (Q(v_0,1)^2 - f(v_0))$. By our assumption we have $Q(v_0,1)^2 - f(v_0) > 0$ for any $v_0 \in \mathbf{R}$ with $v_0 \neq \lambda_0$. Further, by the condition $(*)$ we have $Q(v_0,1) > \sqrt{f(v_0)}$ for $v_0 \neq \lambda_0$ with $f(v_0) \geq 0$. Therefore we have (3) also for real $(v_0, v_2, v_3) \neq (\lambda_0, 0, 0)$ with $f(v_0) \geq 0$. Thus we have $(Z_0)^\sigma = \Phi_0^{-1}(P_0)$ over $y_1 \neq 0$. In the same way we can see that $(Z_0)^\sigma = \Phi_0^{-1}(P_0)$ over $y_0 \neq 0$. So it remains to see that there is no real point over the line $l_\infty = \{y_0 = y_1 = 0\}$. To check this, we introduce a new homogeneous coordinate $(y_0, y_1, y_2 - y_3, y_2 + y_3)$ on $\mathbf{CP}^3$. Then the two subsets $y_2 - y_3 \neq 0$ and $y_2 + y_3 \neq 0$ are real, and it can be easily seen that over the line l_∞, the equation of Z_0 is of the form $z^2 + q^2 = 0$ on these two open subset, where q is non-zero real valued on the real set l_∞^σ. Hence Z_0 does not have real point over l_∞. Thus we have $(Z_0)^\sigma = \Phi_0^{-1}(P_0)$.

Finally we show that $\Gamma_0 = \mu^{-1}(P_0)$ has no real point. To show this, we use a coordinate (w_0, w_2, w_3) (around P_0) defined in (2). Then B is given by $w_1^2 + w_2w_3 = 0$. Further, it is easy to see that the real structure is also given by $\sigma(w_1, w_2, w_3) = (\overline{w}_1, \overline{w}_3, \overline{w}_2)$. Now because Γ_0 is the exceptional curve of a small resolution of an ordinary double point, Γ_0 can be canonically identified with the set of lines contained in the cone $w_1^2 + w_2w_3 = 0$. If $\{(w_1 : w_2 : w_3) = (a_1 : a_2 : a_3)\}$ is a real line, we can suppose $a_1 \in \mathbf{R}$, $a_3 = \overline{a}_2$. It follows that it cannot be contained in the cone. This implies that Γ_0 has no real point. On the other hand, resolutions of the singularities over P_∞ and $\overline{P}_\infty$ do not yield real points. Thus we can conclude that Z has no real point. $\square$

Here we summarize necessary conditions for our threefolds to be (birational to) a twistor space:

Proposition 2.6. *Let Z, B, Q, a and b be as in Proposition 2.1, and put $f(\lambda) = \lambda(\lambda+1)(a\lambda - b)$ as in Proposition 2.3. Then we have: (i) $Q(\lambda,1)^2 - f(\lambda) \geq 0$ holds on $\mathbf{R}$ and the equality holds iff $\lambda = \lambda_0$, (ii) if $f(\lambda) \geq 0$ and $\lambda \neq \lambda_0$ then $Q(\lambda,1) > \sqrt{f(\lambda)}$.*

Proposition 2.1 has the following consequence:

Proposition 2.7. *Let g be a self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature with a non-trivial Killing field, and assume that g is not conformally isometric to LeBrun metric. Then the naturally induced*

$U(1)$-action on $3\mathbf{CP}^2$ is uniquely determined up to equivariant diffeomorphisms.

Proof. Let Z be the twistor space of g. Then Z is as in Proposition 2.1. Let H_i and C_i ($1 \le i \le 4$) be as in the proof of Proposition 2.1. Namely, H_i is a real $U(1)$-invariant plane such that $B|_{H_i}$ is a trope whose reduction is denoted by C_i. Then $\Phi_0^{-1}(H_i)$ consists of two irreducible components, both of which are biholomorphic to H_i ($\simeq \mathbf{CP}^2$). Since $\mu : Z \to Z_0$ is small, $\Phi^{-1}(H_i)$ also consists of two irreducible components, which are denoted by D_i and $\overline{D}_i$. The natural morphism $D_i \to H_i$ is clearly birational. We now claim that the set $\{D_i, \overline{D}_i\}_{i=1}^4$ of smooth $U(1)$-invariant divisors on Z is independent of the choice of small resolutions of Z_0. To see this, recall that if we use an affine coordinate (x_0, x_1, x_2) valid on $\{y_3 \neq 0\}$ as in the proof of Proposition 2.2, Z_0 is defined by $z^2 + (x_2 + Q(x_0, x_1))^2 - x_0x_1(x_0 + x_1)(ax_0 - bx_1) = 0$. So if we put $\xi = z + i(x_2 + Q(x_0, x_1))$ and $\eta = z - i(x_2 + Q(x_0, x_1))$, we get

$$Z_0 : \quad \xi\eta = x_0x_1(x_0 + x_1)(ax_0 - bx_1),$$

and the origin corresponds to a compound A_3-singularity $p_\infty := \Phi_0^{-1}(P_\infty)$. Then the irreducible components of $\Phi_0^{-1}(D_i)$ are defined by $\xi = \ell_i = 0$ and $\eta = \ell_i = 0$, where ℓ_i is one of $x_0, x_1, x_0 + x_1$ and $ax_0 - bx_1$. From this, and from the explicit description of small resolutions which will be explained in §5.3, we can deduce that both of the birational morphisms $D_i \to H_i$ and $\overline{D}_i \to H_i$ are the composition of three blowing-ups, and that each blowing-up is always performed at just one of the two $U(1)$-fixed points on the proper transforms of C_i (or $\Phi_0^{-1}(C_i)$, more precisely). Evidently there are just $2^3 = 8$ choices of blowing-ups satisfying this property in all, and the set of the resulting surfaces is just $\{D_i, \overline{D}_i\}_{i=1}^4$. Thus $\{D_i, \overline{D}_i\}_{i=1}^4$ is independent of the choice of a small resolution of p_∞.

Since $D_i + \overline{D}_i$ is a fundamental divisor, and since we have $-(1/2)K_Z \cdot L = 2$ (L is a twistor line), we have $D_i \cdot L = \overline{D}_i \cdot L = 1$. Hence by a result of Poon [13], $L_i := D_i \cap \overline{D}_i$ is a twistor line which is obviously $U(1)$-invariant, and that L_i is contracted to a point by the twistor fibration $Z \to 3\mathbf{CP}^2$ which is $U(1)$-equivariant. Hence the $U(1)$-action on $3\mathbf{CP}^2$ can be read from that on $3\mathbf{CP}^2$. Therefore the conclusion of the proposition follows. $\square$

The proposition implies that, up to equivariant diffeomorphisms, there are just two effective $U(1)$-actions on $3\mathbf{CP}^2$ which can be the identity component of the isometry group of a self-dual metric whose scalar curvature is positive. One is the semi-free $U(1)$-action, which is the identity component of generic LeBrun metric, and the other is the action obtained in Proposition 2.7. Of course, there are many other differentiable $U(1)$-actions: for example, we can get an infinite number of mutually inequivariant $U(1)$-actions by first taking an effective $U(1) \times U(1)$-action on $3\mathbf{CP}^2$ and then choosing a $U(1)$-subgroup of $U(1) \times U(1)$.

Proposition 2.8. *Let B be the quartic surface defined by the equation (1) and suppose that Q and f satisfy the assumption in Proposition 2.5. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B, and $\mu : Z \to Z_0$ any small resolution (which exists by Propositions 2.2 and 2.3). Put $\Phi = \mu \cdot \Phi_0$. Then we have (i) $K_Z \simeq \Phi^* O(-2)$, (ii) the line bundle $(-1/2)K_Z$ is uniquely determined.*

Proof. Let K_{Z_0} denote the canonical sheaf of Z_0. Then we have $K_{Z_0} \simeq \Phi_0^*(K_{\mathbf{CP}^3} + (1/2)O(B)) \simeq \Phi_0^* O(-2)$. Moreover, since μ is small, we have $K_Z \simeq \Phi^* K_{Z_0}$. Hence we get $K_Z \simeq \Phi^* O(-2)$. For (ii) it suffices to show that $H^1(O_Z) = 0$. Since the singularities of Z_0 are normal, and since the exceptional curves of μ are rational, we get by Leray spectral sequence $H^1(O_Z) \simeq H^1(O_{Z_0})$. Then applying the spectral sequence to Φ_0, and using $\Phi_{0*} O_{Z_0} \simeq O \oplus O(-2)$ and $R^q \Phi_{0*} O_{Z_0} = 0$ for $q \geq 1$, we get $H^1(O_{Z_0}) = 0$. □

3. Defining equations of real touching conics

Let Z, $\Phi : Z \to \mathbf{CP}^3$, $B \subset \mathbf{CP}^3$, $\Phi_0 : Z_0 \to \mathbf{CP}^3$ and $\mu : Z \to Z_0$ have the meaning of the previous section. Recall that B has just three singular points which are denoted by P_0, P_∞ and $\overline{P}_\infty$, where P_0 is a real ordinary double point, and P_∞ and $\overline{P}_\infty$ are conjugate pair of elliptic singularities of type $\tilde{E}_7$. Set $\Gamma_0 = \mu^{-1}(P_0)$, which is a real smooth rational curve. In this section we first study the images of real lines under the map Φ. (We do not assume that Z is a twistor space.)

Definition 3.1. *A real irreducible conic C in $\mathbf{CP}^3$ is called a real touching conic of B if $C \subset B$ or if the intersection number with B is at least two for any intersection points.*

Proposition 3.2. *Let L be a real line in Z, where line means a smooth rational curve whose normal bundle in Z is isomorphic to $O(1)^{\oplus 2}$. Then $\Phi(L)$ is a line in $\mathbf{CP}^3$ iff $L \cap \Gamma_0 \neq \phi$. Otherwise $\Phi(L)$ is a real touching conic of B.*

Proof. By adjunction formula, we have $-2 = K_Z \cdot L + \deg N_{L/Z} = K_Z \cdot L + 2$. Hence we have $(-1/2)K_Z \cdot L = 2$. Therefore $\Phi(L)$ is a curve whose degree is at most two, and $\Phi(L)$ is a line iff $\Phi|_L : L \to \Phi(L)$

is two-to-one. Assume that $\Phi(L)$ is a line, which is necessarily real. Consider the pencil of planes whose base locus is $\Phi(L)$. By Bertini's theorem, general member of this pencil is singular precisely when $\Phi(L)$ goes through the singular point of B. If $P_0 \notin \Phi(L)$, $\Phi(L)$ is the line joining P_∞ and $\overline{P}_\infty$. Namely, $\Phi(L) = l_\infty = \{y_0 = y_1 = 0\}$. As we have already seen in the proof of Proposition 2.5, there is no real point on $\Phi_0^{-1}(l_\infty)$ and hence so is $\Phi^{-1}(l_\infty)$. Therefore $\Phi^{-1}(l_\infty)$ has no real components. But because we have chosen real L, this is a contradiction. Hence we have $P_0 \in \Phi(L)$. It follows that $L \cap \Gamma_0 \neq \phi$. Conversely assume that L is a real line intersecting Γ_0. Then since there are no real points on Γ_0 (Proposition 2.5 (ii)), the intersection is not one point. Because $\Phi(\Gamma_0) = P_0$, this implies that Φ is not one-to-one on L. Hence $\Phi(L)$ must be a line.

Finally suppose that $\Phi(L)$ is a conic. If $(\Phi(L), B)_P = 1$ for some $P \in \Phi(L) \cap B$, then P is a smooth point of B and the intersection is transversal. Therefore $\Phi^{-1}(\Phi(L))$ is locally irreducible near $\Phi^{-1}(P)$. This contradicts to the fact that $\Phi|_L$ is bijective. Therefore we have $(\Phi(L), B)_P \geq 2$ for any $P \in \Phi(L) \cap B$, which implies that $\Phi(L)$ is a touching conic of B, which is necessarily real. $\square$

In general, it seems to be impossible to determine the defining equations of touching conics of a smooth quartic curve. Namely, it seems to be hopeless to write down the equation of touching conics contained in a general plane. In the rest of this section we determine defining equations of real touching conics contained in real $U(1)$-invariant planes. By Proposition 2.1 (iii) such a plane is of the form $H_\lambda = \{y_0 = \lambda y_1\}$ or $H_\infty = \{y_1 = 0\}$, where λ is a real number. As seen in the proof of Proposition 2.5, $B_\lambda = B \cap H_\lambda$ is a union of two $U(1)$-invariant conics, and their intersection is P_∞ and $\overline{P}_\infty$.

Let C be a real touching conic contained in H_λ. Since the two components of B_λ have the same tangent lines at P_∞ and $\overline{P}_\infty$, the local intersection number $(C, B_\lambda)_{P_\infty}$ is zero, two, or four, and by reality, the same holds for $(C, B_\lambda)_{\overline{P}_\infty}$. Correspondingly, real touching conics can be classified into the following three types:

Definition 3.3. Let C be a real touching conic in H_λ. Then (i) C is called *generic type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 0$. (ii) C is called *special type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 2$. (iii) C is called *orbit type* if $(C, B_\lambda)_{P_\infty} = (C, B_\lambda)_{\overline{P}_\infty} = 4$.

If C is a real touching conic of generic type, then $P_\infty, \overline{P}_\infty \notin C$ and $C \cap B_\lambda$ consists of just four points, all satisfying $(C, B_\lambda)_P = 2$. If C is a real touching conic of special type, then C goes through

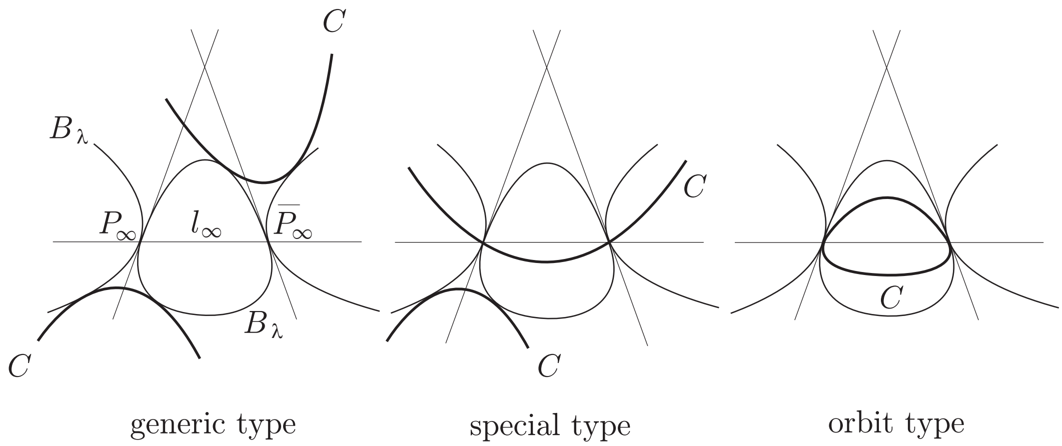


FIGURE 1. real touching conics (C is irreducible)

P_∞ and $\overline{P}_\infty$ but the tangent lines at P_∞ and $\overline{P}_\infty$ are different from those of B_λ. Further, there are other intersection P and $\overline{P}$ satisfying $(C, B_\lambda)_P = (C, B_\lambda)_{\overline{P}} = 2$. If C is a real touching conic of orbit type, there are no other intersection points. In this case, C is the closure of $\mathbf{C}^*$-action, where $\mathbf{C}^*$-action is the complexification of $U(1)$-action.

First we classify real touching conics of generic type:

Proposition 3.4. *(i) If $f(\lambda) > 0$ and $Q(\lambda, 1)^2 > f(\lambda)$, there exists a family of real touching conics of generic type on H_λ, parametrized by a circle. Their defining equations are explicitly given by*

$$2(Q^2 - f)y_1^2 + \sqrt{f}e^{i\theta}y_2^2 + 2Qy_2y_3 + \sqrt{f}e^{-i\theta}y_3^2 = 0, \tag{4}$$

where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$, and $\theta \in \mathbf{R}$. Further, every real touching conic of generic type in H_λ is a member of this family. (ii) If $f(\lambda) < 0$ or $Q(\lambda, 1)^2 = f(\lambda)$, there is no real touching conic of generic type on H_λ. (iii) If $Q(\lambda, 1)^2 > f(\lambda) > 0$, the conic (4) has no real point for any $\theta \in \mathbf{R}$.

We note that $U(1)$ acts transitively (but non-effectively) on the space of these touching conics. To prove the proposition, we need the following

Lemma 3.5. *An equation $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$ has two double roots iff (i) if $a_1 \neq 0$, then $4a_1a_2 = a_1^3 + 8a_3$ and $a_1^2a_4 = a_3^2$ hold, (ii) if $a_1 = 0$, then $a_3 = 0$ and $4a_4 = a_2^2$ hold.*

Proof. Suppose first that the equation has two double roots α and β. Then we have $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = (x - \alpha)^2(x - \beta)^2$. Expanding the right hand side, and comparing the coefficients, we get $a_1 = -2(\alpha+\beta)$, $a_2 = \alpha^2+\beta^2+4\alpha\beta$, $a_3 = -2\alpha\beta(\alpha+\beta)$, and $a_4 = \alpha^2\beta^2$. From the first one, we get $\alpha + \beta = -a_1/2$. If $a_1 \neq 0$, we further get $\alpha\beta = a_3/a_1$. Hence we have $a_2 = (\alpha + \beta)^2 + 2\alpha\beta = a_1^2/4 + 2a_3/a_1$ and $a_4 = (\alpha\beta)^2 = a_3^2/a_1^2$. From these, we get the equalities of (i). If $a_1 = 0$, it immediately follows from $\alpha + \beta = 0$ that $a_3 = 0$, $a_2 = -2\alpha^2$ and $a_4 = \alpha^4$. Hence we get the equalities of (ii). Thus we obtain the necessity. Conversely, assume $a_1 \neq 0$ and the equalities of (i) holds. Put

$$\alpha = -\frac{a_1}{4} + \sqrt{\frac{a_1^2}{16} - \frac{a_3}{a_1}}, \qquad \beta = -\frac{a_1}{4} - \sqrt{\frac{a_1^2}{16} - \frac{a_3}{a_1}}.$$

Then it is a straightforward calculation to see that $(x - \alpha)^2(x - \beta)^2 = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ under our assumptions. Thus the given equation has two double roots α and β. Finally assume $a_1 = a_3 = 0$ and $4a_4 = a_2^2$. Then we have $x^4+a_1x^3+a_2x^2+a_3x+a_4 = x^4+a_2x^2+(a_2^2/4) = (x^2 + (a_2/2))^2$. Hence the equation has two double roots. $\square$

Proof of Proposition 3.4. Let $ay_1^2+by_1y_2+cy_2^2+dy_1y_3+ey_2y_3+hy_3^2 = 0$ be a defining equation of C in H_λ. Then since $B_\lambda \cap \{y_3 = 0\} = \{\overline{P}_\infty\}$ and since C is assumed to be generic type, all of the touching points are on $\{y_3 \neq 0\}$. Putting $x_1 = y_1/y_3$ and $x_2 = y_2/y_3$ on $\{y_3 \neq 0\}$ as before, C is defined by

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + h = 0 \tag{5}$$

and B_λ is defined by (as in the proof of Proposition 2.5)

$$\left(x_2 + \left(Q - \sqrt{f}\right) x_1^2\right) \left(x_2 + \left(Q + \sqrt{f}\right) x_1^2\right) = 0.$$

Let g denote $g_+ := Q + \sqrt{f}$ or $g_- := Q - \sqrt{f}$. Substituting $x_2 = -gx_1^2$ into (5), we get

$$g^2cx_1^4 - gbx_1^3 + (a - ge)x_1^2 + dx_1 + h = 0. \tag{6}$$

If $c = 0$, (6) cannot have two double roots, so we have $c \neq 0$. Suppose $b \neq 0$. Then by Lemma 3.5 (i), (6) has two double roots iff

$$-\frac{4b}{cg} \cdot \frac{a - eg}{cg^2} = -\frac{b^3}{c^3g^3} + \frac{8d}{cg^2} \quad \text{and} \quad \left(-\frac{b}{cg}\right)^2 \cdot \frac{h}{cg^2} = \frac{d^2}{c^2g^4}$$

hold. These can be respectively written

$$4bc(a - eg) = b^3 - 8gc^2d, \quad b^2h = cd^2. \tag{7}$$

Namely, a conic (5) with $b \neq 0$ is a touching conic of generic type iff (7) is satisfied for both of $g = g_+$ and $g = g_-$. In this case, simple calculations show that $4ac = b^2$ and $4ah = d^2$ and $2ae = bd$. From these we get $a = b^2/4c, e = 2cd/b$ and $h = cd^2/b^2$. Substituting these into (5), we get $(bx_1 + 2cx_2 + 2cd/b)^2 = 0$. This implies that C is a double line. Thus contradicting our assumption and we get $b = 0$. Then by Lemma 3.5 (ii), we have $d = 0$ and

$$4g^2ch = (a - ge)^2. \tag{8}$$

If we regard (8) as a homogeneous equation about $(a : c : e : h) \in \mathbf{CP}^3$, (8) is a quadratic cone whose vertex is $(a : c : e : h) = (g : 0 : 1 : 0)$. We need to get the intersection of these two quadrics. Restricting (8) onto the plane $a = \kappa e$, we get

$$4g^2ch = (\kappa - g)^2e^2, \ g = g_{\pm}. \tag{9}$$

It is readily seen that these two conics (for the case $g = g_+$ and $g = g_-$) coincide iff $\kappa = 0$ or $\kappa = (Q^2 - f)/Q$. If $\kappa = 0$, we have $a = 0$, so (5) becomes $cx_2^2 + ex_2 + h = 0$, where the coefficients are subjected to $4ch = e^2$. This implies that C is a union of two lines, contradicting our assumption. Hence we have $\kappa = (Q^2 - f)/Q$. Then (9) becomes

$$4Q^2ch = fe^2. \tag{10}$$

If $e = 0$, it follows that $h = 0$, and hence (5) will again be a union of lines. Therefore we have $e \neq 0$. It is easily seen that the real structure on the space of coefficients is given by $(a : c : e : h) \mapsto (\overline{a} : \overline{h} : \overline{e} : \overline{c})$. Hence if $f = f(\lambda) < 0$, (10) cannot hold for real $(a : c : e : h)$. Namely, on H_λ, there does not exist a real touching conic of generic type if $f(\lambda) < 0$. This implies (ii) of the proposition for the case $f(\lambda) < 0$. If $f = f(\lambda) > 0$, putting $e = 1$ and $h = \overline{c}$ in (10) we get $4Q^2|c|^2 = f$. Hence we can write

$$c = \frac{\sqrt{f}}{2Q}e^{i\theta}$$

for some $\theta \in \mathbf{R}$. Further we have $a = (Q^2 - f)/Q$. Substituting these into (5), we get (4). Then it is immediate to see that the determinant of the matrix defining (4) is $-2(Q^2 - f)^2$. Therefore the conic (4) is irreducible iff $Q^2 - f \neq 0$. Thus we get (i), and also (ii) for the case $Q(\lambda)^2 = f(\lambda)$.

Finally we show (iii). Recall that the real structure on H_λ is given by $(y_1 : y_2 : y_3) \mapsto (\overline{y}_1 : \overline{y}_3 : \overline{y}_2)$. Hence if $(y_1 : y_2 : y_3) \in H_\lambda$ is a real point, we can suppose $y_1 \in \mathbf{R}$ and $y_3 = \overline{y}_2$. Substituting these into (4), we get

$$(Q^2 - f)y_1^2 + Q|y_2|^2 + \sqrt{f} \cdot \mathrm{Re}(e^{i\theta}y_2^2) = 0, \tag{11}$$

where $\mathrm{Re}(z)$ denotes the real part of z. From this it follows that $y_2 = 0$ implies $y_3 = y_1 = 0$, so we have $y_2 \neq 0$. Then recalling that $Q > \sqrt{f}$ (Proposition 2.5 (i)) we have $Q|y_2|^2 > \sqrt{f}|y_2|^2$. Also we have $\mathrm{Re}(e^{i\theta}y_2^2) \leq |y_2|^2$. Therefore we have

$$\begin{aligned}(Q^2 - f)y_1^2 + Q|y_2|^2 + \sqrt{f} \cdot \mathrm{Re}(e^{i\theta}y_2^2) &> (Q^2 - f)y_1^2 + \sqrt{f}|y_2|^2 - \sqrt{f}|y_2|^2 \\ &= (Q^2 - f)y_1^2 \geq 0\end{aligned}$$

This implies that (11) does not hold for any real $(y_1 : y_2 : y_3)$ and $\theta \in \mathbf{R}$. Thus there is no real point on the conic (4) provided $Q^2 > f > 0$, and we get (iii). □

Next we classify real touching conics of special type:

Proposition 3.6. *(i) If $f(\lambda) > 0$, there is no real touching conic of special type on H_λ. (ii) If $f(\lambda) < 0$, there exists a family of real touching conics of special type, parametrized by a circle. Their defining equations are given by*

(12)
$$\sqrt{Q^2 - f}\cdot y_1^2 + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}}\cdot e^{i\theta}y_1y_2 + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}}\cdot e^{-i\theta}y_1y_3 + y_2y_3 = 0,$$

where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$, and $\theta \in \mathbf{R}$ as before. Further, every real touching conic of special type in H_λ is a member of this family. (iii) The conic (12) has no real point for any $\theta \in \mathbf{R}$.

Note again that $U(1)$ acts transitively on the parameter space of these touching conics. Also note that if $f < 0$ we have $Q^2 - f > 0$ and $\sqrt{Q^2 - f} - Q > 0$. Hence every square root in the equation make a unique sense (i.e. we always take the positive root).

Proof. Let C be a real touching conic of special type on H_λ. Then since C goes through P_∞ and $\overline{P}_\infty$, the other two touching points belong to mutually different irreducible component of B_λ. On the other hand, as shown in the proof of Proposition 2.5, each irreducible components of B_λ is real iff $f(\lambda) \geq 0$. Therefore, on H_λ, there does not exist real touching conic of special type if $f(\lambda) > 0$. Thus we get (i). So in the sequel we suppose $f(\lambda) < 0$.

It is again readily seen that touching points are not on the line $\{y_3 = 0\}$. So we still use (x_1, x_2) as a non-homogeneous coordinate on $\mathbf{C}^2 = \{y_3 \neq 0\} \subset H_\lambda$. Then because C contains P_∞ and $\overline{P}_\infty$, an equation of a touching conic C of special type is of the form

$$ax_1^2 + bx_1x_2 + dx_1 + ex_2 = 0. \tag{13}$$

Substituting $x_2 = -gx_1^2$ into (13), we get

$$x_1 \cdot \left(gbx_1^2 + (ge-a)x_1 - d\right) = 0. \tag{14}$$

If $d = 0$, $x_1 = 0$ is a double root of (14). Then the tangent line of C at $P_\infty = (0,0)$ becomes $x_2 = 0$, as is obvious from (13). This implies that C is a touching conic of orbit type, contradicting our assumption.Therefore, we have $d \neq 0$. (14) has a double root other than $x_1 = 0$ iff

$$(ge-a)^2 + 4gbd = 0. \tag{15}$$

Namely (13) is a touching conic of special type iff (15) is satisfied for both of $g = g_+$ and $g = g_-$. (Note that $g_+ \neq g_-$.) If we regard (15) as a homogeneous equation of $(a : b : d : e) \in \mathbf{CP}^3$, (15) is a quadratic cone whose vertex is $(a : c : d : e) = (g : 0 : 0 : 1)$. That is, the parameter space of touching conics of special type is the intersection of two quadratic cones in $\mathbf{CP}^3$. Restricting (15) onto the plane $a = \kappa e$, we get

$$(g-\kappa)^2 e^2 + 4gbd = 0, \;\; g = g_\pm.$$

It is readily seen that these two conics coincide iff $\kappa = \pm\sqrt{g_+ g_-} = \pm\sqrt{Q^2 - f}$. Therefore C is a touching conic of special type iff

$$a = \sqrt{Q^2-f} \cdot e, \;\; \left(Q - \sqrt{Q^2-f}\right) e^2 + 2bd = 0 \tag{16}$$

or

$$a = -\sqrt{Q^2-f} \cdot e, \;\; \left(Q + \sqrt{Q^2-f}\right) e^2 + 2bd = 0 \tag{17}$$

hold. It is easily seen that the real structure on the space of coefficients is given by $(a : b : d : e) \mapsto (\overline{a} : \overline{d} : \overline{b} : \overline{e})$.

Since we have assumed $f < 0$, we have $Q + \sqrt{Q^2 - f} > 0$ and there is no real conic satisfying (17). On the other hand, we have $Q - \sqrt{Q^2 - f} < 0$. Hence by (16) we have

$$2|b|^2 = \left(\sqrt{Q^2-f} - Q\right) e^2,$$

where $b \in \mathbf{C}$ and $e \in \mathbf{R}$. If $e = 0$, then $b = a = 0$, contradicting the assumption that C is a conic. Hence $e \neq 0$, and we may put $e = 1$. Then we can write

$$b = \sqrt{\frac{\sqrt{Q^2-f} - Q}{2}} \cdot e^{i\theta}$$

for some $\theta \in \mathbf{R}$. Also we have $d = \overline{b}$. Thus we obtain (12) of the proposition.

Finally we show (iii). If $(y_1 : y_2 : y_3)$ is a real point of H_λ, we can suppose $y_1 \in \mathbf{R}$ and $y_3 = \overline{y}_2$. Substituting these into (12), we get

$$\sqrt{Q^2 - f} \cdot y_1^2 + \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 \cdot \mathrm{Re}(e^{i\theta} y_2) + |y_2|^2 = 0. \tag{18}$$

If $y_1 = 0$, it follows $y_2 = y_3 = 0$. Hence $y_1 \neq 0$ and we can suppose $y_1 > 0$. Then we have $y_1 \mathrm{Re}(e^{i\theta} y_2) \geq -y_1 |y_2|$. Hence we have

$$\begin{aligned}
& \sqrt{Q^2 - f} \cdot y_1^2 + \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 \mathrm{Re}(e^{i\theta} y_2) + |y_2|^2 \\
\geq \;& \sqrt{Q^2 - f} \cdot y_1^2 - \sqrt{2\left(\sqrt{Q^2 - f} - Q\right)} \cdot y_1 |y_2| + |y_2|^2 \\
= \;& \left(|y_2| - \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot y_1 \right)^2 + \frac{\sqrt{Q^2 - f} + Q}{2} \cdot y_1^2
\end{aligned}$$

Because $y_1 \neq 0$ and $f < 0$, we have $(\sqrt{Q^2 - f} + Q) y_1^2 > 0$. Therefore, the left hand side of (18) is strictly positive. Thus (18) does not hold for any real $(y_1 : y_2 : y_3) \in H_\lambda$ and any $\theta \in \mathbf{R}$. Therefore the conic (12) has no real point for any $\theta \in \mathbf{R}$.

It is immediate to see that the determinant of the matrix defining (12) is $-(Q + \sqrt{Q^2 - f})/8$ and this is negative if $f < 0$. Hence the conic (12) is irreducible $\square$

The case of orbit type is straightforward and need no assumption on the sign of $f(\lambda)$:

Proposition 3.7. *There exists a family of real touching conics of orbit type, parametrized by non-zero real numbers. Their defining equations are*

$$y_2 y_3 = \alpha y_1^2, \ \alpha \in \mathbf{R}^\times. \tag{19}$$

Further, every real touching conic of orbit type in H_λ is contained in this family.

Note that by Lemma 2.4, the conic (19) has no real point iff $\alpha < 0$.

Combining Propositions 3.4, 3.6 and 3.7, we get the following

Proposition 3.8. *Let $\{S_\lambda := \Phi^{-1}(H_\lambda) \,|\, H_\lambda \in \langle l_\infty \rangle^\sigma\}$ be the real members of the pencil of $U(1)$-invariant divisors on Z. Then (i) if $f(\lambda) > 0$, the images of real lines in S_λ are real touching conics of generic type or orbit type, (ii) if $f(\lambda) < 0$, the images of real lines in S_λ are real touching conics of special type or orbit type.*

The ambiguity of the types in this proposition will be excluded in Section 5 (Theorem 5.20), for the images of twistor lines.

4. The inverse images of real touching conics

According to the previous section, real touching conics in fixed H_λ form families parametrized by a circle for generic and special types, or $\mathbf{R}^\times$ for orbit type. In this section we study the inverse images of these touching conics in Z, which are candidates of twistor lines. We begin with the following

Proposition 4.1. *If $\lambda \in \mathbf{R}$ and if $f(\lambda) \neq 0$, then $S_\lambda := \Phi^{-1}(H_\lambda)$ is a smooth rational surface with $c_1^2 = 2$, and is a real $U(1)$-invariant member of $|(-1/2)K_Z|$.*

Proof. Smoothness of S_λ can be checked by giving all small resolutions of the singular threefold Z_0 (which is the double cover branched along B). Indeed, if $f(\lambda) \neq 0$, P_∞ and $\overline{P}_\infty$ are A_3-singularities of the curve $B_\lambda = H_\lambda \cap B$, so that the surface $\Phi_0^{-1}(H_\lambda)$ has A_3-singularities over there, which are minimally resolved through small resolutions of Z_0. If $\lambda = \lambda_0$, $P_0 = (\lambda_0 : 1 : 0 : 0)$ is a node of B_{λ_0}, but this is also resolved by small resolutions of the corresponding ordinary double points of Z_0. See §5.3 and §6, where all resolutions of singularities of Z_0 are concretely given. In order to see the structure of S_λ, note that S_λ is a smooth deformation of a surface $\Phi^{-1}(H)$, where H is a general plane in $\mathbf{CP}^3$. It is easy to that $\Phi^{-1}(H)$ is a smooth rational surface with $c_1^2 = 2$, because $B \cap H$ is a smooth quartic curve for general H. From this it follows the same properties for S_λ,since they can be connected by smooth deformations. Then by Proposition 2.8 we have $S_\lambda \in |\Phi^* O(1)| = |(-1/2)K_Z|$. The reality and the $U(1)$-invariance of S_λ is obvious from our choice of H_λ. □

Next we investigate the inverse images of touching conics of generic type.

Proposition 4.2. *Suppose $f(\lambda) > 0$ and $\lambda \neq \lambda_0$ (namely $Q^2 - f > 0$), and let $C_\theta \subset H_\lambda$ be a real touching conic of generic type defined by the equation (4). Then the following (i)–(iv) hold: (i) $\Phi^{-1}(C_\theta)$ has just two irreducible components, both of which are smooth rational curves that are mapped biholomorphically onto C_θ, (ii) each irreducible component of $\Phi^{-1}(C_\theta)$ has a trivial normal bundle in S_λ, (iii) these two irreducible components of $\Phi^{-1}(C_\theta)$ belong to mutually different pencils on S_λ, (iv) each irreducible component of $\Phi^{-1}(C_\theta)$ is real.*

Proof. Since C_θ and the branch quartic B_λ have the same tangent line at any intersection points, it is obvious that $\Phi_0^{-1}(C_\theta)$ splits into two irreducible components L_1 and L_2 which are mapped biholomorphically onto C_θ. Thus we get (i). For (ii) first note that $\Phi^{-1}(C_\theta) = L_1 + L_2$ belongs to $|-2K|$ of S_λ since we have $\Phi^* O_{H_\lambda}(1) \simeq -K$. Hence we have $(-2K)^2 = (L_1+L_2)^2 = L_1^2 + L_2^2 + 2L_1L_2$ on S_λ. On the other hand, we have $4c_1^2 = 8$ by Proposition 4.1. Hence we get $L_1^2 + L_2^2 + 2L_1L_2 = 8$. Further, since L_1 and L_2 intersect transversally at four points (over the touching points of C_θ with B_λ), we have $L_1L_2 = 4$. Therefore we get $L_1^2 + L_2^2 = 0$. Moreover, by (4), C_θ actually moves in a holomorphic family of curves on H_λ. Hence we have $L_1^2 \geq 0$ and $L_2^2 \geq 0$. Therefore we get $L_1^2 = L_2^2 = 0$. Namely we have (ii). (iii) immediately follows from (ii), since we have $L_1L_2 = 4$ on S_λ. (iv) is harder than one may think at first glance, since there is no real point on C_θ. First we note that it suffices to prove the claim for C_0 (= the curve obtained by setting $\theta = 0$ for C_θ), since $U(1)$ acts transitively on the parameter space of real touching conics of generic type (see Proposition 3.4). The idea of our proof of the reality is as follows: the map $\Phi^{-1}(C_0) \to C_0$ is finite, two sheeted covering whose branch consists of four points. We choose a real simple closed curve $\mathcal{C}$ in C_0 containing all of these branch points, in such a way that over $\mathcal{C}$ we can distinguish two sheets, so that we can explicitly see the reality of each irreducible components. To this end, we still use $(y_1 : y_2 : y_3)$ as a homogeneous coordinate on H_λ and set $U := \{y_1 \neq 0\}$ ($\simeq \mathbf{C}^2$), which is clearly a real subset of H_λ, and use $(v_2, v_3) = (y_2/y_1, y_3/y_1)$ as an affine coordinate on U. Then $Z_0|_U = \Phi_0^{-1}(U)$ is defined by the equation

$$z^2 + (v_2v_3 + Q)^2 - f = 0, \tag{20}$$

where z is a fiber coordinate of $O(2)$, and $Q = Q(\lambda, 1)$, $f = f(\lambda)$ as in the previous section. The real structure is given by $(v_2, v_3, z) \mapsto (\overline{v}_3, \overline{v}_2, \overline{z})$. Then on U, our equation (4) of C_0 becomes $2(Q^2 - f) + \sqrt{f}v_2^2 + 2Qv_2v_3 + \sqrt{f}v_3^2 = 0$. Now we introduce a new coordinate $(u, v) := (v_2 + v_3, v_2 - v_3)$ which is valid on U. Then our real structure is given by $(u, v) \mapsto (\overline{u}, -\overline{v})$, and the defining equation of C_0 becomes $4(Q^2 - f) + \sqrt{f}(u^2 + v^2) + Q(u^2 - v^2) = 0$. From this we immediately have

$$C_0 : \quad v^2 = 4\left(Q + \sqrt{f}\right) + \frac{Q + \sqrt{f}}{Q - \sqrt{f}}u^2. \tag{21}$$

We put $V := \{(u, v) \in U \mid u \in i\mathbf{R}, v \in \mathbf{R}\}$ which is clearly a real subset of U. Then $\mathcal{C} := V \cap C_0$ is a real simple closed curve (an ellipse) in $V \simeq \mathbf{R}^2$. Indeed, putting $u = iw$ ($w \in \mathbf{R}$), we get from (21)

$$C_0 : \quad v^2 + \frac{Q + \sqrt{f}}{Q - \sqrt{f}}w^2 = 4\left(Q + \sqrt{f}\right).$$

(Note that $Q - \sqrt{f} > 0$ by our assumption $f > 0$ and Proposition 2.6.) Substituting $v_2 v_3 = (u^2 - v^2)/4$ into (20), and then using (21), we get

$$(22)\qquad \Phi^{-1}(C_0): \quad 4\left(Q - \sqrt{f}\right)^2 z^2 + fu^2\left(u^2 + 4\left(Q - \sqrt{f}\right)\right) = 0,$$

or, using w above,

$$(23)\qquad \Phi^{-1}(C_0): \quad 4\left(Q - \sqrt{f}\right)^2 z^2 = fw^2\left(4\left(Q - \sqrt{f}\right) - w^2\right).$$

Here note that in (21) u can be used as a coordinate on C_0, only outside the two branch points of $C_\lambda \cap U \to \mathbf{C}$ defined by $(u, v) \mapsto u$. In a neighborhood of these branch points, we have to use v instead of u as a local coordinate on C_0. Then we can see that the inverse image of a neighborhood of $u = \pm 2i(Q - \sqrt{f})^{\frac{1}{2}}$ (i.e. the branch points) also splits into two irreducible components, which is of course as expected. From (22), we easily deduce that the branch points of $\Phi^{-1}(C_0) \to C_0$ are $(u, v) = (0, \pm 2(Q + \sqrt{f})^{\frac{1}{2}})$ and $(u, v) = (\pm 2i(Q - \sqrt{f})^{\frac{1}{2}}, 0)$. All of these four points clearly lie on $\mathcal{C}$, and $\mathcal{C}$ is divided into four segments. It immediately follows from (23) that z always takes real value over $\mathcal{C}$. Moreover, it is clear that the sign of z is constant on each of the four segments in $\mathcal{C}$, and that the sign changes when passing though the branch points. On the other hand, since the real structure is given by $(w, v) \mapsto (-w, -v)$ on V, the real structure on $\mathcal{C}$ sends each segment to another segment which is not adjacent to the original one. From these, and because the real structure on $\Phi^{-1}(\mathcal{C})$ is given by $(w, z) \mapsto (-w, \overline{z}) = (-w, z)$, it follows that each of the two irreducible component of $\Phi^{-1}(\mathcal{C})$ is real. Hence the same is true for $\Phi^{-1}(C_0)$. Thus we have proved (iv) of the proposition. $\square$

We have similar statements for touching conics of special type:

Proposition 4.3. *Assume $f(\lambda) < 0$ and let $C_\theta \subset H_\lambda$ be a real touching conic of special type given by the equation (12). Then (i)–(iv) of Proposition 4.2 hold if we replace $\Phi^{-1}(C_\theta)$ by $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$, where we set $\Gamma := \Phi^{-1}(P_\infty)$ and $\overline{\Gamma} := \Phi^{-1}(\overline{P}_\infty)$.*

Proof. (i) can be proved in the same way as in Proposition 4.2. (But in this case, any small resolution $Z \to Z_0$ gives the normalization of $\Phi_0^{-1}(C_0)$ over P_∞ and $\overline{P}_\infty$, as will be mentioned below.) For (ii) first note that we have $\Phi^{-1}(C_\theta) = L_1 + L_2 + \Gamma + \overline{\Gamma} \in |-2K|$ this time, where L_1 and L_2 are irreducible components of $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$. Γ and $\overline{\Gamma}$ are chains of three (-2)-curves on $S_\lambda = \Phi^{-1}(H_\lambda)$, since they are exceptional curves of the minimal resolution of A_3-singularities of surface. We write $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where Γ_i's are smooth rational curves satisfying $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_3 = 1$ and $\Gamma_1\Gamma_3 = 0$ on S_λ. We then have $\Gamma^2 = \overline{\Gamma}^2 = -2$. Furthermore, as we will see in Lemma 5.10, we have $L_1\Gamma = L_1\overline{\Gamma} = L_2\Gamma = L_2\overline{\Gamma} = 1$. Therefore again by Proposition 4.1, we get $8 = (-2K)^2 = (L_1 + L_2 + \Gamma + \overline{\Gamma})^2 = L_1^2 + L_2^2 + 2L_1L_2 + 4$. But because L_1 and L_2 intersect transversally at two points (over the touching points of C_θ and B_λ), and because L_1 and L_2 do not intersect on $\Gamma \cup \overline{\Gamma}$, we have $L_1L_2 = 2$. Therefore we have $L_1^2 + L_2^2 = 0$. Hence by the same reason in the proof of the previous proposition, we again have $L_1^2 = L_2^2 = 0$. This implies (ii). (iii) follows from (ii), because we have $L_1L_2 = 2$ as is already seen. (iv) can be proved by the same idea as in the previous proposition: first we may assume $\theta = 0$. Then by (12) the equation of C_0 on $U = \{y_1 \neq 0\} = \{(v_2, v_3)\}$ is given by

$$\sqrt{Q^2 - f} + \sqrt{\frac{\sqrt{Q^2 - f} - Q}{2}} \cdot (v_2 + v_3) + v_2v_3 = 0.$$

If we use another coordinate (u, v) defined in the proof of the previous proposition, this can be written as

$$C_0 : \ v^2 = u^2 + 2\sqrt{2}\sqrt{\sqrt{Q^2 - f} - Q} \cdot u + 4\sqrt{Q^2 - f}. \tag{24}$$

Next we introduce a new variable w by setting $u = -\sqrt{2}(\sqrt{Q^2 - f} - Q)^{\frac{1}{2}} + iw$. Then the equation becomes

$$C_0 : \quad v^2 + w^2 = 2\left(\sqrt{Q^2 - f} + Q\right). \tag{25}$$

Put $\mathcal{C} := C_0 \cap \mathbf{R}^2$, where $\mathbf{R}^2 = \{(w, v) \,|\, w \in \mathbf{R}, v \in \mathbf{R}\}$. Then since $\sqrt{Q^2 - f} + Q > 0$, $\mathcal{C}$ is a real circle in $\{(w, v) \in \mathbf{R}^2\}$.

By (20) we have

$$\Phi_0^{-1}(U) : \ z^2 = f - \left(\frac{u^2 - v^2}{4} + Q\right)^2. \tag{26}$$

On the other hand, by (24), we have

$$C_0 : \ u^2 - v^2 = -2\sqrt{2}\sqrt{\sqrt{Q^2 - f} - Q} \cdot u - 4\sqrt{Q^2 - f}$$

on C_0. Substituting this into (26), we get

$$\Phi_0^{-1}(C_0) : \ z^2 = f + \frac{\sqrt{Q^2 - f} - Q}{2} w^2.$$

Then by using (25), we get

$$\Phi_0^{-1}(C_0) : \ z^2 = -\frac{\sqrt{Q^2 - f} - Q}{2} v^2. \tag{27}$$

Therefore, z is pure imaginary over $\mathcal{C}$, so that we can distinguish two sheets by looking the sign of z/i. By (27) and (25), the branch points of $\Phi^{-1}(C_0) \to C_0$ is the two points $(w, v) = (\pm(2(\sqrt{Q^2 - f} + Q))^{\frac{1}{2}}, 0)$ which lie on $\mathcal{C}$. The real structure is given by $(w, v) \mapsto (-\overline{w}, -\overline{v})$ and this is equal to $(-w, -v)$ on $\mathcal{C}$. Thus the real structure on $\mathcal{C}$ interchanges the two segments separated by the two branch points. Moreover, the real structure on the fiber coordinate is given by $z \mapsto \overline{z}$. Therefore, it changes the sign of z/i over $\mathcal{C}$. This implies that each component of $\Phi^{-1}(\mathcal{C})$ is real. Therefore that of $\Phi^{-1}(C_0)$ is also real. $\square$

The situation slightly changes for touching conics of orbit type:

Proposition 4.4. *Suppose $f(\lambda) \neq 0$ and let $C_\alpha \subset H_\lambda$ be a real touching conic of orbit type given by the equation (19). Then we have: (i) C_α is contained in B iff $\alpha = -Q \pm \sqrt{f}$. (ii) $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ has just two irreducible components, both of which are smooth rational curves that are mapped biholomorphically onto C_α. (Here Γ and $\overline{\Gamma}$ are as in Proposition 4.3.) (iii) Each irreducible component of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ has a trivial normal bundle in $S_\lambda = \Phi^{-1}(H_\lambda)$. (iv) The two irreducible components of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ belong to one and the same pencil on S_λ. (v) Each irreducible component of $\Phi^{-1}(C_\alpha) - \Gamma - \overline{\Gamma}$ is real iff $f > 0$ and $-Q - \sqrt{f} \leq \alpha \leq -Q + \sqrt{f}$ are satisfied. (vi) There is no real point on C_α if f and α satisfies the inequalities of (v).*

Note that $f > 0$ implies $Q \geq \sqrt{f}$ by Proposition 2.5 (ii).

Proof. (i) Substituting $y_2 y_3 = \alpha y_1^2$ into the defining equation of B_λ, we get $((\alpha + Q)^2 - f)\, y_1^4 = 0$. Thus if C_α is contained in B iff $(\alpha + Q)^2 = f$, which implies $\alpha = -Q \pm \sqrt{f}$, as desired. (ii) can be seen in the same way as in (i) of Proposition 4.2. (This time, any small resolution $Z \to Z_0$ gives the normalization of $\Phi_0^{-1}(C_0)$.) Next we prove (iii). Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, $\overline{\Gamma} = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$, L_1 and L_2 have the same meaning as in the proof of the last proposition. Then (as will be shown in Lemma 5.16 by explicit calculations) we have $L_1 L_2 = \Gamma\overline{\Gamma} = L_1\Gamma_1 = L_1\overline{\Gamma}_1 = L_2\Gamma_1 = L_2\overline{\Gamma}_1 = L_1\Gamma_3 = L_1\overline{\Gamma}_3 = L_2\Gamma_3 = L_2\overline{\Gamma}_3 = 0$ on S_λ, while $L_1\Gamma_2 = L_1\overline{\Gamma}_2 = L_2\Gamma_2 = L_2\overline{\Gamma}_2 = 1$ (also on S_λ). In particular, we have $L_1\Gamma = L_1\overline{\Gamma} = L_2\Gamma = L_2\overline{\Gamma} = 1$. On the other hand, we still have $8 = (-2K)^2 = (L_1 + L_2 + \Gamma + \overline{\Gamma})^2$ and $L_1^2 \geq 0$ and $L_2^2 \geq 0$. Combining these, we get $L_1^2 = L_2^2 = 0$ on S_λ. Thus we have (iii). (iv) easily follows if we consider the linear systems $|L_1|$ and $|L_2|$, and if we note that $L_1 L_2 = 0$. Next we show (v). Substituting $v_2 v_3 = \alpha$ into (20), we get $z^2 + (\alpha + Q)^2 - f = 0$. From this, the equations of irreducible components of $\Phi_0^{-1}(C_\alpha)$ can be calculated to be

$$z = \pm\sqrt{f - (\alpha + Q)^2} \tag{28}$$

Recalling that the real structure is given by $z \mapsto \overline{z}$, these curves are real iff $f - (\alpha + Q)^2 \geq 0$. In particular, $f \geq 0$ follows. Then we have $-Q - \sqrt{f} \leq \alpha \leq -Q + \sqrt{f}$, and we get (v). Finally, (vi) immediately follows from Lemma 2.4. $\square$

Proposition 4.4 implies that not all touching conics of orbit type can be the image of a twistor line: $f(\lambda) > 0$ is needed, and further, $-Q - \sqrt{f} \leq \alpha \leq -Q + \sqrt{f}$ must be satisfied. But once we know that one of the two irreducible components is a twistor line, it follows that the other component is also a twistor line, since by (iv) these two components can be connected by deformation in S_λ (hence also in Z) preserving the real structures. This is not true for touching conics of generic type and special type, because the two irreducible components of $\Phi^{-1}(C_\theta)$ or $\Phi^{-1}(C_\theta) - \Gamma - \overline{\Gamma}$ intersect as we have already seen in the proofs of Propositions 4.2 and 4.3.

5. The normal bundles of the inverse images of real touching conics

In this section we calculate the normal bundle of L in Z, where L is a real irreducible component of the inverse images of the real touching conics which are determined in Section 3. Roughly speaking, our result states that the normal bundle is isomorphic to $O(1)^{\oplus 2}$ for general L, but sometimes degenerates into $O \oplus O(2)$. We can precisely detect which L has such a degenerate normal bundle, in terms of Q and f which appear in the defining equation of B.

5.1. **Preliminary lemma and notations.** In order to make a distinction between $O(1)^{\oplus 2}$ and $O \oplus O(2)$, we use the following elementary criterion:

Lemma 5.1. *Let $E \to \mathbf{CP}^1$ be a holomorphic line bundle of rank two, and assume that E is isomorphic to $O(1)^{\oplus 2}$ or $O \oplus O(2)$. Let s and t be global sections of E which are linearly independent as global sections. Then $E \simeq O \oplus O(2)$ iff there are constants $a, b \in \mathbf{C}$ such that $as + bt$ has two zeros.*

Proof. It is immediate to see that non-zero sections of $O(1)^{\oplus 2}$ have at most one zero. So sufficiency follows. Conversely let $s = (s_1, s_2)$ and $t = (t_1, t_2)$ be any linearly independent sections of $O \oplus O(2)$, where $s_1, t_1 \in \Gamma(O) = \mathbf{C}$ and $s_2, t_2 \in \Gamma(O(2))$. Then take $a, b \in \mathbf{C}$ such that

$as_1 + bt_1 = 0$. Then $as + bt$ can be regarded as a non-zero section of $O(2)$ so that it has two zeros. $\square$

Next we introduce some notations. As in the previous sections, $\lambda \in \mathbf{R}$ denotes a parameter on the space of real $U(1)$-invariant planes. In other words, λ is a parameter on the real locus l_0^σ of the real line $l_0 := \{y_2 = y_3 = 0\}$. The function $f(\lambda) = \lambda(\lambda+1)(a\lambda - b)$, $(a, b > 0)$ defines four open intervals in the circle l_0^σ:

$$I_1 = (-\infty, -1),\ I_2 = (-1, 0),\ I_3 = (0, b/a) \text{ and } I_4 = (b/a, +\infty).$$

Namely, $I_1 \cup I_3 = \{\lambda \in \mathbf{R} \mid f(\lambda) < 0\}$ and $I_2 \cup I_4 = \{\lambda \in \mathbf{R} \mid f(\lambda) > 0\}$. By Proposition 2.3, the equation $Q(\lambda, 1)^2 - f(\lambda) = 0$ has a unique real solution $\lambda = \lambda_0$ which is necessarily a double root. Since we have $f(\lambda_0) = Q(\lambda_0, 1)^2 > 0$, $\lambda_0 \in I_2 \cup I_4$. By a possible application of a projective transformation with respect to y_0 and y_1, we may suppose that $\lambda_0 \in I_4$. Then we set $I_4^- = (b/a, \lambda_0)$ and $I_4^+ = (\lambda_0, +\infty)$.

Next suppose $\lambda \in I_2 \cup I_4^- \cup I_4^+$, and let

$$\mathcal{C}_\lambda^{\mathrm{gen}} = \{C_\theta \subset H_\lambda \mid C_\theta \text{ is defined by (4)}\}$$

be the set of real touching conics of generic type on H_λ. Note that if $\lambda = \lambda_0$ or if $\lambda \in I_1 \cup I_3$ there is no real touching conic of generic type on H_λ by Proposition 3.4 (ii). Similarly, for $\lambda \in I_1 \cup I_3$, let

$$\mathcal{C}_\lambda^{\mathrm{sp}} = \{C_\theta \subset H_\lambda \mid C_\theta \text{ is defined by (12)}\}$$

be the set of real touching conics of special type on H_λ. Note that if $\lambda \in I_2 \cup I_4$ there is no real touching conic of generic type by Proposition 3.6 (i). Finally for $\lambda \in I_2 \cup I_4$, let

$$\mathcal{C}_\lambda^{\mathrm{orb}} = \left\{C_\alpha \subset H_\lambda \mid -Q - \sqrt{f} \le \alpha \le -Q + \sqrt{f},\ C_\theta \text{ is defined by (19)}\right\}$$

be the set of real touching conics of orbit type on H_λ. Note that the restriction on α implies that the two irreducible components of the inverse image are respectively real (Proposition 4.4 (v)). $\mathcal{C}_\lambda^{\mathrm{gen}}$ and $\mathcal{C}_\lambda^{\mathrm{sp}}$ are parametrized by a circle on which $U(1)$ naturally acts transitively, whereas $\mathcal{C}_\lambda^{\mathrm{orb}}$ is parametrized by a closed interval on which $U(1)$ acts trivially.

In the following three subsections we determine the normal bundles, for each type of real touching conics. These subsections are organized as follows: first we explicitly calculate the intersection of the irreducible components and some curves. Consequently we get a function of λ (which will be written h_i). Second we show that the normal bundle in problem degenerates into $O \oplus O(2)$ precisely when λ is a critical point of this function. Finally we determine the critical points.

The consequences of the results in these three subsections will be postponed until §5.5.

5.2. **The case of generic type.** Suppose $\lambda \in I_2 \cup I_4^- \cup I_4^+$ and take $C_\theta \in \mathcal{C}_\lambda^{\rm gen}$. First we calculate the intersection of C_θ and l_∞, where l_∞ is the real line defined by $y_0 = y_1 = 0$ as before. Let $x_2 = y_2/y_3$ be a non-homogeneous coordinate on l_∞ (around $P_\infty = (0:0:0:1)$).

Lemma 5.2. *The set $\{C_\theta \cap l_\infty \,|\, C_\theta \in \mathcal{C}_\lambda^{\rm gen}\}$ consists of disjoint two circles about P_∞ in l_∞, whose radiuses (with respect to the coordinate x_2 above) are given by*

$$h_0(\lambda) := \frac{Q + \sqrt{Q^2 - f}}{\sqrt{f}} \quad \text{and} \quad h_0(\lambda)^{-1} = \frac{Q - \sqrt{Q^2 - f}}{\sqrt{f}}$$

respectively, where we put $Q = Q(\lambda, 1)$ and $f = f(\lambda)$ as before.

Note that we have $Q^2 - f > 0$ and $Q > \sqrt{f}$ by Proposition 2.6, and therefore $h_0 > 1 > h_0^{-1} > 0$ holds. Moreover, h_0 and h_0^{-1} are differentiable on $I_2 \cup I_4^- \cup I_4^+$.

Proof. On $H_\lambda = \{(y_0 : y_1 : y_2)\}$, l_∞ is defined by $y_1 = 0$. Therefore by (4) we readily have

$$C_\theta \cap l_\infty = \left\{ x_2 = \frac{-Q \pm \sqrt{Q^2 - f}}{\sqrt{f}} \cdot e^{-i\theta} \right\}. \tag{29}$$

This directly implies the claim of the lemma. □

By Proposition 4.2, $\Phi^{-1}(C_\theta)$ consists of two irreducible components, both of which are real rational curves. We denote these components L_θ^+ and L_θ^-, although there is no canonical way of making a distinction of these two. Again by Proposition 4.2, L_θ^+ and L_θ^- respectively form disjoint families

$$\mathcal{L}_\lambda^+ = \{L_\theta^+ \,|\, \theta \in \mathbf{R}\} \quad \text{and} \quad \mathcal{L}_\lambda^- = \{L_\theta^- \,|\, \theta \in \mathbf{R}\}$$

of (real and smooth) rational curves on $S_\lambda = \Phi^{-1}(H_\lambda)$. These are real members of real pencils on S_λ and each member has no real point by Proposition 3.4 (iii). Because $U(1)$ acts also on the parameter spaces $(= S^1)$ of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$, the normal bundles of L_θ^+ and L_θ^- inside Z are independent of the choice of θ. The following proposition plays a key role in determining the normal bundle:

Proposition 5.3. *For any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, the normal bundle of L in Z is isomorphic to either $O(1)^{\oplus 2}$ or $O \oplus O(2)$. Further, the latter holds iff λ is a critical point of $h_0(\lambda)$ defined in Lemma 5.2.*

In particular, members of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ have the same normal bundle in Z.

Proof. By Proposition 4.1, L is contained in the smooth surface $S_\lambda = \Phi^{-1}(H_\lambda)$ and therefore we have an exact sequence $0 \to N_{L/S_\lambda} \to N_{L/Z} \to N_{S_\lambda/Z}|_L \to 0$. By Proposition 4.2 (ii), we have $N_{L/S_\lambda} \simeq O_L$. On the other hand, again by Proposition 4.1, S_λ is a smooth member of $|(-1/2)K_Z|$. Therefore by adjunction formula we have $K_S \simeq K_Z|_{S_\lambda} \otimes N_{S_\lambda/Z} \simeq K_Z|_{S_\lambda} \otimes (-1/2)K_Z|_{S_\lambda}$ and hence $N_{S_\lambda/Z} \simeq (-1/2)K_Z|_{S_\lambda} \simeq -K_{S_\lambda}$. Hence we get $N_{S_\lambda/Z}|_L \simeq -K_{S_\lambda}|_L \simeq -K_L \otimes N_{L/S} \simeq O_L(2)$. Therefore by the short exact sequence above, $N_\lambda := N_{L/Z}$ is isomorphic to either $O \oplus O(2)$ or $O(1)^{\oplus 2}$. Thus we get the first claim of the proposition.

In order to show the second claim, we first explain the natural real structure on $\Gamma(N_\lambda)$, the space of sections of N_λ. Since L is real, σ naturally acts on $\Gamma(N_\lambda)$ as the complex conjugation. For $s \in \Gamma(N_\lambda)$ we denote by $\mathrm{Re}s$ and $\mathrm{Im}s$ the real part and the imaginary part of s respectively. Namely, $\mathrm{Re}s = (s + \overline{\sigma(s)})/2$ and $\mathrm{Im}s = (s - \overline{\sigma(s)})/2$. Next recall that any one-parameter family of holomorphic deformation of L in Z naturally gives rise to a holomorphic section of N_λ. We have the following two one-parameter families of deformations of L in Z: one is obtained by moving L by $\mathbf{C}^*$-action, where the $\mathbf{C}^*$-action is the complexification of the $U(1)$-action. The other is obtained by moving the parameter λ in $\mathbf{C}$, while fixing θ. Let $s \in \Gamma(N_\lambda)$ and $t \in \Gamma(N_\lambda)$ be the holomorphic sections associated to the former and the latter family respectively. These are clearly linearly independent sections. Because the $\mathbf{C}^*$-action preserves S_λ, it follows from Proposition 4.2 (ii) and (iii) that each of the curves of the former family are disjoint. This implies that s is nowhere vanishing. Next we consider the latter family. First, noting $H_\lambda \cap H_{\lambda'} = l_\infty$ for $\lambda \neq \lambda'$, t can be zero only on $\Phi^{-1}(l_\infty)$. Suppose $h_0'(\lambda) = 0$, where the derivative is with respect to $\lambda \in \mathbf{R}$. Then we have $(h_0^{-1})'(\lambda) = 0$. Then, since h is a holomorphic function of λ, it can be easily derived by using the Cauchy-Riemann equation that $\partial h/\partial\lambda = 0$. In the same way, we have $\partial h^{-1}/\partial\lambda = 0$. These imply that t vanishes on $\Phi^{-1}(l_\infty) \cap L =: \{z_\lambda, \overline{z}_\lambda\}$. Therefore by Lemma 5.1, we get $N_\lambda \simeq O \oplus O(2)$.

Next suppose $h_0'(\lambda) \neq 0$, so that $(h_0^{-1})'(\lambda) \neq 0$. We claim that $\mathrm{Re}(as + t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ simultaneously, for any $a \in \mathbf{C}$. Because C_θ intersects l_∞ transversally, t also becomes a nowhere vanishing section under our assumption. Hence the claim is true for $a = 0$. Putting $a = a_1 + ia_2$, $a_1, a_2 \in \mathbf{R}$, we easily get

$$\mathrm{Re}(as + t) = a_1\mathrm{Re}s + (\mathrm{Re}t - a_2\mathrm{Im}s). \tag{30}$$

Since s comes from the $\mathbf{C}^*$-action, and since its real part corresponds to the $U(1)$-action, $(\mathrm{Re}s)(z_\lambda)$ is represented by the tangent vector of the $U(1)$-orbit going through z_λ. On the other hand, (29) implies that $(\mathrm{Re}t)(z_\lambda)$ is represented by a tangent vector which is parallel to $(\mathrm{Im}s)(z_\lambda)$. Hence by (30), we can deduce that $\mathrm{Re}(as+t)(z_\lambda) = 0$ implies $a_1 = 0$ and

$$(\mathrm{Re}t)(z_\lambda) = a_2(\mathrm{Im}s)(z_\lambda). \tag{31}$$

Similarly, $\mathrm{Re}(as+t)(\overline{z}_\lambda) = 0$ implies $a_1 = 0$ and

$$(\mathrm{Re}t)(\overline{z}_\lambda) = a_2(\mathrm{Im}s)(\overline{z}_\lambda). \tag{32}$$

Suppose $a_2 > 0$. Then since $\{\mathrm{Re}s(z_\lambda), \mathrm{Im}s(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$ from the beginning, (31) implies that $\{\mathrm{Re}s(z_\lambda), \mathrm{Re}t(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$. Further, we have

$$(\mathrm{Re}s)(\overline{z}_\lambda) = \sigma_*((\mathrm{Re}s)(z_\lambda)) \quad \text{and} \quad (\mathrm{Re}t)(\overline{z}_\lambda) = \sigma_*((\mathrm{Re}t)(z_\lambda)).$$

Hence we get by (32)

$$(\mathrm{Im}s)(\overline{z}_\lambda) = \frac{1}{a_2}(\mathrm{Re}t)(\overline{z}_\lambda) = \frac{1}{a_2}\sigma_*\left((\mathrm{Re}t)(z_\lambda)\right).$$

So we have

$$\{(\mathrm{Re}s)(\overline{z}_\lambda), (\mathrm{Im}s)(\overline{z}_\lambda)\} = \{\sigma_*((\mathrm{Re}s)(z_\lambda)), \sigma_*((\mathrm{Re}t)(z_\lambda))/a_2\}.$$

But since σ is anti-holomorphic, σ is orientation reversing. Further, as is already seen, $\{\mathrm{Re}s(z_\lambda), \mathrm{Re}t(z_\lambda)\}$ is an oriented basis of $T_{z_\lambda}(\Phi^{-1}(l_\infty))$ (if $a_2 > 0$). This implies that $\{\sigma_*((\mathrm{Re}s)(z_\lambda)), \sigma_*((\mathrm{Re}t)(z_\lambda))/a_2\}$ is an anti-oriented basis of $T_{\overline{z}_\lambda}(\Phi^{-1}(l_\infty))$. This contradicts to the fact that $\{(\mathrm{Re}s)(\overline{z}_\lambda), (\mathrm{Im}s)(\overline{z}_\lambda)\}$ is an oriented basis of $T_{\overline{z}_\lambda}(\Phi^{-1}(l_\infty))$. Therefore, $\mathrm{Re}(as+t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ simultaneously, provided $a_2 > 0$. Parallel arguments show the same claim holds for the case $a_2 < 0$. Thus we have shown that $\mathrm{Re}(as+t)$ cannot vanish at z_λ and $\overline{z}_\lambda$ at the same time, as claimed. On the other hand, it is obvious that $as+t$ does not vanish except $\{z_\lambda, \overline{z}_\lambda\}$. Therefore, the zero locus of $as+t$ consists of at most one point for any $a \in \mathbf{C}$. Since s is a nowhere vanishing section, Lemma 5.1 implies $N_\lambda \simeq O(1)^{\oplus 2}$. □

Lemma 5.4. *Let $h_0 = h_0(\lambda)$ be the positive valued function on $I_2 \cup I_4$ defined in Lemma 5.2, which is differentiable on $I_2 \cup I_4 \backslash \{\lambda_0\}$. Then h_0 has a unique critical point on I_2, and has no critical point on $I_4 \backslash \{\lambda_0\}$. (See Figure 2.)*

Proof. We have $Q(-1) > 0$, $Q(0) > 0$ and $Q(b/a) > 0$ by Proposition 2.5 (i), and

$$h_0 = \frac{Q + \sqrt{Q^2 - f}}{\sqrt{f}} = \frac{Q}{\sqrt{f}} + \sqrt{\frac{Q^2}{f} - 1}. \tag{33}$$

From these, it follows that $\lim_{\lambda \downarrow -1} h_0(\lambda) = \lim_{\lambda \uparrow 0} h_0(\lambda) = +\infty$. Therefor h_0 has at least one critical point on I_2, since h_0 is differentiable on I_2. So to prove the lemma it suffices to show that this is a unique critical point on $I_2 \cup I_4 \backslash \{\lambda_0\}$.

We consider the real valued function $\gamma := Q^2/f$ defined on $I := I_1 \cup I_2 \cup I_3 \cup I_4$, which is clearly differentiable on I. Then $h_0 = \sqrt{\gamma} + \sqrt{\gamma - 1}$ on $I_2 \cup I_4$, and we have

$$h_0' = \gamma' \cdot \left(\frac{1}{2\sqrt{\gamma}} + \frac{1}{2\sqrt{\gamma - 1}} \right),$$

provided $\lambda \neq \lambda_0$. Therefore on $I_2 \cup I_4 \backslash \{\lambda_0\}$, $h_0'(\lambda) = 0$ iff $\gamma'(\lambda) = 0$. It is readily seen that $\lim_{\lambda \downarrow -\infty} \gamma(\lambda) = \lim_{\lambda \uparrow -1} \gamma(\lambda) = -\infty$, $\lim_{\lambda \downarrow -1} \gamma(\lambda) = \lim_{\lambda \uparrow 0} \gamma(\lambda) = \infty$, $\lim_{\lambda \downarrow 0} \gamma(\lambda) = \lim_{\lambda \uparrow (b/a)} \gamma(\lambda) = -\infty$, and $\lim_{\lambda \downarrow (b/a)} \gamma(\lambda) = \lim_{\lambda \uparrow \infty} \gamma(\lambda) = \infty$. Therefore γ has at least one critical point on each I_j, $1 \leq j \leq 4$. We also have

$$\gamma' = Q(2Q'f - Qf')/f^2. \tag{34}$$

Suppose that critical points of γ on I_2 are not unique. Then γ has at least three critical points on I_2. This implies that γ has at least four critical points on $I_2 \cup I_4$. Because $Q > 0$ on $I_2 \cup I_4$ (Proposition 2.5 (i)), these critical points must correspond to zeros of $2Q'f - Qf'$ whose degree is just four. By (34) this implies that the other critical points of γ on I_1 and I_3 must correspond to zeros of Q. But this cannot happen since $Q > 0$ on $I_2 \cup I_4$ and since Q is degree two. Therefore, our assumption fails and it follows that critical points on γ on I_2 is unique. Hence critical points of h_0 on I_2 is also unique. Exactly the same argument shows that γ has a unique critical point on I_4. This critical point must be λ_0, since γ attains the minimal value ($= 1$) there. This implies that g has no critical point on $I_4 \backslash \{\lambda_0\}$. Thus we obtain the claims of the lemma. $\square$

The following is the main result of this subsection:

Proposition 5.5. *(i) If $\lambda \in I_4$ and if $\lambda \neq \lambda_0$, we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. (ii) There is a unique $\lambda \in I_2$ such that $N_{L/Z} \simeq O \oplus O(2)$ for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. For any other $\lambda \in I_2$, we have $N_{L/Z} \simeq O(1)^{\oplus 2}$ for arbitrary $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$. (iii) If $\lambda \in I_2$, any member*

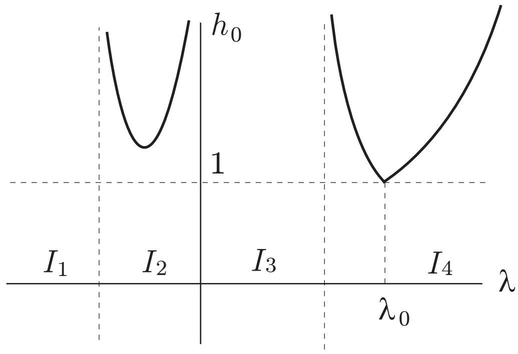


FIGURE 2. behavior of h_0

of $\mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$ is not a twistor line in Z (even if Z is actually a twistor space).

Proof. (i) and (ii) are direct consequences of Proposition 5.3 and Lemma 5.4. To show (iii), let $\lambda' \in I_2$ be the unique critical point of g. Then by (ii), any $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ is not a twistor line. We can see that for any $\lambda \in I_2$ and for any $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, L can be deformed into some $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ preserving the real structure. In fact, we have $\Phi(L) = C_\theta$ for some $C_\theta \in \mathcal{C}_\lambda^{\mathrm{gen}}$. Then since I_2 is an interval in $\mathbf{R}$, C_θ can be canonically deformed into some $C'_\theta \in \mathcal{C}_{\lambda'}^{\mathrm{gen}}$. (The point is that we take a constant θ for any $\lambda \in I_2$.) Correspondingly, we obtain deformation of L into $L' \in \mathcal{L}_{\lambda'}^+ \cup \mathcal{L}_{\lambda'}^-$ such that $\Phi(L') = C'_\theta$. Thus we get an explicit real one-dimensional family of rational curves in Z containing L and L' as its members, as claimed. Any member of this family is real by Proposition 4.2 (iv). Since any deformation of twistor line preserving the real structure is still a twistor line, it follows that L is not a twistor line. □

We note the proof of (iii) does not work for I_4, since as λ goes to λ_0, the curve C_θ (defined by (4)) degenerates into a double line. This is an important point for obtaining a natural compactification of the space of real touching conics of B.

Corollary 5.6. *If $\lambda \in I_2$, only the members of $\mathcal{C}_\lambda^{\mathrm{orb}}$ can be the image of twistor lines. Namely over I_2, members of $\mathcal{C}_\lambda^{\mathrm{gen}}$ cannot be the images of twistor lines.*

Proof. By Proposition 3.8 (i), the image of a twistor line contained in H_λ is either a touching conic of generic type or that of orbit type for $\lambda \in I_2 \cup I_4$. But by Proposition 5.5 (iii) the former cannot be the image of a twistor line if $\lambda \in I_2$. □

Corollary 5.7. *A twistor line of a self-dual 4-manifold (i.e. a fiber of the twistor fibration) is not in general characterized by the property that it is a real smooth rational curve without fixed point whose normal bundle is isomorphic to $O(1)^{\oplus 2}$. More concretely, the twistor space of any non-LeBrun self-dual metric on $3\mathbf{CP}^2$ of positive scalar curvature with a non-trivial Killing field always possesses such a real rational curve.*

Proof. Let Z be a twistor space as in the corollary. Then Z has a structure as in Proposition 2.1, where Q, a and b satisfy the conditions in Proposition 2.6. By (ii) and (iii) of Proposition 5.5, Z always has a real smooth rational curve L satisfying $N_{L/Z} \simeq O(1)^{\oplus 2}$, but which is not a twistor line. This L has no real point by Proposition 3.4 (iii) and the reality of Φ. Existence of Z is proved in [5] and [6]. □

We remark that C. Simpson [14] asked a similar question about a characterization of twistor lines for twistor spaces of hyperKähler manifolds.

Next we give another geometric proof for the fact that L cannot be a twistor line for $\lambda \in I_2$ (although we will not need this result in the sequel).

Proposition 5.8. *If $\lambda \in I_2$ is not a critical point of h_0, there exists a unique $\mu \in I_2$ with $\lambda \neq \mu$ satisfying the following: for any $L \in \mathcal{L}_\lambda^+$ (resp. $L \in \mathcal{L}_\lambda^-$) there exists $L' \in \mathcal{L}_\mu^+$ (resp. $L' \in \mathcal{L}_\mu^-$) such that $L \cap L' \neq \phi$.*

Proof. Let $\lambda' \in I_2$ be the unique critical point of h_0 as before. By our proof of Lemma 5.4 we have $\lim_{\lambda \downarrow -1} h_0(\lambda) = \lim_{\lambda \uparrow 0} h_0(\lambda) = +\infty$ and g is strictly decreasing on $(-1, \lambda')$ and strictly increasing on $(\lambda', 0)$. Suppose $\lambda < \lambda'$. Let l and $\bar{l}$ be the conjugate pair of rational curves which are mapped biholomorphically onto l_∞. (See the proof of Proposition 3.2.) Then by Lemma 5.2, $L \cap l$ is a point which is either $x_2 = h_0(\lambda)e^{i\theta}$ or $x_2 = h_0(\lambda)^{-1}e^{i\theta}$ for some $\theta \in \mathbf{R}$, where we identify l and l_∞ via Φ and use $x_2 = y_2/y_3$ as an affine coordinate on l_∞ as before. If $x_2 = h_0(\lambda)^{-1}e^{i\theta}$, $\bar{l} \cap L$ is a point having $x_2 = h_0(\lambda)e^{i\theta}$. Thus by a possible exchange of l and $\bar{l}$, we may suppose that $l \cap L$ is a point satisfying $x_2 = h_0(\lambda)e^{i\theta}$. Then by the behavior of h_0 mentioned above, there exists a unique $\mu > \lambda'$, $\mu \in I_2$ such that $h_0(\lambda) = h_0(\mu)$. On the other hand, by our choice of L we have $\Phi(L) = C_\theta$ for some $C_\theta \in \mathcal{C}_\lambda^{\mathrm{orb}}$. Then take $L' \in \mathcal{L}_\mu^+$ such that $\Phi(L') = C_\theta \in \mathcal{C}_\mu^{\mathrm{orb}}$. (Although we use the same symbol C_θ, they represent different conics since $\lambda \neq \mu$. The point is that we take the same θ for different λ's.) Then $L \cap L' \cap l$ is a point satisfying $x_2 = h_0(\lambda)e^{i\theta}$. (We also have $L \cap L' \cap \bar{l}$ is a point satisfying $x_2 = h_0(\lambda)^{-1}e^{i\theta}$.) Thus we have proved the claim for $\lambda < \lambda'$. Of course, the case $\lambda > \lambda'$ and the case $L \in \mathcal{L}_\lambda^-$ are similar. $\square$

The proposition shows that when $\lambda \in I_2$ passes through the critical point ($= \lambda'$) of h_0, the local twistor fibration arising from $L \in \mathcal{L}_\lambda^+ \cup \mathcal{L}_\lambda^-$, $\lambda \neq \lambda'$ breaks down. Note also that Proposition 5.8 also holds true for I_4 without any change of the proof, and it implies that if members of $\mathcal{L}_\lambda^+$ (resp. $\mathcal{L}_\lambda^-$) are twistor lines for $\lambda \in I_4^-$, members of $\mathcal{L}_\lambda^-$ (resp. $\mathcal{L}_\lambda^+$) must be twistor lines for $\lambda \in I_4^+$.

5.3. **The case of special type.** In this subsection we calculate the normal bundle of L^+ and L^- in Z, where L^+ and L^- are curves which are mapped biholomorphically onto a real touching conic of special type. Compared to the case of generic type, the problem becomes harder and the result becomes more complicated, since touching conics of special type go through the singular point P_∞ and $\overline{P}_\infty$ of B, so that the situation, and hence the result also, depend on how we resolve the corresponding singularities of Z_0.

First we recall the situation and fix notations. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. Put $p_\infty := \Phi_0^{-1}(P_\infty)$. In a neighborhood of $P_\infty = (0:0:0:1)$, we use (x_0, x_1, x_2) as an affine coordinate by setting $x_i = y_i/y_3$. Then around $P_\infty = (0,0,0)$, B is given by the equation $(x_2 + Q(x_0,x_1))^2 - x_0x_1(x_0+x_1)(ax_0 - bx_1) = 0$. Let z be a fiber coordinate of $O(2) \to \mathbf{CP}^3$. Then Z_0 is given by the equation

$$z^2 + (x_2 + Q(x_0,x_1))^2 - x_0x_1(x_0+x_1)(ax_0 - bx_1) = 0. \tag{35}$$

This can be also written as $\{z+i(x_2+Q(x_0,x_1))\}\{z-i(x_2+Q(x_0,x_1))\} = x_0x_1(x_0+x_1)(ax_0-bx_1)$. Setting $\xi = z + i(x_2 + Q(x_0,x_1))$ and $\eta = z - i(x_2 + Q(x_0,x_1))$, we get

$$Z_0 : \quad \xi\eta = x_0x_1(x_0+x_1)(ax_0 - bx_1). \tag{36}$$

Thus $p_\infty = \{(x_0, x_1, \xi, \eta) = (0,0,0,0)\}$ is a compound A_3-singularity. Small resolutions of p_∞ are explicitly given as follows: first we choose ordered three linear forms $\{\ell_1, \ell_2, \ell_3\} \subset \{x_0, x_1, x_0 + x_1, ax_0 - bx_1\}$. Next blow-up Z_0 along $\{\xi = \ell_1 = 0\}$. Then by setting $\xi = u\ell_1$ we get

$$Z_0' : u\eta = x_0x_1(x_0+x_1)(ax_0 - bx_1)/\ell_1. \tag{37}$$

The exceptional curve of $Z_0' \to Z_0$ is given by $\Gamma_1' := \{(u, \eta, x_0, x_1) \,|\, \eta = x_0 = x_1 = 0\}$. We can use u as an affine coordinate on Γ_1'. Next blowing up Z_0' along $\{u = \ell_2 = 0\}$ and setting $u = v\ell_2$, we get

$$Z_0'' : v\eta = x_0x_1(x_0+x_1)(ax_0 - bx_1)/\ell_1\ell_2. \tag{38}$$

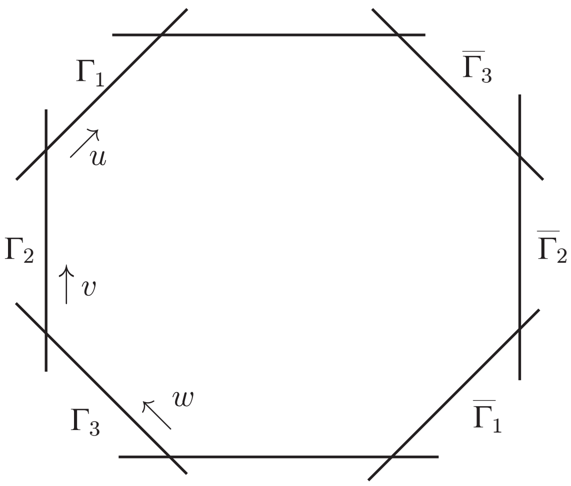


FIGURE 3. $\Phi^{-1}(l_\infty)$

The exceptional curve of $Z_0'' \to Z_0'$ is given by $\Gamma_2'' := \{(v, \eta, x_0, x_1) \mid \eta = x_0 = x_1 = 0\}$, on which we can use v as an affine coordinate. Finally by blowing up along $\{v = \ell_3 = 0\}$ and setting $v = w\ell_3$, we get

$$Z : w\eta = x_0x_1(x_0 + x_1)(ax_0 - bx_1)/\ell_1\ell_2\ell_3, \tag{39}$$

which is clearly smooth in a neighborhood of the origin. The exceptional curve of $Z \to Z_0''$ is given by $\Gamma_3 := \{(w, \eta, x_0, x_1) \mid \eta = x_0 = x_1 = 0\}$, on which we can use w as an affine coordinate.

Once a resolution of p_∞ is given, it naturally determines that of $\overline{p}_\infty$ by reality. Let μ be the resolution of p_∞ and $\overline{p}_\infty$ (for some choice of ℓ_1, ℓ_2 and ℓ_3). Thus a choice of ℓ_1, ℓ_2 and ℓ_3 determines a small resolution of p_∞ and $\overline{p}_\infty$ and there are $4! = 24$ ways of resolutions in all. Let $\Gamma = \mu^{-1}(p_\infty) = \Gamma_1 + \Gamma_2 + \Gamma_3$ and $\overline{\Gamma} = \mu^{-1}(\overline{p}_\infty) = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3$ be the exceptional curves of μ, where Γ_i and $\overline{\Gamma}_i$ are the exceptional curves arising from the i-th blowing-up above. Then we have $\Gamma_1 \cap \Gamma_2 \neq \phi, \Gamma_2 \cap \Gamma_3 \neq \phi$ and $\Gamma_1 \cap \Gamma_3 = \phi$. (See Figure 3.)

In order to calculate the intersection L^+ and L^- with Γ, we need a one-parameter presentation of C_θ, in a neighborhood of p_∞:

Lemma 5.9. *Let $C_\theta \subset H_\lambda$ be a real touching conic of special type whose equation is given by (12), and (x_0, x_1, x_2) the affine coordinate around P_∞ as above. Then in a neighborhood of P_∞, C_θ has a one-parameter* *presentation of the following form:*

(40)

$$\begin{cases} x_0 = \lambda x_1 \\ x_2 = -Be^{-i\theta}x_1 - \dfrac{\sqrt{Q^2-f}+Q}{2}x_1^2 + \dfrac{\sqrt{Q^2-f}+Q}{2}Be^{i\theta}x_1^3 + O(x_1^4), \end{cases}$$

where we put

$$B := B(\lambda) = \left(\frac{\sqrt{Q^2-f}-Q}{2} \right)^{\frac{1}{2}}.$$

Note again that $f < 0$ guarantees $\sqrt{Q^2-f}-Q > 0$.

Proof. By solving (12) with respect to x_2, we get

$$x_2 = -g(x_1) \cdot x_1, \quad g(x_1) := \frac{Be^{-i\theta} + \sqrt{Q^2-f}\ x_1}{1 + Be^{i\theta}x_1}. \tag{41}$$

Calculating the Maclaurin expansion of $g(x_1)$, we get (40). This is a routine work and we omit the detail. □

Lemma 5.10. *In a neighborhood of p_∞, each of the two irreducible components of $\Phi_0^{-1}(C_\theta)$ has a one-parameter presentation with respect to x_1 in the following forms respectively:*

$$\xi = -2iBe^{-i\theta}x_1 + O(x_1^2), \quad \eta = \frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1, \tag{42}$$

and

$$\xi = -\frac{ie^{i\theta}f}{2B}x_1^3 + O(x_1^4), \quad \eta = 2iBx_1e^{-i\theta} + O(x_1^2), \quad x_0 = \lambda x_1. \tag{43}$$

Proof. First by substituting $x_0 = \lambda x_1$ into (35), we get

$$z^2 = (f - Q^2)x_1^4 - 2Qx_1^2x_2 - x_2^2.$$

Substituting (41) into this, we get

$$z^2 = \left\{ (f-Q^2)x_1^2 + 2Qg(x_1)x_1 - g(x_1)^2 \right\} x_1^2.$$

Hence we have

$$z = \pm k(x_1)\, x_1, \quad k(x_1) = \left\{ (f-Q^2)x_1^2 + 2Qg(x_1)x_1 - g(x_1)^2 \right\}^{\frac{1}{2}}.$$

From this we deduce

$$\xi = z + i(x_2 + Qx_1^2) = (\pm k(x_1) - ig(x_1))\, x_1 + iQx_1^2$$

and

$$\eta = z - i(x_2 + Qx_1^2) = (\pm k(x_1) + ig(x_1))\, x_1 - iQx_1^2.$$

Then we get the desired equations by calculating the Maclaurin expansions of $\pm k(x_1) - ig(x_1)$ and $\pm k(x_1) + ig(x_1)$. These are also routine works and we omit the detail. $\square$

Lemma 5.11. *Let L_θ^+ and L_θ^- be the curves in Z which are the proper transforms of the curves (42) and (43) respectively. Then we have: (i) $L_\theta^+ \cap \Gamma_1$ is a point satisfying*

$$u = -2iBe^{-i\theta} \cdot \frac{x_1}{\ell_1} \tag{44}$$

and $L_\theta^+ \cap \Gamma_2$ and $L_\theta^+ \cap \Gamma_3$ are empty, (ii) $L_\theta^- \cap \Gamma_1$ and $L_\theta^- \cap \Gamma_2$ are empty and $L_\theta^- \cap \Gamma_3$ is a point satisfying

$$w = -\frac{ie^{i\theta} f}{2B} \cdot \frac{x_1^3}{\ell_1 \ell_2 \ell_3} \tag{45}$$

where we use u and w as local coordinates on Γ_1 and Γ_3 respectively as explained before, and $B = B(\lambda)$ is as in Lemma 5.9.

Here note that x_1/ℓ_1 and $x_1^3/\ell_1\ell_2\ell_3$ do not depend on x_1, and depend on λ only. Further, we have $B > 0$ since $f < 0$.

Proof. By substituting $\xi = u\ell_1$ into (42), we get the inverse image of (42) to be

$$u\ell_1 = -2iBe^{-i\theta}x_1 + O(x_1^2), \quad \eta = \frac{ie^{i\theta} f}{2B} x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1.$$

Eliminating $\Gamma_1' = \{\eta = x_0 = x_1 = 0\}$, we get the equation of the proper transform in Z_0' to be

$$u = -2iBe^{-i\theta}\frac{x_1}{\ell_1} + O(x_1), \quad \eta = \frac{ie^{i\theta} f}{2B} x_1^3 + O(x_1^4), \quad x_0 = \lambda x_1.$$

By setting $x_1 = 0$, we get $u = -2iBe^{-i\theta} \cdot x_1/\ell_1$. As remarked above, B is non-zero. Therefore the remaining blow-ups $Z_0'' \to Z_0'$ and $Z \to Z_0''$ do not have effect on the intersection. Hence we get (44). Similar calculations show (45). (Note that we have $\xi = w\,\ell_1\ell_2\ell_3$.) $\square$

As in the previous subsection we put $\mathcal{L}_\lambda^+ = \{L_\theta^+ \,|\, \theta \in \mathbf{R}\}$ and $\mathcal{L}_\lambda^- = \{L_\theta^- \,|\, \theta \in \mathbf{R}\}$. (Note that this time we explicitly specified L_θ^+ and L_θ^- respectively in Lemma 5.10.) $U(1)$ again acts transitively on the parameter spaces of these families. By Lemma 5.11, $L_\theta^+ \cap \Gamma_1$ is a point, and $\{L_\theta^+ \cap \Gamma_1 \,|\, \theta \in \mathbf{R}\}$ is a circle in Γ_1 whose radius is

$$h_1(\lambda) := 2B \cdot |x_1/\ell_1|.$$

Similarly, $\{L_\theta^- \cap \Gamma_3 \,|\, \theta \in \mathbf{R}\}$ is a circle in Γ_3 whose radius is

$$h_3(\lambda) := (-f/2B) \cdot |x_1^3/\ell_1\ell_2\ell_3|.$$

(h_2 will appear in the next subsection.) Then we have the following proposition, which implies that the normal bundles of L^+_θ and L^-_θ in Z are determined by the behavior of h_1 and h_3 respectively.

Proposition 5.12. *Assume $\lambda \in I_1 \cup I_3$ and take any $L = L^+_\theta \in \mathcal{L}^+_\lambda$. Then either $N_{L/Z} \simeq O(1)^{\oplus 2}$ or $N_{L/Z} \simeq O \oplus O(2)$ holds. Further, the latter holds iff λ is a critical point of h_1 above. The same claims also hold for $\mathcal{L}^-_\lambda$ if we replace h_1 by h_3.*

Proof. The first claim can be proved in the same way as in Proposition 5.3. The other claims can also be proved in the same manner as in Proposition 5.3: take any $L \in \mathcal{L}^+_\lambda$. Then the two one-parameter families of L in Z in the previous proof make senses also in this case, so that we again have two linearly independent sections s and t of N_λ, $N_\lambda = N_{L/Z}$. Then the previous proof works if we replace h_0 by h_1, $\Phi^{-1}(l_\infty)$ by $\Gamma_1 \cup \overline{\Gamma}_1$, and (29) by (44). For $L \in \mathcal{L}^-_\lambda$, replace h_0 by h_3, $\Phi^{-1}(l_\infty)$ by $\Gamma_3 \cup \overline{\Gamma}_3$, and (29) by (45). $\square$

By definition, h_1 and h_3 depend on the choice of ℓ_1, ℓ_2 and ℓ_3. Therefore, Proposition 5.12 implies that the normal bundles of L^+ and L^- in Z depend on how we resolve p_∞. More precisely, the normal bundle of L^+ depends on the choice of ℓ_1 only, whereas the normal bundle of L^- depends on that of $\{\ell_1, \ell_2, \ell_3\}$.

Thus we need to know the critical points of h_1 and h_3 for every choices of ℓ_1, ℓ_2 and ℓ_3. At first glance there may seem to be too many functions to be investigated, but it is easily seen that h_3 is a reciprocal of h_1 (up to a constant) for some other choice of ℓ_1, ℓ_2 and ℓ_3. Therefore, what we need to know is the behavior of h_1 for the four choices of ℓ_1. (Behavior of these functions near the endpoints of I_1 and I_3 below will be needed in §5.5.)

Lemma 5.13. *(i) If $\ell_1 = x_1$, h_1 has no critical point on I_1, and has a unique critical point on I_3. Further, we have $\lim_{\lambda\downarrow -\infty} h_1(\lambda) = +\infty$ and $h_1(-1) = 0$. (ii) If $\ell_1 = x_0$, h_1 has a unique critical point on I_1, and no critical point on I_3. Further, we have $\lim_{\lambda\downarrow 0} h_1(\lambda) = +\infty$ and $h_1(b/a) = 0$. (iii) If $\ell_1 = x_0 + x_1$, h_1 has no critical point on I_1, and has a unique critical point on I_3. Further, we have $\lim_{\lambda\downarrow -\infty} h_1(\lambda) = 0$ and $\lim_{\lambda\uparrow -1} h_1(\lambda) = +\infty$. (iv) If $\ell_1 = ax_0 - bx_1$, h_1 has a unique critical point on I_1, and no critical point on I_3. Further, we have $h_1(0) = 0$ and $\lim_{\lambda\uparrow b/a} h_1(\lambda) = +\infty$. (See Figure 4.)*

Proof. (i) If $\ell_1 = x_1$, we have $h_1^2 = 2(\sqrt{Q^2 - f} - Q)$. Since $h_1 > 0$ on $I_1 \cup I_3$, the critical points of h_1^2 and h_1 coincide on $I_1 \cup I_3$. We think of h_1^2 as a real valued function defined on the whole of $\mathbf{R}$, but

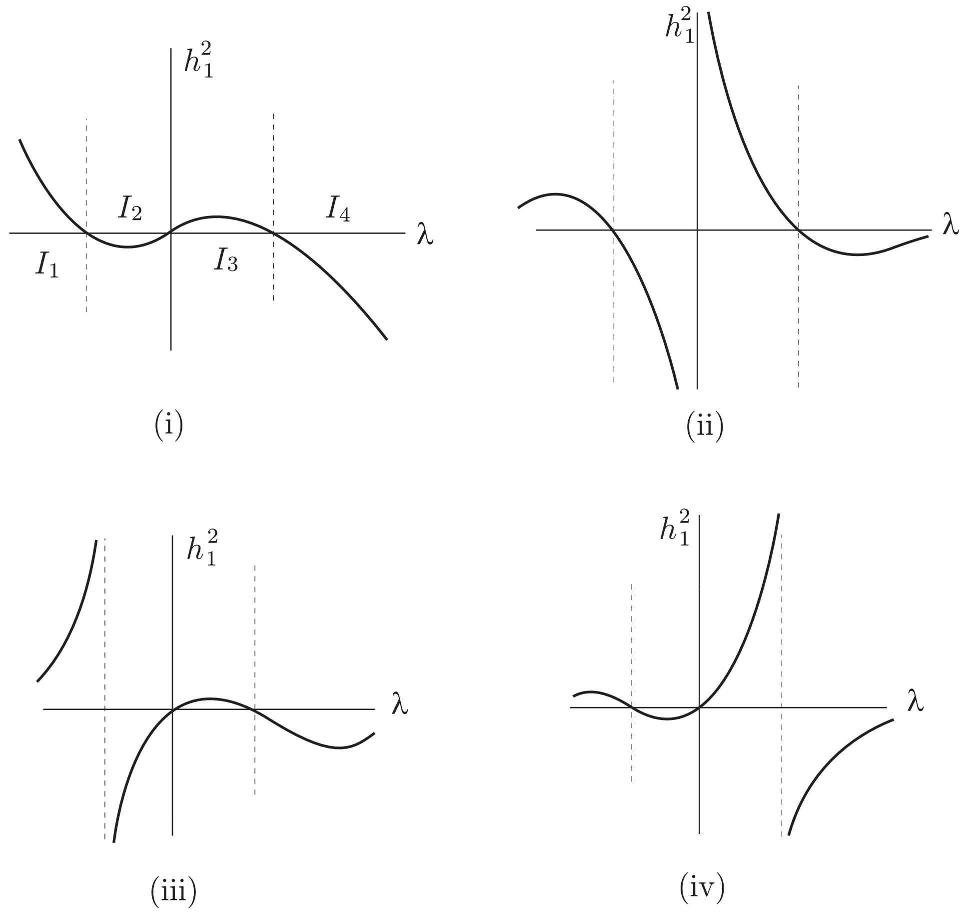


FIGURE 4. behaviors of h_1^2

which is not differentiable at $\lambda = \lambda_0$ in general. It is immediate to see that $h_1^2(-1) = h_1^2(0) = h_1^2(b/a) = 0$, $\lim_{\lambda\downarrow-\infty} h_1^2(\lambda) = +\infty$ and $\lim_{\lambda\uparrow\infty} h_1^2(\lambda) = -\infty$. Then because h_1^2 is differentiable on $\lambda \neq \lambda_0$, h_1^2 has a critical point on I_2 and I_3 respectively. On the other hand, we have

$$\left(\sqrt{Q^2 - f} - Q\right)' = \frac{2QQ' - f' - 2Q'\sqrt{Q^2 - f}}{2\sqrt{Q^2 - f}},$$

and it follows that $(\sqrt{Q^2 - f} - Q)' = 0$ implies

$$(2QQ' - f')^2 = 4Q'^{\,2}(Q^2 - f). \tag{46}$$

It is readily seen that the degree of both hand sides of (46) are six, and that both have λ_0 as a double root. have $(\lambda - \lambda_0)^2$ as a factor. Since we have already got two critical points of h_1^2 other than $\lambda = \lambda_0$, there are at most two solutions of (46) remaining.

We set $g := 2(-\sqrt{Q^2 - f} - Q)$ which is also defined on $\mathbf{R}$ and possibly not differentiable at $\lambda = \lambda_0$. Note that if we replace h_1^2 by g on $\lambda \geq \lambda_0$, then the resulting function is differentiable at $\lambda = \lambda_0$. It is easily verified that $g' = 0$ also implies (46) and it gives a solution not coming from $(h_1^2)' = 0$. Further, we readily have $\lim_{\lambda \downarrow -\infty} g(\lambda) = \lim_{\lambda \uparrow \infty} g(\lambda) = -\infty$.

Suppose that g has a critical point. Then together with the above two critical points of h_1^2 on $I_2 \cup I_3$, we have three solutions of (46) other than $\lambda = \lambda_0$. If h_1^2 has critical points on I_1, its number is at least two. This implies that (46) has five solutions other than $\lambda = \lambda_0$ and this is a contradiction. Therefore h_1^2 has no critical points on I_1, if g has a critical point. Similarly, if the number of the critical points on I_3 is not one, then it must be at least three. This is again a contradiction. Thus if g has a critical point, h_1^2, and hence h_1 has no critical point on I_1 and a unique critical point on I_3.

So suppose that g has no critical point. This happens exactly when g attains the maximal value at $\lambda = \lambda_0$. Then we have $\lim_{\lambda \uparrow \lambda_0} g'(\lambda) > 0$, since otherwise g has a critical point on $\lambda < \lambda_0$. Because we have $\lim_{\lambda \uparrow \lambda_0} g'(\lambda) = \lim_{\lambda \downarrow \lambda_0} (h_1^2)'(\lambda)$, we get $\lim_{\lambda \downarrow \lambda_0} (h_1^2)'(\lambda) > 0$. Since $\lim_{\lambda \uparrow \infty} h_1^2(\lambda) = -\infty$, it follows that h_1^2 has a critical point on I_4. Thus we get three solutions of (46) other than λ_0. Then the same argument in the case that g has a critical point as above, we can deduce that h_1 has no critical point on I_1 and a unique critical point on I_3. Thus we get the claim of (i) concerning critical points of h_1. The remaining claims of (i) immediately follows from the definition of h_1.

Claims of (ii), (iii) and (iv) about critical points can be obtained by applying a projective transformation $\lambda \mapsto 1/\lambda$ for the case (ii), $\lambda \mapsto 1/(\lambda + 1)$ for the case (iii), and $\lambda \mapsto 1/(a\lambda - b)$ for the case (iv) respectively. The other claims are immediate to see. $\square$

As is already mentioned, the behavior of h_3 can be easily seen from that of h_1 for some other choice of ℓ_1, ℓ_2 and ℓ_3. The result is the following:

Lemma 5.14. *(i) If* $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, ax_0 - bx_1\}$*,* h_3 *has no critical point on* I_1*, and has a unique critical point on* I_3*. Further, we have* $\lim_{\lambda \downarrow -\infty} h_3(\lambda) = 0$ *and* $\lim_{\lambda \uparrow -1} h_3(\lambda) = +\infty$*. (ii) If* $\{\ell_1, \ell_2, \ell_3\} = \{x_1, x_0 + x_1, ax_0 - bx_1\}$*,* h_3 *has a unique critical point on* I_1*, and no critical point on* I_3*. Further, we have* $h_3(0) = 0$ *and* $\lim_{\lambda \uparrow b/a} h_3(\lambda) = +\infty$*. (iii) If* $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, ax_0 - bx_1\}$*,* h_3 *has no critical point on* I_1*, and has a unique critical point on* I_3*. Further, we have* $\lim_{\lambda \downarrow -\infty} h_3(\lambda) = +\infty$ *and* $h_3(-1) = 0$*. (iv) If* $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0 + x_1\}$*,* h_3 *has a unique critical point on* I_1*,* *and no critical point on* I_3*. Further, we have* $\lim_{\lambda\downarrow 0} h_3(\lambda) = +\infty$ *and* $h_3(b/a) = 0$*.*

Corollary 5.15. *For any choice of* ℓ_1, ℓ_2 *and* ℓ_3*, the following (i) and (ii) hold: (i) members of* $\mathcal{L}_\lambda^+$ $(\lambda \in I_1)$ *and* $\mathcal{L}_\lambda^+$ $(\lambda \in I_3)$ *cannot be twistor lines at the same time, (ii) the same claim holds also for* $\mathcal{L}_\lambda^-$*.*

Proof. Suppose that $\lambda \in I_1 \cup I_3$ is a critical point of h_1. Then by Proposition 5.12, any member of $\mathcal{L}_\lambda^+$ is not a twistor line because its normal bundle in Z is $O \oplus O(2)$. Then just as in the proof of Proposition 5.5, any member of $\mathcal{L}_\mu^+$ cannot be a twistor line provided that $\mu \in I_1 \cup I_3$ and λ belong to the same interval (I_1 or I_3). By Lemma 5.13, h_1 necessarily has a critical point on just one of I_1 and I_3. Hence (i) holds. The proof is the same for $\mathcal{L}_\lambda^-$ if we use Lemma 5.14 instead. $\square$

Thus together with Proposition 5.12, we have obtained new families of real smooth rational curves which have $O(1)^{\oplus 2}$ as their normal bundles, but which are not twistor lines.

Proposition 5.12 and Lemmas 5.13 and 5.14 enable us to determine the normal bundles of L_θ^+ and L_θ^- in Z for every choices of small resolutions of p_∞. In particular, the normal bundles of L_θ^+ and L_θ^- in Z, and also which component has to be chosen as candidates of twistor lines, depend on the choice made.

5.4. **The case of orbit type.** Suppose $\lambda \in I_2 \cup I_4$. In this subsection we calculate the normal bundles of L_α^+ and L_α^- in Z, where L_α^+ and L_α^- are curves which are mapped biholomorphically onto a real touching conic $C_\alpha \in \mathcal{C}_\lambda^{\rm orb}$ defined by (19). Note again that C_α and $L_\alpha^\pm$ depend not only on α, but also on $\lambda \in I_2 \cup I_4$. Compared to generic type and special type, calculations are much easier since the equations of touching conics of orbit type are much simpler.

First we make a distinction of L_α^+ and L_α^-. We use local coordinates (x_0, x_1, x_2, z) and (x_0, x_1, ξ, η) as in the previous subsection. Recall that $\Phi_0^{-1}(H_\lambda)$ is defined by $\xi\eta = f x_1^4$, and that the equation of irreducible components of $\Phi_0^{-1}(C_\alpha)$ is given by $z = \pm(f - (\alpha + Q)^2)^{\frac{1}{2}} x_1^2$ ((28)). Then we denote by L_α^+ (resp. L_α^-) the components corresponding to $z = (f - (\alpha + Q)^2)^{\frac{1}{2}} x_1^2$ (resp. $z = -(f - (\alpha + Q)^2)^{\frac{1}{2}} x_1^2$). L_α^+ and L_α^- are curves in Z.

Recall that in the previous subsection we have introduced an affine coordinate $v = \xi/\ell_1\ell_2$ on the exceptional curve Γ_2. Points on Γ_2 are indicated by using this v.

Lemma 5.16. *Let* L_α^+ *and* L_α^- *be as above. Then* $L_\alpha^+ \cap \Gamma_1, L_\alpha^+ \cap \Gamma_3, L_\alpha^- \cap \Gamma_1$ *and* $L_\alpha^- \cap \Gamma_3$ *are empty, and* $L_\alpha^+ \cap \Gamma_2$ *and* $L_\alpha^- \cap \Gamma_2$ *are points satisfying* *respectively*

$$L_\alpha^+ \cap \Gamma_2 = \left\{ v = \left(\sqrt{f - (\alpha + Q)^2} + i(\alpha + Q) \right) \frac{x_1^2}{\ell_1 \ell_2} \right\}$$

and

$$L_\alpha^- \cap \Gamma_2 = \left\{ v = \left(-\sqrt{f - (\alpha + Q)^2} + i(\alpha + Q) \right) \frac{x_1^2}{\ell_1 \ell_2} \right\}.$$

Here, note again that $x_1^2/\ell_1\ell_2$ does not depend on x_1.

Proof. Substituting $x_2 = \alpha x_1^2$, we have

$$\xi = z + i(x_2 + Qx_1^2) = \left\{ \pm\sqrt{f - (\alpha + Q)^2} + i(\alpha + Q) \right\} x_1^2$$

and

$$\eta = z - i(x_2 + Qx_1^2) = \left\{ \pm\sqrt{f - (\alpha + Q)^2} - i(\alpha + Q) \right\} x_1^2$$

over C_α. ($\pm$ corresponds to $L_\alpha^\pm$.) From these and from the explicit resolutions of the previous subsection, we can easily see that for any choice of ℓ_1, ℓ_2 and ℓ_3, $L_\alpha^\pm \cap \Gamma_1$ and $L_\alpha^\pm \cap \Gamma_3$ are empty and that $L_\alpha^+ \cap \Gamma_2$ and $L_\alpha^- \cap \Gamma_2$ are points satisfying

$$v = \xi/\ell_1\ell_2 = \left\{ \pm\sqrt{f - (\alpha + Q)^2} + i(\alpha + Q) \right\} \frac{x_1^2}{\ell_1 \ell_2},$$

where $\pm$ corresponds to L_α^+ and L_α^- respectively. Thus we have obtained all of the claims of the lemma. $\square$

Since $L_\alpha^\pm$ and Γ_2 are $U(1)$-invariant, $L_\alpha \cap \Gamma_2$ must be $U(1)$-fixed point. In particular, any points on Γ_2 is $U(1)$-fixed. From these lemmas, we immediately get the following

Lemma 5.17. *Fix $\lambda \in I_2 \cup I_4$. Then the set $\{(L_\alpha^+ \cup L_\alpha^-) \cap \Gamma_2 \mid -Q - \sqrt{f} \le \alpha \le -Q + \sqrt{f}\}$ is a circle in Γ_2 whose center is $\Gamma_2 \cap \Gamma_3$ ($= \{v = 0\}$) and whose radius is $\sqrt{f}\,|x_1^2/\ell_1\ell_2|$.*

The following proposition, which corresponds to Propositions 5.5 (generic type) and 5.12 (special type), can be proved by using the same idea as in Proposition 5.12. So we omit the proof.

Proposition 5.18. *Set $h_2(\lambda) = \sqrt{f}\,(x_1^2/\ell_1\ell_2)$, which is clearly differentiable on $I_2 \cup I_4$. Let N denote the normal bundle of L_α^+ in Z. Then we have either $N \simeq O(1)^{\oplus 2}$ or $N \simeq O \oplus O(2)$, and the latter holds iff λ is a critical point of h_2. The same claim holds also for L_α^-.*

Needless to say, h_2 depends on the choice of ℓ_1 and ℓ_2. Thus as in the case of special type, the normal bundles of L^+_α and L^-_α depend on the choice of small resolution of p_∞. In view of Proposition 5.18, we need to know the critical point of h_2 for each choice of $\{\ell_1, \ell_2\}$. There are $4!/(2!2!) = 6$ choices of $\{\ell_1, \ell_2\}$. If we take $\{\ell_1, \ell_2\} = \{x_0, x_1\}$ for instance, we have $h_2(\lambda)^2 = (\lambda + 1)(a\lambda - b)/\lambda$, and it is elementary to determine the critical points of this function. For any other choices, we always get h_2 in explicit form and it is easy to determine their critical points. So here we only present the result:

Lemma 5.19. *(i) If $\{\ell_1, \ell_2\} = \{x_0, x_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $h_2(-1) = 0, \lim_{\lambda\uparrow 0} h_2(\lambda) = +\infty, h_2(b/a) = 0$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = +\infty$. (ii) If $\{\ell_1, \ell_2\} = \{x_0 + x_1, ax_0 - bx_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $\lim_{\lambda\downarrow -1} h_2(\lambda) = +\infty, h_2(0) = 0, \lim_{\lambda\downarrow b/a} h_2(\lambda) = +\infty$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = 0$. (iii) If $\{\ell_1, \ell_2\} = \{x_1, x_0+x_1\}$, h_2 has no critical point on $I_2\cup I_4$. Further, $\lim_{\lambda\downarrow -1} h_2(\lambda) = +\infty, h_2(0) = 0, h_2(b/a) = 0$ and $\lim_{\lambda\uparrow\infty} h_2(\lambda) = \infty$. (iv) If $\{\ell_1, \ell_2\} = \{x_0, ax_0 - bx_1\}$, h_2 has no critical point on $I_2 \cup I_4$. Further, $h_2(-1) = 0, \lim_{\lambda\uparrow 0} h_2(\lambda) = +\infty, \lim_{\lambda\downarrow b/a} h_2(\lambda) = +\infty$. (v) If $\{\ell_1, \ell_2\} = \{x_0, x_0 + x_1\}$, or if $\{\ell_1, \ell_2\} = \{x_1, ax_0 - bx_1\}$, h_2 has a unique critical point on I_2 and I_4 respectively. (See Figure 5.)*

By Corollary 5.6, if $\lambda \in I_2$, images of twistor lines in H_λ must be of orbit type. Therefore by Proposition 5.18, if a resolution $Z \to Z_0$ yields a twistor space, h_2 does not have critical points on I_2. Hence by Lemma 5.19, we can conclude that $\{\ell_1, \ell_2\} \neq \{x_0, x_0 + x_1\}$ and $\{\ell_1, \ell_2\} \neq \{x_1, ax_0 - bx_1\}$. Namely, our investigation decreases the possibilities of small resolutions. We postpone further consequences until the next subsection.

5.5. **Consequences of the results in §5.2–5.4.** Before stating the results, we again recall our setup. Let B be a quartic surface defined by (1) and assume that Q and f satisfy the necessary conditions as in Proposition 2.6. Let $\Phi_0 : Z_0 \to \mathbf{CP}^3$ be the double covering branched along B. On $\mathbf{CP}^3$ there is a pencil of $U(1)$-invariant planes $\{H_\lambda\}$, where H_λ is defined by $x_0 = \lambda x_1$ which is real iff $\lambda \in \mathbf{R} \cup \{\infty\}$. For any small resolution $\mu : Z \to Z_0$ preserving the real structure, we put $\Phi := \Phi_0\mu$, and let $\{S_\lambda = \Phi^{-1}(H_\lambda)\}$ be (the real part of) a pencil of $U(1)$-invariant divisors on Z, where we put $S_\lambda = \Phi^{-1}(H_\lambda)$ as before.

We start with the following theorem, which uniquely determines the type of real touching conics which can be the images of twistor lines contained in S_λ's above.

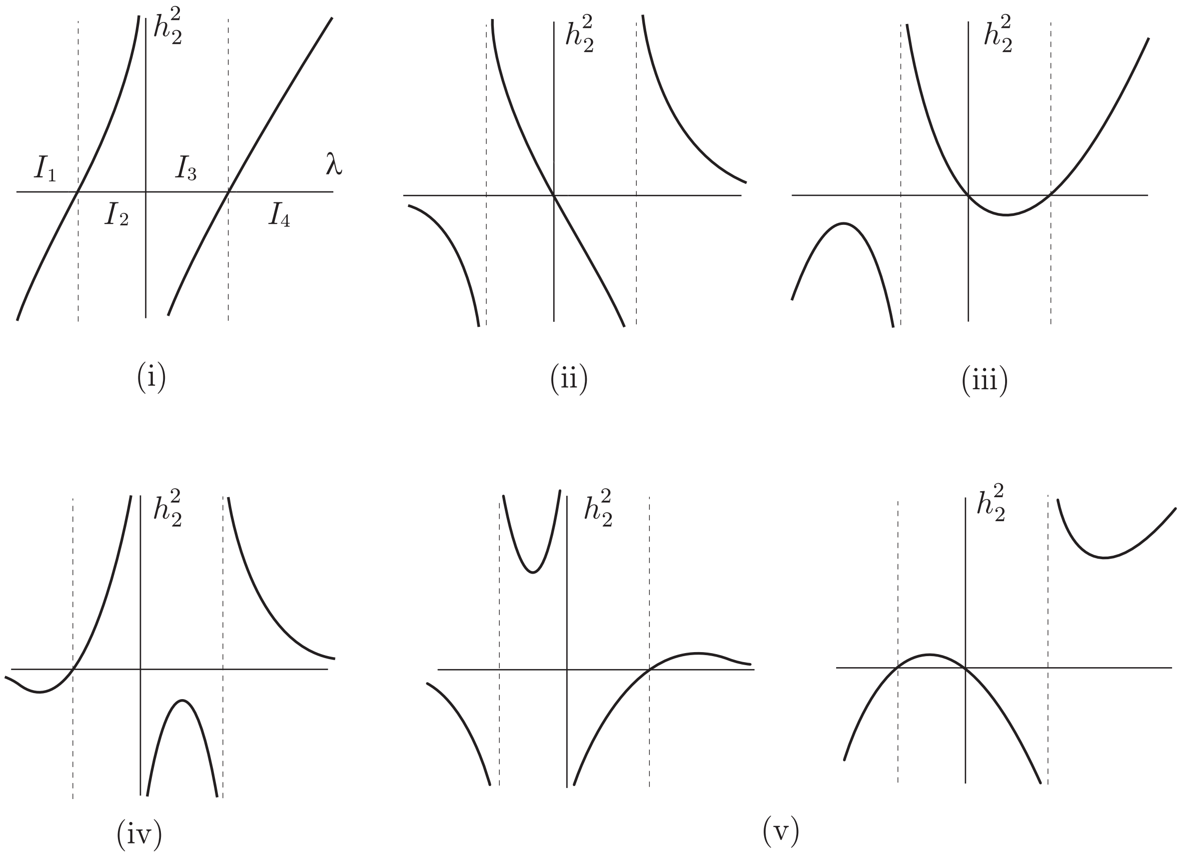


FIGURE 5. behaviors of h_2^2

Theorem 5.20. *Suppose that there is a small resolution $Z \to Z_0$ such that Z is a twistor space. Let L be a twistor line of Z contained in S_λ, $\lambda \in \mathbf{R}$. Then $\Phi(L)$ is a real touching conic of: (i) special type if $\lambda \in I_1 \cup I_3$, (ii) orbit type if $\lambda \in I_2$, (iii) generic type if $\lambda \in I_4$ and if $\lambda \neq \lambda_0$.*

Note that by Proposition 3.2, $\Phi(L) \subset H_\lambda$ is a line if $\lambda = \lambda_0$.

Proof. (i) immediately follows from (ii) of Proposition 3.8 and (v) of Proposition 4.4. (ii) is just Corollary 5.6. Finally we show (iii). By (i) of Proposition 3.8 it suffices to show that if $\lambda \in I_4$, the image cannot be of orbit type. In view of Lemma 5.19, we have $h_2(I_2) = (0, \infty)$ and $h_2(I_4) = (0, \infty)$ for any of the cases (i)–(iv) of the lemma. (We have already seen that the case (v) can be eliminated.) This implies that the circles appeared in Lemma 5.16 sweep out $\Gamma_2 \backslash \{\Gamma_2 \cap \Gamma_1, \Gamma_2 \cap \Gamma_3\}$. Therefore, $L_\alpha^\pm \subset S_\lambda$ with $\lambda \in I_2$, and $L_\alpha^\pm \subset S_\lambda$ with $\lambda \in I_4$ cannot be the images of twistor lines at the same time. Therefore, if $\lambda \in I_4$

and if $\lambda \neq \lambda_0$, the images of twistor lines must be of generic type, as required. $\square$

The following theorem is the main result of this section. Recall that p_∞ is a compound A_3-singularity of Z_0, and there are $4! = 24$ choices of small resolutions of p_∞ (see §5.3). Recall also that once a resolution of p_∞ is given, it naturally induces that of $\overline{p}_\infty$ by reality.

Theorem 5.21. *Among 24 ways of possible small resolutions of p_∞, 22 resolutions do not yield a twistor space. The remaining two resolutions are given by the following two choices of linear forms:*

$$\ell_1 = x_1, \ \ \ell_2 = x_0 + x_1, \ \ \ell_3 = x_0,$$

and

$$\ell_1 = ax_0 - bx_1, \ \ \ell_2 = x_0, \ \ \ell_3 = x_0 + x_1.$$

Here we do not yet claim that the threefolds obtained by these two resolutions are actually twistor spaces.

Proof. By Theorem 5.20 (i), if $\lambda \in I_1$, the images of twistor lines in S_λ are real touching conics of special type. As in Section 5.3, there are two families $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ of real rational curves which are candidates of twistor lines in S_λ. As we have already remarked in Section 4, $L_\theta^+ \in \mathcal{L}_\lambda^+$ and $L_\theta^- \in \mathcal{L}_\lambda^-$ cannot be twistor lines simultaneously. Suppose first that (any of the) members of $\mathcal{L}_\lambda^+$ are twistor lines. Then by Proposition 5.12, the function h_1 does not have critical points on I_1. By Lemma 5.13, this implies that we have either

$$\ell_1 = x_1 \ \text{ or } \ \ell_1 = x_0 + x_1. \tag{47}$$

On the other hand, by Corollary 5.15 (i), under our assumption, members of $\mathcal{L}_\lambda^-$ are twistor lines for $\lambda \in I_3$. Therefore again by Proposition 5.12, h_3 does not have critical points on I_3. Then by Lemma 5.14, the cases (i) and (iii) of the lemma are eliminated and we have either

$$\{\ell_1, \ell_2, \ell_3\} = \{x_1, x_0+x_1, ax_0-bx_1\} \ \text{ or } \ \{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, x_0+x_1\}. \tag{48}$$

(Note here that we do not specify the order.)

Next we consider twistor lines in S_λ for $\lambda \in I_2$. By Theorem 5.20 (ii) the images are real touching conics of orbit type. Then Proposition 5.18 implies that h_2 has no critical point on I_2. Hence by Lemma 5.19, we have either

$$\{\ell_1, \ell_2\} = \{x_0, x_1\} \ \text{or} \ \{x_0+x_1, ax_0-bx_1\} \ \text{or} \ \{x_1, x_0+x_1\} \ \text{or} \ \{x_0, ax_0-bx_1\}. \tag{49}$$

Now we note other restrictions: namely, when λ increases to pass from I_1 to I_2, twistor lines in S_λ varies continuously, so that we have

$$\lim_{\lambda\uparrow -1} h_1(\lambda) = \left(\lim_{\lambda\downarrow -1} h_2(\lambda)\right)^{-1}. \tag{50}$$

(Here the inverse of the right hand side is a consequence of the fact that $\Gamma_1 \cap \Gamma_2 = \{u = 0\} = \{v = \infty\}$.) Similarly, moving λ from I_2 to I_3, we have

$$\lim_{\lambda\uparrow 0} h_2(\lambda) = \left(\lim_{\lambda\downarrow 0} h_3(\lambda)\right)^{-1}. \tag{51}$$

Take $\ell_1 = x_1$ for the first example. Then by Lemma 5.13 (i) we have $h_1(-1) = 0$. Hence it follows from (50) that $\lim_{\lambda\downarrow -1} h_2(\lambda) = \infty$. Then the cases (i) and (iv) of Lemma 5.19 fail and we have $\{\ell_1, \ell_2\} = \{x_0 + x_1, ax_0 - bx_1\}$ ((ii)) or $\{\ell_1, \ell_2\} = \{x_1, x_0 + x_1\}$ ((iii)). The former clearly fails and we get $\ell_2 = x_0 + x_1$. This appears in (49). Then we have from Lemma 5.19 (iii) that $h_2(0) = 0$. Hence by (51), we have $\lim_{\lambda\downarrow 0} h_3(\lambda) = \infty$. It then follows from Lemma 5.14 that $\ell_3 = x_0$. Thus we get $\ell_1 = x_1, \ell_2 = x_0 + x_1, \ell_3 = x_0$.

Next take $\ell_1 = x_0 + x_1$. Then we have $\lim_{\lambda\uparrow -1} h_1(\lambda) = +\infty$ (Lemma 5.13 (iii)), so that $\lim_{\lambda\downarrow -1} h_2(\lambda) = 0$. Then looking (i)–(iv) of Lemma 5.19, this possibility fails. Namely, we have $l_1 \neq x_0 + x_1$. Thus we can conclude that if $L^+_\theta \in \mathcal{L}^+_\lambda$ is a twistor line over I_1, it follows that $\ell_1 = x_1, \ell_2 = x_0 + x_1$ and $\ell_3 = x_0$. This is the former candidate of the theorem.

Next suppose that $L^-_\theta \in \mathcal{L}^-_\lambda$ is a twistor line over I_1 and repeat similar argument above. By Proposition 5.12, h_3 has no critical point on I_1. It then follows from Lemma 5.14 that either $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, ax_0 - bx_1\}$ ((i)) or $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, ax_0 - bx_1\}$ ((iii)) holds. On the other hand, (49) is valid also in this case. Further we have as before

$$\lim_{\lambda\uparrow -1} h_3(\lambda) = \left(\lim_{\lambda\downarrow -1} h_2(\lambda)\right)^{-1} \quad \text{and} \quad \lim_{\lambda\uparrow 0} h_2(\lambda) = \left(\lim_{\lambda\downarrow 0} h_1(\lambda)\right)^{-1}. \tag{52}$$

If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_0 + x_1, ax_0 - bx_1\}$, then $\lim_{\lambda\uparrow -1} h_3(\lambda) = +\infty$ (Lemma 5.14 (i)), so that we have $\lim_{\lambda\downarrow -1} h_2(\lambda) = 0$ by (52). Hence by Lemma 5.19 we have $\{\ell_1, \ell_2\} = \{x_0, ax_0 - bx_1\}$, which implies $\lim_{\lambda\uparrow 0} h_2(\lambda) = \infty$ ((iv) of Lemma 5.19) and $\ell_3 = x_0 + x_1$. Hence $\lim_{\lambda\downarrow 0} h_1(\lambda) = 0$ by (52). It follows from Lemma 5.13 that $\ell_1 = ax_0 - bx_1$, which means $\ell_2 = x_0$. (iv) of Lemma 5.13 says that h_1 has no critical point on I_3, which is consistent with the fact that L^+_θ is a twistor line over I_3.

If $\{\ell_1, \ell_2, \ell_3\} = \{x_0, x_1, ax_0 - bx_1\}$, $h_3(-1) = 0$ (Lemma 5.14 (iii)), so that we have $\lim_{\lambda \downarrow -1} h_2(\lambda) = \infty$ by (52). Therefore we get the two possibilities (ii) and (iii) of Lemma 5.19, but both contain $x_0 + x_1$ which is not compatible with our choice of $\{\ell_1, \ell_2, \ell_3\}$. Thus we have $\{\ell_1, \ell_2, \ell_3\} \neq \{x_0, x_1, ax_0 - bx_1\}$. This implies that if $L_\theta^- \in \mathcal{L}_\lambda^-$ is a twistor line for $\lambda \in I_1$, then $\ell_1 = ax_0 - bx_1, \ell_2 = x_0$, and $\ell_3 = x_0 + x_1$. This is the latter candidate of the theorem, and we have completed the proof. $\square$

At first sight it may not be evident why there are two choices of small resolutions which can yield twistor spaces. But our proof shows the difference of them. To explain this, for each $\lambda \in I_1 \cup I_2 \cup I_3$, let $L_\lambda \subset Z$ be any of the members of $\mathcal{L}_\lambda^\pm$ which are chosen as candidates of twistor lines in the proof of Theorem 5.21. Namely, if $\ell_1 = x_1, \ell_2 = x_0 + x_1, \ell_3 = x_0$ (the former case), members of $\mathcal{L}_\lambda^+$ (resp. $\mathcal{L}_\lambda^-$) must be chosen for $\lambda \in I_1$ (resp. $\lambda \in I_3$). If $\ell_1 = ax_0 - bx_1, \ell_2 = x_0, \ell_3 = x_0 + x_1$ (the latter case), members of $\mathcal{L}_\lambda^-$ (resp. $\mathcal{L}_\lambda^+$) have to be chosen for $\lambda \in I_1$ (resp. $\lambda \in I_3$). (For $\lambda \in I_2$ any members of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ must be chosen simultaneously as in Proposition 4.4.) Our proof shows that when $\lambda \in \mathbf{R}$ increases from $-\infty$ to b/a, the intersection $\Gamma \cap L_\lambda$ moves from Γ_1 to Γ_3 for the former choice, whereas it moves from Γ_3 to Γ_1 for the latter choice. Namely, exchanging the choice reverses the orientation of $\Gamma \cap L_\lambda$ as λ increases.

By similar consideration we can determine which irreducible component must be chosen over $I_4^- \cup I_4^+$. First take the small resolution of p_∞ determined by the first choice of ℓ_1, ℓ_2 and ℓ_3 in Theorem 5.21. Then as mentioned above, over I_3, members of $\mathcal{L}_\lambda^-$ must be chosen. Further, we have $h_3(b/a) = 0$ by Lemma 5.14 (iv). This implies that as λ goes to b/a, the circle of intersection $\cup\{\Gamma_3 \cap L_\theta^- \mid L_\theta^- \in \mathcal{L}_\lambda^-\}$ shrinks to be the $U(1)$-fixed point of Γ_3 which is different from $\Gamma_2 \cap \Gamma_3$. On the other hand, on the irreducible component of $\Phi^{-1}(\ell_\infty)$ which intersects Γ_3, and which is different from Γ_2, one can use $x_2 = y_2/y_3$ as an affine coordinate whose center is the intersection point with Γ_3. Therefore, the circle of intersection appeared in Lemma 5.2 also must shrink to be the origin as λ decreases to be b/a. This uniquely determines which one of $\mathcal{L}_\lambda^+$ and $\mathcal{L}_\lambda^-$ has to be chosen for $\lambda \in I_4^-$. Then for $\lambda \in I_4^+$ another irreducible component must be chosen. The case of the latter choice of ℓ_1, ℓ_2 and ℓ_3 is now obvious. Namely, we have to choose the irreducible component over I_4^- and I_4^+ which are different from the former case.

6. Twistor lines whose images are lines

In this section we study real lines whose images are lines in $\mathbf{CP}^3$. Recall we have shown in Proposition 3.2 that if L is a real line intersecting Γ_0, then $\Phi(L)$ is a line going through P_0. The following proposition is its converse.

Proposition 6.1. *Let $l \subset \mathbf{CP}^3$ be any real line going through P_0. Then $\Phi^{-1}(l)$ has just two irreducible components, both of which are real, smooth and rational. One of the components is the exceptional curve Γ_0 and another component is mapped (2 to 1) onto l. Further, the normal bundle of the latter component in Z is isomorphic to $O(1)^{\oplus 2}$.*

Proof. First we note that if l is a real line, $B \cap l$ consists of just three points, one of which is P_0. This follows from the facts that, B is a quartic, $B \cap l$ is real, P_0 is the unique real point of B (Proposition 2.5), and P_0 is a double point. Therefore $\Phi_0^{-1}(l) \to l$ is two-to-one covering branched at three points. Let P and $\overline{P}$ be the two branch points other than P_0. Because l intersects B transversally at these two points, $\Phi_0^{-1}(P)$ and $\Phi_0^{-1}(\overline{P})$ are smooth points of $\Phi_0^{-1}(l)$. Further, since P_0 is an ordinary double point of B, $\Phi_0^{-1}(P_0)$ is a node of $\Phi_0^{-1}(l)$. From these it follows that $\Phi^{-1}(l)$ has just two irreducible components, one of which is Γ_0. Let L be the irreducible component other than Γ_0. Then L is smooth and $L \to l$ is two-to-one covering whose branch points are P and $\overline{P}$. Therefore by Hurwitz, L is a rational curve. L is real since $\Phi_0^{-1}(l)$ is real.

It remains to show that $N_{L/Z} \simeq O(1)^{\oplus 2}$. The idea is similar to Propositions 5.3, 5.12, and 5.18. We first show that $N_{L/Z} \simeq O(1)^{\oplus 2}$ or $N_{L/Z} \simeq O \oplus O(2)$. By Bertini, $H \cap B$ is smooth outside P_0 for general plane H containing l. Further, since $H \cap B$ is a quartic, $S := \Phi^{-1}(H)$ is a smooth rational surface with $c_1^2 = 2$. Moreover, $\Phi^{-1}(l)$ is an anti-canonical curve of S so that we have $(\Gamma_0 + L)^2 = 2$ on S. Furthermore, it is readily seen that $\Gamma_0^2 = -2$ and $\Gamma_0 \cdot L = 2$ on S. Therefore we have $L^2 = 0$ on S. Then the argument in the proof of Proposition 5.3 implies $N_{L/Z} \simeq O(1)^{\oplus 2}$ or $N_{L/Z} \simeq O(2) \oplus O$. To show that the latter does not hold, we first see that $\Gamma_0/\langle\sigma\rangle$ is canonically identified with the projective space of real lines going through P_0. Concretely, for each real line $l \ni P_0$, we associate the intersection $\Gamma_0 \cap (\Phi^{-1}(l) - \Gamma_0)$ which is a conjugate pair of points. We show by explicit calculation that this correspondence, which we will denote by ψ, is actually an isomorphism. The problem being local, we use a local coordinate (w_1, w_2, w_3) (around P_0) defined in (2). Then in a neighborhood of P_0, Z_0 is given by the equation

$$z^2 + w_1^2 + w_2 w_3 = 0. \tag{53}$$

Small resolutions of the double point $p_0 = \Phi_0^{-1}(P_0) \in Z_0$ are explicitly obtained by blowing-up along $\{z+iw_1 = w_3 = 0\}$ or $\{z+iw_1 = w_2 = 0\}$. In the former case, we can use $(z + iw_1 : w_3) = (-w_2 : z - iw_1)$ as a homogeneous coordinate on Γ_0, whereas in the latter case we can use $(z + iw_1 : w_2) = (-w_3 : z - iw_1)$ instead. We see only in the former case, since the calculation is identical. Let $(w_1 : w_2 : w_3)$ be a real line through P_0. Namely, we assume $w_1 \in \mathbf{R}$, $\overline{w}_2 = w_3$, and $w_1^2 + |w_2|^2 \neq 0$. Then by (53), we have $z = \pm i(w_1^2 + |w_2|^2)^{1/2}$. Hence we get $(z + iw_1 : w_3) = (i(w_1 \pm (w_1^2 + |w_2|^2)^{1/2}) : \overline{w}_2)$. Namely, ψ is explicitly given by

$$\psi : (w_1 : w_2 : \overline{w}_2) \longmapsto \left(i \left(w_1 \pm \sqrt{w_1^2 + |w_2|^2} \right) : \overline{w}_2 \right). \tag{54}$$

(Note that the image of (54) is considered as a point of $\Gamma_0/\langle\sigma\rangle$.) First suppose $w_1 \neq 0$. It is readily seen that we can suppose $w_1 = 1$. Then in (54) the image becomes $(i(1 \pm (1 + |w_2|^2)^{1/2}) : \overline{w}_2)$. Taking the sign '+', (54) can be rewritten as

$$\psi : \mathbf{C} \ni w_2 \mapsto \frac{-i\overline{w}_2}{1 + \sqrt{1 + |w_2|^2}}, \tag{55}$$

where we use (the second entry)/(the first entry) as an affine coordinate on Γ_0. The image of (55) is clearly contained in the unit disk $\{u \in \mathbf{C} \,|\, |u| < 1\}$. We show that (55) give a diffeomorphism between $\mathbf{C} = \mathbf{R}^2$ and the unit disk. Putting $w_2 = re^{i\theta}$, (55) is rewritten as

$$\psi : re^{i\theta} \longmapsto \frac{-ire^{-i\theta}}{1 + \sqrt{1 + r^2}}. \tag{56}$$

It is elementary to show that $k(r) := r/(1+\sqrt{1 + r^2})$ is differentiable on $\{r > 0\}$ and its derivative is always positive, and that $\lim_{r\uparrow\infty} k(r) = 1$ and $\lim_{r\downarrow 0} k(r) = 0$ hold. Hence k gives a bijection between $\{r \geq 0\}$ and $\{0 \leq s < 1\}$. It follows that (55) gives a bijection between $\mathbf{C}$ and the unit disk. Moreover, the positivity of k' implies that (56) is a diffeomorphism on $\mathbf{C}^*$. For $w_2 = 0$, it can be easily checked that $(\partial w_2/\partial \overline{w}_2)(0) \neq 0$. Therefore (55) is a diffeomorphism on $\mathbf{C}$.

Next consider the case $w_1 = 0$. Then we have $w_2 \neq 0$, and the image becomes $(\pm i|w_2| : \overline{w}_2) = (1 : \pm i\overline{w}_2/|w_2|)$. From this, it easily follows that (54) gives a diffeomorphism between the two subsets $\{(0 : 1 : w) \,|\, w \in U(1)\}$ and $\{(1 : u) \in \Gamma_0 \,|\, u \in U(1)\}/\langle\sigma\rangle$. Moreover on $\mathbf{RP}^2 \backslash \mathbf{R}^2$ we can use $(1/r, \theta)$ as a local coordinate on $\mathbf{RP}^2$.

Then we can readily show that $(d/ds)(k(1/s))|_{s=0} \neq 0$. This implies that ψ is diffeomorphic also in a neighborhood of $\mathbf{RP}^2\backslash\mathbf{R}^2$, the infinite circle. Note that the bijectivity of ψ implies that if $l \neq l'$, then the corresponding rational curves in Z are disjoint.

Next take any real plane H containing l. On H there is a one-dimensional family of lines through P_0. Taking the inverse image, we obtain a one-dimensional holomorphic family $\mathcal{L}_H$ of rational curves in Z, containing L as a real member. Any real member of $\mathcal{L}_H$ defines a conjugate pair of points as the intersection with Γ_0. Consequently, $\mathcal{L}_H$ determines a real circle $\mathcal{C}_H$ in Γ_0. (Namely $\mathcal{C}_H = \{L' \cap \Gamma_0 \,|\, L' \in \mathcal{L}_H^{\sigma}\}$.) If s denotes the section of $N = N_{L/Z}$ associated to $\mathcal{L}_H$, then $\mathrm{Re}s(z)$ is non-zero by the diffeomorphicity of ψ, and is represented by a tangent vector of $\mathcal{C}_H$ at z, where we put $\{z, \overline{z}\} = \Gamma_0 \cap L$.

Let $\{v_1, v_2\}$ be any oriented orthogonal basis of $T_z\Gamma_0$, where we take the complex orientation and orthogonality. Then since we have the isomorphism ψ, there is a unique real plane H_i $(i = 1, 2)$ containing l such that v_i is tangent to $\mathcal{C}_{H_i}$. Let s (resp. t) be the global section of $N = N_{L/Z}$ associated to $\mathcal{L}_{H_1}$ (resp. $\mathcal{L}_{H_2}$). We now claim that $as+bt$ does not vanish at z and $\overline{z}$ simultaneously, unless $(a, b) = (0, 0)$. Putting $a = a_1 + ia_2$ and $b = b_1 + ib_2$, we readily have

$$\mathrm{Re}(as + bt) = (a_1\mathrm{Re}s - b_2\mathrm{Im}t) + (b_1\mathrm{Re}t - a_2\mathrm{Im}s). \tag{57}$$

Since $(\mathrm{Re}s)(z)$ and $(\mathrm{Re}s)(\overline{z})$ (resp. $(\mathrm{Re}t)(z)$ and $(\mathrm{Re}t)(\overline{z})$) are represented by tangent vectors of $\mathcal{C}_{H_1}$ (resp. $\mathcal{C}_{H_2}$), our choice of H_1 and H_2 implies that $(\mathrm{Re}s)(z)$ is parallel to $(\mathrm{Im}t)(z)$ and $(\mathrm{Re}t)(z)$ is parallel to $(\mathrm{Im}s)(z)$. The same is true at $\overline{z}$. Hence by (57) if $\mathrm{Re}(as + bt)(z) = 0$, then $a_1\mathrm{Re}s(z) = b_2\mathrm{Im}t(z)$ and $b_1\mathrm{Re}t(z) = a_2\mathrm{Im}s(z)$, and $\mathrm{Re}(as + bt)(\overline{z}) = 0$ implies similar equalities. Since $\mathrm{Re}s, \mathrm{Re}t, \mathrm{Im}s$ and $\mathrm{Im}t$ do not be zero at both of z and $\overline{z}$ as is already mentioned, $a_1 = 0$ iff $b_2 = 0$ and $a_2 = 0$ iff $b_1 = 0$. Therefore either $a_1b_2 \neq 0$ or $b_1a_2 \neq 0$ holds. Suppose $a_1b_2 \neq 0$. Then we show that $a_1\mathrm{Re}s(z) = b_2\mathrm{Im}t(z)$ and $a_1\mathrm{Re}s(\overline{z}) = b_2\mathrm{Im}t(\overline{z})$ cannot hold simultaneously: suppose that $a_1b_2 > 0$. Then $\mathrm{Re}s(z)$ and $\mathrm{Im}t(z)$ have the same direction and it follows that $\{\mathrm{Re}t(z), \mathrm{Re}s(z)\,(= (b_2/a_1)\mathrm{Im}t(z))\}$ is an oriented basis of $T_z\Gamma_0$. On the other hand, we have $\mathrm{Re}s(\overline{z}) = \sigma_*(\mathrm{Re}s(z))$ and $\mathrm{Re}t(\overline{z}) = \sigma_*(\mathrm{Re}t(z))$. Therefore $\{\mathrm{Re}t(\overline{z}), \mathrm{Re}s(\overline{z})\}$ is an anti-oriented basis of $T_{\overline{z}}\Gamma_0$ because σ is orientation reversing. On the other hand, $a_1\mathrm{Re}s(\overline{z}) = b_2\mathrm{Im}t(\overline{z})$ and $a_1b_2 \neq 0$ imply that $\{\mathrm{Re}t(\overline{z}), \mathrm{Re}s(\overline{z})\}$ is an oriented basis of $T_{\overline{z}}\Gamma_0$. This is a contradiction. The case $b_1a_2 > 0$ is similar. Therefore $\mathrm{Re}(as + bt)$ cannot be zero at z and $\overline{z}$ simultaneously provided $a_1b_2 \neq 0$. If $b_1a_2 \neq 0$, then $b_1\mathrm{Re}t(z) = a_2\mathrm{Im}s(z)$ and $b_1\mathrm{Re}t(\overline{z}) = a_2\mathrm{Im}s(\overline{z})$ do not hold at the same time. Thus we have shown that $\mathrm{Re}(as+bt)$ cannot be zero at z and $\overline{z}$ simultaneously for any $(a,b)\neq(0,0)$. Hence so does $as+bt$. Therefore we get $N\simeq O(1)^{\oplus 2}$ by Lemma 5.1. □

Thus in our complex manifold Z there actually exists a connected family of real rational curves parametrized by $\Gamma_0/\langle\sigma\rangle\simeq \mathbf{RP}^2$, whose normal bundle is isomorphic to $O(1)^{\oplus 2}$. By Proposition 3.2, and the canonical isomorphism ψ obtained in the previous proof, all of these real lines must be twistor lines (if Z is a twistor space). Obviously this family is $U(1)$-invariant, although general members are not $U(1)$-invariant:

Proposition 6.2. *Among this family of real lines in Z, just one member is $U(1)$-invariant. Further, the member is fixed by $U(1)$ pointwisely.*

Proof. Recall that in a neighborhood of p_0, Z_0 is defined by the equation $z^2+w_1^2+w_2w_3=0$ ((53)). It is immediate to see that the $U(1)$-action looks like $(w_1,w_2,w_3)\mapsto(w_1,tw_2,t^{-1}w_3)$ for $t\in U(1)$. Thus using homogeneous coordinates used in the last proof, the $U(1)$-action on Γ_0 is given by $(u:v)\mapsto(u:tv)$ or $(u:v)\mapsto(u:t^{-1}v)$, depending on the choice of a small resolution of p_0. Therefore only the real line corresponding to $[(1:0)](=[(0:1)])\in\Gamma_0/\langle\sigma\rangle$ is $U(1)$-fixed. In view of (54) and (2), the equation of this line is explicitly given by $y_2=y_3=0$, which is pointwisely $U(1)$-fixed by Proposition 2.1. Since $\Phi:Z\to\mathbf{CP}^3$ is $U(1)$-equivariant, it follows that the corresponding rational curve in Z is also pointwisely fixed. □

What we have done in this paper can be summarized as follows:

Proposition 6.3. *Let Z_0 be as in Proposition 2.1, where Q and f satisfy the conditions in Proposition 2.6 which is necessary for Z_0 to be birational to a twistor space. Let $\mu:Z\to Z_0$ be one of the small resolutions determined in Theorem 5.21, where the real ordinary double point is resolved in arbitrary way (as a small resolution). Set $\Phi=\mu\Phi_0$ and consider the curve $\Phi^{-1}(l_\infty)$ which is a real cycle of $U(1)$-invariant smooth rational curves consisting of eight irreducible components (cf. Figure 3). Then for any smooth points of $\Phi^{-1}(l_\infty)$, we can explicitly specify a real smooth rational curve in Z going through the points whose normal bundle is $O(1)^{\oplus 2}$. Furthermore, if Z is a twistor space, all the curves we specified must be twistor lines.*

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