NEWFORMS OF HALF-INTEGRAL WEIGHT: THE MINUS SPACE OF $S_{k+1/2}(\Gamma_0(8M))$

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ABSTRACT. We compute generators and relations for a certain 2-adic Hecke algebra of level 8 associated with the double cover of SL_2 and a 2-adic Hecke algebra of level 4 associated with PGL_2. We show that these two Hecke algebras are isomorphic as expected from the Shimura correspondence. We use the 2-adic generators to define classical Hecke operators on the space of holomorphic modular forms of weight $k+1/2$ and level $8M$ where M is odd and square-free. Using these operators and our previous results on half-integral weight forms of level $4M$ we define a subspace of the space of half-integral weight forms as a common -1 eigenspace of certain Hecke operators. Using the relations and a result of Ueda we show that this subspace, which we call the minus space, is isomorphic as a Hecke module under the Ueda correspondence to the space of new forms of weight $2k$ and level $4M$. We observe that the forms in the minus space satisfy a Fourier coefficient condition that gives the complement of the plus space but does not define the minus space.

1. Introduction

It has been observed by Waldspurger [16] that the theory of half-integral weight modular forms is more complicated for levels of the forms $8M$ and $16M$ where M is an odd integer. In particular, one of the Waldspurger's hypothesis, (H2), for his celebrated formula for central twisted L-values for an integral weight newform requires the newform to be either not supercuspidal at 2 or has level divisible by 16, that is, the corresponding half-integral weight "newform" should have level $4M$ or level divisible by 32. Ueda in an unpublished work [14] observed that the levels 2^kM where $2 \leq k \leq 6$ pose greater difficulties than general k. In this work we complete the work of Ueda to obtain a decomposition of the space of holomorphic forms of weight $k+1/2$ and level $8M$ where M is odd and square-free. An attempt at such theory has been made by Manickam, Meher and Ramakrishnan [9] but our results and methods differ completely. For more details on this difference see Remark 2 in Section 5.

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Kohnen [5, 6] defines the plus space $S^+_{k+1/2}(4M)$ to be a subspace of $S_{k+1/2}(\Gamma_0(4M))$ which is the positive eigenspace of a certain Hecke operator. In the case $M = 1$, Loke and Savin [7] gave an interpretation of the Kohnen plus space in representation theory language using a 2-adic Hecke algebra of level 4. Manickam, Ramakrishnan and Vasudevan [8] defined a new subspace inside this space as an orthogonal complement of certain subspaces. In [3] we used the Hecke algebra approach to show that the new subspace is given as a common -1 eigenspace of certain Hecke operators. In this paper we extend this approach by considering 2-adic Hecke algebras of level 8 and describe it using generators and relations. We obtain an isomorphism between certain 2-adic Hecke algebras which is part of the Shimura correspondence. We translate 2-adic Hecke elements into classical Hecke operators that satisfy the relations coming from the 2-adic elements. Once we have these operators and relations we combine it with our results for the case $4M$ to define the minus space of level $8M$ as a common -1 eigenspace of certain classical Hecke operators. Using the Ueda Hecke isomorphism between $S_{k+1/2}(\Gamma_0(8M))$ and $S_{2k}(\Gamma_0(4M))$ we show that the minus space at level $8M$ is Hecke isomorphic to the subspace of new forms in $S_{2k}(\Gamma_0(4M))$ and satisfies the strong multiplicity one. Further, if $f = \sum_{n=1}^{\infty} a_n q^n$ is in the minus space at level $8M$ then $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$. This condition is exactly opposite to Kohnen's plus space Fourier coefficient condition and is contrary to the assertion of [9].

2. Preliminaries and Notation

We will be following the notation of our paper [3]. Let k, N denote positive integers with 4 dividing N. Let $\mathcal{G}$ be the set of all ordered pairs $(\alpha, \phi(z))$ where $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(z)$ is a holomorphic function on the upper half plane $\mathbb{H}$ such that $\phi(z)^2 = t \det(\alpha)^{-1/2}(cz + d)$ with t in the unit circle S^1. For $\zeta = (\alpha, \phi(z)) \in \mathcal{G}$ define the slash operator $|[\zeta]_{k+1/2}$ on functions f on $\mathbb{H}$ by

$$f|[\zeta]_{k+1/2}(z) = f(\alpha z)(\phi(z))^{-2k-1}.$$

For an even Dirichlet character χ modulo N let

$$\Delta_0(N, \chi) := \{\alpha^* = (\alpha, j_\chi(\alpha, z)) \in \mathcal{G} \mid \alpha \in \Gamma_0(N)\} \leq \mathcal{G}.$$

where $j_\chi(\alpha, z) = \chi(d)\varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz + d)^{1/2}$, here $\varepsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$ and $\left(\frac{c}{d}\right)$ is as in Shimura's notation [11]. The space of holomorphic cusp forms $S_{k+1/2}(\Gamma_0(N), \chi)$ satisfies

$$f|[\alpha^*]_{k+1/2}(z) = f(z)$$

for all $\alpha \in \Delta_0(N, \chi)$ and we have the well-known family of Hecke operators $\{T_{p^2}\}_{p\nmid N}$, $\{U_{p^2}\}_{p|N}$ on $S_{k+1/2}(\Gamma_0(N), \chi)$ (please refer to [11] for details).

Let $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ be the non-trivial central extension of $\mathrm{SL}_2(\mathbb{Q}_p)$ by $\mu_2 = \{\pm 1\}$ given by the Kubota-Gelbart 2-cocycle defined below.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, define

$$\tau(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0; \end{cases}$$

if $p = \infty$, set $s_p(g) = 1$ while for a finite prime p

$$s_p(g) = \begin{cases} (c,d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd}, \\ 1 & \text{else}. \end{cases}$$

Define the 2-cocycle σ_p on $\mathrm{SL}_2(\mathbb{Q}_p)$ as follows:

$$\sigma_p(g,h) = \left(\tau(gh)\tau(g), \tau(gh)\tau(h)\right)_p s_p(g) s_p(h) s_p(gh).$$

Then the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is the set $\mathrm{SL}_2(\mathbb{Q}_p) \times \mu_2$ with the group law

$$(g,\ \epsilon_1)(h,\ \epsilon_2) = (gh,\ \epsilon_1\epsilon_2\sigma_p(g,h)).$$

For any subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$, we shall denote by $\overline{H}$ the complete inverse image of H in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$.

We consider the following subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$:

$$K_0^p(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ : \ c \in p^n\mathbb{Z}_p \right\},$$

$$K_1^p(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^p(p^n) \ : \ a \equiv 1 \pmod{p^n\mathbb{Z}_p} \right\}.$$

By [4, Proposition 2.8], $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ splits over $\mathrm{SL}_2(\mathbb{Z}_p)$ for odd primes p, while $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ does not split over $\mathrm{SL}_2(\mathbb{Z}_2)$ and instead splits over the subgroup $K_1^2(4)$. It follows that the center M_2 of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ is a cyclic group of order 4 generated by $(-I, 1)$.

For an open compact subgroup S of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and a genuine character γ of S, let $H(S,\gamma)$ be the subalgebra of $C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$ defined by

$$\{f \in C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)) : f(\tilde{k}\tilde{g}\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),\ \tilde{k},\ \tilde{k}' \in S\}$$

under the usual convolution

$$f_1 * f_2(\tilde{h}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{g}) f_2(\tilde{g}^{-1}\tilde{h}) d\tilde{g} = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{h}\tilde{g}) f_2(\tilde{g}^{-1}) d\tilde{g},$$

Loke and Savin [7] described $H(\overline{K_0^2(4)}, \gamma)$ for a certain γ of order 4 using generators and relations. We in [3] described $H(\overline{K_0^p(p)}, \gamma)$ for odd primes p and certain quadratic characters γ. We translated the elements of the local Hecke algebras to obtain classical operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$, $\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(2^nM))$ for p strictly dividing M, M odd and operators $\widetilde{Q}_2$, $\widetilde{Q}'_2$, $\widetilde{W}_4$ on $S_{k+1/2}(\Gamma_0(4M))$. We shall be using these operators and their properties in Section 5 of this paper.

We now set up some more notation. For $s \in \mathbb{Q}_2$, $t \in \mathbb{Q}_2^\times$ let us define the following elements of $\mathrm{SL}_2(\mathbb{Q}_2)$:

$$x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},\ y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ w(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix},\ h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let $N = \{(x(s), \epsilon) : s \in \mathbb{Q}_2,\ \epsilon = \pm 1\}$, $\bar{N} = \{(y(s), \epsilon) : s \in \mathbb{Q}_2,\ \epsilon = \pm 1\}$ and $T = \{(h(t), \epsilon) : t \in \mathbb{Q}_2^\times,\ \epsilon = \pm 1\}$ be the subgroups of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$. Then the normalizer $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)}(T)$ of T in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ consists of elements $(h(t), \epsilon)$, $(w(t), \epsilon)$ for $t \in \mathbb{Q}_2^\times$.

3. Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ modulo $\overline{K_0^2(8)}$

In this section we shall be describing the local Hecke algebra $H(S, \gamma)$ for $S = \overline{K_0^2(8)}$ and certain genuine characters γ described below.

Let γ be a genuine central character given by sending $(-I, 1)$ to a primitive fourth root of unity. We extend γ to $\overline{K_0^2(8)}$ as follows. Extend γ to $K_1^2(8) \times M_2$ so that it is trivial on $K_1^2(8)$. Since $\overline{K_0^2(8)}/(K_1^2(8) \times M_2)$ is generated by $(h(5), 1)$ it is enough to define it on $(h(5), 1)$. Since $(h(5), 1)$ has order 2 there are two choices for $\gamma((h(5), 1))$. We will denote the resulting two characters of $\overline{K_0^2(8)}$ by χ_1 and χ_2:

$$\chi_1((h(u), 1)) = \begin{cases} 1 & \text{if } u \equiv 1,\ 5 \pmod{8\mathbb{Z}_2} \\ \gamma((-I, 1)) & \text{if } u \equiv 3,\ 7 \pmod{8\mathbb{Z}_2}, \end{cases},$$

$$\chi_2((h(u), 1)) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod{8\mathbb{Z}_2} \\ \gamma((-I, 1)) & \text{if } u \equiv 7 \pmod{8\mathbb{Z}_2} \\ -1 & \text{if } u \equiv 5 \pmod{8\mathbb{Z}_2} \\ -\gamma((-I, 1)) & \text{if } u \equiv 3 \pmod{8\mathbb{Z}_2}. \end{cases}$$

Note that for $\gamma = \chi_1,\ \chi_2$ and $(A, \epsilon) = (x(s), 1)(h(u), 1)(y(t), 1)(I, \epsilon\delta) \in \overline{K_0^2(8)}$ ([3, Lemma A.4]) we have

$$\gamma(A, \epsilon) = \gamma(x(s), 1)\gamma(h(u), 1)\gamma(y(t), 1)\gamma(I, \epsilon\delta) = \gamma(h(u), 1)\gamma(I, \epsilon\delta). \quad (1)$$

From now on we shall denote $K_0^2(8)$ by simply K_0. Also for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ we put $\overline{g} := (g, 1) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$. We shall describe the Hecke algebra $H(\overline{K}_0, \gamma)$ using generators and relations.

We have the following proposition. The proof is a routine calculation.

Proposition 3.1. *A complete set of representatives for the double cosets of* $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ *modulo* $\overline{K}_0$ *consists of* $\overline{g}$, *where* g *varies over the following elements* *of* $\mathrm{SL}_2(\mathbb{Q}_2)$*:*

$$h(2^n),\ w(2^n)\quad \textit{for } n\in\mathbb{Z},\quad h(2^n)y(4), h(2^n)y(2)\qquad \textit{for } n\geq 0,$$

$$y(4)h(2^{-n}), y(2)h(2^{-n}), w(2^{-n})y(2), y(2)w(2^{-n}), y(2)w(2^{-n})y(2)\quad \textit{for } n\geq 1,$$

$$w(2^{-n})y(4),\ y(4)w(2^{-n}),\ y(4)w(2^{-n})y(4),$$
$$y(2)w(2^{-n})y(4), y(4)w(2^{-n})y(2)\qquad \textit{for } n\geq 2$$

and $y(2)w(2^{-1})y(6)$.

We will now compute the support of $H(\overline{K}_0,\chi_1)$ and $H(\overline{K}_0,\chi_2)$. We first have the following lemma on vanishing.

Lemma 3.2. *The Hecke algebra* $H(\overline{K}_0,\chi_1)$ *vanishes on the double cosets of* $\overline{K}_0$ *represented by*

$$\overline{y}(2), \overline{y}(2)\overline{w}(2^{-n}), \overline{w}(2^{-n})\overline{y}(2), \overline{y}(2)\overline{w}(2^{-n})\overline{y}(2),$$
$$\overline{h}(2^n)\overline{y}(2), \overline{y}(2)\overline{h}(2^{-n}), \overline{y}(2)\overline{w}(2^{-1})\overline{y}(6)\qquad \textit{for } n\geq 1$$

and

$$\overline{y}(2)\overline{w}(2^{-n})\overline{y}(4), \overline{y}(4)\overline{w}(2^{-n})\overline{y}(2)\qquad \textit{for } n\geq 2.$$

The Hecke algebra $H(\overline{K}_0,\chi_2)$ *vanishes on the double cosets of* $\overline{K}_0$ *represented by*

$$\overline{y}(4), \overline{y}(4)\overline{h}(2^{-n}), \overline{h}(2^n)\overline{y}(4)\qquad \textit{for } n\geq 1,$$
$$\overline{y}(2)\overline{w}(2^{-1})\overline{y}(6),$$

and

$$\overline{y}(2)\overline{w}(2^{-n})\overline{y}(4), \overline{y}(4)\overline{w}(2^{-n})\overline{y}(2),$$
$$\overline{w}(2^{-n})\overline{y}(4), \overline{y}(4)\overline{w}(2^{-n}), \overline{y}(4)\overline{w}(2^{-n})\overline{y}(4)\qquad \textit{where } n\geq 2.$$

Proof. Recall that ([3, Lemma 3.1]) $H(\overline{K}_0,\gamma)$ is supported on $\tilde{g}$ if and only if for every $\tilde{k}\in K_{\tilde{g}} := \overline{K}_0\cap\tilde{g}\overline{K}_0\tilde{g}^{-1}$ we have $\gamma([\tilde{k}^{-1},\ \tilde{g}^{-1}]) = 1$. So to check the vanishing at $\tilde{g}$ we need to just find suitable $\tilde{k}$.

For example, for $A = y(2)w(2^{-n})$, take $B = \begin{pmatrix}-3 & 2\\ -8 & 5\end{pmatrix}$. Then $\overline{B}\in K_{\overline{A}}$ and

$$[\overline{B}^{-1},\ \overline{A}^{-1}] = \left(\begin{pmatrix}5+2^{2n+2} & -2\\ 8+3\cdot 2^{2n+1} & -3\end{pmatrix},\ -1\right).$$

The above commutator is of the form

$$\left(\begin{pmatrix}-3 & *\\ 0 & -3\end{pmatrix}\pmod{8\mathbb{Z}_2},\ -1\right)$$

and in its triangular decomposition (as in equation (1)) $\delta = 1$. Since χ_1 takes value -1, the vanishing of $H(\overline{K}_0,\chi_1)$ follows on the double coset of A. The vanishing of $H(\overline{K}_0,\chi_1)$, $H(\overline{K}_0,\chi_2)$ at the double cosets listed in the lemma follow similarly. ☐

Lemma 3.3. *$H(\overline{K}_0,\chi_1)$ and $H(\overline{K}_0,\chi_2)$ are supported on the double cosets of $\overline{K}_0$ represented by $\overline{h}(2^n)$ and $\overline{w}(2^{-n})$ for $n\in\mathbb{Z}$.*

$H(\overline{K}_0,\chi_1)$ is supported on $\overline{y}(4)$ while $H(\overline{K}_0,\chi_2)$ is supported on $\overline{y}(2)$.

Proof. The proof is similar to the proof of [3, Lemma 3.5] and mainly uses [3, Lemma 3.1, Lemma 3.2 and Lemma A.3].

We illustrate the case of support of $H(\overline{K}_0,\chi_2)$ on $\overline{y}(2)$. The rest involve similar calculations.

We note that

$$K_{\overline{y}(2)}=\left\{(\begin{pmatrix} a-2b & b\\ c+2(a-d)-4b & 2b+d\end{pmatrix},\pm1):\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in K_0,\ \mathrm{ord}_2(b)\geq 1\right\}$$

has a triangular decomposition $K_{\overline{y}(2)}=N^{K_{\overline{y}(2)}}T^{K_{\overline{y}(2)}}\bar{N}^{K_{\overline{y}(2)}}$ where

$$N^{K_{\overline{y}(2)}}=\{(x(s),\pm1):\mathrm{ord}_2(b)\geq 1\},\quad T^{K_{\overline{y}(2)}}=T^{\overline{K}_0},\quad \bar{N}^{K_{\overline{y}(2)}}=\bar{N}^{\overline{K}_0}.$$

For $B=x(s)$ where $\mathrm{ord}_2(s)\geq 1$ (we may assume $s\neq 0$) we have

$$B^{-1}A^{-1}BA=\begin{pmatrix}1+2s+4s^2 & 2s^2\\ -4s & -2s+1\end{pmatrix}$$

and $s_2(B^{-1}A^{-1}BA)=(-s,2s+1)_2$ when $\mathrm{ord}_2(s)$ is odd, 1 else. Thus for $\mathrm{ord}_2(s)\geq 2$ we have $s_2(B^{-1}A^{-1}BA)=1$. If $s=2u$ with u a unit,

$$\begin{aligned}(-s,-2s+1)_2&=(-2u,-4u+1)_2\\ &=(-2,-4u+1)_2\,(-u,-4u+1)_2\\ &=(-2,-3)_2=-1.\end{aligned}$$

As before, since $\mathrm{ord}_2(s)\geq 1$ the δ-factor in the triangular decomposition of $[(B,\ \epsilon)^{-1},\ \overline{y}(2)]$ is 1 and so

$$\chi_2([(B,\ \epsilon)^{-1},\ \overline{y}(2)])=\begin{cases}\chi_2((h(5),-1))=-1\times-1=1 & \text{if } \mathrm{ord}_2(s)=1,\\ 1 & \text{if } \mathrm{ord}_2(s)\geq 2.\end{cases}$$

For $B=h(u)$ and $B=y(t)$ in $K_{\overline{y}(2)}$ we check that $[(B,\ \epsilon)^{-1},\ \overline{y}(4)]\in K_1^2(8)\times\{1\}$ and so χ_2 takes value 1. □

We note the following decomposition of double cosets into union of single cosets that would be useful in obtaining generators and relations among the Hecke algebra elements.

Lemma 3.4. *(a) For $n\geq 0$,*

$$K_0h(2^n)K_0=\bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2}x(s)h(2^n)K_0=\bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2}K_0h(2^n)y(8s).$$

(b) For $n\geq 1$,

$$K_0h(2^{-n})K_0=\bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2}y(8s)h(2^{-n})K_0=\bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2}K_0h(2^{-n})x(s).$$

(c) *For* $n \geq 2$,

$$K_0 w(2^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} y(8s) w(2^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} K_0 w(2^{-n}) y(8s).$$

(d)

$$K_0 w(2^{-1}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2\mathbb{Z}_2} x(s) w(2^{-1}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2\mathbb{Z}_2} K_0 w(2^{-1}) x(s).$$

(e) *For* $n \geq 0$,

$$K_0 w(2^{n}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n+3}\mathbb{Z}_2} x(s) w(2^{n}) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n+3}\mathbb{Z}_2} K_0 w(2^{n}) x(s).$$

(f) $K_0 y(4) K_0 = K_0 y(4) = y(4) K_0$.

(g) *For* $n \geq 1$,

$$K_0 h(2^{n}) y(4) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n}\mathbb{Z}_2} x(s) h(2^{n}) y(4) K_0 = \bigcup_{s \in \mathbb{Z}_2/2^{2n}\mathbb{Z}_2} K_0 h(2^{n}) y(4+8s).$$

(h) *For* $n \geq 1$,

$$\begin{aligned} K_0 y(4) h(2^{-n}) K_0 &= \bigcup_{s \in \mathbb{Z}_2/2^{2n}\mathbb{Z}_2} y(4+8s) h(2^{-n}) K_0 \\ &= \bigcup_{s \in \mathbb{Z}_2/2^{2n}\mathbb{Z}_2} K_0 y(4) h(2^{-n}) x(s). \end{aligned}$$

(i) *For* $n \geq 2$,

$$\begin{aligned} K_0 w(2^{-n}) y(4) K_0 &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} y(8s) w(2^{-n}) y(4) K_0 \\ &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} K_0 w(2^{-n}) y(4+8s). \end{aligned}$$

(j) *For* $n \geq 2$,

$$\begin{aligned} K_0 y(4) w(2^{-n}) K_0 &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} y(4+8s) w(2^{-n}) K_0 \\ &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} K_0 y(4) w(2^{-n}) y(8s). \end{aligned}$$

(k) *For* $n \geq 2$,

$$\begin{aligned} K_0 y(4) w(2^{-n}) y(4) K_0 &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} y(4+8s) w(2^{-n}) y(4) K_0 \\ &= \bigcup_{s \in \mathbb{Z}_2/2^{2n-3}\mathbb{Z}_2} K_0 y(4) w(2^{-n}) y(4+8s). \end{aligned}$$

(l)

$$K_0 y(2) K_0 = \bigcup_{s \in \mathbb{Z}_2/2\mathbb{Z}_2} x(s) y(2) K_0 = \bigcup_{s \in \mathbb{Z}_2/2\mathbb{Z}_2} K_0 y(2) x(s).$$

(m)

$$K_0y(2)w(2^{-1})K_0 = \bigcup_{s\in\mathbb{Z}_2/2\mathbb{Z}_2} x(s)y(2)w(2^{-1})K_0 = \bigcup_{s\in\mathbb{Z}_2/2\mathbb{Z}_2} K_0y(2)w(2^{-1})x(s).$$

(n)

$$K_0w(2^{-1})y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2\mathbb{Z}_2} x(s)w(2^{-1})y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2\mathbb{Z}_2} K_0w(2^{-1})y(2)x(s).$$

(o)

$$K_0y(2)w(2^{-1})y(2)K_0 = K_0y(2)w(2^{-1})y(2) = y(2)w(2^{-1})y(2)K_0.$$

(p) For $n \geq 2$,

$$K_0y(2)w(2^{-n})K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} y(2+4s)w(2^{-n})K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} K_0y(2)w(2^{-n})y(8s).$$

(q) For $n \geq 2$,

$$K_0w(2^{-n})y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} y(8s)w(2^{-n})y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} K_0w(2^{-n})y(2+4s).$$

(r) For $n \geq 1$,

$$K_0h(2^n)y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n+1}\mathbb{Z}_2} x(s)h(2^n)y(2)K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2} K_0h(2^n)y(2+4s) \bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2} K_0y(4)h(2^n)y(2+4s).$$

(s) For $n \geq 1$,

$$K_0y(2)h(2^{-n})K_0 = \bigcup_{s\in\mathbb{Z}_2/2^{2n+1}\mathbb{Z}_2} K_0y(2)h(2^{-n})x(s) = \bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2} y(2+4s)h(2^{-n})K_0 \bigcup_{s\in\mathbb{Z}_2/2^{2n}\mathbb{Z}_2} y(2+4s)h(2^{-n})y(4)K_0.$$

(t) For $n \geq 2$,

$$\begin{aligned} &K_0y(2)w(2^{-n})y(2)K_0 \\ &= \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} y(2+4s)w(2^{-n})y(2)K_0 \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} y(2+4s)w(2^{-n})y(6)K_0 \\ &= \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} K_0y(2)w(2^{-n})y(2+4s) \bigcup_{s\in\mathbb{Z}_2/2^{2n-2}\mathbb{Z}_2} K_0y(6)w(2^{-n})y(2+4s). \end{aligned}$$

Using Lemma 3.3, [3, Lemma 3.4] and the above decomposition we have the following corollary.

Corollary 3.5. *$H(\overline{K}_0, \chi_1)$ is supported on*

$$\overline{y}(4)\overline{h}(2^{-n}), \overline{h}(2^n)\overline{y}(4) \qquad \text{for } n \geq 1$$

and

$$\overline{y}(4)\overline{w}(2^{-n}), \overline{w}(2^{-n})\overline{y}(4), \overline{y}(4)\overline{w}(2^{-n})\overline{y}(4) \qquad \text{for } n \geq 2.$$

$H(\overline{K}_0, \chi_2)$ is supported on

$$\overline{y}(2)\overline{w}(2^{-n}), \overline{w}(2^{-n})\overline{y}(2) \qquad \text{for } n \geq 2$$

and

$$\overline{h}(2^n)\overline{y}(2), \overline{y}(2)\overline{h}(2^{-n}) \qquad \text{for } n \geq 1.$$

Note that we cannot use the argument in the proof of the above corollary to show support of $H(\overline{K}_0, \chi_2)$ on the double cosets of $\overline{w}(2^{-1})\overline{y}(2)$, $\overline{y}(2)\overline{w}(2^{-1})$ and $\overline{y}(2)\overline{w}(2^{-n})\overline{y}(2)$ for $n \geq 1$. In these cases we check the support directly as in Lemma 3.3.

Thus we have the following proposition.

Proposition 3.6. *$H(\overline{K_0^2(8)}, \chi_1)$ is supported on precisely the double cosets of $\overline{K_0^2(8)}$ represented by*

$$\{\overline{h}(2^n),\ \overline{w}(2^{-n})\}_{n\in\mathbb{Z}} \cup \overline{y}(4) \cup \{\overline{h}(2^n)\overline{y}(4),\ \overline{y}(4)\overline{h}(2^{-n})\}_{n\geq 1}$$
$$\cup\ \{\overline{y}(4)\overline{w}(2^{-n}),\ \overline{w}(2^{-n})\overline{y}(4),\ \overline{y}(4)\overline{w}(2^{-n})\overline{y}(4)\}_{n\geq 2}.$$

$H(\overline{K_0^2(8)}, \chi_2)$ is supported on precisely the double cosets of $\overline{K_0^2(8)}$ represented by

$$\{\overline{h}(2^n),\ \overline{w}(2^{-n})\}_{n\in\mathbb{Z}}\ \cup\ \overline{y}(2)$$
$$\cup\ \{\overline{y}(2)\overline{w}(2^{-n}),\ \overline{w}(2^{-n})\overline{y}(2),\ \overline{y}(2)\overline{w}(2^{-n})\overline{y}(2),\ \overline{h}(2^n)\overline{y}(2),\ \overline{y}(2)\overline{h}(2^{-n})\}_{n\geq 1}.$$

3.1. **Generators and Relations.** Let γ be either χ_1 or χ_2. Following Loke and Savin [7] we extend the character γ on M_2 to the normalizer subgroup $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)}(T)$ of torus T in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ by defining $\gamma(\overline{h}(2^n)) = 1$ for all $n \in \mathbb{Z}$ and

$$\gamma(\overline{w}(1)) = \frac{1 + \gamma((-I, 1))}{\sqrt{2}} =: \varphi_8,$$

an 8-th root of unity.

For $n \in \mathbb{Z}$, define the elements $\mathcal{T}_n$ and $\mathcal{U}_n$ of $H(\overline{K_0^2(8)}, \gamma)$ supported respectively on the $\overline{K_0^2(8)}$ double cosets of $(h(2^n), 1)$ and $(w(2^{-n}), 1)$ such that

$$\mathcal{T}_n(\tilde{k}(h(2^n), 1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((h(2^n), 1))\overline{\gamma}(\tilde{k}'), \tag{2}$$

$$\mathcal{U}_n(\tilde{k}(w(2^{-n}), 1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((w(2^{-n}), 1))\overline{\gamma}(\tilde{k}') \quad \text{for } \tilde{k},\, \tilde{k}' \in \overline{K_0^2(8)}.$$

We use the decomposition Lemma 3.4 and [3, Lemma 3.4] to obtain the following relations in $H(\overline{K_0^2(8)}, \gamma)$.

Lemma 3.7. *(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.*

(2) *For* $n \leq 0$, $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{1+n}$, *and for* $n \geq 0$, $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{U}_{1-n}$.
(3) *For* $n \geq 0$, $\mathcal{U}_2 * \mathcal{T}_n = \mathcal{U}_{2+n}$, *and for* $n \leq 0$, $\mathcal{T}_n * \mathcal{U}_2 = \mathcal{U}_{2-n}$.
(4) *For* $m \geq 2$, $\mathcal{U}_1 * \mathcal{U}_m = \mathcal{T}_{m-1}$ *and* $\mathcal{U}_m * \mathcal{U}_1 = \mathcal{T}_{1-m}$.
(5) *For* $m \leq 1$, $\mathcal{U}_2 * \mathcal{U}_m = \mathcal{T}_{m-2}$ *and* $\mathcal{U}_m * \mathcal{U}_2 = \mathcal{T}_{2-m}$.

3.2. **The algebra** $H(\overline{K_0^2(8)}, \chi_1)$**.** Consider the case when $\gamma = \chi_1$. Since $H(\overline{K_0}, \chi_1)$ is supported on $\overline{K}_0\overline{y}(4)\overline{K}_0$, we define $\mathcal{V}$ to be an element of $H(\overline{K_0}, \chi_1)$ that is supported precisely on $\overline{K}_0\overline{y}(4)\overline{K}_0$ such that $\mathcal{V}(\overline{y}(4)) = 1$. Now since $\mu(\overline{y}(4))\mu(\overline{y}(4)) = \mu(\overline{y}(8))$, using Lemma [3, Lemma 3.4] we get that $\mathcal{V} * \mathcal{V}$ is supported precisely on $\overline{K}_0\overline{y}(8)\overline{K}_0 = \overline{K}_0$ and

$$\mathcal{V} * \mathcal{V}((I, 1)) = \mathcal{V} * \mathcal{V}(\overline{y}(8)) = \mathcal{V}(\overline{y}(4))\mathcal{V}(\overline{y}(4)) = 1,$$

so we get that $\mathcal{V} * \mathcal{V} = 1$.

Similarly, $\mathcal{U}_1 * \mathcal{V}$ is supported precisely at $\overline{K}_0\overline{w}(2^{-1})\overline{y}(4)\overline{K}_0$ and its value at $\overline{w}(2^{-1})\overline{y}(4)$ is $\mathcal{U}_1(\overline{w}(2^{-1}))$ as $\mathcal{V}(\overline{y}(4)) = 1$. But note that

$$\overline{K}_0\overline{w}(2^{-1})\overline{y}(4)\overline{K}_0 = \overline{K}_0\overline{w}(2^{-1})\overline{K}_0,$$

in fact

$$\overline{w}(2^{-1})\overline{y}(4) = \left(\begin{pmatrix} 9 & -1 \\ -8 & 1 \end{pmatrix}, 1\right) \overline{w}(2^{-1}) \left(\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, 1\right),$$

so

$$\begin{aligned} \mathcal{U}_1(\overline{w}(2^{-1})) &= \mathcal{U}_1 * \mathcal{V}(\overline{w}(2^{-1})\overline{y}(4)) \\ &= \mathcal{U}_1 * \mathcal{V}\left(\left(\begin{pmatrix} 9 & -1 \\ -8 & 1 \end{pmatrix}, 1\right) \overline{w}(2^{-1}) \left(\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, 1\right)\right) \\ &= \mathcal{U}_1 * \mathcal{V}(\overline{w}(2^{-1})), \end{aligned}$$

and thus $\mathcal{U}_1 * \mathcal{V} = \mathcal{U}_1$. Similarly we get $\mathcal{V} * \mathcal{U}_1 = \mathcal{U}_1$.

Lemma 3.8. *For* $\mathcal{V}$, $\mathcal{U}_1 \in H(\overline{K_0}, \chi_1)$ *we have following relations:*
(1) $\mathcal{V} * \mathcal{V} = 1$.
(2) $\mathcal{U}_1 * \mathcal{V} = \mathcal{U}_1 = \mathcal{V} * \mathcal{U}_1$.

Proposition 3.9. (1) $\mathcal{U}_2 * \mathcal{U}_2 = 2$.
(2) $\mathcal{U}_1 * \mathcal{U}_1 = 2 + 2\mathcal{V}$.
(3) $\mathcal{U}_2 * \mathcal{V} * \mathcal{U}_2 = \sqrt{2}\, \mathcal{V} * \mathcal{U}_2 * \mathcal{V}$.
(4) $\mathcal{U}_0 * \mathcal{U}_0 = 8 + 2\sqrt{2}\, \mathcal{U}_0 + 8\mathcal{V}$.
(5) $\mathcal{U}_0 * \mathcal{V} = \mathcal{U}_0 = \mathcal{V} * \mathcal{U}_0$ *and consequently* $\frac{\mathcal{U}_0}{\sqrt{2}} * (\frac{\mathcal{U}_0}{\sqrt{2}} - 4) * (\frac{\mathcal{U}_0}{\sqrt{2}} + 2) = 0$.

We shall use the following version of [3, Lemma 3.3].

Lemma 3.10. *Let* f_1, $f_2 \in H(\gamma)$, *and* f_1 *is supported on* $\overline{K}_0\tilde{x}\overline{K}_0 = \bigcup_{i=1}^m \overline{K}_0\tilde{\alpha}_i$ *and* f_2 *is supported on* $\overline{K}_0\tilde{y}\overline{K}_0$, *and let* $\overline{K}_0\tilde{y}^{-1}\overline{K}_0 = \bigcup_{j=1}^n \tilde{\beta}_j\overline{K}_0$. *Then*

$$f_1 * f_2(\tilde{g}) = \sum_{j=1}^n f_1(\tilde{g}\tilde{\beta}_j)f_2(\tilde{\beta}_j^{-1})$$

and the non-zero summands are those for which there exists a j *such that* $\tilde{g}\tilde{\beta}_j \in K_0\tilde{\alpha}_i$.

Proof of Proposition 3.9. We shall prove (3). The proofs of (1) and (2) are similar. Let $\mathcal{W}_2 = \mathcal{U}_2 * \mathcal{V}$ and $\mathcal{Z}_2 = \mathcal{V} * \mathcal{U}_2 * \mathcal{V}$. Then using the decomposition in Lemma 3.4 and [3, Lemma 3.4] we see that $\mathcal{W}_2$, $\mathcal{Z}_2$ are respectively supported on

$$\overline{K}_0\overline{w}(2^{-2})\overline{y}(4)\overline{K}_0 \quad \text{and} \quad \overline{K}_0\overline{y}(4)\overline{w}(2^{-2})\overline{y}(4)\overline{K}_0$$

$$\text{and} \quad \mathcal{W}_2(\overline{w}(2^{-2})\overline{y}(4)) = \overline{\gamma}(\overline{w}(1)) = \mathcal{Z}_2(\overline{y}(4)\overline{w}(2^{-2})\overline{y}(4)).$$

So to get the identity we will first compute the support of $\mathcal{W}_2 * \mathcal{U}_2$.

Using Lemma 3.4,

$$\overline{K}_0\overline{w}(2^{-2})\overline{y}(4)\overline{K}_0 = \bigcup_{s=0,1} \overline{K}_0\tilde{\alpha}_s \quad \text{and} \quad \overline{K}_0\overline{w}(2^{-2})^{-1}\overline{K}_0 = \bigcup_{t=0,1} \tilde{\beta}_t\overline{K}_0$$

where

$$\tilde{\alpha}_s = \overline{w}(2^{-2})\overline{y}(4+8s), \quad \tilde{\beta}_t = \overline{y}(-8t)\overline{w}(-2^{-2}),$$

The matrix part of $\tilde{\beta}_t\tilde{\alpha}_s^{-1}$ are

$$\begin{pmatrix} -1 & -1/4 \\ 0 & -1 \end{pmatrix} \text{ if } t = s = 0, \quad \begin{pmatrix} -1 & -1/4 \\ 8 & 1 \end{pmatrix} \text{ if } t = 1,\ s = 0,$$

$$\begin{pmatrix} -1 & -3/4 \\ 0 & -1 \end{pmatrix} \text{ if } t = 0,\ s = 1, \quad \begin{pmatrix} -1 & -3/4 \\ 8 & 5 \end{pmatrix} \text{ if } t = s = 1.$$

Running $\tilde{g}$ over the double coset representatives we see that $\tilde{g}\tilde{\beta}_t\tilde{\alpha}_s^{-1} \in \overline{K}_0$ implies that $\tilde{g}$ is in the double coset of $\overline{y}(4)\overline{w}(2^{-2})\overline{y}(4)$. Thus $\mathcal{W}_2 * \mathcal{U}_2$ is supported on $\overline{K}_0\overline{y}(4)\overline{w}(2^{-2})\overline{y}(4)\overline{K}_0$. Consequently

$$\mathcal{U}_2 * \mathcal{V} * \mathcal{U}_2 = \mathcal{W}_2 * \mathcal{U}_2 = \alpha\mathcal{Z}_2,$$

where one can compute α by computing $\mathcal{W}_2 * \mathcal{U}_2(\tilde{g})$ with

$$\tilde{g} = \overline{y}(4)\overline{w}(2^{-2})\overline{y}(4) =: (C, \epsilon) \quad \text{and} \quad \epsilon = \sigma_2(y(4), w(2^{-2}))\sigma_2(y(4)w(2^{-2}), y(4)).$$

By Lemma 3.10,

$$\mathcal{W}_2 * \mathcal{U}_2(\tilde{g}) = \sum_{t=0,1} \mathcal{W}_2(\tilde{g}\tilde{\beta}_t)\mathcal{U}_2(\tilde{\beta}_t^{-1}) = \mathcal{U}_2(\overline{w}(2^{-2})) \sum_{t=0,1} \mathcal{W}_2(\tilde{g}\tilde{\beta}_t).$$

Let $A_s = w(2^{-2})y(4+8s)$ and $B_t = y(-8t)w(-2^{-2})$. Then the matrix part of $\tilde{g}\tilde{\beta}_t\tilde{\alpha}_s^{-1}$ is $CB_tA_s^{-1}$ which is

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \text{ if } t = 0,\ s = 1, \qquad \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix} \text{ if } t = 1,\ s = 0,$$

and the sigma-factor of $\tilde{g}\tilde{\beta}_t\tilde{\alpha}_s^{-1}$ is $\epsilon\eta\sigma(C, B_tA_s^{-1})$ where

$$\eta := \sigma(w(2^{-2}), y(4+8s))\sigma(A_s, A_s^{-1})\sigma(y(-8t), w(-2^{-2}))\sigma(B_t, A_s^{-1}).$$

Now η turns out to be -1 when $t = 0,\ s = 1$ and 1 when $t = 1,\ s = 0$. Thus

$$\tilde{g}\tilde{\beta}_0 = (\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, -1)\overline{w}(2^{-2})\overline{y}(4)\overline{y}(8), \quad \tilde{g}\tilde{\beta}_1 = (\begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, 1)\overline{w}(2^{-2})\overline{y}(4).$$

Thus

$$\mathcal{W}_2 * \mathcal{U}_2(\tilde{g}) = \overline{\gamma}(\overline{w}(1))^2(\gamma((-I,1))+1).$$

Hence $\alpha = \sqrt{2}$ and $\mathcal{U}_2 * \mathcal{V} * \mathcal{U}_2 = \sqrt{2}\ \mathcal{Z}_2 = \sqrt{2}\ \mathcal{V} * \mathcal{U}_2 * \mathcal{V}$.

The proofs of (4), (5) now follow using (1), (2), (3) above, Lemma 3.8 and the relation $\mathcal{U}_0 = \mathcal{T}_1 * \mathcal{U}_1 = \mathcal{U}_1 * \mathcal{U}_2 * \mathcal{U}_1$. □

For $n \geq 1$, define

$$\mathcal{R}_n := \mathcal{T}_n * \mathcal{V}, \quad \mathcal{S}_n := \mathcal{V} * \mathcal{T}_{-n},$$

and for $n \geq 2$,

$$\mathcal{W}_n := \mathcal{U}_n * \mathcal{V}, \quad \mathcal{Y}_n := \mathcal{V} * \mathcal{U}_n \quad \text{and} \quad \mathcal{Z}_n := \mathcal{V} * \mathcal{U}_n * \mathcal{V}.$$

Note that by Lemma 3.4 and [3, Lemma 3.4], $\mathcal{R}_n,\ \mathcal{S}_n,\ \mathcal{W}_n,\ \mathcal{Y}_n,\ \mathcal{Z}_n$ are respectively supported on the $\overline{K}_0$ double cosets of

$$\overline{h}(2^n)\overline{y}(4), \quad \overline{y}(4)\overline{h}(2^{-n}), \quad \overline{w}(2^{-n})\overline{y}(4), \quad \overline{y}(4)\overline{w}(2^{-n}) \quad \text{and} \quad \overline{y}(4)\overline{w}(2^{-n})\overline{y}(4).$$

Thus it follows from Proposition 3.6 that $\mathcal{T}_n, \mathcal{U}_n$ for $n \in \mathbb{Z}$, $\mathcal{V}$, $\mathcal{R}_n,\ \mathcal{S}_n$ for $n \geq 1$ and $\mathcal{W}_n,\ \mathcal{Y}_n,\ \mathcal{Z}_n$ for $n \geq 2$ form basis elements of $H(\overline{K}_0, \chi_1)$ as a vector space. Indeed it follows from Lemma 3.7 that $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{V}$ generate $H(\overline{K}_0, \chi_1)$ as an algebra.

Let

$$\widehat{\mathcal{U}_1} = \frac{1}{\sqrt{2}}\mathcal{U}_1, \quad \widehat{\mathcal{U}_2} = \frac{1}{\sqrt{2}}\mathcal{U}_2 \quad \text{and} \quad \widehat{\mathcal{U}_0} = \frac{1}{2\sqrt{2}}\mathcal{U}_0.$$

Using relations above, we obtain the following theorem.

Theorem 1. *The Hecke algebra $H(\overline{K_0^2(8)}, \chi_1)$ is generated by $\widehat{\mathcal{U}_1}$, $\widehat{\mathcal{U}_2}$ and $\mathcal{V}$ modulo the relations:*

(1) $\widehat{\mathcal{U}_1}^2 = 1 + \mathcal{V}$,
(2) $\widehat{\mathcal{U}_2}^2 = 1$,
(3) $\widehat{\mathcal{U}_1}\mathcal{V} = \mathcal{V}\widehat{\mathcal{U}_1} = \widehat{\mathcal{U}_1}$,
(4) $\widehat{\mathcal{U}_2}\mathcal{V}\widehat{\mathcal{U}_2} = \mathcal{V}\widehat{\mathcal{U}_2}\mathcal{V}$.

3.3. **The algebra $H(\overline{K_0^2(8)}, \chi_2)$.** Take $\gamma = \chi_2$, we will similarly get generators and relations for the Hecke algebra $H(\overline{K_0^2(8)}, \chi_2)$.

Define $\mathcal{Z}_1' \in H(\overline{K}_0, \chi_2)$ supported only on the double coset of $\overline{y}(2)\overline{w}(2^{-1})\overline{y}(2)$ such that $\mathcal{Z}_1'(\overline{y}(2)\overline{w}(2^{-1})\overline{y}(2)) = 1$. Note that $\overline{y}(2)\overline{w}(2^{-1})\overline{y}(2) = \overline{x}(1/2)$ and it normalizes $\overline{K}_0$. As before we get that $\mathcal{Z}_1' * \mathcal{Z}_1' = 1$.

Define $\mathcal{V}' \in H(\overline{K_0^2(8)}, \chi_2)$ supported precisely on $\overline{K}_0\overline{y}(2)\overline{K}_0$ such that

$$\mathcal{V}'(\overline{y}(2), 1) = \frac{1 + \gamma((-I,1))}{\sqrt{2}}.$$

We have the following proposition.

Proposition 3.11. *(1)* $\mathcal{Z}_1' * \mathcal{U}_1 * \mathcal{Z}_1' = \mathcal{V}'$.
(2) $\mathcal{U}_2 * \mathcal{Z}_1' = \mathcal{U}_2 = \mathcal{Z}_1' * \mathcal{U}_2$.
(3) $\mathcal{U}_2 * \mathcal{U}_2 = 2 + 2\ \mathcal{Z}_1'$.

(4) $\mathcal{U}_1 * \mathcal{U}_1 = 2$.
(5) $\mathcal{U}_1 * \mathcal{Z}'_1 * \mathcal{U}_1 = \sqrt{2}\ \mathcal{V}' = \sqrt{2}\ \mathcal{Z}'_1 * \mathcal{U}_1 * \mathcal{Z}'_1$.

Proof. For (1) observe that

$$\overline{w}(2^{-1})\overline{x}(1/2) = (\begin{pmatrix} 0 & 1/2 \\ -2 & -1 \end{pmatrix}, -1) = \overline{y}(2)\overline{w}(2^{-1})\overline{x}(1).$$

Thus $\mathcal{Y}'_1 := \mathcal{U}_1 * \mathcal{Z}'_1$ is supported precisely on $\overline{K}_0\overline{y}(2)\overline{w}(2^{-1})\overline{K}_0$ and its value at $\overline{y}(2)\overline{w}(2^{-1})$ is $\frac{1-\gamma((-I,1))}{\sqrt{2}}$. Further, since

$$\overline{x}(1/2)\overline{y}(2)\overline{w}(2^{-1}) = (\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, 1) = \overline{y}(2)(-x(-1), 1)$$

we get that $\mathcal{Z}'_1 * \mathcal{Y}'_1$ is supported precisely at $\overline{K}_0\overline{y}(2)\overline{K}_0$ and that

$$\mathcal{Z}'_1 * \mathcal{Y}'_1(\overline{y}(2)) = \frac{1-\gamma((-I,1))}{\sqrt{2}}\gamma((-I,1)) = \frac{1+\gamma((-I,1))}{\sqrt{2}} = \mathcal{V}'(\overline{y}(2)).$$

Part (2) follows similarly. For (3), (4), (5) we follow as in Proposition 3.9. □

Let $\widehat{\mathcal{U}_1} = \frac{1}{\sqrt{2}}\mathcal{U}_1$ and $\widehat{\mathcal{U}_2} = \frac{1}{\sqrt{2}}\mathcal{U}_2$.

Theorem 2. *The Hecke algebra $H(\overline{K_0^2(8)}, \chi_2)$ is generated by $\widehat{\mathcal{U}_1}$, $\widehat{\mathcal{U}_2}$ and $\mathcal{Z}'_1$ modulo the relations:*
(1) $\widehat{\mathcal{U}_1}^2 = 1$,
(2) $\widehat{\mathcal{U}_2}^2 = 1 + \mathcal{Z}'_1$,
(3) $\widehat{\mathcal{U}_2}\mathcal{Z}'_1 = \widehat{\mathcal{U}_2} = \mathcal{Z}'_1\widehat{\mathcal{U}_2}$,
(4) $\widehat{\mathcal{U}_1}\mathcal{Z}'_1\widehat{\mathcal{U}_1} = \mathcal{Z}'_1\widehat{\mathcal{U}_1}\mathcal{Z}'_1$.

3.4. **Local Shimura correspondence.** Loke and Savin [7] observed an isomorphism between the Hecke algbera $H(\overline{K_0^2(4)}, \gamma)$ (γ a genuine character of $\overline{K_0^2(4)}$ of order 4) and $\mathrm{PGL}_2(\mathbb{Q}_2)$ Iwahori Hecke algebra and called it local Shimura correspondence. In this subsection we prove that the Hecke algebra $H(\overline{K_0^2(8)}, \chi_i)$, $i = 1, 2$, is isomorphic to the Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_2)$ corresponding to $K_0(4)$ modulo scalars (here $K_0(p^n)$ denotes the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ with $(2,1)$-entry in $p^n\mathbb{Z}_p$). We thus verify local Shimura correspondence between level 8 Hecke algebras of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ and the level 4 Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$.

In [2] we give generators and relations for the subalgebra of the Hecke algebra of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to $K_0(p^n)$ that is supported on $\mathrm{GL}_2(\mathbb{Z}_p)$ for any prime p and natural number n but do not consider the full Hecke algebra. We will now describe the full Hecke algebra $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$. In this subsection we will follow the notation of [2].

For $t \in \mathbb{Q}_2^\times$, we consider the following elements of $\mathrm{GL}_2(\mathbb{Q}_2)$:

$$d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad w(2^n) = \begin{pmatrix} 0 & -1 \\ 2^n & 0 \end{pmatrix}, \quad z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Note that we are also using notation $w(t)$ for denoting anti-diagonal elements of $\mathrm{SL}_2(\mathbb{Q}_2)$, but we hope that this abuse of notation is clear from the context.

We have the following lemma.

Lemma 3.12. *A complete set of representatives for the double cosets of $\mathrm{GL}_2(\mathbb{Q}_2)$ mod $K_0(4)$ (up to central elements $z(t)$) consists of:*

$$d(2^n),\ w(2^n) \quad \text{for } n \in \mathbb{Z}, \quad d(2^n)y(2) \quad \text{for } n \geq 0, \quad y(2)d(2^{-n}) \quad \text{for } n \geq 1,$$
$$\text{and} \quad y(2)w(2^n),\ w(2^n)y(2),\ y(2)w(2^n)y(2) \quad \text{for } n \geq 2.$$

We also note the following decomposition of $K_0(4)$ double cosets.

Lemma 3.13. *(a) For $n \geq 0$,*

$$K_0(4)d(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^n\mathbb{Z}_2} x(s)d(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^n\mathbb{Z}_2} K_0(4)d(2^n)y(4s).$$

(b) For $n \geq 1$,

$$K_0(4)d(2^{-n})K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^n\mathbb{Z}_2} y(4s)d(2^{-n})K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^n\mathbb{Z}_2} K_0(4)d(2^{-n})x(s).$$

(c) For $n \geq 2$,

$$K_0(4)w(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^{n-2}\mathbb{Z}_2} y(4s)w(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^{n-2}\mathbb{Z}_2} K_0(4)w(2^n)y(4s).$$

(d) For $n \leq 1$,

$$K_0(4)w(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^{2-n}\mathbb{Z}_2} x(s)w(2^n)K_0(4) = \bigcup_{s \in \mathbb{Z}_2/2^{2-n}\mathbb{Z}_2} K_0(4)w(2^n)x(s).$$

Using the above lemma and since $y(2)$ normalizes $K_0(4)$, we can further obtain decomposition of double cosets $K_0(4)gK_0(4)$ where g varies over all the double coset representatives noted in Lemma 3.12.

Note that in this case Hecke algbera $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$ does not involve any character, so it is trivially supported on all the double cosets. Let X_g be the characteristic function of $K_0(4)gK_0(4)$ and let

$$\mathcal{T}_n = X_{d(2^n)}, \quad \mathcal{U}_n = X_{w(2^n)}, \quad \mathcal{V} = X_{y(2)} \quad \text{and} \quad \mathcal{Z} = X_{z(2)}$$

be elements of the Hecke algebra $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$ (again note that there is a conflict of notation with the Hecke algebra elements of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ but we will see that the elements satisfy exactly the same relations). It is easy to see that $\mathcal{Z}$ is in the center and that $\mathcal{Z}^n = X_{z(2^n)}$.

Using [2, Lemma 3.1] and the above decomposition we obtain the following relations in $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$.

Lemma 3.14. *(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.*

(2) For $n \leq 0$, $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{1+n}$, and for $n \geq 0$, $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{Z}^n\mathcal{U}_{1-n}$.

(3) For $n \geq 0$, $\mathcal{U}_2 * \mathcal{T}_n = \mathcal{U}_{2+n}$, and for $n \leq 0$, $\mathcal{T}_n * \mathcal{U}_2 = \mathcal{Z}^n\mathcal{U}_{2-n}$.

(4) For $m \geq 2$, $\mathcal{U}_1 * \mathcal{U}_m = \mathcal{Z}\mathcal{T}_{m-1}$ and $\mathcal{U}_m * \mathcal{U}_1 = \mathcal{Z}^m\mathcal{T}_{1-m}$.

(5) For $m \leq 1$, $\mathcal{U}_2 * \mathcal{U}_m = \mathcal{Z}^2\mathcal{T}_{m-2}$ and $\mathcal{U}_m * \mathcal{U}_2 = \mathcal{Z}^m\mathcal{T}_{2-m}$.

We have the following proposition.

Proposition 3.15. *(1)* $\mathcal{V} * \mathcal{V} = 1$.
(2) $\mathcal{U}_1 * \mathcal{U}_1 = 2\mathcal{Z}(1+\mathcal{V})$.
(3) $\mathcal{U}_1 * \mathcal{V} = \mathcal{U}_1 = \mathcal{V} * \mathcal{U}_1$.
(4) $\mathcal{U}_2 * \mathcal{U}_2 = \mathcal{Z}^2$.
(5) $\mathcal{U}_2 * \mathcal{V} * \mathcal{U}_2 = \mathcal{Z}\mathcal{V} * \mathcal{U}_2 * \mathcal{V}$.
(6) $\mathcal{U}_0 * \mathcal{U}_0 = 4 + 2\,\mathcal{U}_0 + 4\mathcal{V}$.
(7) $\mathcal{U}_0 * \mathcal{V} = \mathcal{U}_0 = \mathcal{V} * \mathcal{U}_0$.

Proof. Note that $\mathcal{V}$, $\mathcal{U}_0$ are elements of the subalgebra supported on $\mathrm{GL}_2(\mathbb{Z}_p)$ and the relations (1), (6), (7) follow directly from [2, Proposition 3.10, 3.12]. The relations (3), (4) and the braid relation (5) follow easily as the above lemma. For relation (2) we use [2, Lemma 3.2]. For $s = 0, 1$, let $\alpha_s = x(s)w(2)$. Then $\mathcal{U}_1 * \mathcal{U}_1$ is supported on those $g \in \mathrm{GL}_2(\mathbb{Q}_2)$ for which there exists $s,\ t \in \{0, 1\}$ such that

$$(\alpha_s\alpha_t)^{-1}g = \begin{pmatrix} -1/2 & s/2 \\ -t & st-1/2 \end{pmatrix} g \in K_0(4).$$

Checking this for g as it varies over all the double coset representatives we get that the support is precisely on $z(2)$ and $y(2)z(2)$. Further, we get that

$$\begin{aligned} \mathcal{U}_1 * \mathcal{U}_1(y(2)z(2)) &= \sum_{s=0,1} \mathcal{U}_1(\alpha_s)\mathcal{U}_1(\alpha_s^{-1}y(2)z(2)) \\ &= \mathcal{U}_1(\alpha_1 \begin{pmatrix} -1 & 0 \\ -4 & -1 \end{pmatrix}) + \mathcal{U}_1(\alpha_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = 2. \end{aligned}$$

Similarly $\mathcal{U}_1 * \mathcal{U}_1(z(2)) = 2$. Thus we obtain (2). □

The remaining basis elements of $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$ are precisely $\mathcal{T}_n * \mathcal{V}$, $\mathcal{V} * \mathcal{T}_{-n}$ for $n \geq 1$, and $\mathcal{U}_n * \mathcal{V}$, $\mathcal{V} * \mathcal{U}_n$ and $\mathcal{V} * \mathcal{U}_n * \mathcal{V}$ for $n \geq 2$.

We have the following theorem.

Theorem 3. *The Hecke algebra*

$$H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))/\langle \mathcal{Z} \rangle$$

is generated by $\mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{V}$ *with the defining relations:*
(1) $\mathcal{U}_1^2 = 2(1+\mathcal{V})$,
(2) $\mathcal{U}_2^2 = 1$,
(3) $\mathcal{U}_1\mathcal{V} = \mathcal{V}\mathcal{U}_1 = \mathcal{U}_1$,
(4) $\mathcal{U}_2\mathcal{V}\mathcal{U}_2 = \mathcal{V}\mathcal{U}_2\mathcal{V}$.

Corollary 3.16. *We have the following isomorphism of Hecke algebras:*

$$H(\overline{K_0^2(8)}, \chi_1) \cong H(\overline{K_0^2(8)}, \chi_2) \cong H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))/\langle \mathcal{Z} \rangle.$$

The Hecke algebra generators and relations described above allow a study of the representation theory of the maximal compact with $(\overline{K_0^2(8)}, \gamma)$ equivariant vectors and also the infinite-dimensional genuine representations of $\widetilde{\mathrm{SL}}(2)$ with such vectors. We will pursue this study in a subsequent work.

4. Translation of adelic to classical.

We follow the notation as in Section 4 of [3]. Let k be a natural number, M be odd and χ be an even Dirichlet character modulo $8M$. Let $\chi_0 = \chi\left(\frac{-1}{\cdot}\right)^k$. We consider the central character γ of M_2 such that $\gamma((-I,1)) = -i^{2k+1}$ and let $\chi_1,\ \chi_2$ be the extension of γ as in the previous section.

Let $A_{k+1/2}(8M,\chi_0)$ be the set of adelic cuspidal automorphic forms

$$\Phi : \widetilde{\mathrm{SL}_2}(\mathbb{A}) \to \mathbb{C}$$

satisfying certain properties as considered by Waldspurger [16]. By Gelbart-Waldspurger there is an isomorphism between

$$A_{k+1/2}(8M,\chi_0) \to S_{k+1/2}(\Gamma_0(8M),\chi),$$

$\Phi_f \leftrightarrow f$, inducing a ring isomorphism

$$q : \mathrm{End}_{\mathbb{C}}(A_{k+1/2}(8M,\chi_0)) \to \mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(8M),\chi)).$$

We will use q to translate certain elements in $H(\overline{K_0^2(8)},\chi_1)$ and $H(\overline{K_0^2(8)},\chi_2)$ to classical operators on $S_{k+1/2}(\Gamma_0(8M))$ and $S_{k+1/2}(\Gamma_0(8M),\left(\frac{2}{\cdot}\right))$ respectively. Thus the classical operators so obtained satisfy the local Hecke algebra relations noted in the previous section. These relations are crucial for the results obtained in the next section.

Proposition 4.1. *Let $\mathcal{T}_1,\ \mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{V} \in H(\overline{K_0^2(8)},\chi_1)$ and $f \in S_{k+1/2}(\Gamma_0(8M))$.*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{-(2k+1)/2}\sum_{s=0}^{3} f((z+s)/4) = 2^{(3-2k)/2}U_4(f)(z)$.

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2}\left(\frac{2}{M}\right)2f|[W_4,\phi_{W_4}(z)]_{k+1/2}(z)$, where

$$W_4 = \begin{pmatrix} 4n & m \\ 4M & 8 \end{pmatrix}$$

with $n,m\in\mathbb{Z}$ such that $8n - mM = 1$ and $\phi_{W_4}(z) = (2Mz+4)^{1/2}$.

(3) $q(\mathcal{U}_2)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2}\sum_{s=0}^{1} f|[W_8,\phi_{W_8}(z)]_{k+1/2}(z)$, where

$$W_8 = \begin{pmatrix} 16n - 8mMs & m \\ 16M - 128Ms & 16 \end{pmatrix}$$

with $n,m\in\mathbb{Z}$ such that $16n - mM = 1$ and

$$\phi_{W_8}(z) = ((4M - 32Ms)z + 4)^{1/2}.$$

(4) $q(\mathcal{V})(f)(z) = f|[\begin{pmatrix} 1 & 0 \\ 4M & 1 \end{pmatrix}, (4Mz+1)^{1/2}]_{k+1/2}(z)$.

Proof. The proof follows by similar calculations as in [3]. □

We similarly have the following proposition.

Proposition 4.2. *Let $f \in S_{k+1/2}(\Gamma_0(8M),\left(\frac{2}{\cdot}\right))$. Let $W_4,\ W_8$, be as in the above proposition. For $\mathcal{T}_1,\ \mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{Z}_1',\ \mathcal{V}' \in H(\overline{K_0^2(8)},\chi_2)$ we have*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{-(2k+1)/2}\sum_{s=0}^{3} f((z+s)/4)$,

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2} 2f|[W_4, \phi_{W_4}(z)]_{k+1/2}(z)$,
(3) $q(\mathcal{U}_2)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2}\left(\frac{2}{M}\right)\sum_{s=0}^{1} f|[W_8, \phi_{W_8}(z)]_{k+1/2}(z)$,
(4) $q(\mathcal{Z}_1')(f)(z) = f(z-\frac{1}{2})$.

From now on we consider the case of trivial character. Define operators

$$\widetilde{W}_8 := q\left(\frac{\mathcal{U}_2}{\sqrt{2}}\right) \quad \text{and} \quad \widetilde{V}_4 := q(\mathcal{V})$$

on $S_{k+1/2}(\Gamma_0(8M))$ where $\mathcal{U}_2,\ \mathcal{V}$ are elements in $H(\overline{K_0^2(8)}, \chi_1)$. Note that both $\widetilde{V}_4$ and $\widetilde{W}_8$ are involutions. Define $\widetilde{V}_4'$ to be the conjugate of $\widetilde{V}_4$ by $\widetilde{W}_8$.

We have the following corollary from Theorem 1.

Corollary 4.3. $\widetilde{W}_8^2 = 1,\ \widetilde{V}_4^2 = 1$.

Corollary 4.4. *$S_{k+1/2}(\Gamma_0(4M))$ is contained in the $+1$ eigenspace of $\widetilde{V}_4$ and $q(\mathcal{U}_1^2) = 4$ on $S_{k+1/2}(\Gamma_0(4M))$.*

Proof. The first assertion follows directly. For the second one, observe that W_4 in Proposition 4.1 is same as W in [3, Remark 5]. So for $f \in S_{k+1/2}(\Gamma_0(4M))$, we have $q(\mathcal{U}_1)(f) = 2\widetilde{W}_4(f)$. In particular, $q(\mathcal{U}_1^2)(f) = 4f$ as $\widetilde{W}_4$ is an involution on $S_{k/2}(\Gamma_0(4M))$. □

Lemma 4.5. *Let $\mathcal{T},\ \mathcal{T}'$ be elements of $H(\overline{K_0^2(8)}, \chi_1)$ respectively supported on the double cosets of $\tilde{s},\ \tilde{s}^{-1} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ such that $\mathcal{T}'(\tilde{s}^{-1}) = \overline{\mathcal{T}(\tilde{s})}$. Then the* L^2*-inner product $\langle\Phi, \mathcal{T}\Psi\rangle = \langle\mathcal{T}'\Phi, \Psi\rangle$ for any $\Phi,\ \Psi \in A_{k+1/2}(8M, \left(\frac{-1}{\cdot}\right)^k)$.*

Proof. The L^2-inner product

$$\begin{aligned}
\langle\Phi, \mathcal{T}\Psi\rangle &= \int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} \Phi(h)\overline{\mathcal{T}\Psi(h)}dh \\
&= \int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} \Phi(h) \int_{\overline{K_0^2(8)}\tilde{s}\overline{K_0^2(8)}} \overline{\mathcal{T}(x)\Psi(hx)}dxdh \\
&= \int_{\overline{K_0^2(8)}\tilde{s}\overline{K_0^2(8)}} \overline{\mathcal{T}(x)} \int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} \Phi(h)\overline{\Psi(hx)}dhdx \quad \text{(Fubini)} \\
&= \int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} \int_{\overline{K_0^2(8)}\tilde{s}^{-1}\overline{K_0^2(8)}} \overline{\mathcal{T}(x^{-1})}\Phi(hx)dx\ \overline{\Psi(h)}dh \\
&= \int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} \int_{\overline{K_0^2(8)}\tilde{s}^{-1}\overline{K_0^2(8)}} \mathcal{T}'(x)\Phi(hx)dx\ \overline{\Psi(h)}dh \\
&= \langle\mathcal{T}'\Phi, \Psi\rangle.
\end{aligned}$$

□

Proposition 4.6. *The operators $\widetilde{W}_8,\ \widetilde{V}_4,\ \widetilde{V}_4'$ are self-adjoint with respect to the Petersson inner product.*

Proof. By Gelbart [4, (3.10)] for $f, g \in S_{k+1/2}(\Gamma_0(8M))$ the Petersson inner product $\langle f, g\rangle$ equals a constant times the L^2-inner product $\langle\Phi_f, \Phi_g\rangle$. In particular, if T is an operator on $S_{k+1/2}(\Gamma_0(8M))$ such that $T = q(\mathcal{T})$ where $\mathcal{T} \in H(\overline{K_0^2(8)}, \chi_1)$, then $\langle f, Tg\rangle$ equals the constant times $\langle \Phi_f, \mathcal{T}\Phi_g\rangle$.

Since the $\overline{K_0^2(8)}$ double cosets of $(y(4), 1)$ and $(w(2^{-2}), 1)$ equal respectively that of $(y(4), 1)^{-1}$ and $(w(2^{-2}), 1)^{-1}$, by Lemma 4.5 we are done. □

4.1. **Comparison with Kohnen's projection map.** Kohnen [6, page 37] and later Ueda-Yamana [15] define function

$$P_8(f) = f|[\xi + \xi^{-1}]_{k+1/2}$$

where

$$\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}\right).$$

We have the following observation.

Proposition 4.7. *Let* $f \in S_{k+1/2}(\Gamma_0(8M))$. *Then*

$$q(\mathcal{Z}_2)(f) = \left(\frac{2}{2k+1}\right) P_8(f).$$

Proof. Using [15, equation (2.2)], we can write

$$P_8(f) = e^{-(2k+1)\pi i/4} \sum_{s=0}^{1} f|[\begin{pmatrix} 4-8Ms & 1 \\ -32Ms & 4 \end{pmatrix}, (-8Msz+1)^{1/2}]_{k+1/2}.$$

Now the proof essentially follows by observing that $\mathcal{Z}_2$ is precisely supported on the double coset of

$$\overline{K_0^2(8)} \begin{pmatrix} 1 & 1/4 \\ 0 & 1 \end{pmatrix} \overline{K_0^2(8)} = \overline{K_0^2(8)} \begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix} \overline{K_0^2(8)}$$

$$= \bigcup_{s=0}^{1} \begin{pmatrix} 1 & -1/4 \\ 8Ms & 1-2Ms \end{pmatrix} \overline{K_0^2(8)}.$$

Indeed, computing as before we obtain

$$q(\mathcal{Z}_2)(f) = \mathcal{Z}_2\left(\left(\begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix}, 1\right)\right) e^{(2k+1)\pi i/4} P_8(f).$$

Also it is easy to check that

$$\mathcal{Z}_2\left(\left(\begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix}, 1\right)\right) = \mathcal{Z}_2\left(\left(\begin{pmatrix} 1 & 1/4 \\ 0 & 1 \end{pmatrix}, 1\right)\right) \overline{\gamma}(-I, -1) = \overline{\varphi_8}(-i^{2k+1})$$

and that $\left(\frac{2}{2k+1}\right) \overline{\varphi_8}(-i^{2k+1}) = e^{-(2k+1)\pi i/4}$. □

Now using the relation in Proposition 3.9(3) we have

Corollary 4.8. $\frac{1}{\sqrt{2}}\left(\frac{2}{2k+1}\right) P_8 = \widetilde{V}_4\widetilde{W}_8\widetilde{V}_4 = \widetilde{W}_8\widetilde{V}_4\widetilde{W}_8 = \widetilde{V}_4'$.

Extending Kohnen's definition, Ueda-Yamana [15] define the plus space $S^+_{k+1/2}(8M)$ to consist of $f = \sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}(\Gamma_0(8M))$ such that $a_n = 0$ for $(-1)^k n \equiv 2, 3 \pmod 4$.

Corollary 4.9. *$S^+_{k+1/2}(8M)$ is the $+1$ eigenspace of $\widetilde{V}'_4$. The -1 eigenspace of $\widetilde{V}'_4$ consists of f such that $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$.*

Proof. From [6, equation(2)],

$$P_8(f) = \sqrt{2}\left(\frac{2}{2k+1}\right)(\sum_n^{(1)} a_n q^n - \sum_n^{(2)} a_n q^n)$$

where $\sum_n^{(1)}$, respectively $\sum_n^{(2)}$, runs over n with $(-1)^k n \equiv 0, 1 \pmod 4$, respectively $(-1)^k n \equiv 2, 3 \pmod 4$. The result now follows using the above corollary. □

Consider the projection map $\wp_k$ [15] onto the plus space which take $\sum_n a_n q^n$ to $\sum_n^{(1)} a_n q^n$.

Corollary 4.10. *If f belongs to the -1 eigenspace of $\widetilde{V}'_4$ then $\wp_k(f) = 0$.*

5. Minus space of $S_{k+1/2}(\Gamma_0(8M))$

Let M be odd and square-free. In this section we shall define the minus space $S^-_{k+1/2}(8M)$ and show that there is a Hecke algebra isomorphism between $S^-_{k+1/2}(8M)$ and $S^{\text{new}}_{2k}(\Gamma_0(4M))$. We shall give a characterization of the minus space as common -1 eigenspace of certain operators. The method we employ is similar to [3]. The main tools that we use are the generators and relations of Theorem 1 and their translation into classical operators. We also need the operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$, $\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(2^n M))$ for $p \mid M$ and operators $\widetilde{Q}_2$, $\widetilde{Q}'_2$, $\widetilde{W}_4$ on $S_{k+1/2}(\Gamma_0(4M))$ that we defined in [3].

The following proposition is crucial to our study of the minus space. To prove it we will use the relations in Theorem 1 including the crucial braid relation (Theorem 1(4)).

Proposition 5.1. *(1) Let $f \in S_{k+1/2}(\Gamma_0(4M)$. Then*
$f \in S^+_{k+1/2}(4M) \iff \widetilde{W}_8 f = f$.
(2) $S^+_{k+1/2}(4M) + \widetilde{W}_4 S^+_{k+1/2}(4M) + \widetilde{W}_8\widetilde{W}_4 S^+_{k+1/2}(4M)$ is a direct sum.
(3) $S^-_{k+1/2}(4M) + \widetilde{W}_8 S^-_{k+1/2}(4M)$ is a direct sum.

Proof. We first prove (1). For $f \in S_{k+1/2}(\Gamma_0(4M)$ we have

$$q\left(\frac{\mathcal{U}_2}{\sqrt{2}}\right) f = \widetilde{W}_8 f = f \Longrightarrow q\left(\frac{\mathcal{U}_1\mathcal{U}_2}{\sqrt{2}}\right) f = q(\mathcal{U}_1) f \Longrightarrow q\left(\frac{\mathcal{T}_1}{\sqrt{2}}\right) f = q(\mathcal{U}_1) f$$
$$\Longrightarrow q\left(\frac{\mathcal{U}_1\mathcal{T}_1}{\sqrt{2}}\right) f = q(\mathcal{U}_1^2) f = 4f \Longrightarrow \widetilde{W}_4 U_4 f = 2^k f$$
$$\Longrightarrow \widetilde{Q}'_2(f) = 2f \Longrightarrow f \in S^+_{k+1/2}(4M).$$

The second implication follows from Lemma 3.7 while the third and fourth follow from Corollary 4.4. For the last part, see [3, Section 4.3]. Now let $f \in$

$S^+_{k+1/2}(4M)$. Since f satisfies the plus-space Fourier coefficient condition, it follows from Corollary 4.9 that $\widetilde{V}_4'(f) = f$, i.e.,

$$\widetilde{W}_8\widetilde{V}_4\widetilde{W}_8(f) = \widetilde{V}_4\widetilde{W}_8\widetilde{V}_4(f) = f.$$

Using Corollary 4.4 we get that $\widetilde{W}_8 f = f$.

We now prove (2). Let $f,\ g,\ h \in S^+_{k+1/2}(4M)$ be such that

$$f + q(\mathcal{U}_1)g + q\left(\frac{\mathcal{U}_2\mathcal{U}_1}{\sqrt{2}}\right)h = 0$$

(note that $q(\mathcal{U}_1) = 2\widetilde{W}_4$ on $S_{k+1/2}(\Gamma_0(4M))$). Applying $q(\mathcal{V})$ to the above equation and using Corollary 4.4 and Lemma 3.8(2) we get

$$f + q(\mathcal{U}_1)g + q\left(\frac{\mathcal{V}\mathcal{U}_2\mathcal{U}_1}{\sqrt{2}}\right)h = 0.$$

Let $h' = q(\mathcal{U}_1)h \in S_{k+1/2}(\Gamma_0(4M))$. Subtracting the above equations we have $q(\mathcal{U}_2)h' = q(\mathcal{V}\mathcal{U}_2)h'$. Next, applying $q(\mathcal{U}_2)$ to the above and using Proposition 3.9(1),(3) we have $\sqrt{2}h' = q(\mathcal{V}\mathcal{U}_2\mathcal{V})h'$. As $\mathcal{V}^2 = 1$ and using Corollary 4.4, we get $q\left(\frac{\mathcal{U}_2}{\sqrt{2}}\right)h' = h'$. Now part (1) implies that $h' \in S^+_{k+1/2}(4M)$. Thus $h' = 0$ as

$$S^+_{k+1/2}(4M)\bigcap \widetilde{W}_4 S^+_{k+1/2}(4M) = \{0\}$$

(follows as in [3, Proposition 6.17]) and consequently $f = g = h = 0$.

For (3) observe that $S^-_{k+1/2}(4M)$ is contained in the $+1$ eigenspace of $\widetilde{V}_4$ and $\widetilde{W}_8 S^-_{k+1/2}(4M)$ is contained in the $+1$ eigenspace of $\widetilde{V}_4'$. Let $f \neq 0$ belong to the intersection. Then $\widetilde{V}_4 f = f = \widetilde{V}_4' f$. Now using $\widetilde{V}_4' = q\left(\frac{\mathcal{U}_2\mathcal{V}\mathcal{U}_2}{2}\right) = q\left(\frac{\mathcal{V}\mathcal{U}_2\mathcal{V}}{\sqrt{2}}\right)$ (Proposition 3.9(3)) we get $q\left(\frac{\mathcal{U}_2}{\sqrt{2}}\right)(f) = f$. Thus by (1), $f \in S^+_{k+1/2}(4M) \cap S^-_{k+1/2}(4M)$, a contradiction. □

We recall the following theorem of Ueda.

Theorem 4. *(Ueda [12]) Let M be odd and square-free. There exists an isomorphism of vector spaces $\psi : S_{k+1/2}(\Gamma_0(8M)) \to S_{2k}(\Gamma_0(4M))$ satisfying*

$$T_p(\psi(f)) = \psi(T_{p^2}(f)) \ \text{ for all primes } p \text{ coprime to } 2M.$$

We first construct the minus space at level 8. In the above theorem take $M = 1$. It follows using Proposition 5.1, Atkin-Lehner and dimension equality (see [3, Corollary 6.1]) that

Lemma 5.2. *ψ maps $S^+(4) \oplus \widetilde{W}_4 S^+(4) \oplus \widetilde{W}_8\widetilde{W}_4 S^+(4)$ isomorphically onto $S_{2k}(\Gamma_0(1)) \oplus V(2)S_{2k}(\Gamma_0(1)) \oplus V(4)S_{2k}(\Gamma_0(1))$.*

Also, since $S^-(4)$ is Hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(2))$ [3] we have

Lemma 5.3. *ψ maps $S^-(4) \oplus \widetilde{W}_8 S^-(4)$ isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(2)) \oplus V(2)S^{\mathrm{new}}_{2k}(\Gamma_0(2))$.*

Let

$$E := (S^+(4) \oplus \widetilde{W}_4 S^+(4) \oplus \widetilde{W}_8 \widetilde{W}_4 S^+(4)) \ \oplus \ (S^-(4) \oplus \widetilde{W}_8 S^-(4)).$$

Thus ψ maps E Hecke isomorphically onto $S_{2k}^{\mathrm{old}}(\Gamma_0(4))$.

Define $S^-_{k+1/2}(8)$ to be the orthogonal complement of E.

Theorem 5. *$S^-_{k+1/2}(8)$ has a basis of eigenforms for all the operators T_{p^2}, p odd; these eigenforms are also eigenfunctions under U_4. If two eigenforms in $S^-_{k+1/2}(8)$ share the same eigenvalues for all T_{p^2}, then they are a scalar multiple of each other; ψ induces a Hecke algebra isomorphism:*

$$S^-_{k+1/2}(8) \cong S_{2k}^{\mathrm{new}}(\Gamma_0(4)).$$

Proof. The proof uses Lemmas 5.2 and 5.3, Theorem 4 and follows by the argument in [3, Theorem 5]. □

Proposition 5.4. *If $f \in S^-_{k+1/2}(8)$ is a Hecke eigenform for all the Hecke operators T_{p^2}, p odd prime, then $\widetilde{W}_8(f) = \pm f$.*

Further, for any $f \in S^-_{k+1/2}(8)$, we have

$$U_4 f = 0 \quad \text{and} \quad \widetilde{V}_4 f = -f = \widetilde{V}_4' f.$$

Proof. Let $f \in S^-_{k+1/2}(8)$ be a Hecke eigenform under all such T_{p^2}. Let $g = \widetilde{W}_8(f)$. Since $\widetilde{W}_8$ commutes with T_{p^2}, p odd, we get that g is an eigenform for all T_{p^2} with the same eigenvalues as f. Since $F := \psi(f) \in S_{2k}^{\mathrm{new}}(\Gamma_0(4))$ is a newform, by [1] $\psi(g)$ is a scalar multiple of $\psi(f)$. Thus g is a scalar multiple of f. Since $\widetilde{W}_8^2 = 1$, we get the first assertion.

Further, by [1], since F is a newform of level 4, $U_2(F) = 0$. Since the Shimura lift [11], $\mathrm{Sh}_t(f)$, for any square-free t is also an eigenform for all T_p with the same eigenvalues as F, by [1] $\mathrm{Sh}_t\, f$ is a scalar multiple of F. Thus

$$\mathrm{Sh}_t(U_4 f) = U_2(\mathrm{Sh}_t\, f) = 0$$

for all square-free t and hence we get that $U_4 f = 0$.

Now

$$0 = U_4(f) = q(\mathcal{T}_1) f = q(\mathcal{U}_1 \mathcal{U}_2) f.$$

Since $\widetilde{W}_8(f) = \pm f$ we have $q(\mathcal{U}_1) f = 0$. As $\mathcal{U}_1{}^2 = 2 + 2\mathcal{V}$ (Proposition 3.9 (2)) we get $\widetilde{V}_4 f = -f$. Consequently $\widetilde{V}_4' f = -f$.

Since $S^-_{k+1/2}(8)$ has a basis of eigenforms under T_{p^2}, it follows for all $f \in S^-_{k+1/2}(8)$ that we have $U_4 f = 0$ and $\widetilde{V}_4 f = -f = \widetilde{V}_4' f$. □

Theorem 6. *Let $f \in S_{k+1/2}(\Gamma_0(8))$. Then*

$$f \in S^-_{k+1/2}(8) \iff \widetilde{V}_4 f = -f = \widetilde{V}_4' f.$$

Proof. If $f \in S^-_{k+1/2}(8)$ then by Proposition 5.4 the conditions hold.

Conversely, let $\widetilde{V}_4 f = -f = \widetilde{V}'_4 f$. Since

$$S_{k+1/2}(\Gamma_0(4)) = S^+(4) \oplus \widetilde{W}_4 S^+(4) \oplus S^-(4)$$

is contained in the $+1$ eigenspace of $\widetilde{V}_4$ and $\widetilde{W}_8(\widetilde{W}_4 S^+(4) \oplus S^-(4))$ is contained in the $+1$ eigenspace of $\widetilde{V}'_4$ and $\widetilde{V}_4$, $\widetilde{V}'_4$ are self-adjoint, it follows that $f \in S^-_{k+1/2}(8)$. □

Note that since $\widetilde{V}'_4$ is self-adjoint, we can write $S_{k+1/2}(\Gamma_0(8))$ as a direct sum of $+1$ and -1 eigenspaces of $\widetilde{V}'_4$. As noted in Corollary 4.9, $S^+_{k+1/2}(8)$ is the $+1$ eigenspace of $\widetilde{V}'_4$, let us denote by $S^{\min}_{k+1/2}(8)$ the -1 eigenspace of $\widetilde{V}'_4$. In particular, $S^{\min}_{k+1/2}(8)$ is the subspace of $S_{k+1/2}(\Gamma_0(8))$ consisting of forms $\sum_{n=1}^{\infty} a_n q^n$ such that $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$. Further, for a given newform F of level dividing 4, let $S_{k+1/2}(8, F)$ denote the subspace of forms that are Shimura-equivalent to F (i.e., forms f that are eigenforms under T_{p^2} with the same eigenvalues as F under T_p for all odd primes p). Then we have the following simple observation.

Proposition 5.5. *(1)* $S^+_{k+1/2}(8) = S^+(4) \oplus \widetilde{W}_8 A^+(4) \oplus \widetilde{W}_8 S^-(4)$ *where* $A^+(4) = \widetilde{W}_4 S^+(4)$.

(2) Given a newform F *of weight* $2k$ *and level dividing* 4, *there exists a unique Shimura-equivalent form in* $S_{k+1/2}(8, F) \cap S^{\min}_{k+1/2}(8)$.

Proof. Let

$$S := S^+(4) \oplus \widetilde{W}_8 A^+(4) \oplus \widetilde{W}_8 S^-(4), \quad R := A^+(4) \oplus S^-(4) \oplus S^-(8)$$

(here and later $S^-(8)$ is a simplified notation for $S^-_{k+1/2}(8)$). It follows from Corollary 4.4 that $S \subseteq S^+_{k+1/2}(8)$. To prove equality it is enough to show that $R \cap S^+_{k+1/2}(8) = \{0\}$. Let $f + g + h$ belong to the intersection where $f \in A^+(4)$, $g \in S^-(4)$, $h \in S^-(8)$. Thus $\widetilde{V}'_4(f + g + h) = f + g + h$. Since $\widetilde{V}'_4 = \widetilde{V}_4 \widetilde{W}_8 \widetilde{V}_4$, by Corollary 4.4, it follows that $\widetilde{W}_8 f + \widetilde{W}_8 g = f + g + 2\widetilde{V}_4 h$. Since $\widetilde{V}_4$ preserves $S^-(8)$ and as each of the terms in the above relation is in the direct summand, we are done.

For (2), since T_{p^2} for odd prime p commutes with $\widetilde{V}'_4$, we get that $\widetilde{V}'_4$ preserves the space $S_{k+1/2}(8, F)$. Now it follows from (1) and Lemmas 5.2 and 5.3 that for a weight $2k$ newform of level 1 there are two Shimura-equivalent forms in the space $S^+_{k+1/2}(8)$, while for a weight $2k$ newform of level 2 there is precisely one Shimura-equivalent form in $S^+_{k+1/2}(8)$. Consequently, using dimension equality we obtain (2). The case of a newform of level 4 is already considered in Theorem 5. □

We now define the minus space at level $8M$ for M odd square-free. Let $1 \neq M = p_1 p_2 \cdots p_k$, and for each $i = 1, \ldots, k$ define $M_i = M/p_i$. Note that by [3, Corollary 4.3 (4)] $S_{k+1/2}(\Gamma_0(8M_i))$ is contained in the p_i eigenspace of $\widetilde{Q}_{p_i}$. Now following the proof of [3, Proposition 6.4] we obtain

Proposition 5.6. $S_{k+1/2}(\Gamma_0(8M_i)) \bigcap \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(8M_i)) = \{0\}$.

Using Atkin-Lehner [1] and dimension equality we have the following.

Corollary 5.7. *ψ maps $S_{k+1/2}(\Gamma_0(8M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(8M_i))$ isomorphically onto*

$$S_{2k}(\Gamma_0(4M_i)) \oplus V(p_i) S_{2k}(\Gamma_0(4M_i)).$$

Let $S^{+,\mathrm{new}}_{k+1/2}(4M)$ be the new space inside the Kohnen plus subspace of $S_{k+1/2}(4M)$ and $S^{-}_{k+1/2}(4M)$ as defined in [3]. Then by Proposition 5.1 and Atkin-Lehner we have similarly

Corollary 5.8. *ψ maps $S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_8 \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M)$ isomorphically onto*

$$S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \oplus V(2) S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \oplus V(4) S^{\mathrm{new}}_{2k}(\Gamma_0(M)).$$

Corollary 5.9. *ψ maps $S^{-}_{k+1/2}(4M) \oplus \widetilde{W}_8 S^{-}_{k+1/2}(4M)$ isomorphically onto*

$$S^{\mathrm{new}}_{2k}(\Gamma_0(2M)) \oplus V(2) S^{\mathrm{new}}_{2k}(\Gamma_0(2M)).$$

We note the following observation.

Remark 1. *Since $S^{-}_{k+1/2}(4M)$ is contained in the $+1$ eigenspace of $\widetilde{V}_4$, $\widetilde{W}_8 S^{-}_{k+1/2}(4M)$ is contained in the $+1$ eigenspace of $\widetilde{V}'_4$ and hence is contained inside $S^{+}_{k+1/2}(8M)$. In [15], Ueda-Yamana defined a newspace inside $S^{+}_{k+1/2}(8M)$ and proved that it is Hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$. Using the above corollary and following Proposition 5.5 we see that the plus newspace identified by [15] is the space $\widetilde{W}_8 S^{-}_{k+1/2}(4M)$. Note that $\widetilde{V}'_4$ does not preserve the space $S_{k+1/2}(\Gamma_0(4M))$ and so we do not expect a Fourier coefficient condition for $S^{-}_{k+1/2}(4M)$, as also observed in [3].*

Now let

$$B_i = S_{k+1/2}(\Gamma_0(8M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(8M_i)), \quad i = 1, \dots, k.$$

Define

$$E = \sum_{i=1}^{k} B_i \oplus (S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_8 \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M)) \oplus S^{-}_{k+1/2}(4M) \oplus \widetilde{W}_8 S^{-}_{k+1/2}(4M).$$

Proposition 5.10. *Under ψ the space E maps isomorphically onto the old space $S^{\mathrm{old}}_{2k}(\Gamma_0(4M))$.*

Proof. This follows from Corollaries 5.7 and 5.8 and 5.9 and from the decomposition

$$\begin{aligned} S_{2k}^{\mathrm{old}}(\Gamma_0(4M)) = & \left(\sum_{i=1}^{k} S_{2k}(\Gamma_0(4M_i)) \oplus V(p_i) S_{2k}(\Gamma_0(4M_i)) \right) \\ & \oplus (S_{2k}^{\mathrm{new}}(\Gamma_0(M)) \oplus V(2) S_{2k}^{\mathrm{new}}(\Gamma_0(M)) \oplus V(4) S_{2k}^{\mathrm{new}}(\Gamma_0(M))) \\ & \oplus S_{2k}^{\mathrm{new}}(\Gamma_0(2M)) \oplus V(2) S_{2k}^{\mathrm{new}}(\Gamma_0(2M)). \end{aligned}$$

□

We now define the minus space to be the orthogonal complement of E,

$$S^{-}_{k+1/2}(4M) := E^{\perp}.$$

Theorem 7. *The space $S^{-}_{k+1/2}(8M)$ has a basis of eigenforms for all the operators T_{q^2}, where q is an odd prime satisfying $(q, M) = 1$. Under ψ, the space $S^{-}_{k+1/2}(8M)$ maps isomorphically onto the space $S_{2k}^{\mathrm{new}}(\Gamma_0(4M))$. If two forms in $S^{-}_{k+1/2}(8M)$ have the same eigenvalues for all the operators T_{q^2}, $(q, 2M) = 1$, then they are the same up to a scalar factor. Moreover, $S^{-}_{k+1/2}(8M)$ has the strong multiplicity one property in the full space of level $8M$.*

We give the characterization of our minus space. We have the following proposition.

Proposition 5.11. *Let $f \in S^{-}_{k+1/2}(8M)$ be a Hecke eigenform for all the Hecke operators T_{q^2}, q prime and q coprime to $2M$. Then for any prime p dividing M, $\widetilde{W}_{p^2} = \pm f$, $\widetilde{W}_8(f) = \pm f$. Moreover, $U_{p^2}(f) = -p^{k-1}\lambda(p) f$ and $U_4(f) = 0$ where $\lambda(p) = \pm 1$.*

Consequently, for any $f \in S^{-}_{k+1/2}(8M)$ we have $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ for all primes p dividing M and $\widetilde{V}_4 f = -f = \widetilde{V}'_4 f$.

Proof. The proof follows similarly to the proof of Proposition 5.11 and proofs of [3, Propositions 6.12, 6.13, 6.14]. □

Theorem 8. *Let $f \in S_{k+1/2}(8M)$. Then $f \in S^{-}_{k+1/2}(8M)$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ for every prime p dividing M and $\widetilde{V}_4(f) = -f = \widetilde{V}'_4(f)$.*

Proof. One side of the implication follows from Proposition 5.11. For the converse, we use Corollary 4.4, that $S_{k+1/2}(\Gamma_0(8M/p))$ is contained in the p eigenspace of $\widetilde{Q}_p$ for all p dividing M and that the operators are self-adjoint. □

Corollary 5.12. *If $f = \sum_{n=1}^{\infty} a_n q^n \in S^{-}_{k+1/2}(8M)$ then*

$$a_n = 0 \quad \text{for } (-1)^k n \equiv 0, 1 \pmod 4.$$

In particular, the projection map $\wp_k$ is identically zero on the minus space $S^-_{k+1/2}(8M)$.

Proof. The proof follows from the above theorem and Corollaries 4.8 and 4.9. □

Remark 2. *The above corollary contradicts some of the results of* [9]. *In particular, in Section 3 of their paper the authors assert that the projection map $\wp_k$ on the newforms at level $8M$ is itself, i.e. their newspace at level $8M$ (which corresponds to $S^{\mathrm{new}}_{2k}(\Gamma_0(4M))$) satisfies the plus space condition. However, our results above and the example below present a contrary picture : if $f = \sum_{n=1}^{\infty} a_n q^n$ is in the newspace at level $8M$ then $a_n = 0$ for $(-1)^k n \equiv 0, 1 \pmod 4$, i.e., $\wp_k(f) = 0$.*

Remark 3. *We note that Theorems 6 and 8 are analogous to* [2, Theorem 9] *in the integral weight scenario. Indeed Theorem 8 can be restated as $f \in S^-_{k+1/2}(8M)$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ for every prime p dividing M and $q(\mathcal{U}_1)f = 0 = \widetilde{W}_8 q(\mathcal{U}_1)\widetilde{W}_8(f)$.*

Remark 4. *We note that the decomposition of the space $S_{k+1/2}(\Gamma_0(8M))$ is completely analogous to that of $S_{2k}(\Gamma_0(4M))$ when we look at it through the local Hecke algebra. We illustrate this in the case $M = 1$.*

$$\begin{aligned} S_{2k}(\Gamma_0(4)) =& (S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_1)S_{2k}(\Gamma_0(1)) \oplus q(\mathcal{U}_2)S_{2k}(\Gamma_0(1))) \\ & \oplus (S^{\mathrm{new}}_{2k}(\Gamma_0(2)) \oplus q(\mathcal{U}_2)S^{\mathrm{new}}_{2k}(\Gamma_0(2))) \oplus S^{\mathrm{new}}_{2k}(\Gamma_0(4)). \end{aligned}$$

In the above $\mathcal{U}_1$, $\mathcal{U}_2$ are elements in the Hecke algebra $H(\mathrm{GL}_2(\mathbb{Q}_2)//K_0(4))$ coming from the double cosets of $\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right)$ respectively. Also, it follows from [2] *that $q(\mathcal{U}_2)q(\mathcal{U}_1)S_{2k}(\Gamma_0(1)) = q(\mathcal{U}_1)S_{2k}(\Gamma_0(1))$.*

Now let us look at the space $S_{k+1/2}(\Gamma_0(8M))$. We have

$$\begin{aligned} S_{k+1/2}(\Gamma_0(8)) =& (A^+(4) \oplus q(\mathcal{U}_1)A^+(4) \oplus q(\mathcal{U}_2)A^+(4)) \\ & \oplus (S^-(4) \oplus q(\mathcal{U}_2)S^-(4)) \oplus S^-(8). \end{aligned}$$

Here $\mathcal{U}_1$, $\mathcal{U}_2$ are elements in the Hecke algebra $H(K_0^2(8), \chi_1)$ coming from $w(2^{-1})$, $w(2^{-2})$ respectively. Recall from [3] *that*

$$A^+(4) = \widetilde{W}_4 S^+(4) = q(\mathcal{U}_1)S^+(4).$$

Further, by Proposition 5.1,

$$q(\mathcal{U}_2)q(\mathcal{U}_1)A^+(4) = q(\mathcal{U}_2)S^+(4) = S^+(4) = q(\mathcal{U}_1)A^+(4).$$

Example 1. The space $S_{3/2}(\Gamma_0(152))$ is 8-dimensional and there are four primitive Hecke eigenforms of weight 2 and level dividing 76, namely F_{19} of level 19, G_{38}, H_{38} of level 38 and K_{76} of level 76. We have

$$S_{3/2}(\Gamma_0(152)) = S_{3/2}(152, F_{19}) \oplus S_{3/2}(152, G_{38}) \oplus S_{3/2}(152, H_{38}) \oplus S_{3/2}(152, K_{76}).$$

We compute the Shimura decomposition [10]. As we would expect from the above remark, $S_{3/2}(152, F_{19})$ is 3-dimensional space and is spanned by

$$\begin{aligned}
f_1 &= q + q^5 - 2q^6 - q^9 - q^{17} + 2q^{25} + 2q^{30} + 2q^{42} - 3q^{45} + O(q^{50}),\\
f_2 &= q^4 - 2q^{11} - 2q^{16} + 2q^{19} + q^{20} - 2q^{24} + 3q^{28} + 2q^{35} - q^{36} + O(q^{40}),\\
f_3 &= q^7 - q^{11} - 2q^{16} + q^{19} + 2q^{28} + q^{35} - 2q^{39} - q^{43} + 2q^{44} - q^{47} + O(q^{50}),
\end{aligned}$$

$S_{3/2}(152, G_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned}
g_1 &= q - 2q^5 + q^6 + 2q^9 - q^{17} - q^{25} - 3q^{26} - 4q^{30} + 3q^{38} + 5q^{42} + O(q^{50}),\\
g_2 &= q^4 + q^7 - q^{16} - 2q^{20} - 3q^{23} + q^{24} - q^{28} + 2q^{36} + q^{39} + 2q^{47} + O(q^{50}),
\end{aligned}$$

$S_{3/2}(152, H_{38})$ is 2-dimensional space and is spanned by

$$\begin{aligned}
h_1 &= q^2 + 2q^{10} - 3q^{13} - q^{14} - 2q^{18} - q^{21} + 2q^{22} + q^{29} + O(q^{30})\\
h_2 &= q^3 - q^8 + q^{12} - q^{19} - q^{27} - q^{32} - 2q^{40} + q^{48} + O(q^{50})
\end{aligned}$$

and $S_{3/2}(152, K_{76})$ is 1-dimensional space and is spanned by

$$k_1 = q^2 - q^{10} - q^{14} + q^{18} + 2q^{21} - q^{22} - 2q^{29} - 2q^{33} - q^{34} + 2q^{37} + q^{38} - 2q^{41} + O(q^{50}).$$

The Kohnen plus space $S^+_{3/2}(152)$ is 4-dimensional and it is spanned by $\{f_2, f_3, g_2, h_2\}$. We further note that $S_{3/2}(76, F_{19})$ is 2-dimensional and spanned by $\{f_1 + f_3, f_2 - f_3\}$ and $S^-_{3/2}(76)$ is 2-dimensional and spanned by $\{g_1 - g_2, h_1 - h_2\}$. The minus space at level 152, $S^-_{3/2}(152)$, is 1-dimensional and spanned by k_1, and is Shimura equivalent to K_{76}. Note that k_1 satisfies the Fourier coefficient condition as noted in Corollary 5.12.

We finally look at the minus space of level $8M$ with character $\left(\frac{2}{\cdot}\right)$. Following Ueda [13], we define

$$\widetilde{\tau}_8 : S_{k+1/2}(\Gamma_0(8M)) \longrightarrow S_{k+1/2}(\Gamma_0(8M), \left(\frac{2}{\cdot}\right))$$

given by the action

$$\mid \left[\begin{pmatrix} 8a & b \\ 8Mc & 8d \end{pmatrix}, \left(\frac{Mc}{d}\right) 8^{1/4}(i(Mcz + d))^{1/2} \right]_{k+1/2}$$

where $a,\ b,\ c,\ d$ are such that $8ad - Mbc = 1$ and $b \equiv d \equiv 1 \pmod 8$. The above action is independent of the choice of $a,\ b,\ c,\ d$ satisfying the conditions. It is routine to check that

$$\widetilde{\tau}_8 \Delta_0(8M) \widetilde{\tau}_8^{-1} = \Delta_0(8M, \left(\frac{2}{\cdot}\right))$$

and that $\widetilde{\tau}_8^2 = 1$ on $S_{k+1/2}(\Gamma_0(8M))$. Further, we check that $\widetilde{\tau}_8$ commutes with Hecke operators T_{p^2} for all p odd, giving a Hecke isomorphism from $S_{k+1/2}(\Gamma_0(8M))$ to $S_{k+1/2}(\Gamma_0(8M), \left(\frac{2}{\cdot}\right))$.

Define the minus space

$$S^{-}_{k+1/2}(8M, \left(\frac{2}{\cdot}\right)) := \widetilde{\tau}_8 S^{-}_{k+1/2}(8M).$$

It follows that $S^{-}_{k+1/2}(8M, \left(\frac{2}{\cdot}\right))$ maps Hecke isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(4M))$. Further, $S^{-}_{k+1/2}(8M, \left(\frac{2}{\cdot}\right))$ has a similar characterization: $g \in S^{-}_{k+1/2}(8M, \left(\frac{2}{\cdot}\right))$ if and only if $\widetilde{\tau}_8 \widetilde{Q}_p \widetilde{\tau}_8^{-1}(g) = -g = \widetilde{\tau}_8 \widetilde{Q}'_p \widetilde{\tau}_8^{-1}(g)$ for every prime p dividing M and $\widetilde{\tau}_8 \widetilde{V}_4 \widetilde{\tau}_8^{-1}(g) = -g = \widetilde{\tau}_8 \widetilde{V}'_4 \widetilde{\tau}_8^{-1}$.

Let $\mathcal{Z}'_1 \in H(\overline{K_0^2(8)}, \chi_2)$.

Proposition 5.13. *The action of $\widetilde{\tau}_8 \widetilde{V}_4 \widetilde{\tau}_8^{-1}$ equates the action of $q(\mathcal{Z}'_1)$ on $S_{k+1/2}(\Gamma_0(8M), \left(\frac{2}{\cdot}\right))$. In particular, $S^{-}_{k+1/2}(8M, \left(\frac{2}{\cdot}\right))$ is contained in the -1 eigenspace of $q(\mathcal{Z}'_1)$.*

Proof. By Propositions 4.1 and 4.2, $\widetilde{V}_4$ acts by $|[\left(\begin{smallmatrix} 1 & 0 \\ 4M & 1 \end{smallmatrix}\right), 1]_{k+1/2}$ and $q(\mathcal{Z}'_1)$ acts by $|[\left(\begin{smallmatrix} 1 & -1/2 \\ 0 & 1 \end{smallmatrix}\right), 1]_{k+1/2}$. We check that

$$\widetilde{\tau}_8^{-1}\left(\begin{pmatrix} 1 & 0 \\ 4M & 1 \end{pmatrix}, 1\right) \widetilde{\tau}_8 \left(\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, 1\right) = \left(\begin{pmatrix} * & * \\ C_1 & D_1 \end{pmatrix}, (C_1 z + D_1)^{1/2}\right)$$

where

$$C_1 = 32Ma^2 \quad \text{and} \quad D_1 = 1 + 4Mab + 16Ma^2.$$

As $D_1 \equiv 1 \pmod 4$, $\epsilon_{D_1} = 1$. Note that $\left(\frac{M}{D_1}\right) = 1$ and hence $\left(\frac{C_1}{D_1}\right) = \left(\frac{2M}{D_1}\right) = \left(\frac{2}{D_1}\right)$. Thus the right hand side belongs to $\Delta_0(8M, \left(\frac{2}{\cdot}\right))$ and we are done. □

Remark 5. *We note that when $M = 1$, the action of $\widetilde{\tau}_8$ is same as that of $|[\left(\begin{smallmatrix} 0 & -1 \\ 8 & 0 \end{smallmatrix}\right), 8^{1/4}(-iz)^{1/2}]_{k+1/2}$ and we can check that*

$$\widetilde{\tau}_8 \widetilde{W}_8 \widetilde{\tau}_8^{-1} = q\left(\frac{\mathcal{U}_1}{\sqrt{2}}\right)$$

(Proposition 4.2) on $S_{k+1/2}(\Gamma_0(8), \left(\frac{2}{\cdot}\right))$; recall that $\frac{\mathcal{U}_1}{\sqrt{2}} \in H(\overline{K_0^2(8)}, \chi_2)$ is an involution. Thus $g \in S^{-}_{k+1/2}(8, \left(\frac{2}{\cdot}\right))$ if and only if

$$q(\mathcal{Z}'_1)(g) = -g = q\left(\frac{\mathcal{U}_1 \mathcal{Z}'_1 \mathcal{U}_1}{2}\right)(g).$$

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