NEWFORMS OF HALF-INTEGRAL WEIGHT: THE MINUS SPACE COUNTERPART

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Abstract. We study genuine local Hecke algebras of the Iwahori type of the double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ and translate the generators and relations to classical operators on the space $S_{k+1/2}(\Gamma_0(4M))$, M odd and square-free. In [9] Manickam, Ramakrishnan and Vasudevan defined the new space of $S_{k+1/2}(\Gamma_0(4M))$ that maps Hecke isomorphically onto the space of newforms of $S_{2k}(\Gamma_0(2M))$. We characterize this newspace as a common -1-eigenspace of certain pair of conjugate operators that are coming from local Hecke algebras. We use the classical Hecke operators and relations that we obtain to give a new proof of the results in [9] and to prove our characterization result.

1. Introduction

Let M be odd and square-free and k be a positive integer. In a remarkable work, Niwa [10] comparing the traces of Hecke operators proved existence of Hecke isomorphism between $S_{k+1/2}(\Gamma_0(4M))$, the space of holomorphic cusp forms of weight $k+1/2$ on the congruence subgroup $\Gamma_0(4M)$ and $S_{2k}(\Gamma_0(2M))$, the space of weight $2k$ cusp forms on $\Gamma_0(2M)$. In [5, 6] Kohnen considers a certain Hecke operator on $S_{k+1/2}(\Gamma_0(4M))$ which is an analogue of Niwa's operator at level 4. This operator has two eigenvalues, one positive and one negative and the Kohnen plus space is the eigenspace of the positive eigenvalue. Kohnen considers a new space $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M))$ inside his plus space and proves that this new subspace is Hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(M))$, the space of newforms of weight $2k$ and level M. From Kohnen's results, it is clear that the Niwa map sends the Kohnen plus space to a subspace of old forms inside $S_{2k}(\Gamma_0(2M))$. In a subsequent work, Manickam, Ramakrishnan and Vasudevan [9] define the newspace of $S_{k+1/2}(\Gamma_0(4M))$ that maps Hecke isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$, the space of newforms of weight $2k$ and level $2M$. Our main objective in this paper is to give a common eigenspace characterization for this newspace of $S_{k+1/2}(\Gamma_0(4M))$ in terms of certain finitely many pairs of cojugate operators.

This is a continuation of our earlier work in [2] where we use local Hecke algebras to give an eigenspace characterization of the space of integral weight newforms. The local Hecke algebra method allows us to obtain the newspace

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of Manickam et al in a different way and we show that it is the common -1-eigenspace of Kohnen's operator, a conjugate of Kohnen's operator and pairs of p-adic analogues of Kohnen's operator and their conjugates for each prime dividing M. We call this newspace the minus space at level $4M$.

Our results are motivated by the results of Loke and Savin [8] who interpreted the Kohnen plus space in representation theory language. For the case $M = 1$, Loke and Savin defined another space of half-integer weight forms which they showed is "conjugate" to the Kohnen plus space. This means that it is an image of the Kohnen plus space by an invertible Hecke operator and is isomorphic to the Kohnen plus space as a Hecke module. We show that the Kohnen plus space and the space considered by Loke and Savin do not intersect and that their sum maps isomorphically to the space of old forms $S_{2k}^{\mathrm{old}}(\Gamma_0(2))$ under the Niwa map. We define the minus space at level 4 to be the orthogonal complement of the direct sum under the Petersson inner product and show that it is mapped isomorphically under the Niwa map to $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$, the space of newforms on $\Gamma_0(2)$. We characterize this space as a common eigenspace of two Hecke operators: the Niwa operator used by Kohnen to define the Kohnen plus space and a conjugate of the Niwa operator which was considered by Loke and Savin. The minus space is the intersection of the negative eigenspace of both operators. We normalize the negative eigenvalue to be -1 as in [2]. Our description of the minus space at level 4 is completely analogous to our description of the new space $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$ in [2] where we showed that $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$ is the common -1-eigenspace of two Hecke operators. To summarize the case of $M = 1$: We show that the space $S_{k+1/2}(\Gamma_0(4))$ decomposes into a direct sum of three spaces: the Kohnen plus space, a "conjugate" of the Kohnen plus space given by Loke and Savin and the minus space. The Kohnen plus space and its conjugate are indistinguishable as Hecke modules which is the same as saying that they are mapped under the Niwa map to "conjugate" spaces of old forms. The minus space is different as a Hecke module from both spaces.

In order to generalize this result for M odd and square-free we consider certain p-adic Hecke algebras for every prime p dividing M. Our work follows that of Loke and Savin who studies a certain 2-adic Hecke algebra which allowed them to give a representation theoretic interpretation of the Kohnen plus space and to introduce the operator which is conjugate of Niwa's operator and the space which is a "conjugate" to Kohnen's plus space.

We compute genuine local Hecke algebras, of the Iwahori type with genuine quadratic central character, for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$, the double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ and prove that this is isomorphic to the Iwahori Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_p)$. In [13], Savin obtained description of Iwahori-type Hecke algebras for coverings of simply connected Chevally group $G \neq \mathrm{SL}_2$. We are not aware of any such results for SL_2 apart from the work of Loke-Savin [8] for the 2-adic case which, we generalize for any odd prime p.

In our p-adic Hecke algebra, we consider two p-adic operators that give rise to conjugate classical Hecke operators which when we use along with

Niwa's operator and its conjugate allow us to define our minus space at level $4M$. We note that these two p-adic operators are p-adic analogues of Niwa's operator and its conjugate. We give two descriptions of the minus space: one description as an orthogonal complement of a certain sum of subspaces and another description as a common -1-eigenspace of the Niwa operator, its conjugate and a pair of conjugate operators for each prime dividing M. This again is completely analogous to our description of the space of newforms of weight $2k$ for $\Gamma_0(2M)$ given in [2, Theorem 1]. We show that the minus space of weight $k+1/2$ at level $4M$ is isomorphic as a Hecke module to the space of newforms of weight $2k$ at level $2M$.

Due to the Hecke isomorphism and multiplicity one it is clear that the minus space we define is identical to the newspace of [9]. In particular we obtain a new proof of the Hecke isomorphism in [9]. We note that our description of the minus space as an orthogonal complement differs from the description of the newspace in [9]. We elaborate this point in Remark 6.

Our paper is divided as follows. We set up notation following Shimura's work on half-integral weight forms and recall Gelbart's theory of the double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$. In Section 3 we define a genuine Hecke algebra of the double cover of $\mathrm{SL}_2(\mathbb{Q}_p)$ modulo certain subgroups and a genuine central character and give its presentation using generators and relations. In particular we recall the work of Loke and Savin when $p = 2$. In Section 4 we translate certain elements in our p-adic Hecke algebra to classical Hecke operators on $S_{k+1/2}(\Gamma_0(4M))$. We obtain two classical operators: $\widetilde{Q}_p$ with eigenvalues p and -1 and an involution $\widetilde{W}_{p^2}$. We further consider $\widetilde{Q}'_p$ which is conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_{p^2}$. We check that these operators are self-adjoint with respect to the Petersson inner product. We recall Kohnen's classical operator Q on $S_{k+1/2}(\Gamma_0(4M))$ which he uses to describe his plus space. We show that his operator Q comes from the 2-adic Hecke algebra considered by Loke and Savin. Let $\widetilde{Q}'_2 := \left(\frac{2}{2k+1}\right) Q/\sqrt{2}$ and $\widetilde{Q}_2$ be conjugate of $\widetilde{Q}'_2$ by an involution $\widetilde{W}_4$. The operators $\widetilde{Q}'_p$ and $\widetilde{Q}_p$ are p-adic analogues of Kohnen's operator Q and its conjugate. In Section 5 we define our minus space $S^-_{k+1/2}(\Gamma_0(4M))$ and prove our main result:

Theorem. *Let $S^-_{k+1/2}(\Gamma_0(4M)) \subseteq S_{k+1/2}(\Gamma_0(4M))$ be the common -1-eigenspace of operators $\widetilde{Q}_p$ and $\widetilde{Q}'_p$ for all primes p dividing $2M$. Then $S^-_{k+1/2}(\Gamma_0(4M))$ has a basis of eigenforms for all the operators T_{q^2} where q is a prime coprime to $2M$ and all the operators U_{p^2} where p is a prime dividing $2M$, and maps isomorphically under the Niwa map onto the space $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$.*

We are certain that the Hecke algebra approach can be employed to give a newform theory for the space of half-integral weight forms of a general level. Indeed in [3] we use the methods developed in this paper to define the minus space at level $8M$, M odd and square-free and show that the minus space at level $8M$ is Hecke isomorphic to $S_{2k}^{\mathrm{new}}(\Gamma_0(4M))$. This generalizes Ueda-Yamana's work in [17]. Please refer to Remark 7 for more details. We plan to use the results in this paper to study Whittaker functions associated with automorphic forms coming from Hecke eigenforms in the minus space. As an application we plan to generalize the Kohnen-Zagier formula for the twisted central L-values of an integer weight modular form of level $2M$.

2. Preliminaries and Notation

Let $k,\ N$ denote positive integers. Let $\Gamma_0(N)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N$. We denote by $S_k(\Gamma_0(N))$ the space of holomorphic cusp forms of weight k on the group $\Gamma_0(N)$. For each prime p not dividing N we have the Hecke operator T_p on $S_k(\Gamma_0(N))$ whose action on q-expansion can be given as follows: if $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ then $T_p(f) = \sum_{n=1}^{\infty}(a_{pn} + p^{k-1}a_{n/p})q^n$.

For $m \in \mathbb{N}$, let U_m, $V(m)$ be given by the following action on any formal q-series:

$$U_m(\sum_{n=1}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} a_{mn} q^n, \quad V(m)(\sum_{n=1}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} a_n q^{mn}.$$

It is well known that $V(m)$ maps $S_k(\Gamma_0(N))$ to $S_k(\Gamma_0(mN))$ and if $m \mid N$ then U_m is an operator on $S_k(\Gamma_0(N))$.

We briefly recall the theory of half-integral weight modular forms [14]. Let $\mathcal{G}$ be the set of all ordered pairs $(\alpha, \phi(z))$ where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(z)$ is a holomorphic function on the upper half plane $\mathbb{H}$ such that $\phi(z)^2 = t \det(\alpha)^{-1/2}(cz+d)$ with t in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Then $\mathcal{G}$ is a group under the following operation:

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

Let $P : \mathcal{G} \to \mathrm{GL}_2^+(\mathbb{R})$ be the homomorphism given by the projection map onto the first coordinate.

Let $\zeta = (\alpha, \phi(z)) \in \mathcal{G}$. Define the slash operator $|[\zeta]_{k+1/2}$ on functions f on $\mathbb{H}$ by $f|[\zeta]_{k+1/2}(z) = f(\alpha z)(\phi(z))^{-2k-1}$.

Let N be divisible by 4 and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Define the automorphy factor

$$j(\alpha, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2},$$

where $\varepsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$ and $\left(\frac{c}{d}\right)$ is as in Shimura's notation. Let

$$\Delta_0(N) := \{\alpha^* = (\alpha, j(\alpha, z)) \in \mathcal{G} \mid \alpha \in \Gamma_0(N)\} \leq \mathcal{G}.$$

The map $L : \Gamma_0(N) \to \mathcal{G}$ given by $\alpha \mapsto \alpha^*$ defines an isomorphism onto $\Delta_0(N)$. Thus $P|_{\Delta_0(N)}$ and L are inverse of each other. Denote by $\Delta_1(N)$ the image of $\Gamma_1(N)$.

Let χ be an even Dirichlet character modulo N. Let $S_{k+1/2}(\Gamma_0(N),\chi)$ be the space of cusp forms of weight $k+1/2$, level N and character χ consisting of $f \in S_{k+1/2}(\Delta_1(N))$ such that $f|[\alpha^*]_{k+1/2}(z) = \chi(d)f(z)$ for all $\alpha \in \Gamma_0(N)$. In particular when χ is trivial $S_{k+1/2}(\Gamma_0(N),\chi) = S_{k+1/2}(\Delta_0(N))$, in this case we will simply denote the space by $S_{k+1/2}(\Gamma_0(N))$.

Let ξ be an element of $\mathcal{G}$ such that $\Delta_0(N)$ and $\xi^{-1}\Delta_0(N)\xi$ are commensurable. Then we have an operator $|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2}$ on $S_{k+1/2}(\Gamma_0(N))$ defined by

$$f|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2} = \det(\xi)^{(2k-3)/4}\sum_v f|[\xi_v]_{k+1/2}$$

where $\Delta_0(N)\xi\Delta_0(N) = \bigcup_v \Delta_0(N)\xi_v$.

Let $\xi = (\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, p^{1/2})$. If p is a prime dividing N, then by [14, Proposition 1.5],

$$f|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2} = p^{(2k-3)/2}\sum_{s=0}^{p^2-1} f|[(\begin{pmatrix} 1 & s \\ 0 & p^2 \end{pmatrix}, p^{1/2})]_{k+1/2}(z),$$

thus if $f = \sum_{n=1}^{\infty} a_n q^n$ then $f|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2} = \sum_{n=1}^{\infty} a_{p^2n}q^n = U_{p^2}(f)$. If p is a prime such that $(p,N)=1$ then the Hecke operator T_{p^2} is defined by

$$T_{p^2}(f) = f|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2}.$$

We shall be studying local Hecke algebra of the double cover of SL_2. We next recall Gelbart's [4] description of the double cover. Let p be any prime (including the infinite prime). The group $\mathrm{SL}_2(\mathbb{Q}_p)$ has a non-trivial central extension by $\mu_2 = \{\pm 1\}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) & \longrightarrow & \mathrm{SL}_2(\mathbb{Q}_p) & \longrightarrow & 1 \\ & & \{(I,\pm 1)\} & & (g,\pm 1) & \longmapsto & g & & \end{array}$$

We use the 2-cocycle defined below to determine the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Let $(\cdot,\cdot)_p$ be the Hilbert symbol over $\mathbb{Q}_p$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, define

$$\tau(g) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases};$$

if $p=\infty$, set $s_p(g)=1$ while for a finite prime p

$$s_p(g) = \begin{cases} (c,d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd} \\ 1 & \text{else.} \end{cases}$$

Define the 2-cocycle σ_p on $\mathrm{SL}_2(\mathbb{Q}_p)$ as follows:

$$\sigma_p(g,h) = (\tau(gh)\tau(g), \tau(gh)\tau(h))_p\, s_p(g)s_p(h)s_p(gh).$$

Then the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is the set $\mathrm{SL}_2(\mathbb{Q}_p)\times\mu_2$ with the group law:

$$(g,\ \epsilon_1)(h,\ \epsilon_2)=(gh,\ \epsilon_1\epsilon_2\sigma_p(g,h)).$$

For any subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$, we shall denote by $\overline{H}$ the complete inverse image of H in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$.

We consider the following subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$:

$$K_0^p(p^n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}_p)\ :\ c\in p^n\mathbb{Z}_p\right\},$$

$$K_1^p(p^n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}_p)\ :\ c\in p^n\mathbb{Z}_p,\ a\equiv 1 \pmod{p^n\mathbb{Z}_p}\right\}.$$

By [4, Proposition 2.8] for odd primes p, $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ splits over $\mathrm{SL}_2(\mathbb{Z}_p)$. Thus $\overline{\mathrm{SL}_2(\mathbb{Z}_p)}$ is isomorphic to the direct product $\mathrm{SL}_2(\mathbb{Z}_p)\times\mu_2$ and $\overline{K_0^p(p)}$ is isomorphic to $K_0^p(p)\times\mu_2$. It follows from [4, Corollary 2.13] that the center M_p of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is simply the direct product $\{\pm I\}\times\mu_2$. Thus any genuine central character is given by a non-trivial character of $\mu_2\times\mu_2$.

However $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ does not split over $\mathrm{SL}_2(\mathbb{Z}_2)$ but instead splits over the subgroup $K_1^2(4)$. In this case the center M_2 of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ is a cyclic group of order 4 generated by $(-I,1)$ and so a genuine central character is given by sending $(-I,1)$ to a primitive fourth root of unity.

We set up a few more notations. For $s\in\mathbb{Q}_p$, $t\in\mathbb{Q}_p^\times$ let us define the following elements of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$x(s)=\begin{pmatrix} 1 & s\\ 0 & 1\end{pmatrix},\ y(s)=\begin{pmatrix} 1 & 0\\ s & 1\end{pmatrix},\ w(t)=\begin{pmatrix} 0 & t\\ -t^{-1} & 0\end{pmatrix},\ h(t)=\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}.$$

Let $N=\{(x(s),\epsilon): s\in\mathbb{Q}_p,\ \epsilon=\pm1\}$, $\bar{N}=\{(y(s),\epsilon): s\in\mathbb{Q}_p,\ \epsilon=\pm1\}$ and $T=\{(h(t),\epsilon): t\in\mathbb{Q}_p^\times,\ \epsilon=\pm1\}$ be the subgroups of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Then the normalizer $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ of T in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ consists of elements $(h(t),\epsilon)$, $(w(t),\epsilon)$ for $t\in\mathbb{Q}_p^\times$. We note the following useful relations: for $s,\ t\in\mathbb{Q}_p^\times$ and $u,\ v\in\mathbb{Q}_p$, we have

$$\begin{aligned}(h(s),1)(h(t),1)&=(h(st),(s,t)_p),\\ (w(s),1)(w(t),1)&=(h(-st^{-1}),(s,t)_p),\\ (h(s),1)(w(t),1)&=(w(st),(s,-t)_p),\\ (w(s),1)(h(t),1)&=(w(st^{-1}),(-s,t)_p),\end{aligned}\tag{1}$$

$$\begin{aligned}
(h(s),1)(x(u),1) &= \left(\begin{pmatrix} s & su \\ 0 & s^{-1}\end{pmatrix}, 1\right),\\
(x(u),1)(h(s),1) &= \left(\begin{pmatrix} s & s^{-1}u \\ 0 & s^{-1}\end{pmatrix}, 1\right),\\
(h(s),1)(y(u),1) &= \left(\begin{pmatrix} s & 0 \\ s^{-1}u & s^{-1}\end{pmatrix}, \sigma_p(h(s),y(u))\right),\\
(y(u),1)(h(s),1) &= \left(\begin{pmatrix} s & 0 \\ su & s^{-1}\end{pmatrix}, \sigma_p(y(u),h(s))\right),
\end{aligned} \tag{2}$$

where

$$\sigma_p(h(s),y(u)) = \sigma_p(y(u),h(s)) = \begin{cases} 1 & \text{if } u=0 \\ (s,u)_p & \text{if } u\neq 0,\ \mathrm{ord}_p(su) \text{ even} \\ (s,s)_p & \text{if } u\neq 0,\ \mathrm{ord}_p(su) \text{ odd,} \end{cases}$$

$$\begin{aligned}
(w(t),1)(x(u),1) &= \left(\begin{pmatrix} 0 & t \\ -t^{-1} & -t^{-1}u\end{pmatrix}, \sigma_p(w(t),x(u))\right),\\
(x(u),1)(w(t),1) &= \left(\begin{pmatrix} -ut^{-1} & t \\ -t^{-1} & 0\end{pmatrix}, 1\right),\\
(w(t),1)(y(v),1) &= \left(\begin{pmatrix} tv & t \\ -t^{-1} & 0\end{pmatrix}, 1\right),\\
(y(v),1)(w(t),1) &= \left(\begin{pmatrix} 0 & t \\ -t^{-1} & tv\end{pmatrix}, \sigma_p(y(v),w(t))\right),
\end{aligned} \tag{3}$$

where

$$\sigma_p(w(t),x(u)) = \begin{cases} (-t,-u)_p & \text{if } u\neq 0,\ \mathrm{ord}_p(t) \text{ odd} \\ 1 & \text{else,} \end{cases}$$

and

$$\sigma_p(y(v),w(t)) = \begin{cases} (-t,v)_p & \text{if } u\neq 0,\ \mathrm{ord}_p(t) \text{ odd} \\ 1 & \text{else,} \end{cases}$$

and

$$\begin{aligned}
(x(u),1)(y(v),1) &= \left(\begin{pmatrix} 1+uv & u \\ v & 1\end{pmatrix}, 1\right),\\
(y(v),1)(x(u),1) &= \left(\begin{pmatrix} 1 & u \\ v & uv+1\end{pmatrix}, \sigma_p(y(v),x(u))\right),\\
(x(u),1)(x(v),1) &= \left(\begin{pmatrix} 1 & u+v \\ 0 & 1\end{pmatrix}, 1\right),\\
(y(v),1)(y(u),1) &= \left(\begin{pmatrix} 1 & 0 \\ u+v & 1\end{pmatrix}, 1\right),
\end{aligned} \tag{4}$$

where

$$\sigma_p(y(v), x(u)) = \begin{cases} (v, uv+1)_p & \text{if } v(uv+1) \neq 0, \ \mathrm{ord}_p(v) \text{ odd} \\ 1 & \text{else.} \end{cases}$$

For any subgroup S of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$, we further let $N^S = N \cap S$, $T^S = T \cap S$ and $\bar{N}^S = \bar{N} \cap S$.

3. A Local Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$

Loke and Savin [8] studied a genuine local Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ corresponding to $\overline{K_0^2(4)}$ and a genuine central character, and gave an interpretation of Kohnen's plus space at level 4 in terms of certain elements in this 2-adic Hecke algebra. In this section we shall recall their work on the 2-adic Hecke algebra. We shall then study genuine Iwahori Hecke algebra for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ corresponding to $\overline{K_0^p(p)}$ and a genuine character of M_p for a general odd prime p.

Let p be any finite prime and $C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$ be the space of locally constant, compactly supported complex-valued functions on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. For an open compact subgroup S of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and a genuine character γ of S (that is, a character of S that acts nontrivially on μ_2), let $H(S,\gamma)$ be the subalgebra of $C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$ defined as follows:

$$\{f \in C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)) : f(\tilde{k}\tilde{g}\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p), \ \tilde{k}, \ \tilde{k}' \in S\}.$$

Then $H(S,\gamma)$ is a $\mathbb{C}$-algebra under the convolution which, for any $f_1, f_2 \in H(S,\gamma)$, is defined by

$$f_1 * f_2(\tilde{h}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{g})f_2(\tilde{g}^{-1}\tilde{h})d\tilde{g} = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{h}\tilde{g})f_2(\tilde{g}^{-1})d\tilde{g},$$

where $d\tilde{g}$ is the Haar measure on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ such that the measure of S is one. We call $H(S,\gamma)$ the genuine Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ with respect to S and γ. We may sometime denote $f_1 * f_2$ simply by $f_1 f_2$.

For certain S and γ, we would like to describe the algebra $H(S,\gamma)$ using generators and relations. In order to do so we need to first compute the support of $H(S,\gamma)$. We say that $H(S,\gamma)$ is supported on $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ if there exists $f \in H(S,\gamma)$ such that $f(\tilde{g}) \neq 0$. We shall use the following lemmas to compute the support.

Lemma 3.1. *Let $S_{\tilde{g}} = S \cap \tilde{g}S\tilde{g}^{-1}$. Then $H(S,\gamma)$ is supported on $\tilde{g}$ if and only if for every $\tilde{k} \in S_{\tilde{g}}$ we have $\gamma([\tilde{k}^{-1}, \ \tilde{g}^{-1}]) = 1$, where $[\cdot,\cdot]$ is the usual commutator bracket.*

Lemma 3.2. *The function $\alpha_{\tilde{g}} : S_{\tilde{g}} \longrightarrow \mathbb{C}$ defined by $\alpha_{\tilde{g}}(\tilde{k}) = \gamma([\tilde{k}^{-1}, \ \tilde{g}^{-1}])$ is a character of $S_{\tilde{g}}$.*

In order to compute the support using above lemmas we shall need certain results on cocycle multiplication. We note them in the appendix.

We also note the following well-known lemmas that will be useful in computing convolutions.

Lemma 3.3. *Let $f_1,\ f_2 \in H(S,\gamma)$ such that f_1 is supported on $S\tilde{x}S = \bigcup_{i=1}^{m} \tilde{\alpha}_i S$ and f_2 is supported on $S\tilde{y}S = \bigcup_{j=1}^{n} \tilde{\beta}_j S$. Then*

$$f_1 * f_2(\tilde{h}) = \sum_{i=1}^{m} f_1(\tilde{\alpha}_i) f_2(\tilde{\alpha}_i^{-1}\tilde{h})$$

where the nonzero summands are precisely for those i for which there exist a j such that $\tilde{h} \in \tilde{\alpha}_i \tilde{\beta}_j S$.

For $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ let $\mu(\tilde{g})$ denote the number of disjoint left (right) S cosets in the decomposition of the double coset $S\tilde{g}S$.

Lemma 3.4. *Let $\tilde{g},\ \tilde{h} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ be such that $\mu(\tilde{g})\mu(\tilde{h}) = \mu(\tilde{g}\tilde{h})$. Let f_1, $f_2 \in H(S,\gamma)$ be respectively supported on $S\tilde{g}S$ and $S\tilde{h}S$. Then $f_1 * f_2$ is precisely supported on $S\tilde{g}\tilde{h}S$ and $f_1 * f_2(\tilde{g}\tilde{h}) = f_1(\tilde{g}) f_2(\tilde{h})$.*

3.1. **Local Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ modulo $\overline{K_0^2(4)}$.** Let $S = \overline{K_0^2(4)}$ and let γ be a genuine character of M_2 determined by its value on $(-I,1)$. Since $\overline{K_0^2(4)}$ is the direct product $K_1^2(4) \times M$, we can extend γ to a genuine character of $\overline{K_0^2(4)}$ by setting it trivial on $K_1^2(4)$. Loke and Savin described $H(S,\gamma)$ for the above choice of S and γ as follows.

Using relations in (1), extend γ to the normalizer $N_{\widetilde{\mathrm{SL}}(\mathbb{Q}_2)}(T)$ by defining $\gamma((h(2^n),1)) = 1$ for all integers n and $\gamma((w(1),1)) = (1+\gamma((-I,1)))/\sqrt{2}$, a primitive 8th root of unity. For $n \in \mathbb{Z}$, define the elements $\mathcal{T}_n$ and $\mathcal{U}_n$ of $H(\overline{K_0^2(4)},\gamma)$ supported respectively on the $\overline{K_0^2(4)}$ double cosets of $(h(2^n),1)$ and $(w(2^{-n}),1)$ such that

$$\mathcal{T}_n(\tilde{k}(h(2^n),1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((h(2^n),1))\overline{\gamma}(\tilde{k}'),$$
$$\mathcal{U}_n(\tilde{k}(w(2^{-n}),1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((w(2^{-n}),1))\overline{\gamma}(\tilde{k}') \quad \text{for } \tilde{k},\ \tilde{k}' \in \overline{K_0^2(4)}.$$

Theorem 1. *(Loke-Savin [8]) For $m,n \in \mathbb{Z}$,*

(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.
(2) $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{U}_{1-n}$.
(3) $\mathcal{U}_1 * \mathcal{U}_n = \mathcal{T}_{n-1}$ and $\mathcal{U}_n * \mathcal{U}_1 = \mathcal{T}_{1-n}$.

The Hecke algebra $H(\overline{K_0^2(4)},\gamma)$ is generated by $\mathcal{U}_0$ and $\mathcal{U}_1$ modulo relations $(\mathcal{U}_0 - 2\sqrt{2})(\mathcal{U}_0 + \sqrt{2}) = 0$ and $\mathcal{U}_1^2 = 1$.

3.2. **Iwahori Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ modulo $\overline{K_0^p(p)}$, p odd.** Fix an odd prime p. Let $S = \overline{K_0^p(p)}$. Let γ be a character of $K_0^p(p)$ such that it is trivial on $K_1^p(p)$. Since $\frac{K_0^p(p)}{K_1^p(p)} \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$, we can define γ by a character of $(\mathbb{Z}/p\mathbb{Z})^\times$. We shall use the same symbol γ to denote a genuine character of S

by defining $\gamma(A,\epsilon)=\epsilon\gamma(A)$ for $A\in K_0^p(p)$. We call $H(S,\gamma)$ with the above choice of S and γ to be the genuine Iwahori Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ with central character γ. Our main result in this subsection is to describe this Iwahori Hecke algebra using generators and relations when γ is quadratic.

In the rest of this subsection we shall denote $K_0^p(p)$ simply by K_0. We first note the following lemma.

Lemma 3.5. *A complete set of representatives for the double cosets of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ mod $\overline{K_0}$ is given by $(h(p^n),1)$, $(w(p^{-n}),1)$ where n varies over integers.*

We need to compute the support of $H(\overline{K_0},\gamma)$. Fix an integer n. Let $A=h(p^n)$ and $\tilde{A}=(A,\epsilon_1)$. We shall show that $H(\overline{K_0},\gamma)$ is supported on $\tilde{A}$. We have

$$S_{\tilde{A}}=\Big\{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\pm1\right)\in\overline{\mathrm{SL}_2(\mathbb{Z}_p)}\ :\mathrm{ord}_p(c)\geq\max\{-2n+1,1\},\\ \mathrm{ord}_p(b)\geq\max\{2n,0\}\Big\}.$$

We check that $S_{\tilde{A}}$ has a triangular decomposition $S_{\tilde{A}}=N^{S_{\tilde{A}}}T^{S_{\tilde{A}}}\bar{N}^{S_{\tilde{A}}}$ where $T^{S_{\tilde{A}}}=T^{\overline{K_0}}$, $N^{S_{\tilde{A}}}=\{(x(s),\pm1):\mathrm{ord}_p(s)\geq\max\{2n,0\}\}$ and $\bar{N}^{S_{\tilde{A}}}=\{(y(t),\pm1):\mathrm{ord}_p(t)\geq\max\{-2n+1,1\}\}$.

By Lemma 3.1 and 3.2, it is enough to check that the value of γ on the commutator $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]$ is 1 for any $(B,\ \epsilon_2)$ in $N^{S_{\tilde{A}}}$, $T^{S_{\tilde{A}}}$ and $\bar{N}^{S_{\tilde{A}}}$ respectively.

By Lemma A.3, for $B=(x(s),\epsilon_2)\in N^{S_{\tilde{A}}}$, we get $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]=(\begin{pmatrix}1&sp^{-2n}-s\\0&1\end{pmatrix},1)$; for $B=(h(u),\epsilon_2)\in T^{S_{\tilde{A}}}$, $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]=(I,1)$; and for $B=(y(t),\epsilon_2)\in N^{S_{\tilde{A}}}$, we get that $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]=(\begin{pmatrix}1&0\\(p^{2n}-1)t&1\end{pmatrix},1)$. Since each of them belongs to $K_1^p(p)\times\{1\}$, we are done.

Next let $A=w(p^{-n})$. We show that $H(\overline{K_0},\gamma)$ is supported on $\tilde{A}=(A,\epsilon_1)$ provided $\gamma(u^2)=1$ for all units u in $\mathbb{Z}_p$. In this case we have

$$S_{\tilde{A}}=\Big\{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\pm1\right)\in\overline{\mathrm{SL}_2(\mathbb{Z}_p)}\ :\mathrm{ord}_p(c)\geq\max\{2n,1\},\\ \mathrm{ord}_p(b)\geq\max\{-2n+1,0\}\Big\}$$

and $S_{\tilde{A}}$ has a triangular decomposition $S_{\tilde{A}}=N^{S_{\tilde{A}}}T^{S_{\tilde{A}}}\bar{N}^{S_{\tilde{A}}}$ where $T^{S_{\tilde{A}}}=T^{\overline{K_0}}$, $N^{S_{\tilde{A}}}=\{(x(s),\pm1):\mathrm{ord}_p(s)\geq\max\{-2n+1,0\}\}$, $\bar{N}^{S_{\tilde{A}}}=\{(y(t),\pm1):\mathrm{ord}_p(t)\geq\max\{2n,1\}\}$.

By Lemma A.3, for $B=(x(s),\epsilon_2)\in N^{S_{\tilde{A}}}$, we get $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]=(\begin{pmatrix}1+s^2p^{2n}&-s\\-sp^{2n}&1\end{pmatrix},1)$, so γ takes value 1 on this commutator. In the case

$B = (y(t), \epsilon_2) \in N^{S_{\tilde{A}}}$, we have

$$B^{-1}A^{-1}BA = \begin{pmatrix} 1 & -p^{-2n}t \\ -t & 1+p^{-2n}t^2 \end{pmatrix}, \text{ where } \mathrm{ord}_p(t) \geq \max\{2n, 1\},$$

so $s_p(B^{-1}A^{-1}BA) = 1$ if either $-t(1+p^{-2n}t^2) = 0$ or $\mathrm{ord}_p(t)$ is even. Assume that $-t(1+p^{-2n}t^2) \neq 0$ and $\mathrm{ord}_p(t)$ is odd. Then $s_p(B^{-1}A^{-1}BA) = \left(-t, 1+p^{-2n}t^2\right)_p = \left(-p, 1+p^{-2n}t^2\right)_p$. Let $u = 1+p^{-2n}t^2$. Since $\mathrm{ord}_p(t) \geq \max\{2n, 1\}$, we have $u \equiv 1 \pmod{p\mathbb{Z}_p}$. Hence $s_p(B^{-1}A^{-1}BA) = (-p, u)_p = \left(\frac{u}{p}\right) = 1$. So in this case also γ takes value 1.

For $B = (h(u), \epsilon_2) \in T^{S_{\tilde{A}}}$, $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}] = (\begin{pmatrix} 1/u^2 & 0 \\ 0 & u^2 \end{pmatrix}, 1)$, so $\gamma([(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]) = \gamma(u^2)$.

Thus if $\gamma(u^2) = 1$ for all units u in $\mathbb{Z}_p$ then $H(\overline{K_0}, \gamma)$ is supported on $(w(p^{-n}), \epsilon)$. In particular this holds if our choice of γ is quadratic. Thus we have

Proposition 3.6. *If γ is a quadratic character then $H(\overline{K_0}, \gamma)$ is supported on the double cosets of $\overline{K_0}$ represented by $(h(p^n), 1)$ and $(w(p^{-n}), 1)$ as n varies over integers.*

We now obtain the generators and relations in $H(\overline{K_0}, \gamma)$ when γ is quadratic.

We consider the character γ of $\overline{K_0}$ to be the genuine character of the center M_p and extend it to the normalizer group $N_{\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}}(T)$ as follows.

Let $\varepsilon_p = 1$ or i depending on whether $p \equiv 1$ or $3 \pmod 4$, thus $\varepsilon_p^2 = \left(\frac{-1}{p}\right)$. Let $t = p^n u \in \mathbb{Q}_p^\times$ where $n \in \mathbb{Z}$ and u is a unit in $\mathbb{Z}_p$. Define

$$\gamma((h(t), 1)) = \begin{cases} \gamma((h(u), 1)) & \text{if } n \text{ is even} \\ \varepsilon_p \left(\frac{u}{p}\right) \gamma((h(u), 1)) & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that γ thus defined is a character of T.

We now extend γ to the normalizer $N_{\widetilde{\mathrm{SL}_2(\mathbb{Q}_p)}}(T)$ by defining $\gamma((w(1), 1)) := 1$ and using the relation

$$(w(t), 1) = (h(t), 1)(w(1), 1)(I, \left(-1, t^{-1}\right)_p).$$

Thus for $t = p^n u$ as above,

$$\gamma((w(t), 1)) = \begin{cases} \gamma((h(u), 1)) & \text{if } n \text{ is even} \\ \varepsilon_p \left(\frac{-u}{p}\right) \gamma((h(u), 1)) & \text{if } n \text{ is odd.} \end{cases}$$

We define the elements $\mathcal{T}_n$ and $\mathcal{U}_n$ of $H(\overline{K_0}, \gamma)$ supported respectively on the double cosets of $(h(p^n), 1)$ and $(w(p^{-n}), 1)$ such that

$$\mathcal{T}_n(\tilde{k}(h(p^n), 1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((h(p^n), 1))\overline{\gamma}(\tilde{k}'),$$

$$\mathcal{U}_n(\tilde{k}(w(p^{-n}), 1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((w(p^{-n}), 1))\overline{\gamma}(\tilde{k}') \quad \text{for } \tilde{k},\ \tilde{k}' \in \overline{K_0}.$$

Thus Proposition 3.6 implies that $\mathcal{T}_n$, $\mathcal{U}_n$ as n varies over integers form a $\mathbb{C}$-basis for $H(\overline{K_0}, \gamma)$ when γ is quadratic.

In order to obtain relations amongst $\mathcal{T}_n$ and $\mathcal{U}_n$, we note the following lemma which can be obtained by using the triangular decomposition of K_0.

Lemma 3.7. *1. For $n \geq 0$,*

$$K_0 h(p^n) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} x(s) h(p^n) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} K_0 h(p^n) y(ps).$$

2. For $n \geq 1$,

$$K_0 h(p^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} y(ps) h(p^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} K_0 h(p^{-n}) x(s).$$

3. For $n \geq 1$,

$$K_0 w(p^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n-1}\mathbb{Z}_p} y(ps) w(p^{-n}) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n-1}\mathbb{Z}_p} K_0 w(p^{-n}) y(ps).$$

4. For $n \geq 0$,

$$K_0 w(p^n) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n+1}\mathbb{Z}_p} x(s) w(p^n) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^{2n+1}\mathbb{Z}_p} K_0 w(p^n) x(s).$$

Proposition 3.8. *We have the following relations:*

(1) If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.
(2) For $n \geq 0$, $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_{-n} * \mathcal{U}_1 = \mathcal{U}_{n+1}$.
(3) For $n \geq 0$, $\mathcal{U}_0 * \mathcal{T}_{-n} = \mathcal{U}_{-n}$ and $\mathcal{T}_n * \mathcal{U}_0 = \mathcal{U}_{-n}$.
(4) For $n \geq 1$, $\mathcal{U}_0 * \mathcal{U}_n = \overline{\gamma}((-I, 1))\mathcal{T}_n$ and $\mathcal{U}_n * \mathcal{U}_0 = \overline{\gamma}((-I, 1))\mathcal{T}_{-n}$.

Proof. We prove (1) and the second part of (4). The rest are similar.

For (1) let $mn \geq 0$. We may assume both $m,\ n \geq 0$. It follows from Lemma 3.7 and 3.4 that $\mathcal{T}_m * \mathcal{T}_n$ is precisely supported on the double coset $\overline{K_0}(h(p^{n+m}), 1)\overline{K_0}$ and that

$$\mathcal{T}_m * \mathcal{T}_n((h(p^m), 1)(h(p^n), 1)) = \mathcal{T}_m((h(p^m), 1))\mathcal{T}_n((h(p^n), 1)).$$

Let m and n both be even. Then $(h(p^m), 1)(h(p^n), 1) = (h(p^{n+m}), 1)$ and so

$$\begin{aligned}\mathcal{T}_m * \mathcal{T}_n((h(p^{n+m}), 1)) &= \mathcal{T}_m((h(p^m), 1))\mathcal{T}_n((h(p^n), 1)) \\ &= \overline{\gamma}((h(p^m), 1))\overline{\gamma}((h(p^n), 1)) = 1 = \mathcal{T}_{m+n}((h(p^{n+m}), 1)),\end{aligned}$$

hence $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$. Next suppose both m and n are odd, so $m + n$ is even. Then $(h(p^m), 1)(h(p^n), 1) = (h(p^{n+m}), 1)(I, (p, p)_p)$ and so

$$\mathcal{T}_m * \mathcal{T}_n((h(p^{n+m}), 1)) = \overline{\gamma}((I, (p, p)_p))\mathcal{T}_m((h(p^m), 1))\mathcal{T}_n((h(p^n), 1))$$

$$= \left(\frac{-1}{p}\right)\overline{\gamma}((h(p^m), 1))\overline{\gamma}((h(p^n), 1)) = \left(\frac{-1}{p}\right)\overline{\varepsilon_p^2} = 1 = \mathcal{T}_{m+n}((h(p^{n+m}), 1)).$$

Now suppose m is odd and n is even (or vice versa), so $m+n$ is odd. In this case $(h(p^m), 1)(h(p^n), 1) = (h(p^{n+m}), 1)$ and so

$$\mathcal{T}_m * \mathcal{T}_n((h(p^{n+m}), 1)) = \overline{\varepsilon_p} = \mathcal{T}_{m+n}((h(p^{n+m}), 1))$$

and we are done.

For (4), let $n \geq 1$. As before using Lemma 3.7 and 3.4 we know that $\mathcal{U}_n * \mathcal{U}_0$ is supported on the double coset $\overline{K_0}(h(p^{-n}),1)\overline{K_0}$ and that

$$\mathcal{U}_n * \mathcal{U}_0((w(p^{-n}),1)(w(1),1)) = \mathcal{U}_n((w(p^{-n}),1))\mathcal{U}_0((w(1),1)).$$

We have $(w(p^{-n}),1)(w(1),1) = (h(p^{-n}),1)(-I,(p^{-n},-1)_p)$ and so

$$\overline{\gamma}((-I,1))\,\mathcal{U}_n * \mathcal{U}_0((h(p^{-n}),1)) = \left(p^{-n},-1\right)_p \mathcal{U}_n((w(p^{-n}),1))\mathcal{U}_0((w(1),1))$$

$$= \begin{cases} \left(\frac{-1}{p}\right)\overline{\varepsilon_p}\left(\frac{-1}{p}\right) = \overline{\varepsilon_p} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} = \mathcal{T}_{-n}((h(p^{-n}),1)),$$

and thus $\mathcal{U}_n * \mathcal{U}_0 = \overline{\gamma}((-I,1))\mathcal{T}_{-n}$. □

We shall consider two choices for γ as a character of $(\mathbb{Z}/p\mathbb{Z})^*$, either γ is trivial or γ is given by the Kronecker symbol $\gamma = \left(\frac{\cdot}{p}\right)$. Then we have the following proposition.

Proposition 3.9. *(1)* $\mathcal{U}_0^2 = \begin{cases} (p-1)\mathcal{U}_0 + p & \text{if } \gamma \text{ is trivial} \\ \left(\frac{-1}{p}\right)p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$

(2) $\mathcal{U}_1^2 = \begin{cases} p & \text{if } \gamma \text{ is trivial} \\ \varepsilon_p(p-1)\mathcal{U}_1 + \left(\frac{-1}{p}\right)p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$

(3) If γ is trivial, then $\mathcal{T}_1 * \mathcal{U}_1 = p\,\mathcal{U}_0$ and $\mathcal{T}_{-1} = (1/p)\,\mathcal{U}_1 * \mathcal{T}_1 * \mathcal{U}_1$.

Proof. For (1) we use Lemma 3.3 to check that $\mathcal{U}_0^2$ is at most supported on the double cosets $\overline{K_0}$ and $\overline{K_0}(w(1),1)\overline{K_0}$. Thus we need to only compute the values of $\mathcal{U}_0^2$ at $(I,1)$ and $(w(1),1)$. Using Lemma 3.7 and 3.3 we have

$$\begin{aligned} \mathcal{U}_0^2((I,1)) &= \sum_{s=0}^{p-1} \mathcal{U}_0((x(s),1)(w(1),1))\mathcal{U}_0((w(1),1)^{-1}(x(s),1)^{-1}) \\ &= \sum_{s=0}^{p-1} \mathcal{U}_0((w(1),1))\mathcal{U}_0((w(-1),1)(x(-s),1)) \\ &= \sum_{s=0}^{p-1} \mathcal{U}_0((h(-1),1)(w(1),1)(x(-s),1)) \\ &= \sum_{s=0}^{p-1} \overline{\gamma}(-1) = \begin{cases} p & \text{if } \gamma \text{ is trivial} \\ \left(\frac{-1}{p}\right)p & \text{if } \gamma = \left(\frac{\cdot}{p}\right), \end{cases} \end{aligned}$$

where the third equality follows from the relation $(h(-1),1)(w(1),1) = (w(-1),(-1,1)_p)$ by Equation (1).

Similarly, we get that

$$\begin{aligned}
\mathcal{U}_0^2((w(1),1)) &= \sum_{s=0}^{p-1} \mathcal{U}_0((x(s),1)(w(1),1))\mathcal{U}_0((w(1),1)^{-1}(x(s),1)^{-1}(w(1),1)) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0((\begin{pmatrix} 0 & -1 \\ 1 & -s \end{pmatrix}, 1)(w(1),1)) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0((y(s),1)) = \sum_{s=1}^{p-1} \mathcal{U}_0((y(s),1)),
\end{aligned}$$

since $\mathcal{U}_0((I,1)) = 0$ (as $(I,1)$ is not in the support of $\mathcal{U}_0$). It is easy to check that for $1 \leq s \leq p-1$,

$$(y(s),1) = (\begin{pmatrix} 1 & 1/s \\ 0 & 1 \end{pmatrix}, 1)(w(1),1)(\begin{pmatrix} -s & -1 \\ 0 & -1/s \end{pmatrix}, 1) \in \overline{K_0}(w(1),1)\overline{K_0}$$

and hence

$$\mathcal{U}_0^2((w(1),1)) = \sum_{s=1}^{p-1} \overline{\gamma}(-1/s) = \sum_{s=1}^{p-1} \gamma(s) = \begin{cases} p-1 & \text{if } \gamma \text{ is trivial} \\ \sum_{s=1}^{p-1} \left(\frac{s}{p}\right) = 0 & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$$

Thus if we write $\mathcal{U}_0^2 = c_1\mathcal{U}_0 + c_2$, we get that

$$c_1 = \begin{cases} p-1 & \text{if } \gamma \text{ trivial} \\ 0 & \text{if } \gamma = \left(\frac{\cdot}{p}\right) \end{cases}, \quad c_2 = \begin{cases} p & \text{if } \gamma \text{ trivial} \\ \left(\frac{-1}{p}\right) p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$$

Now we prove (2). Again using Lemma 3.3 we see that $\mathcal{U}_1^2$ is at most supported on the double cosets $\overline{K_0}$ and $\overline{K_0}(w(p^{-1}),1)\overline{K_0}$. So we need to find the values of $\mathcal{U}_1^2$ at $(I,1)$ and $(w(p^{-1}),1)$. Using Lemma 3.7 and 3.3,

$$\begin{aligned}
\mathcal{U}_1^2((I,1)) &= \sum_{s=0}^{p-1} \mathcal{U}_1((y(ps),1)(w(p^{-1}),1))\mathcal{U}_1((w(p^{-1}),1)^{-1}(y(ps),1)^{-1}) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_1((w(p^{-1}),1))\mathcal{U}_1((w(-p^{-1}),1)(y(-ps),1)) \\
&= \sum_{s=0}^{p-1} \overline{\varepsilon_p} \left(\frac{-1}{p}\right) \mathcal{U}_1((h(-1),(-p,-1)_p)(w(p^{-1}),1)) \\
&= \sum_{s=0}^{p-1} \overline{\varepsilon_p} \left(\frac{-1}{p}\right) \gamma(-1) \left(\frac{-1}{p}\right) \overline{\varepsilon_p} \left(\frac{-1}{p}\right) \\
&= \gamma(-1)p = \begin{cases} p & \text{if } \gamma \text{ trivial} \\ \left(\frac{-1}{p}\right) p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\mathcal{U}_1^2((w(p^{-1}),1))\\
&=\sum_{s=0}^{p-1}\mathcal{U}_1((y(ps),1)(w(p^{-1}),1))\mathcal{U}_1((w(-p^{-1}),1)(y(-ps),1)(w(p^{-1}),1))\\
&=\sum_{s=0}^{p-1}\overline{\varepsilon_p}\left(\frac{-1}{p}\right)\mathcal{U}_1((\begin{pmatrix} s & -p^{-1}\\ p & 0\end{pmatrix},(p^2,-p^2s)_p)(w(p^{-1}),1))\\
&=\sum_{s=0}^{p-1}\overline{\varepsilon_p}\left(\frac{-1}{p}\right)\mathcal{U}_1((x(s/p),(p,-p)_p))=\sum_{s=1}^{p-1}\overline{\varepsilon_p}\left(\frac{-1}{p}\right)\mathcal{U}_1((x(s/p),1)).
\end{aligned}$$

Now we check that for $1\le s\le p-1$

$$(x(s/p),1)(I,\left(\frac{s}{p}\right))=(\begin{pmatrix} s & 0\\ p & 1/s\end{pmatrix},1)(w(p^{-1}),1)(\begin{pmatrix} 1 & 0\\ p/s & 1\end{pmatrix},1)$$

and so

$$\begin{aligned}
\mathcal{U}_1^2((w(p^{-1}),1))&=\sum_{s=1}^{p-1}\overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{s}{p}\right)\overline{\gamma}(1/s)\overline{\varepsilon_p}\left(\frac{-1}{p}\right)\\
&=\sum_{s=1}^{p-1}\left(\frac{-s}{p}\right)\overline{\gamma}(1/s)\\
&=\begin{cases}\sum_{s=1}^{p-1}\left(\frac{-s}{p}\right)=0 & \text{if } \gamma \text{ trivial}\\ \sum_{s=1}^{p-1}\left(\frac{-s}{p}\right)\left(\frac{s^{-1}}{p}\right)=\left(\frac{-1}{p}\right)(p-1) & \text{if } \gamma=\left(\frac{\cdot}{p}\right).\end{cases}
\end{aligned}$$

Thus if we write $\mathcal{U}_1^2=c_1\mathcal{U}_1+c_2$, we get that

$$c_1=\begin{cases}0 & \text{if } \gamma \text{ trivial}\\ \varepsilon_p(p-1) & \text{if } \gamma=\left(\frac{\cdot}{p}\right)\end{cases},\quad c_2=\begin{cases}p & \text{if } \gamma \text{ trivial}\\ \left(\frac{-1}{p}\right)p & \text{if } \gamma=\left(\frac{\cdot}{p}\right).\end{cases}$$

For (3) let γ be a trivial character. From Proposition 3.8(4), we have $\mathcal{U}_0*\mathcal{U}_1=\mathcal{T}_1$. Right multiplication by $\mathcal{U}_1$ on both sides and using (2) above give $\mathcal{T}_1*\mathcal{U}_1=p\,\mathcal{U}_0$. Further, using the same proposition we get that $\mathcal{T}_{-1}=\mathcal{U}_1*\mathcal{U}_0=(1/p)\,\mathcal{U}_1*\mathcal{T}_1*\mathcal{U}_1$. □

Remark 1. *We compare the p-adic operator $\mathcal{U}_1$ with Ueda's classical operator Y_p* [16, Proposition 1.27] *which satisfies a similar relation. In particular if we consider operator $\mathcal{U}_1'=\overline{\varepsilon_p}\mathcal{U}_1$, then in the case γ is trivial we have*

$$(\mathcal{U}_1')^2=(\overline{\varepsilon_p}\mathcal{U}_1)^2=\varepsilon_p^2p=\left(\frac{-1}{p}\right)p,$$

while in the case $\gamma=\left(\frac{\cdot}{p}\right)$ we have

$$(\mathcal{U}_1')^2=(\overline{\varepsilon_p}\mathcal{U}_1)^2=\overline{\varepsilon_p}^2\left(\varepsilon_p(p-1)\mathcal{U}_1+p\left(\frac{-1}{p}\right)\right)=(p-1)\mathcal{U}_1'+p.$$

Thus $\mathcal{U}_1'$ satisfies exactly the same relations as the operator Y_p.

Theorem 2. *The "genuine" Iwahori Hecke algebra $H(\overline{K_0^p(p)}, \gamma)$ for γ trivial or $\left(\frac{\cdot}{p}\right)$ is generated as a $\mathbb{C}$-algebra by $\mathcal{U}_0$ and $\mathcal{U}_1$ with the defining relations given by the above proposition.*

Proof. We let $\mathcal{A}$ be an abstract algebra generated by $\tilde{\mathcal{U}}_0$ and $\tilde{\mathcal{U}}_1$ with defining relations as (1) and (2) of Proposition 3.9. We have a homomorphism from $\tilde{A}$ to $H(\gamma)$ mapping $\tilde{\mathcal{U}}_0$ to $\mathcal{U}_0$ and $\tilde{\mathcal{U}}_1$ to $\mathcal{U}_1$. It follows from Proposition 3.8 that this homomorphism is onto. We let M be the kernel of this homomorphism. Using relations (1) and (2) it follows that M is a linear combination of words of the form $\tilde{\mathcal{U}}_0\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_0$.... and $\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_0\tilde{\mathcal{U}}_1$..... There are four possibilities for the beginning and ending of such a word and each one is mapped by the homomorphism to a different basis element (again using Proposition 3.8). It follows that $M = 0$. □

Remark 2. *We note that the Hecke algebras $H(\overline{K_0^p(p)}, \gamma)$ for γ trivial or $\left(\frac{\cdot}{p}\right)$ are isomorphic (with roles of $\tilde{\mathcal{U}}_0$, $\tilde{\mathcal{U}}_1$ switched after suitable normalization). Further, these are isomorphic to the Iwahori Hecke algebra of* $\mathrm{PGL}_2(\mathbb{Q}_p)$*, giving, what Loke-Savin called, local Shimura correspondence at odd primes.*

The Hecke algebra generators and relations described above allow a study of the representation theory of the maximal compact with $(\overline{K_0^p(p)}, \gamma)$ equivariant vectors and also the infinite dimensional genuine representations of $\widetilde{\mathrm{SL}}(2)$ with such vectors. We will pursue this study in a subsequent work.

4. Translation of adelic to classical.

In this section, following Gelbart [4] and Waldspurger [18], we review the connection between automorphic forms on $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ and classical modular forms of half-integral weight. We use this connection to translate certain elements in the p-adic Hecke algebra described in the previous section into classical operators and thus obtain relations satisfied by these classical operators.

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the adele ring of $\mathbb{Q}$ and $\widetilde{\mathrm{SL}}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ with the group law: for $g = (g_\nu),\ h = (h_\nu) \in \mathrm{SL}_2(\mathbb{A})$ and $\epsilon_1,\ \epsilon_2 \in \{\pm 1\}$

$$(g, \epsilon_1)(h, \epsilon_2) = (gh,\ \epsilon_1\epsilon_2\sigma(g,h)),\ \text{where } \sigma(g,h) = \prod_\nu \sigma_\nu(g_\nu, h_\nu).$$

The group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ splits over $\mathrm{SL}_2(\mathbb{Q})$ and the splitting is given by

$$s_{\mathbb{Q}} : \mathrm{SL}_2(\mathbb{Q}) \longrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}),\ g \mapsto (g, s_{\mathbb{A}}(g)) \text{ where } s_{\mathbb{A}}(g) = \prod_\nu s_\nu(g).$$

By [4, Proposition 2.16], for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, $s_{\mathbb{A}}(\alpha) = \left(\frac{c}{d}\right)_s$ unless $c = 0$ in which case $s_{\mathbb{A}}(\alpha) = 1$. Here $\left(\frac{c}{d}\right)_s = \left(\frac{c}{d}\right)(c,d)_\infty$.

Lemma 4.1. *Let* $4 \mid N$*. For* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*, we have*

$$s_{\mathbb{A}}(\alpha) = \begin{cases} \left(\frac{c}{d}\right)_s (c,d)_2 & \textit{if } c \neq 0 \textit{ and } \mathrm{ord}_2(c) \textit{ is even} \\ \left(\frac{c}{d}\right)_s & \textit{if } c \neq 0 \textit{ and } \mathrm{ord}_2(c) \textit{ is odd} \\ 1 & \textit{if } c = 0. \end{cases}$$

Proof. If $c = 0$ then $s_\nu(\alpha) = 1$ for all places ν and so $s_{\mathbb{A}}(\alpha) = 1$.

Suppose $c \neq 0$. Since $\alpha \in \Gamma_0(N)$ and $4 \mid N$, d is odd and coprime to c. By definition, for any finite prime q, we have $s_q(\alpha) = (c,d)_q$ if $\mathrm{ord}_q(c)$ is odd and is 1 else. Hence

$$s_{\mathbb{A}}(\alpha) = \prod_{q \text{ finite}} s_q(\alpha) = \prod_{\mathrm{ord}_q(c) \text{ odd}} (c,d)_q .$$

It follows from the proof of [4, Proposition 2.16] (the proof only uses that d is odd and coprime to c), that $\left(\frac{c}{d}\right)_s = \prod_{q|c} (c,d)_q$. Now

$$\prod_{\mathrm{ord}_q(c) \text{ odd}} (c,d)_q = \prod_{q|c} (c,d)_q \prod_{\mathrm{ord}_q(c) \text{ even}>0} (c,d)_q = \left(\frac{c}{d}\right)_s \prod_{\mathrm{ord}_q(c) \text{ even}>0} (c,d)_q .$$

So we just need to show that $\prod_{\mathrm{ord}_q(c) \text{ even}>0} (c,d)_q$ is $(c,d)_2$ if $\mathrm{ord}_2(c)$ is even and is 1 if $\mathrm{ord}_2(c)$ is odd (note that $\mathrm{ord}_2(c) \geq 2$). Let p be any odd prime such that $\mathrm{ord}_p(c)$ is even and > 0. Let $c = p^{2n}u$ where u is unit in $\mathbb{Z}_p$. Then $(c,d)_p = (u,d)_p = 1$ as both d, u are units in $\mathbb{Z}_p$. Hence we are done. □

For $\tilde{g} = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $z \in \mathbb{H}$, define

$$\tilde{g}(z) = \frac{az+b}{cz+d} \qquad \text{and} \qquad J(\tilde{g}, z) = \epsilon(cz+d)^{1/2}.$$

By [4, Lemma 3.3], $J(\tilde{g}, z)$ satisfies the automorphy condition i.e.,

$$J(\tilde{g}\tilde{h}, z) = J(\tilde{g}, \tilde{h}z) J(\tilde{h}, z).$$

Let $f \in S_{k+1/2}(\Gamma_0(N))$ where $4 \mid N$ and $\alpha \in \Gamma_0(N)$. Then considering $\overline{\alpha} = (\alpha, s_{\mathbb{A}}(\alpha)) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$, using above lemma we have,

$$\begin{aligned} f(\overline{\alpha}z) &= \left(\frac{c}{d}\right) (\varepsilon_d^{-1})^{2k+1} (cz+d)^{k+1/2} f(z) \\ &= \left(\frac{c}{d}\right) (\varepsilon_d^{-1})^{2k+1} s_{\mathbb{A}}(\alpha) J(\overline{\alpha}, z)^{2k+1} f(z) \\ &= \begin{cases} (\varepsilon_d^{-1} J(\overline{\alpha}, z))^{2k+1} f(z) & \text{if } c = 0 \\ (c,d)_\infty (\varepsilon_d^{-1} J(\overline{\alpha}, z))^{2k+1} f(z) & \text{if } c \neq 0 \text{ and } \mathrm{ord}_2(c) \text{ is odd} \\ (c,d)_\infty \, (c,d)_2 \, (\varepsilon_d^{-1} J(\overline{\alpha}, z))^{2k+1} f(z) & \text{if } c \neq 0 \text{ and } \mathrm{ord}_2(c) \text{ is even.} \end{cases} \end{aligned}$$

For $\theta \in \mathbb{R}$, let $k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Define $\tilde{K}_\infty := \{\tilde{k}(\theta) : \theta \in (-2\pi, 2\pi]\}$ where

$$\tilde{k}(\theta) = \begin{cases} (k(\theta), 1) & \text{if } -\pi < \theta \le \pi, \\ (k(\theta), -1) & \text{if } -2\pi < \theta \le -\pi \text{ or } \pi < \theta \le 2\pi. \end{cases}$$

Then $\tilde{K}_\infty$ is a maximal compact subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $\tilde{k}(\theta) \mapsto e^{i\frac{2k+1}{2}\theta}$ is a genuine character of $\tilde{K}_\infty$. Let

$$K_1(N) := \prod_{q<\infty} \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_q) \ : \ c \equiv 0, \text{ and } a, d \equiv 1 \pmod{N\mathbb{Z}_q}\}.$$

Recall the strong approximation theorem for $\widetilde{\mathrm{SL}}_2(\mathbb{A})$: every element $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$ can be written as

$$\tilde{g} = (\alpha, s_{\mathbb{A}}(\alpha))\tilde{g}_\infty(k_1, 1),$$

where $(\alpha, s_{\mathbb{A}}(\alpha)) \in s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))$, $k_1 \in K_1(N)$ and $\tilde{g_\infty} \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ determined up to left multiplication by elements in $s_{\mathbb{Q}}(\Gamma_1(N))$.

We follow the notation of Waldspurger [18]. Let χ be an even Dirichlet character modulo N. Write $\chi_0 = \chi\left(\frac{-1}{\cdot}\right)^k$. Define $\tilde{\gamma_2}$ on $\mathbb{Z}_2^\times$ as

$$\tilde{\gamma_2}(t) = \begin{cases} 1 & \text{if } t \equiv 1 \pmod{4\mathbb{Z}_2} \\ -i & \text{if } t \equiv 3 \pmod{4\mathbb{Z}_2}, \end{cases}$$

and for $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^2(4)$, define

$$\tilde{\epsilon}_2(k_0) = \begin{cases} \tilde{\gamma_2}(d)^{-1}\,(c,d)_2\, s_2(k_0) & \text{if } c \neq 0 \\ \tilde{\gamma_2}(d) & \text{if } c = 0. \end{cases}$$

Let χ_0 also denote the idelic character (of $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times$) corresponding to the Dirichlet character χ_0 (it will be clear from the context when we consider χ_0 to be idelic or Dirichlet character) and $\chi_{0,p}$ be the p-component of χ_0. Let $A_{k+1/2}(N, \chi_0)$ denote the set of functions $\Phi : \widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathbb{C}$ satisfying the following properties:

(1) $\Phi(s_{\mathbb{Q}}(\alpha)\tilde{g}(k_1, 1)) = \Phi(\tilde{g})$ for all $k_1 \in \prod_{q\nmid N} \mathrm{SL}_2(\mathbb{Z}_q)$, $\alpha \in \mathrm{SL}_2(\mathbb{Q})$, $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$.
(2) Φ is genuine, that is, $\Phi((I, \zeta)\tilde{g}) = \zeta\Phi(\tilde{g})$ for $\zeta \in \mu_2$.
(3) For odd primes p such that $p^n \| N$, $\Phi(\tilde{g}(k_0, 1)) = \chi_{0,p}(d)\Phi(\tilde{g})$ for all $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^p(p^n)$.
(4) If $2^n \| N$ $(n \geq 2)$, $\Phi(\tilde{g}(k_0, 1)) = \tilde{\epsilon}_2(k_0)\chi_{0,2}(d)\Phi(\tilde{g})$ for all $k_0 \in K_0^2(2^n)$.
(5) $\Phi(\tilde{g}\tilde{k}(\theta)) = e^{i\frac{2k+1}{2}\theta}\Phi(\tilde{g})$ for all $\tilde{k}(\theta) \in \tilde{K}_\infty$.

(6) Φ is smooth as a function of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and satisfies the differential equation $\Delta\Phi = -[(2k+1)(2k-3)/16]\Phi$ where Δ is the Casimir operator.

(7) Φ is square integrable, that is, $\int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} |\Phi(\tilde{g})|^2 d\tilde{g} < \infty$.

(8) Φ is cuspidal, that is, $\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \Phi\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\tilde{g}\right) da = 0$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$.

By [18, Proposition 3] there exists an isomorphism between

$$A_{k+1/2}(N,\chi_0) \to S_{k+1/2}(\Gamma_0(N),\chi)$$

given by $\Phi \mapsto f_\Phi$ where for $z \in \mathbb{H}$,

$$f_\Phi(z) = \Phi(\tilde{g}_\infty)J(\tilde{g}_\infty, i)^{2k+1}$$

where $\tilde{g}_\infty \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is such that $\tilde{g}_\infty(i) = z$. The inverse map is given by $f \mapsto \Phi_f$ where for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$ if $\tilde{g} = (\alpha, s_{\mathbb{A}}(\alpha))\tilde{g}_\infty(k_1, 1)$,

$$\Phi_f(\tilde{g}) = f(\tilde{g}_\infty(i))J(\tilde{g}_\infty, i)^{-2k-1}.$$

This isomorphism induces a ring isomorphism of spaces of linear operators,

$$q : \mathrm{End}_{\mathbb{C}}(A_{k+1/2}(N,\chi_0)) \to \mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(N),\chi))$$

given by

$$q(\mathcal{T})(f) = f_{\mathcal{T}(\Phi_f)}.$$

4.1. $N = 4M$**,** M **odd and** $p\|M$**.** Let p be an odd prime and let $N = 4M$ with M odd such that p strictly divides M. In this subsection we translate the elements $\mathcal{T}_1$, $\mathcal{U}_1$, $\mathcal{U}_0$ and $\mathcal{T}_{-1}$ in the p-adic Hecke algebra to certain classical operators on $S_{k+1/2}(\Gamma_0(4M),\chi)$. We restrict ourselves to χ being the trivial character modulo $4M$. In this case $\chi_0 = \left(\frac{-1}{\cdot}\right)^k$ has conductor either 1 or 4 and so $\chi_{0,p}$ is trivial on $\mathbb{Z}_p^\times$ while $\chi_{0,2}$ acts by $\chi_0^{-1} = \chi_0$ on $\mathbb{Z}_2^\times$.

Let γ be the character on $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ induced by $\chi_{0,p}|\mathbb{Z}_p^\times$ (so in the current case γ is trivial). Then Iwahori Hecke algebra $H(\overline{K_0^p(p)},\gamma)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A_{k+1/2}(N,\chi_0))$ via the following action: for $\mathcal{T} \in H(\overline{K_0^p(p)},\gamma)$ and $\Phi \in A_{k+1/2}(N,\chi_0)$,

$$\mathcal{T}(\Phi)(\tilde{g}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} \mathcal{T}(\tilde{x})\Phi(\tilde{g}\tilde{x})d\tilde{x}.$$

Proposition 4.2. *Let M be positive integer such that p strictly divides M. Let χ be the trivial character modulo $4M$ and let γ be induced by $\chi_{0,p}$. Let $\mathcal{T}_1, \mathcal{U}_1, \mathcal{U}_0, \mathcal{T}_{-1} \in H(\overline{K_0^p(p)},\gamma)$ and $f \in S_{k+1/2}(\Gamma_0(4M),\chi)$. Then,*

$$\text{(1)}\quad \left(\frac{-1}{p}\right)^k q(\mathcal{T}_1)(f)(z) = p^{-k-1/2}\sum_{s=0}^{p^2-1} f\left(\frac{z+s}{p^2}\right) = p^{(3-2k)/2}U_{p^2}(f).$$

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f|[(\alpha_s, \phi_{\alpha_s})]_{k+1/2}(z)$, *where* $\alpha_s = \begin{pmatrix} p^2n - 4Mms & m \\ 4pM(1-s) & p \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$ *is of determinant* p^2 *and* $m,\ n \in \mathbb{Z}$ *are such that* $pn-(4M/p)m = 1$, *and* $\phi_{\alpha_s}(z) = (4M(1-s)z+1)^{1/2}$.

(3) $q(\mathcal{U}_0)(f)(z) = \sum_{s=0}^{p-1} f|[(\beta_s, \phi_{\beta_s})]_{k+1/2}(z)$, *where* $\beta_s = \begin{pmatrix} 1 & m-s \\ 4M_1 & np-4M_1s \end{pmatrix} \in \Gamma_0(4M_1)$ *with* $M_1 = M/p$ *and* $m,\ n \in \mathbb{Z}$ *are such that* $pn - 4M_1m = 1$ *and* $\phi_{\beta_s} = (4M_1z + (np-4M_1s))^{1/2}$.

(4) $q(\mathcal{T}_{-1})(f)(z) = \left(\frac{-1}{p}\right)^k \sum_{s=0}^{p^2-1} f|[(\gamma_s, \phi_{\gamma_s}(z))]_{k+1/2}(z)$, *where* $\gamma_s = \begin{pmatrix} p^2 & 0 \\ -4Ms & 1 \end{pmatrix}$ *and* $\phi_{\gamma_s}(z) = (-4(M/p)sz + p^{-1})^{1/2}$.

Proof. For (1), let $\tilde{g}_\infty = (g_\infty, 1) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ such that $\tilde{g}_\infty i = z$. Then using decomposition in Lemma 3.7 we have

$$\begin{aligned}\mathcal{T}_1(\Phi_f)(\tilde{g}_\infty) &= \sum_{s=0}^{p^2-1} \overline{\gamma}(h(p),1)\Phi_f(\tilde{g}_\infty(x(s),1)(h(p),1)) \\ &= \overline{\varepsilon_p}\sum_{s=0}^{p^2-1} \Phi_f(\tilde{g}_\infty(x(s),1)(h(p),1)) \\ &= \overline{\varepsilon_p}\sum_{s=0}^{p^2-1} \Phi_f(s_{\mathbb{Q}}(A_s)\tilde{g}_\infty(x(s),1)(h(p),1))\end{aligned}$$

where for each $0 \le s \le p^2-1$, we take $A_s = h(p^{-1})x(-s) = \begin{pmatrix} p^{-1} & -p^{-1}s \\ 0 & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$ (note that $\Phi_f(s_{\mathbb{Q}}(\alpha)\tilde{g}) = \Phi_f(\tilde{g})$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Q})$, $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$). Clearly $s_\nu(A_s) = 1$ for all primes ν, so $s_{\mathbb{Q}}(A_s) = (A_s, 1)$. The ∞-component of

$$\underbrace{(A_s,1)}_{\text{diagonal}}\ \underbrace{\tilde{g}_\infty}_{\infty\text{ place}}\ \underbrace{(x(s),1)(h(p),1)}_{p\text{ place}}$$

is $(A_s,1)\tilde{g}_\infty$, for a prime q such that $(q, 2M) = 1$ the q-component is $(A_s, 1) \in \mathrm{SL}_2(\mathbb{Z}_q)\times\{1\}$, for a prime r such that $(r, 2p) = 1$ and $r^b \| M$, the r-component is $(A_s, 1) \in K_0^r(r^b) \times \{1\}$, the 2-component is $(A_s, 1) \in K_0^2(4) \times \{1\}$ and the p-component is $(A_s, 1)(x(s), 1)(h(p), 1) = (I, (p, p)_p) = (I, \left(\frac{-1}{p}\right))$.

Since χ is trivial, $\chi_{0,2} = \left(\frac{-1}{\cdot}\right)^k$ while $\chi_{0,p}$ and $\chi_{0,r}$ are trivial. So the 2-component acts by $\tilde{\epsilon}_2(A_s)\chi_{0,2}(p) = \tilde{\gamma_2}(p)\chi_{0,2}(p) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)^k$ and the p-component acts by $\left(\frac{-1}{p}\right)$. Thus,

$$\begin{aligned}\mathcal{T}_1(\Phi_f)(\tilde{g}_\infty) &= \overline{\varepsilon_p}\sum_{s=0}^{p^2-1}\Phi_f(s_{\mathbb{Q}}(A_s)\tilde{g}_\infty(x(s),1)(h(p),1)) \\ &= (\overline{\varepsilon_p})^2\left(\frac{-1}{p}\right)^k\left(\frac{-1}{p}\right)\sum_{s=0}^{p^2-1}\Phi_f(A_sg_\infty,1) \\ &= \left(\frac{-1}{p}\right)^k\sum_{s=0}^{p^2-1}f(A_sg_\infty(i))J((A_sg_\infty,1),i)^{-2k-1}.\end{aligned}$$

Consequently,

$$q(\mathcal{T}_1)(f)(z) = \mathcal{T}_1(\Phi_f)(\tilde{g}_\infty)J((g_\infty,1),i)^{2k+1} = \left(\frac{-1}{p}\right)^k p^{-k-1/2}\sum_{s=0}^{p^2-1}f\left(\frac{z+s}{p^2}\right).$$

For (2) we need the following decomposition (we use $(4,M)=1$)

$$K_0w(p^{-1})K_0 = \bigcup_{s\in\mathbb{Z}_p/p\mathbb{Z}_p} y(4Ms)w(p^{-1})K_0.$$

Taking $\tilde{g}_\infty$ such that $\tilde{g}_\infty i = z$ we have

$$\mathcal{U}_1(\Phi_f)(\tilde{g}_\infty) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\sum_{s=0}^{p-1}\Phi_f(\tilde{g}_\infty(y(4Ms),1)(w(p^{-1}),1)).$$

Since p is coprime to $4M/p$, we fix $m,\ n\in\mathbb{Z}$ such that $pn-(4M/p)m=1$. For $0\le s\le p-1$, take

$$A_s = \begin{pmatrix} pn & \frac{m}{p} \\ 4M & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4Ms & 1\end{pmatrix} = \begin{pmatrix} pn-4ms\frac{M}{p} & \frac{m}{p} \\ 4M(1-s) & 1\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Q}).$$

Since $s_\nu(A_s)=1$ for all primes ν we have $s_{\mathbb{Q}}(A_s)=(A_s,1)$. As before the ∞-component of

$$s_{\mathbb{Q}}(A_s)\ \tilde{g}_\infty\ (y(4Ms),1)(w(p^{-1}),1)$$

is $(A_s,1)\tilde{g}_\infty$, for a prime q such that $(q,2M)=1$ the q-component is $(A_s,1)\in \mathrm{SL}_2(\mathbb{Z}_q)\times\{1\}$, for a prime r such that $(r,2p)=1$ and $r^b\| M$, the r-component is $(A_s,1)\in K_0^r(r^b)\times\{1\}$ and the 2-component is $(A_s,1)\in K_1^2(4)\times\{1\}$ (as $(2,2)$-th entry of A_s is 1). For the p-component we check that $(A_s,1) = (\begin{pmatrix} pn & m/p \\ 4M & 1\end{pmatrix},1)(y(-4Ms),1)$ and

$$(A_s,1)(y(4Ms),1)(w(p^{-1}),1) = (\begin{pmatrix} -m & n \\ -p & 4M/p\end{pmatrix},\left(\frac{M/p}{p}\right)).$$

Since χ is trivial, as before the r-component act trivially, the p-component acts by $\left(\frac{M/p}{p}\right)$ (as $\chi_{0,p}(4M/p) = 1$) and the 2-component by $\tilde{\epsilon}_2(A_s)\chi_{0,2}(1) = 1$.

Thus we have

$$\mathcal{U}_1(\Phi_f)(\tilde{g}_\infty) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\sum_{s=0}^{p-1}\Phi_f(s_{\mathbb{Q}}(A_s)\tilde{g}_\infty(y(4Ms),1)(w(p^{-1}),1))$$
$$= \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1}\Phi_f((A_s,1)(g_\infty,1))$$
$$= \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f((A_s,1)z)J((A_s,1),z)^{-2k-1}J((g_\infty,1),i)^{-2k-1}.$$

So we have

$$q(\mathcal{U}_1)(f)(z) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f((A_s,1)z)J((A_s,1),z)^{-2k-1}.$$

Let $\alpha_s = A_s\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}$ and $\phi_{\alpha_s}(z) = (4M(1-s)z+1)^{1/2}$. Then

$q(\mathcal{U}_1)(f)(z)$

$$= \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f\left(\frac{(p^2n-4mMs)z+m}{4pM(1-s)z+p}\right)(4M(1-s)z+1)^{-k-1/2}$$
$$= \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f|[(\alpha_s,\phi_{\alpha_s})]_{k+1/2}(z).$$

For (3), using Lemma 3.7 we have

$$\mathcal{U}_0(\Phi_f)(\tilde{g}_\infty) = \sum_{s=0}^{p-1}\Phi_f(\tilde{g}_\infty(x(s),1)(w(1),1)).$$

Let $m,\ n \in \mathbb{Z}$ such that $pn - (4M/p)m = 1$ and let $M_1 = M/p$. For $0 \le s \le p-1$, take

$$A_s = \begin{pmatrix} 1 & -s+m \\ 4M_1 & -4M_1 s + np\end{pmatrix} \in \Gamma_1(4M_1).$$

By Lemma 4.1 we have $s_{\mathbb{A}}(A_s) = \left(\frac{4M_1}{-4M_1s+np}\right) = 1$. Thus the ∞-component of $s_{\mathbb{Q}}(A_s)\ \tilde{g}_\infty\ (x(s),1)(w(1),1)$ is $(A_s,1)(g_\infty,1)$, for a prime q such that $(q,2M)=1$ the q-component is $(A_s,1) \in \mathrm{SL}_2(\mathbb{Z}_q)\times\{1\}$, if r is an odd prime such that $r^b \| M$ then the r-component is $(A_s,1) \in K_0^r(r^b)\times\{1\}$ and the 2-component is $(A_s,1) \in K_1^2(4)\times\{1\}$. For p-component, since $\mathrm{ord}_p(4M_1) = 0$, we have $(\begin{pmatrix} 1 & m \\ 4M_1 & np\end{pmatrix},1)(x(-s),1) = (A_s,1)$ and $(\begin{pmatrix} 1 & m \\ 4M_1 & np\end{pmatrix},1)(w(1),1) =$

$(\begin{pmatrix} -m & 1 \\ -np & 4M_1 \end{pmatrix}, \beta)$ where β is either $(4M_1, -1)_p$ or $(4M_1, np)_p$ depending on whether $\mathrm{ord}_p(np)$ is odd or even. In either case it is clear that β is 1. Thus the p-component is $(\begin{pmatrix} -m & 1 \\ -np & 4M_1 \end{pmatrix}, 1) \in K_0 \times \{1\}$.

Since χ is trivial, the p-component and r-component act trivially, and the 2-component acts by $\tilde{\epsilon}_2(A_s)\chi_{0,2}(-4M_1 s + np) = (4M_1, -4M_1 s + np)_2 s_2(A_s)$ which clearly equals 1. Thus

$$\begin{aligned} \mathcal{U}_0(\Phi_f)(\tilde{g}_\infty) &= \sum_{s=0}^{p-1} \Phi_f((A_s, 1)\tilde{g}_\infty) \\ &= \sum_{s=0}^{p-1} f(A_s z) J((A_s, 1), z)^{-2k-1} J((g_\infty, 1), i)^{-2k-1} \end{aligned}$$

and consequently

$$q(\mathcal{U}_0)(f)(z) = \sum_{s=0}^{p-1} f\left(\frac{z + (m-s)}{4M_1 z + (np - 4M_1 s)}\right)(4M_1 z + (np - 4M_1 s))^{-k-1/2}.$$

For (4), using $K_0 h(p^{-1}) K_0 = \bigcup_{s \in \mathbb{Z}_p/p^2\mathbb{Z}_p} y(4Ms)h(p^{-1})K_0$ we have

$$\mathcal{T}_{-1}(\Phi_f)(\tilde{g}_\infty) = \overline{\varepsilon_p} \sum_{s=0}^{p^2-1} \Phi_f(\tilde{g}_\infty(y(4Ms), 1)(h(p^{-1}), 1)).$$

For $0 \le s \le p^2 - 1$, take $A_s = h(p)y(-4Ms) = \begin{pmatrix} p & 0 \\ -4(M/p)s & p^{-1} \end{pmatrix}$, then $s_{\mathbb{Q}}(A_s) = (A_s, \xi_s)$ where

$$\xi_s := \begin{cases} 1 & \text{if } s = 0 \\ 1 & \text{if } \mathrm{ord}_p(s) = 1 \text{ and } \mathrm{ord}_2(s) \text{ odd} \\ \left(\frac{-1}{p}\right)(\frac{Ms}{p}, p)_2 & \text{if } \mathrm{ord}_p(s) = 1 \text{ and } \mathrm{ord}_2(s) \text{ even} \\ \left(\frac{-1}{p}\right)(\frac{Ms}{p}, p)_p & \text{if } (s,p) = 1 \text{ and } \mathrm{ord}_2(s) \text{ odd} \\ (\frac{Ms}{p}, p)_2\ (\frac{Ms}{p}, p)_p & \text{if } (s,p) = 1 \text{ and } \mathrm{ord}_2(s) \text{ even.} \end{cases}$$

We verify the above formula for ξ_s in the case $\mathrm{ord}_p(s) = 1$ and $\mathrm{ord}_2(s)$ is even, the other cases follow similarly. Clearly $\mathrm{ord}_p(s) = 1$ and $\mathrm{ord}_2(s)$ even imply that $\mathrm{ord}_p(-4(M/p)s) = 1$ and $\mathrm{ord}_2(-4(M/p)s)$ is even. So we have $s_2(A_s) = 1$, $s_p(A_s) = (-\frac{4Ms}{p}, p^{-1})_p$ and by definition, $s_\infty(A_s) = 1$. For any prime q, note that $(-\frac{4Ms}{p}, p^{-1})_q = (-Msp, p)_q = (Ms, p)_q$. So

$$\xi_s = \prod_\nu s_\nu(A_s) = (Ms, p)_p \prod_{\substack{q\,:\,(q,\,2p)=1,\\ \mathrm{ord}_q(4Ms)\text{ odd}}} (Ms, p)_q$$

If $\mathrm{ord}_q(4Ms)$ is even, then $(Ms,p)_q = (u,p)_q$ for some unit u in $\mathbb{Z}_q$, so $(Ms,p)_q = 1$. Thus using the product formula $\prod_\nu (Ms,p)_\nu = 1$ we have

$$\xi_s = \prod_\nu s_\nu(A_s) = (Ms,p)_p \prod_{q\,:\,(q,\,2p)=1} (Ms,p)_q = (Ms,p)_2.$$

Since $(p,p)_2 = \left(\frac{-1}{p}\right)$ we get that $\left(\frac{-1}{p}\right)(\frac{Ms}{p},p)_2 = (Ms,p)_2$ and we are done.

Thus we have

$$\mathcal{T}_{-1}(\Phi_f)(\tilde{g}_\infty) = \overline{\varepsilon_p}\sum_{s=0}^{p^2-1} \xi_s \Phi_f((A_s,1)\tilde{g}_\infty(y(4Ms),1)(h(p^{-1}),1)).$$

Now the ∞-component of $(A_s,1)\ \tilde{g}_\infty\ (y(4Ms),1)(h(p^{-1}),1)$ is $(A_s,1)\tilde{g}_\infty$, for a prime q such that $(q,2M)=1$ the q-component is $(A_s,1)\in \mathrm{SL}_2(\mathbb{Z}_q)\times\{1\}$, if r is an odd prime coprime to p such that $r^b \| M$ then the r-component belongs to $K_0^r(r^b)\times\{1\}$, the 2-component is $(\begin{pmatrix} p & 0 \\ -4(M/p)s & p^{-1}\end{pmatrix},1) \in K_0^2(4)\times\{1\}$ and the p-component is $(A_s,1)(y(4Ms),1)(h(p^{-1}),1)$ which is precisely equal to (I,η_s) where $\eta_s := \begin{cases} \left(\frac{-1}{p}\right) & \text{if } s=0 \\ 1 & \text{if } \mathrm{ord}_p(s)=1 \\ \left(\frac{-1}{p}\right)(\frac{Ms}{p},p)_p & \text{if } (s,p)=1. \end{cases}$

Since χ is trivial, $\chi_{0,p}$ is trivial and so the p-component acts on Φ_f simply by multiplication by η_s. Next we look at how the 2-component acts on Φ_f. Since $\chi_{0,2} = \left(\frac{-1}{\cdot}\right)^k$ we get that

$$\tilde{\epsilon}_2(A_s)\chi_{0,2}(p^{-1}) = \begin{cases} \tilde{\gamma_2}(p^{-1})\chi_{0,2}(p^{-1}) & \text{if } s=0 \\ \tilde{\gamma_2}(p^{-1})^{-1}(-4\frac{M}{p}s,p^{-1})_2 s_2(A_s)\chi_{0,2}(p^{-1}) & \text{if } s\neq 0 \end{cases}$$

$$= \vartheta_s := \begin{cases} \overline{\varepsilon_p}\left(\frac{-1}{p}\right)^k & \text{if } s=0 \\ \varepsilon_p\left(\frac{-1}{p}\right)^k & \text{if } s\neq 0 \text{ and } \mathrm{ord}_2(s) \text{ odd} \\ \varepsilon_p\left(\frac{-1}{p}\right)^{k+1}(\frac{Ms}{p},p)_2 & \text{if } s\neq 0 \text{ and } \mathrm{ord}_2(s) \text{ even.} \end{cases}$$

One can check that

$$\vartheta_s\eta_s = \varepsilon_p\left(\frac{-1}{p}\right)^k \xi_s,$$

and so

$$\mathcal{T}_{-1}(\Phi_f)(\tilde{g}_\infty) = \overline{\varepsilon_p}\sum_{s=0}^{p^2-1}\xi_s\vartheta_s\eta_s\Phi_f((A_s,1)\tilde{g}_\infty) = \left(\frac{-1}{p}\right)^k \sum_{s=0}^{p^2-1}\Phi_f((A_s,1)\tilde{g}_\infty).$$

Thus

$$q(\mathcal{T}_{-1})(f)(z) = \left(\frac{-1}{p}\right)^k \sum_{s=0}^{p^2-1} f\left(\frac{p^2 z}{-4Msz+1}\right)\left(\frac{-4Msz+1}{p}\right)^{-k-1/2}$$

$$= \left(\frac{-1}{p}\right)^k \sum_{s=0}^{p^2-1} f|[(\gamma_s, \phi_{\gamma_s}(z))]_{k+1/2}(z)$$

where $\gamma_s = \begin{pmatrix} p^2 & 0 \\ -4Ms & 1 \end{pmatrix}$ and $\phi_{\gamma_s}(z) = (-4(M/p)sz + p^{-1})^{1/2}$. $\square$

Let $\widetilde{Q}_p := q(\mathcal{U}_0)$ and $\widetilde{W}_{p^2} := q(p^{-1/2}\mathcal{U}_1)$. Then we have the following

Corollary 4.3. *On $S_{k+1/2}(\Gamma_0(4M))$,*

(1) $\widetilde{W}_{p^2}$ is an involution,
(2) $(\widetilde{Q}_p - p)(\widetilde{Q}_p + 1) = 0$,
(3) $\widetilde{Q}_p = \left(\frac{-1}{p}\right)^k p^{1-k} U_{p^2} \widetilde{W}_{p^2}$,
(4) If $f \in S_{k+1/2}(\Gamma_0(4M/p))$ then $\widetilde{Q}_p(f) = pf$.

Proof. The proof of (1) to (3) follows by using Proposition 3.9 and 4.2. For (4) we use Proposition 4.2(3). $\square$

We further define an operator $\widetilde{Q}'_p$ on $S_{k+1/2}(\Gamma_0(4M))$ to be the conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_{p^2}$, i.e., $\widetilde{Q}'_p = \widetilde{W}_{p^2}\widetilde{Q}_p\widetilde{W}_{p^2}$. Thus $\widetilde{Q}'_p$ satisfies the same quadratic relation as $\widetilde{Q}_p$ and we have $\widetilde{Q}'_p = \left(\frac{-1}{p}\right)^k p^{1-k}\widetilde{W}_{p^2}U_{p^2}$.

Remark 3. *We note that for a prime q such that $(q, 2M) = 1$, one can similarly obtain the usual Hecke operator T_{q^2} on $S_{k+1/2}(\Gamma_0(4M))$. In particular, if we take $\mathcal{T}_1 := X_{(h(q),1)} \in H(\overline{\mathrm{SL}_2(\mathbb{Z}_q)}, \gamma_q)$ then $q(\mathcal{T}_1) = \left(\frac{-1}{p}\right)^k p^{(3-2k)/2} T_{q^2}$.*

Moreover if p and q are distinct primes such that p^n, q^m strictly divide N then the operators $\mathcal{S} \in H(\overline{K_0^p(p^n)}, \gamma_p)$ and $\mathcal{T} \in H(\overline{K_0^q(q^m)}, \gamma_q)$ in $\mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(N)))$ commute.

In particular the operators $\widetilde{Q}_p$, $\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4M))$ that we defined above commute with T_{q^2} for primes q coprime to $2M$.

Remark 4. *Let $f \in S_{k+1/2}(\Gamma_0(2^\nu M))$ where $\nu \geq 2$. Then we have exactly the same statement as Proposition 4.2 for the action on f with M replaced by $2^\nu M$. In particular, if $f \in S_{k+1/2}(\Gamma_0(2^\nu M/p))$ then $\widetilde{Q}_p(f) = pf$. The results of the next section on self-djointness also hold similarly.*

4.2. **Self-adjointness.** Let M be odd such that $p \| M$. In this subsection we check that the operators $\widetilde{W}_{p^2}$, $\widetilde{Q}_p$ and $\widetilde{Q}'_p$ are self-adjoint operators on $S_{k+1/2}(\Gamma_0(4M))$. The property of self-adjointness will be used to give a description of our minus space in terms of common eigenspaces.

Proposition 4.4. *The operator $\widetilde{W}_{p^2}$ is self-adjoint with respect to the Petersson inner product.*

Proof. We write

$$\widetilde{W}_{p^2}(f) = \frac{\overline{\varepsilon_p}}{\sqrt{p}}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\mathcal{S}_p(f), \quad \mathcal{S}_p(f) := \sum_{s\in\mathbb{Z}/p\mathbb{Z}} f|[(\alpha_s, \phi_{\alpha_s}(z))]_{k+1/2}$$

where $(\alpha_s, \phi_{\alpha_s}(z)) = \left(\begin{pmatrix} p^2n - 4mMs & m \\ 4pM(1-s) & p\end{pmatrix}, (4M(1-s)z+1)^{1/2}\right) \in \mathcal{G}$ and n, m are integers such that $pn - (4M/p)m = 1$.

We will show that $\langle \mathcal{S}_p(f), g\rangle = \left(\frac{-1}{p}\right)\langle f, \mathcal{S}_p(g)\rangle$. We write $\mathcal{S}_p = S_{1,p} + S_{2,p}$ where $S_{1,p}$ consists of the $s = 0$ term and $S_{2,p}$ consists of rest of the terms. Also, let $M_1 = M/p$.

We first consider $S_{2,p}$. For $s \neq 0$, as $pn - 4M_1ms = 1 + 4M_1m(1-s)$ it is clear that $pn - 4M_1ms$ and $4M(1-s)$ are relatively coprime, hence there exists integers u, v such that $u(pn - 4M_1ms) + v4M(1-s) = 1$. In particular, this implies that $-4M_1msu \equiv 1 \pmod p$. Since $-4M_1m \equiv 1 \pmod p$, we get that $su \equiv 1 \pmod p$.

We take $X = \begin{pmatrix} u & v \\ -4M(1-s) & pn - 4M_1ms\end{pmatrix} \in \Gamma_0(4M)$, then $X^* = (X, j(X,z))$ where $j(X,z) = \left(\frac{-4M(1-s)}{pn-4M_1ms}\right)(-4M(1-s)z + (pn - 4M_1ms))^{1/2}$, as $pn - 4M_1ms \equiv 1 \pmod 4$. Since f has level $4M$ we have

$$f|[(\alpha_s, \phi_{\alpha_s}(z))]_{k+1/2} = f|[X^*]_{k+1/2}|[(\alpha_s, \phi_{\alpha_s}(z))]_{k+1/2}.$$

We claim that in $\mathcal{G}$,

$$X^*(\alpha_s, \phi_{\alpha_s}(z)) = \left(\begin{pmatrix} p & um+vp \\ 0 & p\end{pmatrix}, \left(\frac{u}{p}\right)\right).$$

It is easy to see equality in the matrix component, also, $j(X, \alpha_s z)\phi_{\alpha_s}(z)$ simplifies to just $\left(\frac{-4M(1-s)}{pn-4M_1ms}\right)$. So we only need to check equality of the Kronecker symbols $\left(\frac{-4M(1-s)}{pn-4M_1ms}\right) = \left(\frac{u}{p}\right)$. While making a choice of m, n so that $pn - 4M_1m = 1$, we may choose m to be a negative integer so that for $1 \le s \le p-1$, $pn - 4M_1ms = 1 + 4M_1m(1-s) > 0$. So we have

$$\begin{aligned}\left(\tfrac{-4M(1-s)}{pn-4M_1ms}\right) &= \left(\tfrac{-4M_1m(1-s)}{1+4M_1m(1-s)}\right)\left(\tfrac{p}{1+4M_1m(1-s)}\right)\left(\tfrac{m}{1+4M_1m(1-s)}\right) \\ &= \left(\tfrac{p}{1+4M_1m(1-s)}\right)\left(\tfrac{m}{1+4M_1m(1-s)}\right).\end{aligned}$$

Note that $\left(\frac{p}{1+4M_1m(1-s)}\right) = \left(\frac{1+4M_1m(1-s)}{p}\right) = \left(\frac{pn-4M_1ms}{p}\right) = \left(\frac{u}{p}\right)$. If m is odd clearly $\left(\frac{m}{1+4M_1m(1-s)}\right) = 1$. Also, if $m = 2^\nu m'$ where $\nu \ge 1$ and m' is odd, then $\left(\frac{m}{1+4M_1m(1-s)}\right) = \left(\frac{2}{1+4M_1m(1-s)}\right)^\nu\left(\frac{m'}{1+4M_1m(1-s)}\right) = 1$ since in this case we have $1 + 4M_1m(1-s) \equiv 1 \pmod 8$. Thus our claim is proved.

Consequently, we have $S_{2,p}(f) = \sum_{u\in(\mathbb{Z}/p\mathbb{Z})^\times} f|[(\begin{pmatrix} p & um \\ 0 & p \end{pmatrix}, \left(\frac{u}{p}\right))]_{k+1/2}$. Since the adjoint of $|[(\begin{pmatrix} p & um \\ 0 & p \end{pmatrix}, \left(\frac{u}{p}\right))]_{k+1/2}$ is $|[(\begin{pmatrix} p & -um \\ 0 & p \end{pmatrix}, \left(\frac{u}{p}\right))]_{k+1/2}$, the adjoint of $S_{2,p}$ is $\left(\frac{-1}{p}\right) S_{2,p}$, i.e., $\langle S_{2,p}(f), g\rangle = \left(\frac{-1}{p}\right)\langle f, S_{2,p}(g)\rangle$.

Next we consider the term $S_{1,p}(f) = f|[(\begin{pmatrix} p^2n & m \\ 4pM & p \end{pmatrix}, (4Mz+1)^{1/2})]_{k+1/2}$. For this case we may choose m to be a positive integer. Let $\gamma_p := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_p \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pmod p$ and $\gamma_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{8M_1m}$ (this is possible since $(p, 8M_1m) = 1$). We may also choose c, d above so that $c < 0$ and $d > 0$. We claim that

$$S_{1,p}(f) = f|[(\begin{pmatrix} pa & b \\ p^2c & pd \end{pmatrix}, \left(\frac{M_1}{p}\right)\left(\frac{c}{d}\right)(cpz+d)^{1/2})]_{k+1/2}.$$

Let $Y = \begin{pmatrix} a-4bM_1 & \frac{-ma+bpn}{p} \\ pc-4Md & -mc+dpn \end{pmatrix}$. Then $Y \in \mathrm{SL}_2(\mathbb{Z})$ and $pc-4Md \equiv 0 \pmod{4M}$, so $Y \in \Gamma_0(4M)$. We further note that $-mc+dpn \equiv 1 \pmod 4$, $dpn - mc = d(1+4M_1m) - mc > 0$. To prove the claim we need to check that

$$Y^*(\begin{pmatrix} p^2n & m \\ 4pM & p \end{pmatrix}, (4Mz+1)^{1/2}) = (\begin{pmatrix} pa & b \\ p^2c & pd \end{pmatrix}, \left(\frac{M_1}{p}\right)\left(\frac{c}{d}\right)(cpz+d)^{1/2}).$$

As before, matrix equality is easy to check and the automorphy factor of the left hand side equals kronecker symbol $\left(\frac{pc-4Md}{-cm+dpn}\right)$ times $(pcz+d)^{1/2}$. So we need to show that $\left(\frac{pc-4Md}{-cm+dpn}\right) = \left(\frac{M_1}{p}\right)\left(\frac{c}{d}\right)$. Now

$$\begin{aligned}\left(\frac{pc-4Md}{-cm+dpn}\right) &= \left(\frac{p}{-cm+dpn}\right)\left(\frac{c-4M_1d}{-cm+dpn}\right) = \left(\frac{-cm+dpn}{p}\right)\left(\frac{c-4M_1d}{-cm+dpn}\right) \\ &= \left(\frac{-m}{p}\right)\left(\frac{c-4M_1d}{-cm+dpn}\right) = \left(\frac{M_1}{p}\right)\left(\frac{c-4M_1d}{-cm+dpn}\right).\end{aligned}$$

Since $(m, -cm+dpn) = 1$ we can write $\left(\frac{c-4M_1d}{-cm+dpn}\right) = \left(\frac{d+cm-dpn}{-cm+dpn}\right)\left(\frac{m}{-cm+dpn}\right)$. We have

$$\left(\frac{d+cm-dpn}{-cm+dpn}\right)\left(\frac{m}{-cm+dpn}\right) = \left(\frac{d}{-cm+dpn}\right)\left(\frac{m}{-cm+dpn}\right) = \left(\frac{c}{d}\right)\left(\frac{m}{d}\right)\left(\frac{m}{-cm+dpn}\right).$$

We finally check that $\left(\frac{m}{d}\right)\left(\frac{m}{-cm+dpn}\right) = 1$. If m is odd,

$$\left(\frac{m}{-cm+dpn}\right) = \left(\frac{dpn}{m}\right) = \left(\frac{d}{m}\right)\left(\frac{pn}{m}\right) = 1 = \left(\frac{m}{d}\right).$$

If $m = 2^\nu m'$, $\nu \geq 1$ then $dpn - cm \equiv 1 \pmod 8$ and so

$$\left(\frac{m}{-cm+dpn}\right) = \left(\frac{2}{-cm+dpn}\right)^\nu \left(\frac{m'}{-cm+dpn}\right) = \left(\frac{dpn}{m'}\right) = 1 = \left(\frac{m}{d}\right).$$

Thus our claim is proved.

Next we note that
$$(\begin{pmatrix} pa & b \\ p^2c & pd \end{pmatrix}, \left(\frac{c}{d}\right)(cpz+d)^{1/2}) = (\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})\gamma_p^* \cdot (\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}) =: \varsigma_p,$$
and so $S_{1,p}(f) = \left(\frac{M_1}{p}\right) f|[\varsigma_p]_{k+1/2}$.

We check similarly that
$$(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})(\gamma_p^*)^2(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}) = (\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \left(\frac{-1}{p}\right))Z^*$$
where $Z = \begin{pmatrix} a^2+bc & \frac{ab+bd}{p} \\ pc(a+d) & bc+d^2 \end{pmatrix} \in \Gamma_0(4M)$ and so
$$f|[(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})(\gamma_p^*)^2(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4})]_{k+1/2} = \left(\frac{-1}{p}\right) f.$$
Note that
$$\begin{aligned}(\varsigma_p)^2 &= (\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})\gamma_p^* \cdot (\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4})(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})\gamma_p^* \cdot (\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}) \\ &= (\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})\gamma_p^* \cdot (\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, 1)\gamma_p^* \cdot (\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}) \\ &= (\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, 1)(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4})(\gamma_p^*)^2 \cdot (\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/4}).\end{aligned}$$
Thus
$$f|[(\varsigma_p)^2]_{k+1/2} = \left(\frac{-1}{p}\right) f, \quad \text{i.e., } f|[\varsigma_p^{-1}]_{k+1/2} = \left(\frac{-1}{p}\right) f|[\varsigma_p]_{k+1/2}.$$
Since the adjoint of ς_p is ς_p^{-1} we get $\langle S_{1,p}(f), g\rangle = \left(\frac{-1}{p}\right)\langle f, S_{1,p}(g)\rangle$.

Thus $\langle \mathcal{S}_p(f), g\rangle = \left(\frac{-1}{p}\right)\langle f, \mathcal{S}_p(g)\rangle$. So $\langle \widetilde{W}_{p^2}(f), g\rangle = \frac{\overline{\varepsilon_p}}{\sqrt{p}}\left(\frac{-M_1}{p}\right)\langle \mathcal{S}_p(f), g\rangle$ $= \frac{\overline{\varepsilon_p}}{\sqrt{p}}\left(\frac{M_1}{p}\right)\langle f, \mathcal{S}_p(g)\rangle = \langle f, \frac{\varepsilon_p}{\sqrt{p}}\left(\frac{M_1}{p}\right)\mathcal{S}_p(g)\rangle = \langle f, \widetilde{W}_{p^2}(g)\rangle$. Hence we are done. □

Next we want to show that $\widetilde{Q}_p = q(\mathcal{U}_0)$ is self-adjoint. We use the relations $\mathcal{U}_1\mathcal{T}_1\mathcal{U}_1 = p\mathcal{T}_{-1}$ and $\mathcal{T}_1\mathcal{U}_1 = p\,\mathcal{U}_0$ (Proposition 3.9(3)). Thus we have
$$\langle q(\mathcal{U}_0)f, g\rangle = \frac{1}{p}\,\langle q(\mathcal{T}_1)q(\mathcal{U}_1)f, g\rangle.$$
Since by the above theorem $q(\mathcal{U}_1)$ is self-adjoint we get that
$$\langle f, q(\mathcal{U}_0)g\rangle = \frac{1}{p}\,\langle f, p\; q(\mathcal{U}_0)g\rangle = \frac{1}{p}\,\langle f, q(\mathcal{T}_1)q(\mathcal{U}_1)g\rangle$$
$$= \frac{1}{p}\,\langle f, \frac{1}{p}\; q(\mathcal{U}_1)^2 q(\mathcal{T}_1)q(\mathcal{U}_1)g\rangle = \frac{1}{p}\,\langle q(\mathcal{U}_1)f, \frac{1}{p}\; q(\mathcal{U}_1)q(\mathcal{T}_1)q(\mathcal{U}_1)g\rangle$$
$$= \frac{1}{p}\,\langle q(\mathcal{U}_1)f, q(\mathcal{T}_{-1})g\rangle.$$

Since $q(\mathcal{U}_1)$ is surjective it follows that $q(\mathcal{U}_0)$ is self-adjoint if and only if the adjoint of $q(\mathcal{T}_{-1})$ is $q(\mathcal{T}_1)$. We now show that the adjoint of $q(\mathcal{T}_{-1})$ is $q(\mathcal{T}_1)$.

Consider elements $\xi = (\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, p^{1/2})$ and $\eta = (\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, p^{-1/2})$ in $\mathcal{G}$. We can choose β_s such that $\Gamma_0(4M) \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \Gamma_0(4M) = \bigcup \Gamma_0(4M)\beta_s = \bigcup \beta_s \Gamma_0(4M)$. So by [14, Propositions 1.1, 1.2] we have $\Delta_0(4M)\xi\Delta_0(4M) = \bigcup \Delta_0(4M)\xi_s = \bigcup \xi_s\Delta_0(4M)$ where $P(\xi_s) = \beta_s$.

Since $\Delta_0(4M)\eta\Delta_0(4M) = \Delta_0(4M)\xi^{-1}\Delta_0(4M)(\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix}, 1)$, it follows that $\Delta_0(4M)\eta\Delta_0(4M) = \bigcup \Delta_0(4M)\xi_s^{-1}(\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix}, 1)$.

Thus for f, $g \in S_{k+1/2}(\Gamma_0(4M))$, we have

$$\begin{aligned} &\langle f|[\Delta_0(4M)\xi\Delta_0(4M)]_{k+1/2},\ g\rangle = \langle p^{(2k-3)/2} \sum_s f|[\xi_s]_{k+1/2},\ g\rangle \\ &= \langle f,\ p^{(2k-3)/2} \sum_s g|[\xi_s^{-1}]_{k+1/2}\rangle = \langle f,\ g|[\Delta_0(4M)\eta\Delta_0(4M)]_{k+1/2}\rangle \end{aligned} \tag{5}$$

as elements of the type $(aI, 1)$ belong to the center of $\mathcal{G}$ and act trivially via the slash operator.

Using the triangular decomposition we check that

$$\Gamma_0(4M) \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4M) = \bigcup_{s=0}^{p^2-1} \Gamma_0(4M) \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4Ms & 1 \end{pmatrix}$$

and so

$$\Delta_0(4M)\eta\Delta_0(4M) = \bigcup_{s=0}^{p^2-1} \Delta_0(4M)\ \eta\ (\begin{pmatrix} 1 & 0 \\ -4Ms & 1 \end{pmatrix}, (-4Msz+1)^{1/2})$$

$$= \bigcup_{s=0}^{p^2-1} \Delta_0(4M)(\begin{pmatrix} p^2 & 0 \\ -4Ms & 1 \end{pmatrix}, (-4(M/p)sz + p^{-1})^{1/2}).$$

Thus it follows from parts (1) and (4) of Proposition 4.2 that

$$g|[\Delta_0(4M)\eta\Delta_0(4M)]_{k+1/2} = \left(\frac{-1}{p}\right)^k p^{(2k-3)/2} q(\mathcal{T}_{-1})(g),$$

and $f|[\Delta_0(4M)\xi\Delta_0(4M)]_{k+1/2} = \left(\frac{-1}{p}\right)^k p^{(2k-3)/2} q(\mathcal{T}_1)(f)$. Thus by equation (5) we obtain the following

Proposition 4.5. *The operator $q(\mathcal{T}_{-1})$ is adjoint of $q(\mathcal{T}_1)$ and consequently $\widetilde{Q}_p$ is self-adjoint with respect to the Petersson inner product.*

4.3. **Translating elements of 2-adic Hecke algebra and Kohnen's plus space.** Following Niwa and Kohnen's work, Loke and Savin gave an interpretation of Kohnen's plus space at level 4 in terms of certain elements in the 2-adic Hecke algebra described previously. In this subsection we shall describe Kohnen's plus space at level $4M$ for M odd in a similar way.

Let χ be the trivial character modulo 4, thus $\chi_0 = \left(\frac{-1}{\cdot}\right)^k$. Let γ be a character of M_2 such that $\gamma((-I,1)) = -i^{2k+1}$ and let $\varphi_8 := \gamma((w(1),1))$. Then, for any $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^2(4)$ we have $\tilde{\epsilon}_2(k_0)\chi_{0,2}(d) = \gamma((k_0,1))$.

Proposition 4.6. *(Loke-Savin [8]) For $\mathcal{T}_1$, $\mathcal{U}_1 \in H(\overline{K_0^2(4)}, \gamma)$ and $f \in S_{k+1/2}(\Gamma_0(4), \chi)$,*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{(3-2k)/2}U_4(f)(z)$.
(2) $q(\mathcal{U}_1)(f)(z) = \left(\frac{2}{2k+1}\right) W_4(f)(z)$ where the operator W_4 is given by $W_4(f)(z) = (-2iz)^{-k-1/2}f(-1/4z)$ and $\left(\frac{2}{2k+1}\right)$ is the usual Kronecker symbol.

Niwa [10] considered operator $R = W_4U_4$ on $S_{k+1/2}(\Gamma_0(4), \chi)$, proved that it is self-adjoint and that $(R-\alpha_1)(R-\alpha_2) = 0$ where $\alpha_1 = \left(\frac{2}{2k+1}\right) 2^k$, $\alpha_2 = -\frac{\alpha_1}{2}$. Kohnen [5] defined his plus space $S^+_{k+1/2}(\Gamma_0(4))$ at level 4 to be the α_1-eigenspace of R in $S_{k+1/2}(\Gamma_0(4))$. It follows from the above proposition that $S^+_{k+1/2}(\Gamma_0(4))$ is the 2-eigenspace of $q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$ and hence that of $q(\mathcal{U}_2)/\sqrt{2}$.

In the case of level $4M$ with M odd and χ a trivial character modulo $4M$, Kohnen [6] defines a classical operator Q on $S_{k+1/2}(\Gamma_0(4M))$ in order to obtain his plus space. The operator Q is defined by

$$Q := [\Delta_0(4M,\chi)\rho\Delta_0(4M,\chi)] \quad \text{where } \rho = (\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}).$$

By [6, Proposition 1] Q is self-adjoint and satisfies $(Q-\alpha)(Q-\beta) = 0$ where $\alpha = (-1)^{[(k+1)/2]}2\sqrt{2}$, $\beta = -\alpha/2$, and the plus space $S^+_{k+1/2}(\Gamma_0(4M))$ is precisely the α-eigenspace of Q.

Proposition 4.7. *Let $f \in S_{k+1/2}(\Gamma_0(4M))$ with M odd. Then we have*

$$Q(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_2)(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_1)q(\mathcal{T}_1)(f).$$

Consequently $S^+_{k+1/2}(\Gamma_0(4M))$ is the 2-eigenspace of $q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$.

Proof. Following [6, Proposition 1] we can write

$$Q(f)=\sum_{s=0}^{4} f|[\rho]_{k+1/2}|[\begin{pmatrix}1 & 0\\ 4Ms & 1\end{pmatrix},(4Msz+1)^{1/2}]_{k+1/2}$$
$$=e^{-(2k+1)\pi i/4}\sum_{s=0}^{4} f|[\begin{pmatrix}4+4Ms & 1\\ 16Ms & 4\end{pmatrix},(4Msz+1)^{1/2}]_{k+1/2}$$

and its adjoint

$$\tilde{Q}(f)=\sum_{s=0}^{4} f|[\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix},e^{-\pi i/4}]_{k+1/2}|[\begin{pmatrix}1 & 0\\ 4Ms & 1\end{pmatrix},(4Msz+1)^{1/2}]_{k+1/2}$$
$$=e^{(2k+1)\pi i/4}\sum_{s=0}^{4} f|[\begin{pmatrix}4-4Ms & -1\\ 16Ms & 4\end{pmatrix},(4Msz+1)^{1/2}]_{k+1/2}.$$

Since Q is self-adjoint, $Q=\tilde{Q}$.

We now compute $q(\mathcal{U}_2)(f)$. Let $\tilde{g}_\infty\in\widetilde{\mathrm{SL}_2}(\mathbb{R})$ be such that $\tilde{g}_\infty i=z$. Using $K_0^2(4)w(2^{-2})K_0^2(4)=\bigcup_{s\in\mathbb{Z}/4\mathbb{Z}}y(4M(1-s))w(2^{-2})K_0^2(4)$ (from [8, Proposition 3]), we get

$$\mathcal{U}_2(\Phi_f)(\tilde{g}_\infty)=\overline{\varphi_8}\sum_{s=0}^{3}\Phi_f(\tilde{g}_\infty(y(4M(1-s)),1)(w(2^{-2}),1)).$$

Take $A_s=\begin{pmatrix}1-\left(\frac{-1}{M}\right)Ms & -\left(\frac{-1}{M}\right)/4\\ 4Ms & 1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Q})$, so $s_{\mathbb{Q}}(A_s)=(A_s,1)$. The ∞-component of

$$s_{\mathbb{Q}}(A_s)\ \tilde{g}_\infty\ (y(4M(1-s)),1)(w(2^{-2}),1)$$

is $(A_s,1)\tilde{g}_\infty$, for a prime q such that $(q,2M)=1$ the q-component is $(A_s,1)\in\mathrm{SL}_2(\mathbb{Z}_q)\times\{1\}$, for an odd prime p such that $p^b\|M$, the p-component is $(A_s,1)\in K_0^p(p^b)\times\{1\}$ while the 2-component is

$$(A_s,1)(y(4M(1-s)),1)(w(2^{-2}),1))=(\begin{pmatrix}\left(\frac{-1}{M}\right) & \frac{1-M\left(\frac{-1}{M}\right)}{4}\\ -4 & M\end{pmatrix},1).$$

Since M is odd, it is clear that $\frac{1-M\left(\frac{-1}{M}\right)}{4}\in\mathbb{Z}_2$ and so the 2-component is in $K_0^2(4)\times\{1\}$. The p-component acts trivially while the 2-component acts by $(\tilde{\gamma_2}(M))^{-1}(-1,M)_2\chi_{0,2}(M)=:\omega_M$. Hence

$$q(\mathcal{U}_2)(f)(z)=\overline{\varphi_8}\ \omega_M\sum_{s=0}^{3}f(A_sz)J(A_s,z)^{-2k-1}$$
$$=\overline{\varphi_8}\ \omega_M\sum_{s=0}^{3}f\left(\frac{(4-4M\left(\frac{-1}{M}\right)sz)-\left(\frac{-1}{M}\right)}{16Msz+4}\right)(4Msz+1)^{-k-1/2}.$$

We note that $e^{(2k+1)\pi i/4} = \left(\frac{2}{2k+1}\right)\frac{1+i^{2k+1}}{\sqrt{2}} = \left(\frac{2}{2k+1}\right)\overline{\varphi_8}$. Thus when $M \equiv 1 \pmod 4$ since $\omega_M = 1$, comparing the expression of $\tilde{Q}$ and $q(\mathcal{U}_2)$ we see that $\tilde{Q}(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_2)(f)$. In the case $M \equiv 3 \pmod 4$ we get that $\omega_M = -i(-1)^k$, so $\left(\frac{2}{2k+1}\right)\overline{\varphi_8}\omega_M = e^{-(2k+1)\pi i/4}$ and consequently $Q(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_2)(f)$. Since by Theorem 1, $\mathcal{U}_2 = \mathcal{U}_1 * \mathcal{T}_1$ we get that $Q(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_1)q(\mathcal{T}_1)(f)$. Hence we are done.

The last statement follows since $(-1)^{[(k+1)/2]} = \left(\frac{2}{2k+1}\right)$. $\square$

As before we can translate $\mathcal{T}_1$, $\mathcal{U}_1$, $\mathcal{U}_0 \in H(\overline{K_0^2(4)}, \gamma)$ to classical operators on $S_{k+1/2}(\Gamma_0(4M))$.

Proposition 4.8. *For $f \in S_{k+1/2}(\Gamma_0(4M))$,*

(1) $q(\mathcal{T}_1)(f)(z) = 2^{(3-2k)/2} U_4(f)(z) = \sum_{s=0}^{3} f|[\begin{pmatrix} 1 & -s \\ 0 & 4 \end{pmatrix}, 2^{1/2}]_{k+1/2}(z).$

(2) $q(\mathcal{U}_1)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2} f|[W, \phi_W(z)]_{k+1/2}(z)$ *where* $W = \begin{pmatrix} 4n & m \\ 4M & 4 \end{pmatrix}$ *with* $m,\ n \in \mathbb{Z}$ *such that* $4n - mM = 1$ *and* $\phi_W(z) = (2Mz+2)^{1/2}$.

(3) $q(\mathcal{U}_0)(f)(z) = \overline{\varphi_8}\left(\frac{-1}{M}\right)^{k+3/2} \sum_{s=0}^{3} f|[A_s, \phi_{A_s}(z)]_{k+1/2}(z)$ *where* $A_s = \begin{pmatrix} n & -ns+m \\ M & -Ms+4 \end{pmatrix}$ *with* $m,\ n \in \mathbb{Z}$ *such that* $4n - mM = 1$ *and* $\phi_{A_s}(z) = (Mz+4-Ms)^{1/2}$.

Define $\widetilde{Q}_2 := q(\mathcal{U}_0)/\sqrt{2}$. It follows from the relation $\mathcal{U}_0 = \mathcal{T}_1\mathcal{U}_1$ that $\widetilde{Q}_2 = q(\mathcal{T}_1)q(\mathcal{U}_1)/\sqrt{2}$. One can also observe it directly from the above proposition. Let $\widetilde{W}_4 := q(\mathcal{U}_1)$. Thus $\widetilde{W}_4$ is an involution. Let $\widetilde{Q}'_2$ be the conjugate of $\widetilde{Q}_2$ by $\widetilde{W}_4$. Thus $\widetilde{Q}_2 = 2^{1-k}U_4\widetilde{W}_4$ and $\widetilde{Q}'_2 = 2^{1-k}\widetilde{W}_4U_4$. The Kohnen's plus space at level $4M$ is the 2-eigenspace of $\widetilde{Q}'_2$. Note that $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ are self-adjoint with respect to the Petersson inner product. The operators $\widetilde{Q}'_p$ and $\widetilde{Q}_p$ are p-adic analogues of Kohnen's operator $\widetilde{Q}'_2$ and its conjugate $\widetilde{Q}_2$.

Remark 5. *We note that $q(\mathcal{U}_1)$ in the above proposition can also be given by the following expression:*

$$q(\mathcal{U}_1)(f)(z) = \overline{\varphi_8}\left(\frac{2}{M}\right)\left(\frac{-1}{M}\right)^{k+3/2} f|[W, \phi_W(z)]_{k+1/2}(z)$$

where $W = \begin{pmatrix} 4n & m \\ 4M & 8 \end{pmatrix}$ *with* $m,\ n \in \mathbb{Z}$ *such that* $8n - mM = 1$ *and* $\phi_W(z) = (2Mz+4)^{1/2}$. *We shall use this expression of* $q(\mathcal{U}_1)$ *in* [3].

5. Eigenvalues of U_p

For every positive integer n and a modular form F, let $F_n(z) := V(n)F(z) = F(nz)$. Let M be a positive integer such that $p \nmid M$. If $F \in S_{2k}(\Gamma_0(M))$, then by the well-known action of T_p and U_p we have

$$U_p(F)(z) = T_p(F)(z) - p^{2k-1}F_p(z). \tag{6}$$

Assume that $F \in S_{2k}(\Gamma_0(M))$ is a primitive Hecke eigenform and a_p is the p-th Fourier coefficient of F. Then $T_p(F) = a_pF$. It is known that a_p is real and by the Ramanujan conjecture proved by Deligne we have that $|a_p| \le 2p^{(2k-1)/2}$.

Lemma 5.1. *(a) If* $(p,n) = 1$ *then* $U_p(F_n) = a_pF_n - p^{2k+1}F_{np}$.
(b) If $p \mid n$ *then* $U_p(F_n) = F_{n/p}$.

Proof. It is well known that if $(p,n) = 1$ then $V(n)T_p(F) = T_pV(n)F$. Hence using (6) and that F is a primitive Hecke eigenform we get that

$$\begin{aligned} U_p(F_n) &= T_p(F_n) - p^{2k-1}F_{np} = V(n)T_p(F) - p^{2k-1}F_{np} \\ &= V(n)a_pF - p^{2k-1}F_{np} = a_pF_n - p^{2k-1}F_{np}. \end{aligned}$$

For (b) write $n = mp$. Then

$$U_p(F_n)(z) = \frac{1}{p}\sum_{k=0}^{p-1} F_{mp}\left(\frac{z+k}{p}\right) = \frac{1}{p}\sum_{k=0}^{p-1} F_m(z+k) = F_{n/p}(z).$$

□

Thus U_p stabilizes the two dimensional subspace spanned by F_n and F_{np} for $(p,n) = 1$. We will compute the eigenvalues of U_p on this space. If $G = \lambda F_n + \beta F_{np}$ is an eigenfunction of U_p then it follows from part (b) of the above lemma that $\lambda \neq 0$. Hence we can assume that $\lambda = 1$. We have

$$U_p(F_n + \beta F_{np}) = (a_p + \beta)F_n - p^{2k-1}F_{np}.$$

It is clear from above that β cannot be zero and that G is an eigenfunction if and only if $a_p + \beta = -p^{2k-1}/\beta$ with eigenvalue $a_p + \beta$. Hence $\beta^2 + a_p\beta + p^{2k-1} = 0$ and we have

$$\beta = \frac{-a_p \pm \sqrt{a_p^2 - 4p^{2k-1}}}{2}.$$

The eigenvalues of U_p on the subspace $\langle F_n, F_{np}\rangle$ are

$$a_p + \beta = \frac{a_p \pm \sqrt{a_p^2 - 4p^{2k-1}}}{2}.$$

Proposition 5.2. *If an eigenvalue λ of $(U_p)^2$ on the two dimensional subspace spanned by F_n and F_{np} is real then $\lambda = \pm p^{2k-1}$.*

Proof. Using the Ramanujan conjecture we can see that the eigenvalues of U_p are real or purely imaginary if and only if $a_p = \pm 2p^{k-1/2}$ or $a_p = 0$. In those cases the eigenvalue of $(U_p)^2$ are precisely $\pm p^{2k-1}$. □

6. The minus space of half-integral weight forms

Let M be odd and square-free. In this section we use the operators and relations that we obtain in Section 4 to define the minus space $S^-_{k+1/2}(\Gamma_0(4M))$ of weight $k + 1/2$ and level $4M$. We show that there is an Hecke algebra isomorphism between $S^-_{k+1/2}(\Gamma_0(4M))$ and $S^{\text{new}}_{2k}(\Gamma_0(2M))$ and we give a common eigenspace characterization of $S^-_{k+1/2}(\Gamma_0(4M))$. It follows that this minus space is identical to the newspace in [9].

For the sake of clarity we start by defining the minus space at level 4 and at level $4p$ for p an odd prime. After that we treat the general case of level $4M$.

6.1. Minus space for $\Gamma_0(4)$.

We recall the following theorem of Niwa which was obtained by proving equality of traces of Hecke operators.

Theorem 3. *(Niwa [10]) Let M be odd and square-free. There exists an isomorphism of vector spaces $\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ satisfying*

$$T_p(\psi(f)) = \psi(T_{p^2}(f)) \quad \textit{for all primes } p \textit{ coprime to } 2M.$$

Moreover if $f \in S_{k+1/2}(\Gamma_0(4))$ then we further have $U_2(\psi(f)) = \psi(U_4(f))$.

We also recall the Shimura lift [14]: For t a positive square-free integer, there is a linear map $\mathrm{Sh}_t : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ given by

$$\mathrm{Sh}_t\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty}\left(\sum_{\substack{d|n\\(d,2M)=1}} \left(\frac{-1}{d}\right)^k \left(\frac{t}{d}\right) d^{k-1} a\left(t\frac{n^2}{d^2}\right)\right) q^n.$$

We note the following observations [11]:

(a) Sh_t need not be injective but if $\mathrm{Sh}_t(f) = 0$ for all square-free t, then $f = 0$.

(b) Sh_t commutes with all Hecke operators, i.e., $T_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(T_{p^2}(f))$ for all primes p coprime to $2M$ and $U_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(U_{p^2}(f))$ for all primes p dividing $2M$.

We denote $S^+_{k+1/2}(\Gamma_0(4))$ simply by $S^+(4)$. We note the following theorem of Kohnen.

Theorem 4. *(Kohnen [5])*

(1) $\dim(S^+(4)) = \dim(S_{2k}(\Gamma_0(1)))$.

(2) *$S^+(4)$ has a basis of eigenforms for all the operators T_{p^2}, p odd.*

(3) If f is such an eigenform then $\psi(f)$ is an old form and $\psi(f) = \lambda F + \beta F_2$ where $F \in S_{2k}(\Gamma_0(1))$ is a primitive eigenform determined by the eigenvalues of f.

Define $A^+_{k+1/2}(\Gamma_0(4)) := \widetilde{W}_4 S^+_{k+1/2}(\Gamma_0(4))$, which we shall simply denote by $A^+(4)$. We know that $S^+(4)$ is the 2-eigenspace of $\widetilde{Q}'_2$, hence $A^+(4)$ is the 2-eigenspace of $\widetilde{Q}_2$. Since $\widetilde{W}_4$ is invertible we can use the above theorem of Kohnen to get that $\dim(A^+(4)) = \dim(S_{2k}(\Gamma_0(1)))$ and

Corollary 6.1.

(1) $A^+(4)$ has a basis of eigenforms under T_{p^2} for all p odd.

(2) ψ maps $A^+(4)$ into the space of old forms in $S_{2k}(\Gamma_0(2))$.

Proof. Let $f \in S^+(4)$ be an eigenform under T_{p^2} for all p odd satisfying $T_{p^2}(f) = \lambda_p f$. Since $\widetilde{W}_4$ commutes with all such T_{p^2}, we get that $g = \widetilde{W}_4 f \in A^+(4)$ is also an eigenform under all T_{p^2} with eigenvalues λ_p. By Theorem 3, $\psi(f)$ and $\psi(g)$ are eigenforms in $S_{2k}(\Gamma_0(2))$ under all T_p with the same set of eigenvalues λ_p. Since $\psi(f)$ is an old form it follows from Atkin-Lehner [1] that $\psi(g)$ is also an old form (belonging to the same two dimensional subspace spanned by F and F_2). $\square$

We note the following key proposition which shows that the sum $S^+(4) + A^+(4)$ is a direct sum. We shall see analogues of this result in Subsection 6.2 and 6.3.

Proposition 6.2. *$S^+(4) \bigcap A^+(4) = \{0\}$.*

Proof. Suppose there is a nonzero $f \in S^+(4) \bigcap A^+(4)$. We can assume that f is an eigenform under T_{p^2} for all p odd (since T_{p^2} stabilizes the intersection $S^+(4) \bigcap A^+(4)$). Since $A^+(4)$ and $S^+(4)$ are respectively the 2-eigenspaces of $\widetilde{Q}_2$ and $\widetilde{Q}'_2$, we have $\widetilde{Q}_2(f) = 2f = \widetilde{Q}'_2(f)$. Using the relations $\widetilde{Q}_2 = 2^{1-k} U_4 \widetilde{W}_4$, $\widetilde{Q}'_2 = 2^{1-k} \widetilde{W}_4 U_4$ and $\widetilde{W}_4^2 = 1$, we get that $U_4^2 = 2^{2k-2} \widetilde{Q}_2 \widetilde{Q}'_2$ and thus

$$(U_4)^2(f) = 2^{2k} f.$$

Applying ψ to the above equation we get that $(U_2)^2(\psi(f)) = 2^{2k}\psi(f)$. Now $\psi(f)$ belongs to the subspace spanned by F and F_2 for some primitive form $F \in S_{2k}(\Gamma_0(1))$ and by Proposition 5.2, the eigenvalues of $(U_2)^2$ on this subspace are either non-real or $\pm 2^{2k-1}$. This is a contradiction. $\square$

Define $S^-_{k+1/2}(\Gamma_0(4))$ to be the orthogonal complement of $S^+(4) \oplus A^+(4)$. Since $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ are Hermitian it follows that $S^-_{k+1/2}(\Gamma_0(4))$ is the common eigenspace with the eigenvalue -1 of the operators $\widetilde{Q}_2$ and $\widetilde{Q}'_2$. We shall write $S^-_{k+1/2}(\Gamma_0(4))$ simply by $S^-(4)$. So we have

$$S_{k+1/2}(\Gamma_0(4)) = S^+(4) \oplus A^+(4) \oplus S^-(4). \tag{7}$$

Theorem 5. *$S^-(4)$ has a basis of eigenforms for all the operators T_{p^2}, p odd; these eigenforms are also eigenfunctions under U_4. If two eigenforms in $S^-(4)$ share the same eigenvalues for all T_{p^2} then they are scalar multiples of each other. ψ induces a Hecke algebra isomorphism:*

$$S^-(4) \cong S_{2k}^{\mathrm{new}}(\Gamma_0(2)).$$

Proof. Since ψ maps $S^+(4) \oplus A^+(4)$ into $S_{2k}^{\mathrm{old}}(\Gamma_0(2))$ and $\dim(S^+(4) \oplus A^+(4)) = 2\dim(S_{2k}(\Gamma_0(1))) = \dim(S_{2k}^{\mathrm{old}}(\Gamma_0(2)))$, we get that ψ maps this direct sum onto $S_{2k}^{\mathrm{old}}(\Gamma_0(2))$.

Now T_{p^2} commutes with $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ for every odd prime p so we get that T_{p^2} stabilizes $S^-(4)$, hence it has a basis of eigenforms for all T_{p^2} with p odd.

If f is such an eigenform then $F := \psi(f)$ is an eigenform in $S_{2k}(\Gamma_0(2))$ under all T_p, p odd. By Atkin-Lehner [1], F is either an old form or a newform. Since ψ is injective, it follows that F must be a newform. So ψ maps the space $S^-(4)$ into the space $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$. By equality of dimensions, we get that ψ is an isomorphism of $S^-(4)$ onto $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$. Consequently by [1] an eigenform in $S^-(4)$ under all T_{p^2} for p odd is uniquely determined up to scalar multiplication.

Further, for such an eigenform f, by [1, Theorem 3], $U_2(F) = -2^{k-1}\lambda(2)F$ where $\lambda(2) = \pm 1$. Thus $\psi(U_4(f)) = U_2(F) \in S_{2k}^{\mathrm{new}}(\Gamma_0(2))$, so $U_4(f)$ belongs to $S^-(4)$. Since U_4 commutes with T_{p^2} for all p odd, we get that $U_4(f)$ is an eigenform under all T_{p^2} with the same eigenvalues as f and hence is a scalar multiple of f. ☐

6.2. **Minus space for $\Gamma_0(4p)$ for p an odd prime.** In this subsection we need the involution $\widetilde{W}_{p^2}$ and the operators U_{p^2}, $\widetilde{Q}_p$ and $\widetilde{Q}'_p = \widetilde{W}_{p^2}\widetilde{Q}_p\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4p))$ that we defined in Section 4.

Consider the subspace $\mathcal{V}(1)$ of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level 1, that is,

$$\mathcal{V}(1) = S_{2k}(\Gamma_0(1)) \oplus V(2)S_{2k}(\Gamma_0(1)) \oplus V(p)S_{2k}(\Gamma_0(1)) \oplus V(2p)S_{2k}(\Gamma_0(1)).$$

We consider the eigenvalues of $(U_p)^2$ on $\mathcal{V}(1)$.

Lemma 6.3. *The operator U_p stabilizes $\mathcal{V}(1)$. If an eigenvalue λ of $(U_p)^2$ on this space is real then $\lambda = \pm p^{2k-1}$.*

Proof. For a primitive Hecke eigenform F in $S_{2k}(\Gamma_0(1))$ consider the four dimensional subspace spanned by F, F_2, F_p, F_{2p}. Then $\mathcal{V}(1)$ is a direct sum of such four dimensional subspaces. By Lemma 5.1, U_p preserves the two dimensional subspace spanned by F and F_p and the two dimensional subspace spanned by F_2 and F_{2p}. It follows by Proposition 5.2, that the eigenvalues of $(U_p)^2$ on these two dimensional subspaces are either non-real or $\pm p^{2k-1}$. ☐

Let $R := S^+_{k+1/2}(\Gamma_0(4)) \oplus A^+_{k+1/2}(\Gamma_0(4))$. Then we have

Proposition 6.4. $R \bigcap \widetilde{W}_{p^2} R = \{0\}$.

Proof. Let $f \neq 0$ belong to the intersection. We can again assume that f is an eigenform under T_{q^2} for all primes q coprime to $2p$. Since by Corollary 4.3(4), $S_{k+1/2}(\Gamma_0(4))$ is contained in the p-eigenspace of $\widetilde{Q}_p$ and so $\widetilde{W}_{p^2} S_{k+1/2}(\Gamma_0(4))$ is contained in the p-eigenspace of $\widetilde{Q}'_p$ we have $\widetilde{Q}_p(f) = pf = \widetilde{Q}'_p(f)$. Using $\widetilde{Q}_p = \left(\frac{-1}{p}\right)^k p^{1-k} U_{p^2} \widetilde{W}_{p^2}$, we get that $(U_{p^2})^2 = p^{2k-2} \widetilde{Q}_p \widetilde{Q}'_p$ and thus

$$(U_{p^2})^2(f) = p^{2k} f.$$

Since $f \neq 0$, there exists a square-free integer t such that the Shimura lift $\mathrm{Sh}_t(f) \neq 0$. Applying this Sh_t to the above equation we get that $(U_p)^2(\mathrm{Sh}_t(f)) = p^{2k}\mathrm{Sh}_t(f)$. Since Sh_t commutes with all the Hecke operators we get that $\mathrm{Sh}_t(f) \in \mathcal{V}(1)$. But by Lemma 6.3, the eigenvalues of $(U_p)^2$ on $\mathcal{V}(1)$ are either non-real or $\pm p^{2k-1}$ leading to a contradiction. $\square$

Corollary 6.5. *Niwa's map ψ maps $R \oplus \widetilde{W}_{p^2} R$ isomorphically onto $\mathcal{V}(1)$.*

Proof. As before (see Corollary 6.1(2)) ψ maps $R \oplus \widetilde{W}_{p^2} R$ into $\mathcal{V}(1)$. It follows from the equality of dimensions that the map is onto. $\square$

Next we consider the following subspace of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level 2,

$$\mathcal{V}(2) = S_{2k}^{\mathrm{new}}(\Gamma_0(2)) \oplus V(p) S_{2k}^{\mathrm{new}}(\Gamma_0(2)).$$

This space is a direct sum of two dimensional subspaces spanned by F and F_p where F is a primitive Hecke eigenform in $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$. Using Proposition 5.2 we have the following lemma.

Lemma 6.6. *If an eigenvalue λ of $(U_p)^2$ on $\mathcal{V}(2)$ is real then $\lambda = \pm p^{2k-1}$.*

Since (by Theorem 5) ψ maps $S_{k+1/2}^{-}(\Gamma_0(4))$ isomorphically onto $S_{2k}^{\mathrm{new}}(\Gamma_0(2))$, it follows that ψ maps $\widetilde{W}_{p^2} S_{k+1/2}^{-}(\Gamma_0(4))$ into the space $\mathcal{V}(2)$. The proof of the following is identical to that of Proposition 6.4.

Proposition 6.7. $S_{k+1/2}^{-}(\Gamma_0(4)) \bigcap \widetilde{W}_{p^2} S_{k+1/2}^{-}(\Gamma_0(4)) = \{0\}$.

Corollary 6.8. *ψ maps $S_{k+1/2}^{-}(\Gamma_0(4)) \oplus \widetilde{W}_{p^2} S_{k+1/2}^{-}(\Gamma_0(4))$ isomorphically onto $\mathcal{V}(2)$.*

Finally, we consider the following subspace of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level p,

$$\mathcal{V}(p) = S_{2k}^{\mathrm{new}}(\Gamma_0(p)) \oplus V(2) S_{2k}^{\mathrm{new}}(\Gamma_0(p)).$$

This space is a direct sum of two dimensional subspaces spanned by F and F_2 where F is a primitive Hecke eigenform in $S_{2k}^{\mathrm{new}}(\Gamma_0(p))$. We have

Lemma 6.9. *If an eigenvalue λ of $(U_2)^2$ on $\mathcal{V}(p)$ is real then $\lambda = \pm 2^{2k-1}$.*

Let $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ be the new space inside the plus space in $S_{k+1/2}(\Gamma_0(4p))$. Kohnen [6, Theorem 2] proved that ψ maps $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ into $\mathcal{V}(p)$ and the dimension of $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ equals the dimension of $S^{\mathrm{new}}_{2k}(\Gamma_0(p))$. Then as before ψ maps $\widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4))$ into $\mathcal{V}(p)$ and we have the following proposition and corollary.

Proposition 6.10. $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)) \bigcap \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)) = \{0\}$.

Corollary 6.11. *ψ maps $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ isomorphically onto $\mathcal{V}(p)$.*

We define the following subspace of $S_{k+1/2}(\Gamma_0(4p))$,

$$\begin{aligned} E := R \oplus \widetilde{W}_{p^2} R \,\oplus\, & S^{-}_{k+1/2}(\Gamma_0(4)) \oplus \widetilde{W}_{p^2} S^{-}_{k+1/2}(\Gamma_0(4)) \\ & \oplus\, S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)). \end{aligned}$$

By Corollary 6.5, 6.8 and 6.11, we get that ψ maps the space E isomorphically onto the old space $S^{\mathrm{old}}_{2k}(\Gamma_0(2p))$. We define the minus space to be the orthogonal complement of E under the Petersson inner product. That is,

$$S^{-}_{k+1/2}(\Gamma_0(4p)) := E^{\perp}.$$

Theorem 6. *$S^{-}_{k+1/2}(\Gamma_0(4p))$ has a basis of eigenforms for all the operators T_{q^2} where q is a prime coprime to $2p$, uniquely determined up to scalar multiplication. ψ maps the space $S^{-}_{k+1/2}(\Gamma_0(4p))$ isomorphically onto the space $S^{\mathrm{new}}_{2k}(\Gamma_0(2p))$.*

Proof. Since the operators T_{q^2} with $(q,2p)=1$ stabilize the space E and since they are self-adjoint with respect to the Petersson inner product, it follows that they stabilize the space $S^{-}_{k+1/2}(\Gamma_0(4p))$, hence $S^{-}_{k+1/2}(\Gamma_0(4p))$ has a basis of eigenforms for all such operators T_{q^2}. If f is such an eigenform then $\psi(f) \in S_{2k}(\Gamma_0(2p))$ is also an eigenform for all the operators T_q, $(q,2p)=1$ and thus (by [1]) $\psi(f)$ is either an old form or a newform. Since ψ is injective and maps E onto $S^{\mathrm{old}}_{2k}(\Gamma_0(2p))$, it follows that $\psi(f)$ is a newform. Thus ψ maps the space $S^{-}_{k+1/2}(\Gamma_0(4p))$ into the space $S^{\mathrm{new}}_{2k}(\Gamma_0(2p))$. By equality of dimensions, we get that ψ maps the space $S^{-}_{k+1/2}(\Gamma_0(4p))$ isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(2p))$. Consequently an eigenform in $S^{-}_{k+1/2}(\Gamma_0(4p))$ is uniquely determined up to multiplication by a scalar. $\square$

Corollary 6.12. *Let $f \in S^{-}_{k+1/2}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators T_{q^2}, q prime and $(q,2p)=1$. Then $\widetilde{W}_{p^2} f = \beta(p) f$, $\widetilde{W}_4 f = \beta(2) f$ where $\beta(p) = \pm 1$, $\beta(2) = \pm 1$.*

Proof. Let $g = \widetilde{W}_{p^2} f$. Since $\widetilde{W}_{p^2}$ commutes with all the operators T_{q^2} for $(q,2p)=1$ we get that g is an eigenform for all the operators T_{q^2} with the same eigenvalues as f. Since $\psi(f)$ is a newform, it follows by [1] that $\psi(g)$

is a scalar multiple of $\psi(f)$. Since ψ is an isomorphism we get that g is a scalar multiple of f. Since $\widetilde{W}_{p^2}$ is an involution we get that the scalar is ± 1. The same proof applies to $\widetilde{W}_4$. $\square$

Let $f \in S^-_{k+1/2}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators T_{q^2} as above. It follows that $F := \psi(f)$ is a Hecke eigenform in $S^{\mathrm{new}}_{2k}(\Gamma_0(2p))$ for all the operators T_q, $(q, 2p) = 1$. Since the Shimura lift $\mathrm{Sh}_t(f)$ is also an eigenform for all the operators T_q with the same eigenvalues as F, it follows from [1] that $\mathrm{Sh}_t(f)$ is a scalar multiple of F (which could be zero). Also, $U_p(F) = -p^{k-1}\lambda(p)F$ where $\lambda(p) = \pm 1$ and $U_2(F) = -2^{k-1}\lambda(2)F$ where $\lambda(2) = \pm 1$.

Proposition 6.13. *Let $f \in S^-_{k+1/2}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators T_{q^2}, q prime and $(q, 2p) = 1$. Then*

$$U_{p^2}(f) = -p^{k-1}\lambda(p)f, \quad U_4(f) = -2^{k-1}\lambda(2)f$$

where $\lambda(p) = \pm 1$ and $\lambda(2) = \pm 1$ are defined as above.

Proof. Let $g = U_{p^2}f$. Then $\mathrm{Sh}_t(g) = U_p\mathrm{Sh}_t(f) = -p^{k-1}\lambda(p)\mathrm{Sh}_t(f)$ for every positive square-free integer t. It follows that $\mathrm{Sh}_t(g - p^{k-1}\lambda(p)f) = 0$ for all such t implying $g - p^{k-1}\lambda(p)f = 0$ which is what we need. For the prime 2, the proof is the same. $\square$

Proposition 6.14. *Let $f \in S^-_{k+1/2}(\Gamma_0(4p))$. Then $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

Proof. Let $f \in S^-_{k+1/2}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators T_{q^2}, $(q, 2p) = 1$. Since $\widetilde{Q}_p = \left(\frac{-1}{p}\right)^k p^{1-k}U_{p^2}\widetilde{W}_{p^2}$ and $\widetilde{Q}_2 = 2^{1-k}U_4\widetilde{W}_4$ it follows from Corollary 6.12 and Proposition 6.13 that f is an eigenform for the operators $\widetilde{Q}_p$, $\widetilde{Q}'_p$, $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ with eigenvalues ± 1. However, the eigenvalues of $\widetilde{Q}_p$, $\widetilde{Q}'_p$ are p and -1 and the eigenvalues of $\widetilde{Q}_2$ and $\widetilde{Q}'_2$ are 2 and -1 hence the eigenvalues have to be -1. Since $S^-_{k+1/2}(\Gamma_0(4p))$ has a basis of such eigenforms we get the result. $\square$

Theorem 7. *Let $f \in S_{k+1/2}(\Gamma_0(4p))$. Then $f \in S^-_{k+1/2}(\Gamma_0(4p))$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

Proof. If $f \in S^-_{k+1/2}(\Gamma_0(4p))$ then by Proposition 6.14 the conditions hold. Now assume that $f \in S_{k+1/2}(\Gamma_0(4p))$ is in the intersection of -1-eigenspaces of $\widetilde{Q}_p$, $\widetilde{Q}'_p$, $\widetilde{Q}_2$ and $\widetilde{Q}'_2$. For every $g \in S_{k+1/2}(\Gamma_0(4))$ we have $\widetilde{Q}_p(g) = pg$. Since $\widetilde{Q}_p$ is self-adjoint,

$$-\langle f, g\rangle = \langle \widetilde{Q}_p f, g\rangle = \langle f, \widetilde{Q}_p g\rangle = p\langle f, g\rangle$$

implying $\langle f, g\rangle = 0$. Thus f is orthogonal to $R \oplus S^-_{k+1/2}(\Gamma_0(4))$. For every $g \in \widetilde{W}_{p^2}S_{k+1/2}(\Gamma_0(4))$ we have $\widetilde{Q}'_p(g) = pg$ and the same argument shows that $\langle f, g\rangle = 0$ implying f is orthogonal to $\widetilde{W}_{p^2}(R \oplus S^{-}_{k+1/2}(\Gamma_0(4)))$. Since Kohnen's plus space is the 2-eigenspace of $\widetilde{Q}'_2$, for $g \in S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ we have $\widetilde{Q}'_2(g) = 2g$, consequently for $g \in \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$ we have $\widetilde{Q}_2(g) = 2g$. Hence $\langle f, g\rangle = 0$ for such g, that is, f is orthogonal to $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p)) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4p))$. It follows that $f \in S^{-}_{k+1/2}(\Gamma_0(4p))$. □

6.3. **Minus space for $\Gamma_0(4M)$ for M odd and square-free.** Let $M \neq 1$ be an odd and square-free natural number. Write $M = p_1 p_2 \cdots p_k$. For each $i = 1, \dots k$ let $M_i = M/p_i$. Since $S_{k+1/2}(\Gamma_0(4M_i))$ is contained in the p_i-eigenspace of $\widetilde{Q}_{p_i}$ (Corollary 4.3(4)), following the proof of Proposition 6.4 we obtain that

Proposition 6.15. $S_{k+1/2}(\Gamma_0(4M_i)) \bigcap \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i)) = \{0\}$.

Corollary 6.16. *The Niwa map* $\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ *maps* $S_{k+1/2}(\Gamma_0(4M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i))$ *isomorphically onto* $S_{2k}(\Gamma_0(2M_i)) \oplus V(p_i) S_{2k}(\Gamma_0(2M_i))$.

Let $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M))$ be the new space inside the Kohnen plus subspace of $S_{k+1/2}(\Gamma_0(4M))$. Then similarly we have

Proposition 6.17. $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M)) \bigcap \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M)) = \{0\}$.

Corollary 6.18. ψ *maps* $S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M)) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M))$ *isomorphically onto* $S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \oplus V(2) S^{\mathrm{new}}_{2k}(\Gamma_0(M))$.

We let $B_i = S_{k+1/2}(\Gamma_0(4M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i))$, $i = 1, \dots k$. Define

$$E := \sum_{i=1}^{k} B_i \oplus S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M)) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(\Gamma_0(4M)).$$

Proposition 6.19. *Under* ψ *the space* E *maps isomorphically onto the old space* $S^{\mathrm{old}}_{2k}(\Gamma_0(2M))$.

Proof. This follows from Corollary 6.16 and 6.18 and from the decomposition

$$S^{\mathrm{old}}_{2k}(\Gamma_0(2M)) = \left(\sum_{i=1}^{k} S_{2k}(\Gamma_0(2M_i)) \oplus V(p_i) S_{2k}(\Gamma_0(2M_i)) \right) \oplus \left(S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \oplus V(2) S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \right).$$

□

We now define the minus space to be the orthogonal complement of E, under the Petersson inner product, that is,

$$S^{-}_{k+1/2}(\Gamma_0(4M)) := E^{\perp}.$$

Let $f \in S^-_{k+1/2}(\Gamma_0(4M))$ be a Hecke eigenform for all the operators T_{q^2} where q is an odd prime satisfying $(q, M) = 1$. Let $\psi(f) = F$. The proofs of the following results are identical to the proofs in the previous subsections.

Proposition 6.20. *F is up to a scalar a primitive Hecke eigenform in $S^{new}_{2k}(\Gamma_0(2M))$.*

Theorem 8. *The space $S^-_{k+1/2}(\Gamma_0(4M))$ has a basis of eigenforms for all the operators T_{q^2} where q is an odd prime satisfying $(q, M) = 1$. Under ψ, the space $S^-_{k+1/2}(\Gamma_0(4M))$ maps isomorphically onto the space $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$. If two forms in $S^-_{k+1/2}(\Gamma_0(4M))$ have the same eigenvalues for all the operators T_{q^2}, $(q, 2M) = 1$, then they are same up to a scalar factor.*

In particular the minus space $S^-_{k+1/2}(\Gamma_0(4M))$ has strong multiplicity one property in the full space, that is, if f_1 and f_2 are Hecke eigenforms in $S_{k+1/2}(\Gamma_0(4M))$ with the same eigenvalues for all T_{q^2}, $(q, 2M) = 1$ and if f_1 is a nonzero element of the minus space $S^-_{k+1/2}(\Gamma_0(4M))$ then f_2 is a scalar multiple of f_1.

Remark 6. *Our results in Theorem 5, 7 and 8 give an another proof of Theorem 5 of* [9]. *We note that in* [9] *the old space is defined using the operators U_{p^2} for $p \mid 2M$ while our definition uses Atkin-Lehner type operators $\widetilde{W}_{p^2}$. The operators U_{p^2}, $\widetilde{W}_{p^2}$ and $\widetilde{Q}_p$ come from the local Hecke algebra element corresponding to the double cosets of $(h(p), 1)$, $(w(p^{-1}), 1)$ and $(w(1), 1)$ respectively and our proofs essentially depend on relations among these operators that we derive from the local Hecke algebra. Since $S^+(4)$ is the 2-eigenspace of $\widetilde{Q}'_2$ we indeed have $S^+(4) = \widetilde{Q}'_2 S^+(4) = \widetilde{W}_4 U_4 S^+(4)$ which implies equality of spaces, $U_4 S^+(4) = \widetilde{W}_4 S^+(4) = A^+(4)$. Thus $U_4 \widetilde{W}_4 A^+(4) = A^+(4)$. However $U_4 A^+(4)$ need not equal $S^+(4)$ as noted in Example 2 in the next subsection. In the case of odd primes p_i dividing M the space $S_{k+1/2}(\Gamma_0(4M_i))$ is contained in the p_i-eigenspace of $\widetilde{Q}_{p_i}$, which in particular implies that $U_{p_i^2} \widetilde{W}_{p_i^2} S^-_{k+1/2}(\Gamma_0(4M_i)) = S^-_{k+1/2}(\Gamma_0(4M_i))$, but as before we do not expect the spaces $U_{p_i^2} S^-_{k+1/2}(\Gamma_0(4M_i))$ and $\widetilde{W}_{p_i^2} S^-_{k+1/2}(\Gamma_0(4M_i))$ to be equal inside $S_{k+1/2}(\Gamma_0(4M))$. We illustrate this by the following reasoning which needs to be proved. Consider the simple case $M = 4p$, p an odd prime. In this case if $U_{p^2} S^-(4) = \widetilde{W}_{p^2} S^-(4)$ then the corresponding picture in the integral weight should be $U_p S^{\mathrm{new}}_{2k}(\Gamma_0(2)) = W_p S^{\mathrm{new}}_{2k}(\Gamma_0(2)) = V_p S^{\mathrm{new}}_{2k}(\Gamma_0(2))$ (where the last equality was shown in* [2]*). If $S^{\mathrm{new}}_{2k}(\Gamma_0(2))$ is non-zero then the action of U_p (see Lemma 5.1) and the fact that $S^{\mathrm{new}}_{2k}(\Gamma_0(2)) \cap V_p S^{\mathrm{new}}_{2k}(\Gamma_0(2)) = \{0\}$ leads to a contradiction. Since representation theoretically $A^+(4)$ corresponds to $S_{2k}(\Gamma_0(1))$ using the same reasoning we do not expect the spaces $U_4 A^+(4)$ and $S^+(4)$ to be equal.*

Let $f \in S^-_{k+1/2}(\Gamma_0(4M))$ be a Hecke eigenform for all the operators T_{q^2}, $(q, 2M) = 1$. Then $\psi(f) = F$ is a Hecke eigenform in $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$ for all operators T_q, $(q, 2M) = 1$. By [1], for all primes p such that $p|M$, $U_p(F) = -p^{k-1}\lambda(p)F$ where $\lambda(p) = \pm 1$ and $U_2(F) = -2^{k-1}\lambda(2)F$ where $\lambda(2) = \pm 1$.

Proposition 6.21. *Let $f \in S^-_{k+1/2}(\Gamma_0(4M))$ be a Hecke eigenform for all the operators T_{q^2}, q prime, $(q, 2M) = 1$. Then for all primes p such that $p|M$*

$$U_{p^2}(f) = -p^{k-1}\lambda(p)f \quad \text{and} \quad U_4(f) = -2^{k-1}\lambda(2)f$$

where $\lambda(p) = \pm 1$ and $\lambda(2) = \pm 1$ are defined as above.

Following [14, Theorem 1.9] we have

Corollary 6.22. *Let $f = \sum_{n=0}^{\infty} a_n q^n \in S^-_{k+1/2}(\Gamma_0(4M))$ be a Hecke eigenform for all Hecke operators, i.e, $T_{q^2}(f) = \omega_q f$ for all primes $(q, 2M) = 1$ and $U_{p^2}(f) = \omega_p f$ for all primes $p \mid 2M$. Let $F = \sum_{n=0}^{\infty} A_n q^n \in S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$ be the unique normalized primitive form determined by f, i.e. $A_p = \omega_p$ for all primes p. Then for a fundamental discriminant D such that $(-1)^k D > 0$,*

$$\mathrm{L}\left(s - k + 1, \left(\frac{D}{\cdot}\right)\right) \sum_{n=1}^{\infty} a_{|D|n^2} \; n^{-s} = a(|D|) \sum_{n=1}^{\infty} A_n n^{-s}.$$

We finally give the characterization of our minus space. The proofs of the following proposition and theorem are as before.

Proposition 6.23. *Let $f \in S^-_{k+1/2}(\Gamma_0(4M))$. Then for every prime p dividing M we have $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

Theorem 9. *Let $f \in S_{k+1/2}(\Gamma_0(4M))$. Then $f \in S^-_{k+1/2}(\Gamma_0(4M))$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ for every prime p dividing M and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

6.4. **Some examples.** We complete this section by giving two examples. For simplicity we shall denote plus and minus spaces $S^+_{k+1/2}(\Gamma_0(4M))$ and $S^-_{k+1/2}(\Gamma_0(4M))$ by $S^+_{k+1/2}(4M)$ and $S^-_{k+1/2}(4M)$.

We shall use Shimura decomposition [15] and we recall the following notation: for a primitive Hecke eigenform F of weight $2k$ and level dividing $2M$, $S_{k+1/2}(4M, F)$ denotes the subspace of $S_{k+1/2}(\Gamma_0(4M))$ consisting of forms that are Shimura-equivalent to F (i.e., forms f that are eigenforms under T_{p^2} with the same eigenvalues as F under T_p for almost all odd primes p coprime to M).

Example 1. The space $S_{3/2}(\Gamma_0(28))$ is one dimensional and is spanned by

$$f = q - q^2 - q^4 + q^7 + q^8 - q^9 + q^{14} - 2q^{15} + q^{16} + 3q^{18} - 2q^{21} + \dots$$

Then by Shimura decomposition,

$$S_{3/2}(\Gamma_0(28)) = \bigoplus_{\substack{F\in S_2^{\mathrm{new}}(\Gamma_0(M))\\ \text{prim.},\ M|14}} S_{3/2}(28,F) = S_{3/2}(28,F_{14})$$

as there are no primitive Hecke eigenforms of weight 2 at level 1, 2, 7 and $F_{14} \in S_2^{\mathrm{new}}(\Gamma_0(14))$ is the only primitive Hecke eigenform at level 14. In particular, we have $S^+_{3/2}(28) = \{0\}$ and $S^-_{3/2}(28) = S_{3/2}(\Gamma_0(28)) = \langle f\rangle$.

Example 2. The space $S_{17/2}(\Gamma_0(12))$ is 13-dimensional. We first give the Shimura decomposition of $S_{17/2}(\Gamma_0(12))$. We note that there are seven primitive Hecke eigenforms of weight 16 and level dividing 6, namely, F_1 of level 1, G_2 of level 2, H_3, K_3 of level 3 each and L_6, M_6, N_6 each of level 6. Using Shimura decomposition algorithm in [12] we have

$$\begin{aligned}S_{17/2}(\Gamma_0(12)) = S_{17/2}(12,F_1) \oplus S_{17/2}(12,G_2) \oplus S_{17/2}(12,H_3) \oplus S_{17/2}(12,K_3)\\ \oplus S_{17/2}(12,L_6) \oplus S_{17/2}(12,M_6) \oplus S_{17/2}(12,N_6),\end{aligned} \tag{8}$$

where $S_{17/2}(12,F_1)$ is the 4-dimensional space spanned by

$$\begin{aligned}
f_1 &= q + 88q^4 + 513q^9 + 3024q^{12} - 4368q^{13} - 13760q^{16} + 33264q^{21} + \cdots\\
f_2 &= 11q^2 + 64q^4 + 232q^7 - 1408q^8 + 4608q^9 + 190q^{10} - 6578q^{11} + \cdots\\
f_3 &= 9q^3 - 64q^4 + 189q^6 - 232q^7 - 190q^{10} + 1152q^{12} - 3328q^{13} + \cdots\\
f_4 &= q^5 - 11q^8 + 18q^9 - 9q^{12} - 116q^{17} + 344q^{20} - 99q6^{21} - 189q^{24} + \cdots ;
\end{aligned}$$

the space $S_{17/2}(12,G_2)$ is 2-dimensional and is spanned by

$$\begin{aligned}
g_1 &= q + 21q^3 - 128q^4 - 609q^6 + 3192q^7 + 5313q^9 - 12810q^{10} + \cdots\\
g_2 &= 3q^2 + 7q^3 - 203q^6 - 384q^8 - 416q^9 + 2706q^{11} - 896q^{12} + \cdots ;
\end{aligned}$$

the space $S_{17/2}(12,H_3)$ is 2-dimensional and is spanned by

$$\begin{aligned}
h_1 &= q^5 + 7q^8 - 27q^{12} - 80q^{17} + 56q^{20} + 189q^{21} + 81q^{24} + 231q^{29} + \cdots\\
h_2 &= 7q^2 - 27q^3 + 81q^6 - 896q^8 + 854q^{11} + 3456q^{12} - 1876q^{14} + \cdots ;
\end{aligned}$$

the space $S_{17/2}(12,K_3)$ is 2-dimensional and is spanned by

$$\begin{aligned}
k_1 &= q - 362q^4 - 2187q^9 - 11826q^{12} + 19032q^{13} + 51940q^{16} + \cdots\\
k_2 &= 1971q^3 + 13184q^4 + 31266q^6 - 20158q^7 + 271340q^{10} + \cdots ;
\end{aligned}$$

the last three summands in (8) are 1-dimensional each with $S_{17/2}(12,L_6)$ spanned by

$$l_1 = 13q^2 + 129q^3 + 736q^5 + 1323q^6 + 1664q^8 + 5918q^{11} + 16512q^{12} + \cdots ;$$

the space $S_{17/2}(12,M_6)$ spanned by

$$m_1 = q^3 - 18q^6 - 42q^7 - 12q^{10} + 128q^{12} + 384q^{13} - 126q^{15} - 1074q^{19} + 896q^{21} + \cdots ;$$

and the space $S_{17/2}(12, N_6)$ spanned by

$n_1 = 16q - 1539q^3 - 2048q^4 - 5994q^6 - 50178q^7 - 34992q^9 - 2460q^{10} + \cdots.$

We can also check (using bound in [7]) that the Kohnen's plus space $S^+_{17/2}(12)$ is 4-dimensional. Indeed

$$S^+_{17/2}(12) = \langle f_1,\ f_4,\ h_1,\ k_1\rangle = S^+_{17/2}(4) \oplus \widetilde{W}_9 S^+_{17/2}(4) \oplus S^{+,\mathrm{new}}_{17/2}(12)$$

with $S^+_{17/2}(4) = \langle f_1 - 336f_4\rangle$ and $S^{+,\mathrm{new}}_{17/2}(12) = \langle h_1,\ k_1\rangle$. Note that from Remark 6, $A^+_{17/2}(4) = U_4(S^+_{17/2}(4))$, so $A^+_{17/2}(4) = \langle U_4(f_1 - 336f_4)\rangle = \langle 88f_1 + 336f_2 + 672f_3 - 115584f_4\rangle$ and $S^-_{17/2}(4) = \langle g_1 + 3g_2\rangle$ (again we use Shimura decomposition algorithm to get the explicit forms in $S^+_{17/2}(4)$ and $S^-_{17/2}(4)$). One can further check that $U_4(A^+_{17/2}(4))$ does not equal $S^+_{17/2}(4)$, indeed $A^+_{17/2}(4)$ is spanned by a form with q-expansion given by

$88q + 3696q^2 + 6048q^3 - 13760q^4 - 115584q^5 + 127008q^6 - 77952q^7 + 798336q^8 + \cdots$

and so

$$U_4(A^+_{17/2}(4)) = \langle -13760q + 798336q^2 + 1306368q^3 - 5855744q^4 + \cdots\rangle$$

which is clearly not equal to $S^+_{17/2}(4)$.

Thus we have

$$S_{17/2}(12, F_1) = R \oplus \widetilde{W}_9 R \text{ where } R = S^+_{17/2}(4) \oplus A^+_{17/2}(4),$$

$$S_{17/2}(12, G_2) = S^-_{17/2}(4) \oplus \widetilde{W}_9 S^-_{17/2}(4),$$

$$S_{17/2}(12, H_3) \oplus S_{17/2}(12, K_3) = S^{+,\mathrm{new}}_{17/2}(12) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{17/2}(12)$$

and

$$S_{17/2}(12, L_6) \oplus S_{17/2}(12, M_6) \oplus S_{17/2}(12, N_6) = \langle l_1,\ m_1,\ n_1\rangle = S^-_{17/2}(12).$$

Remark 7. (*i*) *In general,* $S^-_{k+1/2}(\Gamma_0(4M)) = \bigoplus_F S_{k+1/2}(4M, F)$ *where* F *runs through all primitive Hecke eigenforms of weight* $2k$ *and level* $2M$*.*
(*ii*) *The Kohnen plus space is given by a well-known Fourier coefficient condition. But we do not expect any such Fourier coefficient condition for forms in our minus space as is also evident from the above examples. We note that in* [17]*, Ueda and Yamana define generalized Kohnen plus space of level* $8M$ *and show that the newspace inside this plus space is Hecke isomorphic to* $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$*. In* [3]*, we obtain a self-adjoint involution on* $S_{k+1/2}(\Gamma_0(8M))$ *coming from an element in a certain* 2*-adic Hecke algebra of* $\widetilde{\mathrm{SL}}_2$ *of level* 8 *that is not inside the corresponding* 2*-adic Hecke algebra of* $\widetilde{\mathrm{SL}}_2$ *of level* 4*. We observe that the plus space defined by Ueda-Yamana is precisely the* $+1$*-eigenspace of this involution and that their plus newspace is a "conjugate" of* $S^-_{k+1/2}(\Gamma_0(4M))$*. We define the minus space at level* $8M$ *and show that this space is contained inside the* -1*-eigenspace of the involution and hence satisfy a Fourier coefficient condition that is exactly opposite to the Kohnen's plus*

space Fourier coefficient condition. Since this involution on $S_{k+1/2}(\Gamma_0(8M))$ *does not preserve the space* $S_{k+1/2}(\Gamma_0(4M))$*, we do not expect Fourier coefficient condition for* $S^-_{k+1/2}(\Gamma_0(4M))$*. For more details, please refer to* [3].

Appendix A. Some observations on cocycle multiplication

Let p denote any prime. In this appendix we note down some useful observations on the multiplication in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ by cocycle σ_p.

Recall the Hilbert symbol $(\cdot,\cdot)_p$ defined on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. For an odd prime p it can be given by the following formula: For a, b coprime to p,

$$\left(p^s a, p^t b\right)_p = \left(\frac{-1}{p}\right)^{st}\left(\frac{a}{p}\right)^{t}\left(\frac{b}{p}\right)^{s}.$$

Thus $(p,p)_p = \left(\frac{-1}{p}\right)$ and $(-p,u)_p = (p,u)_p = \left(\frac{u}{p}\right)$ where u is a unit in $\mathbb{Z}_p$. For the prime 2, if a, b are odd, then

$$\left(2^s a, 2^t b\right)_2 = (-1)^{\frac{(a-1)(b-1)}{4}}\left(\frac{2}{|a|}\right)^{t}\left(\frac{2}{|b|}\right)^{s}.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$. For $(A,\epsilon_1) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$, $(A,\ \epsilon_1)^{-1} = (A^{-1},\ \epsilon_1\sigma_p(A,A^{-1}))$ where

(i) If $c=0$ then $\sigma_p(A,A^{-1}) = (a,a)_p = (d,d)_p$.
(ii) If $c \neq 0$ and $\mathrm{ord}_p(c)$ is even then $\sigma_p(A,A^{-1}) = 1$.
(iii) If $c \neq 0$ and $\mathrm{ord}_p(c)$ is odd then

$$\sigma_p(A,A^{-1}) = \begin{cases} (c,d)_p\,(-c,a)_p & \text{if } d\neq 0,\ a\neq 0 \\ (c,d)_p & \text{if } d\neq 0,\ a=0 \\ (-c,a)_p & \text{if } d=0,\ a\neq 0 \\ 1 & \text{if } d=0,\ a=0. \end{cases}$$

In particular if $A \in \{x(p^n),\ y(p^n),\ w(p^n)\}_{n\in\mathbb{Z}}$, then $\sigma_p(A,A^{-1}) = 1$. For $A = h(p^n)$ with $n \in \mathbb{Z}$, if $p=2$ then $\sigma_p(A,A^{-1}) = 1$, however if p is an odd prime then

$$\sigma_p(A,A^{-1}) = \begin{cases} 1 & \text{if } n \text{ even,} \\ \left(\frac{-1}{p}\right) & \text{else.} \end{cases}$$

Let (A,ϵ_1), $(B,\ \epsilon_2) \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. The following lemmas can be easily obtained using the cocycle formula.

Lemma A.1. *We have* $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}] = (B^{-1}A^{-1}BA, \xi)$ *where* $\xi = \sigma_p(A,A^{-1})\sigma_p(B,B^{-1})\sigma_p(B,A)\sigma_p(A^{-1},BA)\sigma_p(B^{-1},A^{-1}BA)$.

Lemma A.2. *The σ_p-factor (ξ factor above) of $[(B,\ \epsilon_2)^{-1},\ (A,\ \epsilon_1)^{-1}]$ equals the product*

$$\Big(\tau(B),\ \tau(B^{-1})\Big)_p \cdot \Big(\tau(A),\ \tau(A^{-1})\Big)_p \cdot \Big(\tau(BA)\tau(B),\ \tau(BA)\tau(A)\Big)_p$$
$$\cdot \Big(\tau(A^{-1}BA)\tau(A^{-1}),\ \tau(A^{-1}BA)\tau(BA)\Big)_p$$
$$\cdot \Big(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}),\ \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\Big)_p \cdot s_p(B^{-1}A^{-1}BA).$$

In the proofs for checking the support of our local Hecke algebra (section 3) we need the following lemma.

Lemma A.3. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$. Then*

(a) If $B = x(s)$ where $s \neq 0$, then σ_p-factor is

$$\begin{cases} \big(-sc^2,\ 1-cds\big)_p & \textit{if } sc^2(1-cds) \neq 0 \textit{ and } \mathrm{ord}_p(s) \textit{ is odd,} \\ 1 & \textit{else.} \end{cases}$$

(b) If $B = h(u)$ where $u \neq \pm 1$, then σ_p-factor is

$$\begin{cases} \big(ac(1-u^2),\ 1+(1-u^2)bc\big)_p & \textit{if } ac(1-u^2)(1+(1-u^2)bc) \neq 0 \\ & \textit{and } \mathrm{ord}_p(ac(1-u^2)) \textit{ is odd,} \\ 1 & \textit{else.} \end{cases}$$

(c) If $B = y(t)$ where $t \neq 0$, then σ_p-factor is

$$\begin{cases} \big((a^2-1)t+abt^2, 1+abt+b^2t^2\big)_p & \textit{if } ((a^2-1)t+abt^2)(1+abt+b^2t^2) \neq 0 \\ & \textit{and } \mathrm{ord}_p((a^2-1)t+abt^2) \textit{ is odd,} \\ 1 & \textit{else.} \end{cases}$$

In each of the above cases the σ_p-factor is simply $s_p(B^{-1}A^{-1}BA)$.

Proof. For (a) let $B = x(s)$ where $s \neq 0$. Then we have

$$BA = \begin{pmatrix} a+sc & b+sd \\ c & d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} 1+cds & sd^2 \\ -sc^2 & 1-cds \end{pmatrix},$$

$$B^{-1}A^{-1}BA = \begin{pmatrix} 1+cds+s^2c^2 & sd^2-s+cds^2 \\ -sc^2 & 1-cds \end{pmatrix}.$$

It is easy to see that $\Big(\tau(B),\ \tau(B^{-1})\Big)_p = 1$ and that

$$\Big(\tau(A),\ \tau(A^{-1})\Big)_p = \Big(\tau(A^{-1}BA)\tau(A^{-1}),\ \tau(A^{-1}BA)\tau(BA)\Big)_p$$
$$= \begin{cases} 1 & \text{if } c \neq 0 \\ (d,a)_p & \text{else.} \end{cases}$$

Further, one can check that $\Big(\tau(BA)\tau(B), \tau(BA)\tau(A)\Big)_p = 1$ and also

$$\Big(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}),\ \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\Big)_p = 1.$$

Finally we have $s_p(B^{-1}A^{-1}BA)$

$$= \begin{cases} \big(-sc^2,\ 1-cds\big)_p & \text{if } sc^2(1-cds)\neq 0 \text{ and } ord_p(s) \text{ is odd,} \\ 1 & \text{else.} \end{cases}$$

By using Lemma A.2, multiplying all the above terms we get the required σ_p-factor.

For (b) we proceed similarly. Let $B = h(u)$ where $u \neq \pm 1$. Then

$$BA = \begin{pmatrix} ua & ub \\ u^{-1}c & u^{-1}d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} uad - u^{-1}bc & bd(u-u^{-1}) \\ ac(u^{-1}-u) & u^{-1}ad - ubc \end{pmatrix},$$

$$B^{-1}A^{-1}BA = \begin{pmatrix} 1+(1-u^{-2})bc & bd(1-u^{-2}) \\ ac(1-u^2) & 1+(1-u^2)bc \end{pmatrix}.$$

We have $\Big(\tau(B),\ \tau(B^{-1})\Big)_p = \big(u, u^{-1}\big)_p$. Also, $\Big(\tau(A),\ \tau(A^{-1})\Big)_p = 1$ if $c \neq 0$ and $(d,a)_p$ else. We check that

$$\Big(\tau(BA)\tau(B),\ \tau(BA)\tau(A)\Big)_p = \begin{cases} \big(c, u^{-1}\big)_p & \text{if } c\neq 0 \\ \big(d, u^{-1}\big)_p & \text{else,} \end{cases}$$

$$\Big(\tau(A^{-1}BA)\tau(A^{-1}),\ \tau(A^{-1}BA)\tau(BA)\Big)_p$$

$$= \begin{cases} \big(-a(u^{-1}-u), u^{-1}\big)_p & \text{if } ac \neq 0 \\ (bu, -b)_p & \text{if } a = 0 \text{ and } c \neq 0 \\ \big(du^{-1}, a\big)_p & \text{if } a \neq 0 \text{ and } c = 0, \end{cases}$$

$$\Big(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}),\ \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\Big)_p$$

$$= \begin{cases} \big(ac(u^{-1}-u), u^{-1}\big)_p & \text{if } ac \neq 0 \\ (bc, u)_p = (-1, u)_p & \text{if } a = 0 \text{ and } c \neq 0 \\ (-ad, u)_p & \text{if } a \neq 0 \text{ and } c = 0, \end{cases}$$

and $s_p(B^{-1}A^{-1}BA) =$

$$\begin{cases} \big(ac(1-u^2),\ 1+(1-u^2)bc\big)_p & \text{if } ac(1-u^2)(1+(1-u^2)bc) \neq 0 \\ & \text{and } \mathrm{ord}_p(ac(1-u^2)) \text{ is odd,} \\ 1 & \text{else.} \end{cases}$$

Again by multiplying all the above terms we get the required σ_p-factor.

For (c), let $B = y(t)$ where $t \neq 0$. Then

$$BA = \begin{pmatrix} a & b \\ at+c & bt+d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} 1-abt & -b^2t \\ a^2t & 1+abt \end{pmatrix},$$

$$B^{-1}A^{-1}BA = \begin{pmatrix} 1-abt & -b^2t \\ (a^2-1)t+abt^2 & 1+abt+b^2t^2 \end{pmatrix}.$$

As before, $\Big(\tau(B),\ \tau(B^{-1})\Big)_p = (t,-t)_p = 1$, and $\Big(\tau(A),\ \tau(A^{-1})\Big)_p = 1$ if $c \neq 0$ and $(d,a)_p$ else. One can compute (using $ad-bc=1$ in the Hilbert symbol calculations) that

$$\Big(\tau(BA)\tau(B),\ \tau(BA)\tau(A)\Big)_p = \begin{cases} (t(at+c),-ct)_p & \text{if } a \neq -c/t \text{ and } c \neq 0 \\ (-c,a)_p & \text{if } a = -c/t \text{ and } c \neq 0 \\ (a,-dt)_p & \text{if } c = 0, \end{cases}$$

and

$$\Big(\tau(A^{-1}BA)\tau(A^{-1}),\, \tau(A^{-1}BA)\tau(BA)\Big)_p$$
$$= \begin{cases} (t(at+c),-ct)_p & \text{if } a \neq -c/t \text{ and } c \neq 0 \text{ and } a \neq 0 \\ 1 & \text{if } a \neq -c/t \text{ and } c \neq 0 \text{ and } a = 0 \\ (-c,a)_p & \text{if } a = -c/t \text{ and } c \neq 0 \\ (a,at)_p & \text{if } c = 0. \end{cases}$$

All the above factors clearly multiply to 1. Also, it turns out that

$$\Big(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}),\ \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\Big)_p = 1,$$

so we get the required σ_p-factor. □

We also note the triangular decomposition of $\overline{K_0^p(p^n)}$.

Lemma A.4. *We have a triangular decomposition*

$$\overline{K_0^p(p^n)} = N^{\overline{K_0^p(p^n)}} T^{\overline{K_0^p(p^n)}} \bar{N}^{\overline{K_0^p(p^n)}}.$$

More precisely for $(A,\epsilon) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon) \in \overline{K_0^p(p^n)}$,

$$(A,\epsilon) = (x(s),1)(h(u),1)(y(t),1)(I,\epsilon\delta)$$

where

$$u = d^{-1},\ s = d^{-1}b,\ t = d^{-1}c,$$

and

$$\delta = \begin{cases} 1 & c = 0 \\ (d,-1)_p & c \neq 0,\ \operatorname{ord}_p(c) \text{ is odd} \\ (-c,d)_p & c \neq 0,\ \operatorname{ord}_p(c) \text{ is even.} \end{cases}$$

Proof. Clearly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}.$$

Let $u = d^{-1},\ s = bd^{-1},\ t = cd^{-1}$. Since

$$x(s)h(u)y(t) = \begin{pmatrix} u & su^{-1} \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} u + su^{-1}t & su^{-1} \\ tu^{-1} & u^{-1} \end{pmatrix},$$

we get that

$$(x(s), 1)(h(u), 1)(y(t), 1) = (x(s)h(u)y(t), \delta) = (A, \delta)$$

where

$$\delta = \sigma(x(s), h(u))\sigma(x(s)h(u), y(t)) = \begin{cases} 1 & t = 0 \\ (u, -1)_p & t \neq 0,\ \mathrm{ord}_p(t) \text{ is odd} \\ (t, u)_p & t \neq 0,\ \mathrm{ord}_p(t) \text{ is even.} \end{cases}$$

Substituting u, s, t in terms of b, c, d we get δ as in the statement. $\square$

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