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Hecke algebras, new vectors and new forms on $\Gamma_0(m)$

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Abstract We characterize the space of new forms for $\Gamma_0(m)$ as a common eigenspace of certain Hecke operators which depend on primes p dividing the level m. To do that we find generators and relations for a p-adic Hecke algebra of functions on $K = \mathrm{GL}_2(\mathbb{Z}_p)$. We explicitly find the $n+1$ irreducible representations of K which contain a vector of level n including the unique representation that contains the "new vector" of level n. After translating the p-adic Hecke operators that we obtain into classical Hecke operators we obtain the results about the new space mentioned above.

Keywords Hecke algebras · Hecke operators · New forms · New vectors

Mathematics Subject Classification Primary 11F11; Secondary 11F41 · 11F70

1 Introduction

The theory of Hecke operators and new forms of integer weight for $\Gamma_0(m)$ was developed by Atkin and Lehner for the case of trivial central character [1] and by Atkin–Lehner–Li–Miyake for arbitrary central characters [1,9,11]. Atkin and Lehner define Hecke operators T_q for primes q not dividing m and operators U_p for primes p dividing m. They define the new space of cusp forms on $\Gamma_0(m)$ as the space orthogonal under the Petersson inner product to all the old forms on $\Gamma_0(m)$ which are forms that come from lower levels m' dividing m. They show that all the Hecke operators stabilize the new space, that they commute and are diagonalizable. Further, there is a common basis of eigenforms where each eigenspace is one

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dimensional and spanned by a primitive eigenform, a form whose first Fourier coefficient is one. A basic tool in the discussion is a certain involution on the whole space called the Atkin–Lehner involution. Atkin and Lehner remark that the definition of the new space as an orthogonal complement does not give enough information on this space. In this paper we will show how to characterize the new space using eigenvalues of Hecke operators. In particular when a prime p divides m or p^2 divides m but p^3 does not divide m we will use a certain product of the Atkin–Lehner involution and the operator U_p. When p^3 divides m, the information on the new space can not be obtained using the operators considered by Atkin and Lehner and we will introduce a family of Hecke operators which "capture" the various spaces of old forms on $\Gamma_0(m)$.

In their remarkable work, Niwa [13] and Kohnen [7] considered an operator Q, a certain product of classical Hecke operators, on the space of half-integral weight modular forms of level 4. Kohnen defined the plus space to be a particular eigenspace of this operator. Loke and Savin [10] interpreted Kohnen's definition representation theoretically in the context of a Hecke algeba for the double cover of $\mathrm{SL}_2(\mathbb{Q}_2)$ and used this Hecke algebra to classify the representations that contain maximal level vectors fixed by a certain congruence subgroup. Using similar methods we will study a Hecke algebra of functions on $\mathrm{GL}_2(\mathbb{Q}_p)$ which are compactly supported and bi-invariant with respect to an open compact subgroup $K_0(p^n)$ which is defined below. In the case $n = 1$, this is the usual Iwahori Hecke algebra and has a well-known presentation by generators and relations. For $n \geq 2$, we will restrict our study to a subalgebra of the above Hecke algebra consisting of functions that are supported on $K = \mathrm{GL}_2(\mathbb{Z}_p)$. We will find generators and relations for this subalgebra and show that it is commutative. Casselman [3,4] showed that there is a unique irreducible representation of K which contains a $K_0(p^n)$ fixed vector but does not contain a $K_0(p^k)$ fixed vector for $k < n$. Such a vector is called a new vector. Casselman [3] showed that every irreducible admissible representation of $\mathrm{GL}(2, F)$ where F is a p-adic field contains a unique new vector of minimal level. Schmidt [14] used the classification of irreducible admissible representations of $\mathrm{GL}(2, F)$ to describe the new vectors in these representations. We will use the relations in the subalgebra mentioned above to study the finite dimensional representations of K containing a $K_0(p^n)$ fixed vector and thereby explicitly describe Casselman's new vectors in terms of Hecke algebra elements. Using our Hecke algebras we will construct classical Hecke operators that are needed to classify the new space. We view our paper as a connection between the theory of new vectors described by Casselman and the theory of newforms by Atkin and Lehner.

Our work is motivated by the results of Niwa, Kohnen and Loke and Savin mentioned above on the Shimura correspondence. In particular, the work of Kohnen on the Kohnen plus space showed that the definition of a subspace of half-integral weight modular forms as an eigenspace of a Hecke operator was essential in studying the Shimura correspondence and has many applications. The work of Loke and Savin gave a representation theoretical interpretation of this definition and opened the way to the study of such subspaces in a more general context. In our recent work [2] on half-integral weight modular forms we apply this approach to extend the work of Kohnen [8] on the plus space to another space of half-integral weight forms which we call the minus space. This space is a subspace of cuspidal modular forms of weight $k + 1/2$ and level $4M$ where M is odd and square-free and it is defined as a common eigenspace of $2l$ Hecke operators where l is the number of primes in the decomposition of $2M$. The motivation for defining this subspace comes from Theorem 1 below where we characterize the new space of level $N = 2M$ (or general N square-free) as the common eigenspace of $2l$ Hecke operators. Moreover, we show in [2] that our minus space is isomorphic to the space of newforms of weight $2k$ and level $2M$ under the Shimura-Niwa correspondence. Thus the minus space complements Kohnen's plus new space at level $4M$, giving a counterpart of Kohnen's newform theory of half-integral weight. We are certain that our more general description of the space of new forms of any level N in this paper will allow to define similar subspaces of half-integral weight forms and also play a role in the representation theory of integral and half-integral weight forms via the Waldspurger correspondence.

2 The main results

Let $S_{2k}(\Gamma_0(m))$ be the space of cusp forms of weight $2k$ on $\Gamma_0(m)$. The space of old forms $S_{2k}^{\text{old}}(\Gamma_0(m))$ is defined to be the space spanned by all the forms $f(lz)$ where $f \in S_{2k}(\Gamma_0(m_1))$ and $l,\ m_1 \in \mathbb{N}$, with $lm_1|m$ and $m_1 \neq m$. The space of new forms $S_{2k}^{\text{new}}(\Gamma_0(m))$ is the space orthogonal to the space of old forms under the Petersson inner product. Let $\mathrm{GL}_2(\mathbb{R})^+$ be the group of 2×2 real matrices with positive determinant and $\mathbb{H}$ be the upper half plane. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ and $z \in \mathbb{H}$ define

$$j(g, z) = det(g)^{-1/2}(cz + d),$$

and for functions f on $\mathbb{H}$ define the slash operator $|_{2k}g$ by

$$f|_{2k}g = j(g, z)^{-2k} f\left(\frac{az + b}{cz + d}\right).$$

Let p be a prime dividing m. Assume that $p^n|m$ and $p^{n+1} \nmid m$, we denote this by $p^n \| m$. We define the following operators:

$$\tilde{U}_p(f)(z) = p^{-k} \sum_{s=0}^{p-1} f((z + s)/p),$$

$$W_{p^n}(f)(z) = f|_{2k} \begin{pmatrix} p^n\beta & 1 \\ m\gamma & p^n \end{pmatrix}(z) \quad \text{where } p^{2n}\beta - m\gamma = p^n.$$

Let $m = p^n m'$ with $p \nmid m'$ and $n \geq 2$. We fix j such that $1 \leq j \leq n - 1$. Let

$$L_j(f) = \sum_{s \in (\mathbb{Z}/p^{n-j}\mathbb{Z})^*} f|_{2k} A_s$$

where $A_s \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix of the form $\begin{pmatrix} a_s & b_s \\ p^j m' & p^{n-j} - sm' \end{pmatrix}$. In this case we define for $1 \leq r \leq n - 1$ the operators

$$S_{p^n,r} = I + \sum_{j=r}^{n-1} L_j.$$

We also define

$$S'_{p^n,r} = W_{p^n} S_{p^n,r} W_{p^n}^{-1}.$$

Remark 1 (i) The operator $\tilde{U}_p$ is denoted by $U_p^* = p^{1-k}U_p$ in Atkin and Lehner [1, Lemma 14] where U_p is the usual Hecke operator, sometimes also denoted as T_p [12]. The operator W_{p^n} is the usual Atkin–Lehner involution [1].

(ii) The operators $S_{p^n,r}$ that are defined in the case $n \geq 2$ did not appear in [1]. Despite their complicated form, these operators come naturally from the local Hecke algebra. We will see later that the operator L_j above comes from the characteristic function of the double coset $K_0(p^n)\begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix}K_0(p^n)$ and that $S_{p^n,r}$ satisfies a simple quadratic relation: $S_{p^n,r}(S_{p^n,r} - p^{n-r}) = 0$.

Our main theorems characterize the space of new forms as a common eigenspace of above defined operators:

Theorem 1 *Let N be a square-free positive number. For any prime $p \mid N$, let $Q_p = \tilde{U}_p W_p$ and $Q'_p = W_p \tilde{U}_p$. Then the space of new forms $S_{2k}^{\text{new}}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_p and Q'_p as p varies over the prime divisors of N. That is, $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes $p \mid N$.*

Theorem 2 *Let $N = M_1^2 M$ where M_1 and M are square-free and coprime. For any prime p dividing M_1, let $Q_{p^2} = (\tilde{U}_p)^2 W_{p^2}$ and $Q'_{p^2} = W_{p^2}(\tilde{U}_p)^2$. Then $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p dividing M and $Q_{p^2}(f) = 0 = Q'_{p^2}(f)$ for all primes p dividing M_1.*

Theorem 2′ *Let N be as in Theorem 2. Then $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p dividing M and $S_{p^2,1}(f) = 0 = S'_{p^2,1}(f)$ for all primes p dividing M_1.*

Theorem 3 *Let N be a positive integer. Then the space of new forms $S_{2k}^{\text{new}}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_p and Q'_p where p varies over the primes such that $p \| N$ and the 0 eigenspaces of $S_{p^\gamma,\gamma-1}$ and $S'_{p^\gamma,\gamma-1}$ for primes p such that $p^\gamma \| N$ with $\gamma \geq 2$. That is, $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p such that $p \| N$ and $S_{p^\gamma,\gamma-1}(f) = 0 = S'_{p^\gamma,\gamma-1}(f)$ for all primes p such that $p^\gamma \| N$ for $\gamma \geq 2$.*

Let $q = e^{2\pi i z}$ and $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}(\Gamma_0(m))$. Let p be an odd prime. Define

$$R_p(f)(z) = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) a_n q^n, \qquad R_\chi(f)(z) = \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) a_n q^n.$$

By [1, Lemma 33], R_p and R_χ are operators on $S_{2k}(\Gamma_0(m))$ provided that $p^2 \mid m$ and $16 \mid m$ respectively.

Theorem 4 *Let $N = 2^\beta M_1 M_2$ where $M_1 M_2$ is odd such that M_1 is square-free and any prime divisor of M_2 divides it with a power at least 2. Let $\beta \geq 4$. Then $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p dividing M_1, $(R_\chi)^2(f) = f$ and $(R_p)^2(f) = f$ for all primes p dividing M_2, and $S_{p^\gamma,\gamma-1}(f) = 0$ for all primes p such that $p^\gamma \| 2^\beta M_2$.*

3 *p*-adic Hecke algebras and the representations of *K*

In this section we will find generators and relations for a Hecke algebra of functions on $K = \mathrm{GL}_2(\mathbb{Z}_p)$ which are bi-invariant with respect to $K_0(p^n)$. We will use these results to classify smooth irreducible finite dimensional representations of K which have $K_0(p^n)$ fixed vectors.

Denote by G the group $\mathrm{GL}_2(\mathbb{Q}_p)$. Let $K_0\left(p^n\right)$ be the subgroup of K defined by

$$K_0(p^n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \ : \ c \in p^n\mathbb{Z}_p\right\}.$$

The subgroup $K_0(p)$ denotes the usual Iwahori subgroup. In this section we shall consider the Hecke algebra of G with respect to $K_0(p^n)$.

It is well known that the space $C_c^\infty(G)$, the space of locally constant, compactly supported complex-valued functions on G, forms a $\mathbb{C}$-algebra under convolution which, for any $f_1, f_2 \in C_c^\infty(G)$, is defined by

$$f_1 * f_2(h) = \int_G f_1(g) f_2\left(g^{-1}h\right) dg = \int_G f_1(hg) f_2\left(g^{-1}\right) dg,$$

where dg is the Haar measure on G such that the measure of $K_0(p^n)$ is one. The Hecke algebra corresponding to $K_0(p^n)$, denoted by $H(G//K_0(p^n))$, is the subalgebra of $C_c^\infty(G)$ consisting of $K_0(p^n)$ bi-invariant functions:

$$H\left(G//K_0\left(p^n\right)\right) = \{f \in C_c^\infty(G) : f(kgk') = f(g) \text{ for } \ g \in G,\ k,\ k' \in K_0\left(p^n\right)\}.$$

Let X_g denote the characteristic function of the double coset $K_0\left(p^n\right) g K_0\left(p^n\right)$. Then $H(G//K_0\left(p^n\right))$ as a $\mathbb{C}$-vector space is spanned by X_g as g varies over the double coset representatives of G modulo $K_0(p^n)$.

Let $\mu(K_0\left(p^n\right) g K_0\left(p^n\right))$ denote the number of disjoint left (right) $K_0\left(p^n\right)$ cosets in the double coset $K_0\left(p^n\right) g K_0\left(p^n\right)$. Then the following lemmas are well-known [6, Corollary 1.1].

Lemma 3.1 *If* $\mu\left(K_0\left(p^n\right) g K_0\left(p^n\right)\right) \mu\left(K_0\left(p^n\right) h K_0\left(p^n\right)\right) = \mu\left(K_0\left(p^n\right) gh K_0\left(p^n\right)\right)$ *then* $X_g * X_h = X_{gh}$.

Lemma 3.2 *Let* $f_1,\ f_2 \in H\left(G//K_0\left(p^n\right)\right)$ *such that* f_1 *is supported on* $K_0(p^n)xK_0(p^n) = \bigcup_{i=1}^m \alpha_i K_0(p^n)$ *and* f_2 *is supported on* $K_0(p^n)yK_0(p^n) = \bigcup_{j=1}^n \beta_j K_0(p^n)$. *Then*

$$f_1 * f_2(h) = \sum_{i=1}^m f_1(\alpha_i) f_2\left(\alpha_i^{-1}h\right)$$

where the nonzero summands are precisely for those i *for which there exist a* j *such that* $h \in \alpha_i\beta_j K_0(p^n)$.

For $t \in \mathbb{Q}_p$ we shall consider the following elements:

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ y(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\ w(t) = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix},$$
$$d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},\ z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Let $N = \{x(t) : t \in \mathbb{Q}_p\}$, $\bar{N} = \{y(t) : t \in \mathbb{Q}_p\}$ and A be the group of diagonal matrices of G. Let $Z_G = \{z(t) : t \in \mathbb{Q}_p^*\}$ denote the center of G.

3.1 The Iwahori Hecke algebra

Lemma 3.3 *A complete set of representatives for the double cosets of G mod $K_0(p)$ are given by $d(p^n)z(m)$, $w(p^n)z(m)$ where n, m varies over integers.*

Proof For proof refer to [6, Sect. 2.3]. □

Lemma 3.4 *(1) For $n \geq 0$ we have*

$$K_0(p)d(p^n)K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} x(s)d(p^n)K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} K_0(p)d(p^n)y(ps).$$

(2) For $n \geq 1$ we have

$$K_0(p)d(p^{-n})K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} y(ps)d(p^{-n})K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^n\mathbb{Z}_p} K_0(p)d(p^{-n})x(s).$$

(3) For $n \geq 1$ we have

$$K_0(p)w(p^n)K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^{n-1}\mathbb{Z}_p} y(ps)w(p^n)K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^{n-1}\mathbb{Z}_p} K_0(p)w(p^n)y(ps).$$

(4) For $n \geq 0$ we have

$$K_0(p)w(p^{-n})K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p} x(s)w(p^{-n})K_0(p) = \bigsqcup_{s\in\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p} K_0(p)w(p^{-n})x(s).$$

Proof The proof easily follows from the triangular decomposition

$$K_0(p) = (N \cap K_0(p))(A \cap K_0(p))(\bar{N} \cap K_0(p)).$$

□

Let $\mathcal{T}_n = X_{d(p^n)}$, $\mathcal{U}_n = X_{w(p^n)}$ and $\mathcal{Z} = X_{z(p)}$ be elements of the Hecke algebra $H(G//K_0(p))$. It is easy to see that $\mathcal{Z}$ commutes with every $f \in H(G//K_0(p))$ and that $\mathcal{Z}^n = X_{z(p^n)}$. We have the following well-known lemma.

Lemma 3.5 *(1) If $n, m \geq 0$ or $n, m \leq 0$, then $\mathcal{T}_n * \mathcal{T}_m = \mathcal{T}_{n+m}$.*
(2) If $n \geq 0$ then $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{Z}^n * \mathcal{U}_{1-n}$.
(3) If $n \geq 0$ then $\mathcal{U}_1 * \mathcal{T}_{-n} = \mathcal{U}_{1-n}$ and $\mathcal{T}_{-n} * \mathcal{U}_1 = \mathcal{Z}^{-n} * \mathcal{U}_{1+n}$.
(4) If $n \geq 0$ then $\mathcal{U}_0 * \mathcal{T}_{-n} = \mathcal{U}_{-n}$ and $\mathcal{T}_n * \mathcal{U}_0 = \mathcal{Z}^n * \mathcal{U}_{-n}$.
(5) For $n \in \mathbb{Z}$, $\mathcal{U}_1 * \mathcal{U}_n = \mathcal{Z} * \mathcal{T}_{n-1}$ and $\mathcal{U}_n * \mathcal{U}_1 = \mathcal{Z}^n * \mathcal{T}_{1-n}$.
(6) For $n \geq 1$, $\mathcal{U}_0 * \mathcal{U}_n = \mathcal{T}_n$ and $\mathcal{U}_n * \mathcal{U}_0 = \mathcal{Z}^n * \mathcal{T}_{-n}$.
(7) $\mathcal{U}_0 * \mathcal{U}_0 = (p-1)\mathcal{U}_0 + p$.

Proof The parts (1) to (6) follows from Lemmas 3.1 and 3.4.

Using Lemma 3.2 it is easy to see that $\mathcal{U}_0 * \mathcal{U}_0$ is supported only on the double cosets $K_0(p)$ and $K_0(p)w(1)K_0(p)$, so to obtain (7) it is enough to find the values of $\mathcal{U}_0 * \mathcal{U}_0$ on the elements $w(1)$ and 1. Using Lemmas 3.2 and 3.4,

$$\mathcal{U}_0 * \mathcal{U}_0(w(1)) = \sum_{s=0}^{p-1} \mathcal{U}_0(x(s)w(1))\mathcal{U}_0(w(1)x(-s)w(1)) = \sum_{s=0}^{p-1} \mathcal{U}_0(y(-s)).$$

For each $1 \leq s \leq p-1$ we have $y(-s) \in K_0 w(1) K_0$ while $y(0) \notin K_0 w(1) K_0$, hence $\mathcal{U}_0 * \mathcal{U}_0(w(1)) = p-1$. Further,

$$\mathcal{U}_0 * \mathcal{U}_0(1) = \sum_{s=0}^{p-1} \mathcal{U}_0(x(s)w(1))\mathcal{U}_0(w(1)x(-s)) = \sum_{s=0}^{p-1} \mathcal{U}_0(w(1)) = p.$$

□

Thus we obtain the following well-known theorem.

Theorem 5 *The Iwahori Hecke Algebra $H(G//K_0(p))$ is generated by $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{Z}$ with the relations:*

(1) $\mathcal{U}_1^2 = \mathcal{Z}$,
(2) $(\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0$,
(3) *$\mathcal{Z}$ commutes with $\mathcal{U}_0$ and $\mathcal{U}_1$.*

Remark 2 The algebra $H(G//K_0(p))/\langle \mathcal{Z} \rangle$ is generated by $\mathcal{U}_0$ and $\mathcal{U}_1$ with the relations $\mathcal{U}_1^2 = 1$ and $(\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0$.

3.2 A subalgebra of $H(G//K_0(p^n))$, $n \geq 2$

It is difficult to compute generators and relations for $H(G//K_0(p^n))$ for general n as the double coset representatives of G modulo $K_0(p^n)$ becomes more intricate as n increases. So instead of the full algebra $H(G//K_0(p^n))$ we consider a subalgebra $H(K//K_0(p^n))$ consisting of functions that are supported on K. In this section we compute a basis of this finite dimensional subalgebra and the relations between the basis elements. In Sect. 3.3, we will use the relations in $H(K//K_0(p^n))$ to explicitly describe the finite dimensional representations of K containing a $K_0(p^n)$ fixed vector in terms of certain elements in $H(K//K_0(p^n))$ and thereby obtain a description for Casselman's new vectors in terms of Hecke algebra elements. We first note the following lemma [4, Lemma 1].

Lemma 3.6 *A complete set of representatives for the double cosets of K mod $K_0(p^n)$ are given by* $1,\ w(1),\ y(p),\ y(p^2),\ \ldots\ y\left(p^{n-1}\right)$.

For simplicity, we shall write K_0 for $K_0(p^n)$.

Let $\mathcal{U}_0 = X_{w(1)}$ and $\mathcal{V}_r = X_{y(p^r)}$ for $1 \leq r \leq n-1$ be the elements of $H(G//K_0)$. Then by the above lemma, $H(K//K_0)$ is spanned by $1, \mathcal{U}_0$ and $\mathcal{V}_r$ where $1 \leq r \leq n-1$.

We shall need the following lemmas.

Lemma 3.7 *Assume that r satisfies $n > r \geq n/2$. Then*

$$K_0 y(p^r) K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} d(s) y\left(p^r\right) K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} K_0 y\left(p^r\right) d(s).$$

Proof Since $K_0 = N'A'\bar{N}'$ where $N' = N \cap K_0$, $A' = A \cap K_0$ and $\bar{N}' = \bar{N} \cap K_0$, and $A' = DZ'$ where D consists of matrices $d(a) \in K$ and $Z' = Z_G \cap K$, we have

$$K_0 y\left(p^r\right) K_0 = N'A'\bar{N}' y\left(p^r\right) K_0 = N'A' y\left(p^r\right) K_0 = N'D y\left(p^r\right) K_0.$$

Now any $a \in \mathbb{Z}_p^*$ can be written as $a = sa'$ where $a' \in 1 + p^{n-r}\mathbb{Z}_p$ and $s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$. Since

$$y\left(-p^r\right) d(a') y\left(p^r\right) = \begin{pmatrix} a' & 0 \\ p^r(1-a') & 1 \end{pmatrix} \in K_0$$

we get that

$$K_0 y\left(p^r\right) K_0 = \bigcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} N' d(s) y\left(p^r\right) K_0.$$

We obtain the decomposition since

$$N'd(s) = d(s)N' \quad \text{and} \quad y\left(-p^r\right) x(u) y(p^r) = \begin{pmatrix} 1+up^r & u \\ -up^{2r} & 1-up^r \end{pmatrix} \in K_0.$$

Now we show that the union is disjoint. Let $g_1 = d(s_1)y(p^r)$ and $g_2 = d(s_2)y(p^r)$. Assume $g_1^{-1}g_2 \in K_0$ then

$$y(-p^r)d\left(s_1^{-1}s_2\right)y(p^r) = \begin{pmatrix} s_1^{-1}s_2 & 0 \\ \left(1-s_1^{-1}s_2\right)p^r & 1 \end{pmatrix} \in K_0,$$

hence $s_1^{-1}s_2 \in 1 + p^{n-r}\mathbb{Z}_p$. □

Lemma 3.8 *Assume that $0 < r < n/2$. Let $K_0^{y(p^r)} = y(p^r)K_0 y(p^r)^{-1} \cap K_0$. Then an element of $K_0^{y(p^r)}$ can be written as $y(v)z(t)d(s)x(u)$ where $v \in p^n\mathbb{Z}_p$, $t,\ s \in \mathbb{Z}_p^*$, $u \in \mathbb{Z}_p$ and $s - 1 - p^r u \in p^{n-r}\mathbb{Z}_p$.*

Lemma 3.9 *Assume that r satisfies $0 < r < n/2$. Then*

$$K_0 y(p^r) K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} d(s)y(p^r)K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} K_0 y(p^r) d(s).$$

Proof As in Lemma 3.7, an element of $K_0 y(p^r) K_0$ can be written as $g = d(s)x(u)y(p^r)k_0$ where $s \in \mathbb{Z}_p^*$, $u \in \mathbb{Z}_p$ and $k_0 \in K_0$. Now

$$g = d(s)d(1+p^r u)^{-1} d(1+p^r u)x(u)y(p^r)k_0,$$

it follows from Lemma 3.8 that $d(1+p^r u)x(u) \in K_0^{y(p^r)}$. Let $s_1 = s(1+p^r u)^{-1} \in \mathbb{Z}_p^*$. Then we get that $g = d(s_1)y(p^r)k_1$ for some $k_1 \in K_0$ hence we get the decomposition as in the statement. The disjointness follows as in Lemma 3.7. □

Proposition 3.10 *We have the following relations in $H(K//K_0)$:*

(1) $\mathcal{V}_r^2 = p^{n-r-1}(p-1)(I + \sum_{j=r+1}^{n-1} \mathcal{V}_j) + p^{n-r-1}(p-2)\mathcal{V}_r$.
(2) $\mathcal{V}_r * \mathcal{V}_j = (p-1)p^{n-j-1}\mathcal{V}_r = \mathcal{V}_j * \mathcal{V}_r$ for $r+1 \leq j \leq n-1$.
(3) Let $\mathcal{Y}_{r+1} = I + \sum_{j=r+1}^{n-1} \mathcal{V}_j$. Then

$$\mathcal{V}_r * \mathcal{Y}_{r+1} = p^{n-r-1}\mathcal{V}_r = \mathcal{Y}_{r+1} * \mathcal{V}_r,$$

and so,

$$(\mathcal{V}_r - p^{n-r-1}(p-1))(\mathcal{V}_r + \mathcal{Y}_{r+1}) = 0.$$

Proof For (1), we first compute the support of $\mathcal{V}_r * \mathcal{V}_r$. By Lemmas 3.7 and 3.9,

$$K_0 y(p^r) K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \alpha_s K_0 \quad \text{where } \alpha_s = d(s)y(p^r),$$

so using Lemma 3.2 we get that $\mathcal{V}_r * \mathcal{V}_r$ is supported on those $g \in G$ for which there exists $s,\ t \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$ such that

$$(\alpha_s\alpha_t)^{-1}g = \begin{pmatrix} \frac{1}{st} & 0 \\ \frac{-p^r(t+1)}{st} & 1 \end{pmatrix} g \in K_0.$$

It is enough to check the support on $g = 1,\ w(1),\ y(p^j)$ for $1 \le j \le n-1$. Note that $(\alpha_s\alpha_t)^{-1}w(1) = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix} \notin K_0$. For $g = 1$ taking $s = 1$ and $t = p^{n-r} - 1 \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$ we get that $\mathcal{V}_r * \mathcal{V}_r$ is supported on K_0. For $g = y(p^j)$,

$$(\alpha_s\alpha_t)^{-1}g \in K_0 \iff p^j st - p^r(t+1) \in p^n\mathbb{Z}_p.$$

If $j < r$, this is impossible. First assume that $r < j < n$, then the above equation holds if and only if $p^{j-r}st - (t+1) \in p^{n-r}\mathbb{Z}_p$. Taking $t = p^{j-r} - 1$ and $s = (1 + p^{n-j})t^{-1} \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$, we are done. Now assume $j = r$. If $p > 2$ then taking $t = p^{n-r} - 2$ and $s = -1/t$ we are done. If $p = 2$ then no choice of $s,\ t$ works. Thus we get that $\mathcal{V}_r * \mathcal{V}_r$ is supported on K_0 and $K_0 y(p^j)K_0$ where if $p > 2$ then $r \le j < n$ while for $p = 2$ we have $r < j < n$. Since $y(-p^r) \in K_0 y(p^r)K_0$,

$$\mathcal{V}_r * \mathcal{V}_r(1) = \sum_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \mathcal{V}_r(y(-p^r)) = p^{n-r-1}(p-1).$$

For $r \le j < n$,

$$\mathcal{V}_r * \mathcal{V}_r(y(p^j)) = \sum_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \mathcal{V}_r(y(-p^r)d(s)y(p^j)).$$

We want to check for which s there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0$ such that $y(-p^r)d(s)y(p^j)Ay(-p^r) \in K_0$, i.e., $(p^{j-r} - s^{-1})(a - bp^r) - d \in p^{n-r}\mathbb{Z}_p$. If $r < j$ then for any $s \in \mathbb{Z}_p^*$ take $b = c = 0$, $a = \frac{p^{n-r}-1}{p^{j-r}-s^{-1}}$, $d = -1$, thus $\mathcal{V}_r * \mathcal{V}_r(y(p^j)) = p^{n-r-1}(p-1)$. If $p > 2$ and $j = r$, it is easy to see that such an A exists if and only if $s \notin 1 + p\mathbb{Z}_p$, in this case take $b = c = 0$ and $a = \frac{p^{n-r}-1}{1-s^{-1}}$, $d = -1$. The number of $s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$ such that $s \notin 1 + p\mathbb{Z}_p$ is equal to $p^{n-r-1}(p-2)$ and so $\mathcal{V}_r * \mathcal{V}_r(y(p^r)) = p^{n-r-1}(p-2)$.

For (2), for $r + 1 \le j < n$, we get that $\mathcal{V}_r * \mathcal{V}_j$ is supported at $g \in G$ if and only if there exists $s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p$ and $t \in \mathbb{Z}_p^*/1 + p^{n-j}\mathbb{Z}_p$ such that

$$\begin{pmatrix} \frac{1}{st} & 0 \\ \frac{-(p^r t + p^j)}{st} & 1 \end{pmatrix} g \in K_0.$$

It is easy to check that the above does not hold for $g = 1,\ w(1),\ y(p^i)$ for $i \ne r$. If $i = r$, taking $s = p^{j-r} + 1$, $t = 1$ we are done. Similarly $\mathcal{V}_j * \mathcal{V}_r$ is supported only on $K_0 y(p^r)K_0$. Now

$$\mathcal{V}_r * \mathcal{V}_j\left(y\left(p^r\right)\right) = \sum_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \mathcal{V}_j\left(y\left(-p^r\right)d(s^{-1})y(p^r)\right),$$

so we want to count s for which there exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0$ such that $y\left(-p^r\right) d\left(s^{-1}\right) y\left(p^r\right) A y\left(-p^j\right) \in K_0$, i.e., $\left(1 - s^{-1}\right)\left(a - bp^j\right) - dp^{j-r} \in p^{n-r}\mathbb{Z}_p$, which holds if and only if $s - 1 \in p^{j-r}\mathbb{Z}_p^*$, in which case if $s - 1 = p^{j-r}u$ then taking $b = 0$, $a = s$, $d = u$ we are done. Thus $\mathcal{V}_r * \mathcal{V}_j = C_j \mathcal{V}_r$ where for $r + 1 \leq j < n$,

$$C_j = \#\{s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p : s - 1 \in p^{j-r}\mathbb{Z}_p^*\} = (p-1)p^{n-j-1}.$$

For $\mathcal{V}_j * \mathcal{V}_r(y(p^r))$ we use that $K_0 y(-p^r) K_0 = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} d(s) y(-p^r) K_0$ to get

$$\mathcal{V}_j * \mathcal{V}_r(y(p^r)) = \sum_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \mathcal{V}_j\left(y\left(p^r\right) d(s) y\left(-p^r\right)\right),$$

the calculations now follow as above.

For (3),

$$\begin{aligned} \mathcal{V}_r * \mathcal{Y}_{r+1} &= \mathcal{V}_r + (p-1)\mathcal{V}_r + (p-1)p\mathcal{V}_r + \cdots + (p-1)p^{n-r-2}\mathcal{V}_r \\ &= \mathcal{V}_r + \left(p^{n-r-1} - 1\right)\mathcal{V}_r = p^{n-r-1}\mathcal{V}_r, \end{aligned}$$

the rest follows from (1). □

For $1 \leq r \leq n-1$, let $\mathcal{Y}_r$ be as before, i.e., $\mathcal{Y}_r = I + \sum_{j=r}^{n-1} \mathcal{V}_j$. Let $\mathcal{Y}_n = I$. We have the following corollary.

Corollary 3.11 *(1)* $\mathcal{Y}_{n-r}^2 = p^r \mathcal{Y}_{n-r}$ *for all* $0 \leq r \leq n-1$.
(2) $\mathcal{Y}_r * \mathcal{Y}_l = p^{n-r}\mathcal{Y}_l = \mathcal{Y}_l * \mathcal{Y}_r$ *for* $r \geq l$.

Proof Note that $\mathcal{V}_{n-r} = \mathcal{Y}_{n-r} - \mathcal{Y}_{n-r+1}$ for all $1 \leq r \leq n-1$. Clearly (1) holds for $r = 0$. Assume that $\mathcal{Y}_{n-(a-1)}^2 = p^{a-1}\mathcal{Y}_{n-(a-1)}$. Then using Lemma 3.10

$$\begin{aligned} \mathcal{Y}_{n-a}^2 &= (\mathcal{Y}_{n-(a-1)} + \mathcal{V}_{n-a})(\mathcal{Y}_{n-(a-1)} + \mathcal{V}_{n-a}) \\ &= \mathcal{Y}_{n-(a-1)}^2 + 2\mathcal{Y}_{n-(a-1)}\mathcal{V}_{n-a} + \mathcal{V}_{n-a}^2 \\ &= p^{a-1}\mathcal{Y}_{n-(a-1)} + 2p^{a-1}\mathcal{V}_{n-a} + (p-1)p^{a-1}\mathcal{Y}_{n-(a-1)} + (p-2)p^{a-1}\mathcal{V}_{n-a} \\ &= p^a\mathcal{Y}_{n-(a-1)} + p^a\mathcal{V}_{n-(a-1)} \\ &= p^a\mathcal{Y}_{n-a}. \end{aligned}$$

Similarly for (2), let $r = l + m$ for some $m \geq 0$. Then

$$\mathcal{Y}_r * \mathcal{Y}_l = \mathcal{Y}_r * (\mathcal{V}_l + \mathcal{V}_{l+1} + \mathcal{V}_{l+2} + \cdots \mathcal{V}_{l+m-1} + \mathcal{Y}_r).$$

Now for $0 \leq j \leq m-1$,

$$\begin{aligned} \mathcal{Y}_r * \mathcal{V}_{l+j} &= \mathcal{V}_{l+j} + \sum_{i=r}^{n-1} \mathcal{V}_i * \mathcal{V}_{l+j} = \mathcal{V}_{l+j} + \sum_{i=r}^{n-1} (p-1)p^{n-i-1}\mathcal{V}_{l+j} \\ &= \mathcal{V}_{l+j} + \mathcal{V}_{l+j}(p^{n-r} - 1) = p^{n-r}\mathcal{V}_{l+j}. \end{aligned}$$

Hence

$$\mathcal{Y}_r * \mathcal{Y}_l = p^{n-r}(\mathcal{V}_l + \mathcal{V}_{l+1} + \cdots + \mathcal{V}_{l+m-1} + \mathcal{Y}_r) = p^{n-r}\mathcal{Y}_l.$$

□

In the next proposition, we obtain relations for $\mathcal{U}_0$.

Proposition 3.12 *(1)* $\mathcal{U}_0 * \mathcal{U}_0 = p^{n-1}(p-1)\mathcal{U}_0 + p^n\mathcal{Y}_1$.
(2) $\mathcal{U}_0 * \mathcal{Y}_r = p^{n-r}\mathcal{U}_0 = \mathcal{Y}_r * \mathcal{U}_0$ *for all* $1 \le r \le n$.
(3) $\mathcal{U}_0 * (\mathcal{U}_0 - p^n) * (\mathcal{U}_0 + p^{n-1}) = 0$.

Proof Note that

$$K_0 w(1) K_0 = \bigsqcup_{s \in \mathbb{Z}_p/p^n Z_p} \alpha_s K_0 \quad \text{where } \alpha_s = x(s)w(1).$$

To compute $\mathcal{U}_0 * \mathcal{U}_0$ we need to check if it is supported on $g = 1$, $w(1)$ and $y(p^j)$ for $1 \le j \le n-1$, i.e., we need to check if there exists $s,\ t$ such that

$$(\alpha_s\alpha_t)^{-1} g = \begin{pmatrix} -1 & s \\ -t & st-1 \end{pmatrix} g \in K_0.$$

For $g = 1$ taking $s = t = 0$, for $g = w(1)$ taking $s = t = 1$ and for $g = y(p^j)$, taking $s = p^{n-j}, t = -p^j$ we get that $\mathcal{U}_0 * \mathcal{U}_0$ is supported on K_0, $K_0 w(1) K_0$ and $K_0 y(p^j) K_0$ for all $1 \le j \le n-1$. Clearly $\mathcal{U}_0 * \mathcal{U}_0(1) = p^n$. Doing similar calculations as before we get that

$$\mathcal{U}_0 * \mathcal{U}_0(w(1)) = \#\{s \in \mathbb{Z}_p/p^n Z_p : s \notin p\mathbb{Z}_p\} = p^{n-1}(p-1),$$

and

$$\mathcal{U}_0 * \mathcal{U}_0(y(p^j)) = p^n \quad \text{for } 1 \le j \le n-1.$$

Thus

$$\mathcal{U}_0 * \mathcal{U}_0 = p^{n-1}(p-1)\mathcal{U}_0 + p^n(I + \mathcal{V}_1 + \cdots \mathcal{V}_{n-1}) = p^{n-1}(p-1)\mathcal{U}_0 + p^n\mathcal{Y}_1.$$

Similarly we can check that for each $1 \le j \le n-1$, $\mathcal{U}_0 * \mathcal{V}_j$ and $\mathcal{V}_j * \mathcal{U}_0$ are supported only on $K_0 w(1) K_0$ and that

$$\mathcal{U}_0 * \mathcal{V}_j = \mathcal{V}_j * \mathcal{U}_0 = (p-1)p^{n-j-1}\mathcal{U}_0$$

which implies (2).

The statement (3) now follows using (1) and (2). $\square$

Thus we have the following theorem.

Theorem 6 *The algebra $H(K//K_0(p^n))$ is an $n+1$ dimensional commutative algebra with generators $\{\mathcal{U}_0,\ \mathcal{Y}_1,\ \mathcal{Y}_2,\ \ldots,\ \mathcal{Y}_n\}$ and relations given by Corollary 3.11 and Proposition 3.12.*

As mentioned before we have not yet found an analogue of Theorem 5 for $H(G//K_0(p^n))$ for $n \ge 2$. However we shall need the following relation. Let $\mathcal{T}_m = X_{d(p^m)}$, $\mathcal{U}_m = X_{w(p^m)}$, $\mathcal{Z} = X_{z(p)}$ be the elements in $H(G//K_0(p^n))$. Then

Lemma 3.13 $(\mathcal{T}_1)^m * \mathcal{U}_m = \mathcal{T}_m * \mathcal{U}_m = \mathcal{Z}^m * \mathcal{U}_0$ *for all* $m \le n$.

Proof The proof follows as before by using Lemma 3.1 and since

$$K_0(p^n) d(p^m) K_0(p^n) = \bigsqcup_{s \in \mathbb{Z}_p/p^m\mathbb{Z}_p} x(s) d(p^m) K_0(p^n) \quad \text{for } m \ge 0,$$

and

$$K_0(p^n)w(p^r)K_0(p^n) = \bigsqcup_{s\in\mathbb{Z}_p/p^{n-r}\mathbb{Z}_p} x(s)w(p^r)K_0(p^n) \quad \text{for } r \leq n.$$

□

3.3 Representations of K having a $K_0(p^n)$ fixed vector

In this section we recall some results of Casselman [3,4]. We are interested in irreducible representations of K having a $K_0(p^n)$ fixed vector. Let

$$I(n) := Ind_{K_0(p^n)}^K 1 = \{\phi : K \to \mathbb{C} : \phi(k_0k) = \phi(k) \text{ for } k_0 \in K_0(p^n),\ k \in K\}.$$

Then $I(n)$ is a right representation of K, denoted by π_R, where $\pi_R(k)(\phi)(k') = \phi(k'k)$. The dimension of this representation is $[K : K_0(p^n)] = p^{n-1}(p+1)$. It follows from Frobenius Reciprocity that every (smooth) irreducible representation of K which has a nonzero $K_0(p^n)$ fixed vector is isomorphic to a subrepresentation of $I(n)$. We shall therefore decompose $I(n)$ into sum of irreducible representations.

The following lemma is clear.

Lemma 3.14 *We have $I(n)^{K_0(p^n)} = H(K//K_0(p^n))$ and consequently the dimension of $I(n)^{K_0(p^n)}$ is $n+1$.*

Using induction argument and Frobenius reciprocity we obtain the following well-known results.

Proposition 3.15 *The representation $I(n)$ is a sum of $n+1$ distinct irreducible representations.*

Corollary 3.16 *Let $n \geq 0$. There exists a unique irreducible representation $\sigma(n)$ of K such that $\sigma(n)$ has a $K_0(p^n)$ fixed vector and such that $\sigma(n)$ does not have a $K_0(p^k)$ fixed vector for $k < n$. Further, $\sigma(n)$ has a unique $K_0(p^n)$ fixed vector up to scalar multiplication and the dimension of $\sigma(n)$ is given by: $dim(\sigma(0)) = 1$, $dim(\sigma(1)) = p$ and $dim(\sigma(n)) = p^{n-2}(p^2-1)$ for $n \geq 2$.*

We note the following theorem of Casselman.

Theorem 7 (Casselman [3]) *Let (π, V) be an irreducible admissible representation of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ with trivial central character. Let n be the minimal integer such that there exists a nonzero $K_0(p^n)$ fixed vector in V. Then this vector is unique up to a scalar.*

We shall now explicitly describe the irreducible subrepresentations of $I(n)$. Let us consider the action π_L of $H(K//K_0(p^n))$ on $I(n)$: for $f \in H(K//K_0(p^n))$ and $\phi \in I(n)$ set

$$\pi_L(f)(\phi)(g) = \int_K f(k)\phi(k^{-1}g)dk \quad \text{for all } g \in K.$$

In particular, if $\phi \in I(n)^{K_0(p^n)}$ which by Lemma 3.14 is same as the algebra $H(K//K_0(p^n))$ then we have $\pi_L(f)(\phi) = f * \phi$. It is easy to check that the action π_L commutes with the action π_R. It now follows by Schur's Lemma that for each $f \in H(K//K_0(p^n))$ the operator $\pi_L(f)$ acts as a scalar operator on an irreducible subrepresentation of $I(n)$. We shall use this to distinguish the irreducible components of $I(n)$ as follows.

If σ is any irreducible subrepresentation of $I(n)$ then σ contains a $K_0(p^n)$ fixed vector, that is, there exists a non-zero vector $v_\sigma \in \sigma \cap I(n)^{K_0(p^n)}$. Thus v_σ is a linear combination

Table 1 Action of $\mathcal{U}_0$, $\{\mathcal{Y}_r\}_{1 \le r \le n}$ on eigenvectors v_i, w_k

	$\mathcal{U}_0$	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\dots$	$\mathcal{Y}_k$	$\dots$	$\mathcal{Y}_{n-1}$	$\mathcal{Y}_n$
v_1	p^n	p^{n-1}	p^{n-2}	p^{n-3}	$\dots$	p^{n-k}	$\dots$	p	1
v_2	$-p^{n-1}$	p^{n-1}	p^{n-2}	p^{n-3}	$\dots$	p^{n-k}	$\dots$	p	1
w_1	0	0	p^{n-2}	p^{n-3}	$\dots$	p^{n-k}	$\dots$	p	1
w_2	0	0	0	p^{n-3}	$\dots$	p^{n-k}	$\dots$	p	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_k	0	0	0	0	$\dots$	p^{n-k}	$\dots$	p	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{n-2}	0	0	0	0	$\dots$	0	$\dots$	p	1
w_{n-1}	0	0	0	0	$\dots$	0	$\dots$	0	1

of $\mathcal{U}_0$ and $\mathcal{Y}_r$ for $1 \le r \le n$. Since $\pi_L(f)$ acts as a scalar for every $f \in H(K//K_0(p^n))$ the vector v_σ will be an eigenvector under the action of $\pi_L(\mathcal{U}_0)$ and $\pi_L(\mathcal{Y}_r)$ for all $1 \le r \le n$. For each σ we can compute these eigenvectors v_σ and their corresponding eigenvalues using the relations in Corollary 3.11 and Proposition 3.12. In fact we obtain the following proposition.

Proposition 3.17 *A basis of eigenvectors for $H(K//K_0(p^n))$ under the above action is given by:*

$$\begin{aligned} v_1 &= \mathcal{U}_0 + \mathcal{Y}_1, \\ v_2 &= \mathcal{U}_0 - p\mathcal{Y}_1, \\ w_k &= \mathcal{Y}_k - p\mathcal{Y}_{k+1} \ for\ 1 \le k \le n-1, \end{aligned}$$

with eigenvalues given by Table 1 where each entry of the table at the intersection of row v and column F stands for the eigenvalue of the action of F on v, for example, $\mathcal{U}_0 * v_1 = p^n v_1$.

We have the following corollary to the above proposition.

Corollary 3.18 *The representation $I(n)$ is a sum of $n+1$ irreducible subspaces given by: $S_1 = \mathrm{Span}(\pi_R(K)v_1)$, $S_2 = \mathrm{Span}(\pi_R(K)v_2)$ and $T_k = \mathrm{Span}(\pi_R(K)w_k)$ where $1 \le k \le n-1$ such that $\dim(S_1) = 1$, $\dim(S_2) = p$, $\dim(T_k) = p^{k-1}(p^2-1)$. By Corollary 3.16, $T_{n-1} = \sigma(n)$ and hence is the unique irreducible representation of K such that T_{n-1} has a $K_0(p^n)$ fixed vector w_{n-1} but does not have $K_0(p^k)$ fixed vector for $k < n$.*

Proof It follows from Table 1 that the set of eigenvalues for vectors v_i for $i = 1,\ 2$ and w_k for $1 \le k \le n-1$ are distinct and hence each of them lies in an irreducible component. To finish the proof we need to compute the dimensions, for which we shall need the following lemma. A statement similar to this lemma appears in [10].

Lemma 3.19 *The operators $\pi_L(\mathcal{U}_0)$ and $\pi_L(\mathcal{V}_r)$ for $1 \le r \le n-1$ have trace zero.*

Proof For $g \in K$, let ϕ_g be the characteristic function of $K_0(p^n)g$, then $I(n)$ as a complex vector space has a basis consisting of ϕ_g as g varies over the right coset representatives of K modulo $K_0(p^n)$. Thus to prove the lemma it is enough to show that $\pi_L(\mathcal{U}_0)(\phi_g)(g) = \pi_L(\mathcal{V}_r)(\phi_g)(g) = 0$. We will show it for $\mathcal{V}_r$, for $\mathcal{U}_0$ the same argument works. It is easy to

Table 2 Action of $\mathcal{U}_0$, $\{\mathcal{Y}_r\}_{1 \le r \le n-1}$ on eigenvectors v_i, w_k

	$\mathcal{U}_0$	$\mathcal{V}_1$	$\dots$	$\mathcal{V}_k$	$\dots$	$\mathcal{V}_{n-2}$	$\mathcal{V}_{n-1}$
v_1	p^n	$p^{n-2}(p-1)$	$\dots$	$p^{n-k-1}(p-1)$	$\dots$	$p(p-1)$	$p-1$
v_2	$-p^{n-1}$	$p^{n-2}(p-1)$	$\dots$	$p^{n-k-1}(p-1)$	$\dots$	$p(p-1)$	$(p-1)$
w_1	0	$-p^{n-2}$	$\dots$	$p^{n-k-1}(p-1)$	$\dots$	$p(p-1)$	$(p-1)$
w_2	0	0	$\dots$	$p^{n-k-1}(p-1)$	$\dots$	$p(p-1)$	$(p-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_k	0	0	$\dots$	$-p^{n-k-1}$	$\dots$	$p(p-1)$	$(p-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_{n-2}	0	0	$\dots$	0	$\dots$	$-p$	$(p-1)$
w_{n-1}	0	0	$\dots$	0	$\dots$	0	-1

see that $\pi_L(\mathcal{V}_r)(\phi_g)$ is supported on $K_0(p^n)y(p^r)K_0(p^n)g$. So if $\pi_L(\mathcal{V}_r)(\phi_g)(g) \neq 0$ then $g \in K_0(p^n)y(p^r)K_0(p^n)g$ which is impossible as $K_0(p^n) \neq K_0(p^n)y(p^r)K_0(p^n)$. □

Using Table 1 in Proposition 3.17, it is easy to obtain Table 2 where we consider the action of $\mathcal{U}_0,\ \mathcal{V}_1,\ \mathcal{V}_2,\ \dots,\ \mathcal{V}_{n-1}$ instead.

Let $d_1, d_2,\ \dots,\ d_{n+1}$ be the dimension of $S_1,\ S_2,\ \dots,\ T_{n-1}$ respectively. Then using Lemma 3.19 and the above table we have the following system of linear equations:

$$\begin{aligned}
p^n d_1 - p^{n-1} d_2 &= 0\\
p^{n-2}(p-1)d_1 + p^{n-2}(p-1)d_2 - p^{n-2}d_3 &= 0\\
&\vdots\\
p^{n-k-1}(p-1)(d_1+d_2+d_3+\cdots+d_{k+1}) - p^{n-k-1}d_{k+2} &= 0\\
&\vdots\\
(p-1)(d_1+d_2+d_3+\cdots+d_n) - d_{n+1} &= 0\\
d_1+d_2+\cdots+d_{n-1} &= p^{n-1}(p+1)
\end{aligned}$$

solving which we get the dimensions. □

4 Translation from the adelic setting to the classical setting

In this section following Gelbart [5] we shall review the connection between automorphic forms and classical modular forms and use this connection to translate p-adic operators of the previous section into their classical counterparts and thereby obtain relations satisfied by them.

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the adele ring of $\mathbb{Q}$ and $Z_{\mathbb{A}}$ denote the center of $\mathrm{GL}_2(\mathbb{A})$. Let $G_\infty = \mathrm{GL}_2(\mathbb{R})^+$. Let N be a positive integer. We let $K_l = \mathrm{GL}_2(\mathbb{Z}_l)$ for a prime l not dividing N and let $K_p = K_0(p^\alpha)$ for a prime p such that $p^\alpha \| N$. Let K_f be the subgroup of $\mathrm{GL}_2(\mathbb{A})$ defined by

$$K_f(N) = \prod_{q<\infty} K_q.$$

By the strong approximation theorem we have

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) G_\infty K_f(N).$$

We denote by $A_{2k}(N)$ the space of functions $\Phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying the following properties:

(1) $\Phi(\gamma z g k) = \Phi(g)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Q}),\ z \in Z_{\mathbb{A}},\ g \in \mathrm{GL}_2(\mathbb{A}),\ k \in K_f(N)$.

(2) $\Phi(gr(\theta)) = e^{-i2k\theta}\Phi(g)$ where $r(\theta) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \in \mathrm{SO}(2)$.

(3) Φ is smooth as a function of G_∞ and satisfies the differential equation $\Delta\Phi = -k(k-1)\Phi$ where Δ is the Casimir operator.

(4) $\Phi \in \mathrm{L}^2(Z_{\mathbb{A}}\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$.

(5) Φ is cuspidal, that is, $\int_{\mathbb{Q}\backslash\mathbb{A}} \Phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A})$.

By Gelbart [5, Proposition 3.1] there exists an isomorphism

$$A_{2k}(N) \to S_{2k}(\Gamma_0(N))$$

given by $\Phi \mapsto f_\Phi$ where for $z \in \mathbb{H}$,

$$f_\Phi(z) = \Phi(g_\infty) j(g_\infty, i)^{2k}$$

where $g_\infty \in G_\infty$ is such that $g_\infty(i) = z$. The inverse map is given by $f \mapsto \Phi_f$ where for $g \in \mathrm{GL}_2(\mathbb{A})$ if $g = \gamma g_\infty k$ (using strong approximation),

$$\Phi_f(g) = f(g_\infty(i)) j(g_\infty, i)^{-2k}.$$

This isomorphism induces a ring isomorphism of spaces of linear operators,

$$q : \mathrm{End}_{\mathbb{C}}(A_{2k}(N)) \to \mathrm{End}_{\mathbb{C}}(S_{2k}(\Gamma_0(N)))$$

given by

$$q(\mathcal{T})(f) = f_{\mathcal{T}(\Phi_f)}.$$

Let $N = p^n M$ where p is a prime coprime to M and $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We note that the $H\,(G//K_0\,(p^n))$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A_{2k}(N))$ via the following action:

$$\text{for } \mathcal{T} \in H(G//K_0\left(p^n\right)) \text{ and } \Phi \in A_{2k}(N),\ \ \mathcal{T}(\Phi)(g) = \int_G \mathcal{T}(x)\Phi(gx)dx.$$

Remark 3 We note that if p_1 and p_2 are distinct primes then the operators $\mathcal{T}_1 \in H(G//K_0(p_1^n))$ and $\mathcal{T}_2 \in H(G//K_0(p_2^n))$ in $\mathrm{End}_{\mathbb{C}}(A_{2k}(N))$ commute, that is, $\mathcal{T}_1 \circ \mathcal{T}_2 = \mathcal{T}_2 \circ \mathcal{T}_1$.

We have the following proposition.

Proposition 4.1 *Let $N = p^n M$ where $n \geq 1$ and $p \nmid M$. Let $f \in S_{2k}(\Gamma_0(N))$. Consider operators $\mathcal{T}_1,\ \mathcal{U}_m \in H(G//K_0(p^n))$ where $m \leq n$. If $n \geq 2$, further consider $\mathcal{V}_r \in H(G//K_0(p^n))$ where $1 \leq r \leq n-1$. Then,*

(1) $q(\mathcal{T}_1)(f)(z) = p^{-k}\sum_{s=0}^{p-1} f((z+s)/p) = \tilde{U}_p(f)(z)$.
(2) *If* $f \in S_{2k}(\Gamma_0(p^r M))$ *where* $r \le n$ *then* $q(\mathcal{U}_r)(f)(z) = p^{n-r} f|_{2k} W_{p^r}(z)$ *where* $W_{p^r} = \begin{pmatrix} p^r\beta & 1 \\ p^r M\gamma & p^r \end{pmatrix}$ *is an integer matrix of determinant* p^r. *In particular,* $q(\mathcal{U}_n)(f)(z) = f|_{2k} W_{p^n}(z)$.
(3) $q(\mathcal{V}_r)(f)(z) = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} f|_{2k} A_s$ *where* $A_s \in \mathrm{SL}_2(\mathbb{Z})$ *is any matrix of the form* $\begin{pmatrix} a_s & b_s \\ p^r M & p^{n-r} - sM \end{pmatrix}$.
(4) *If* $f \in S_{2k}(\Gamma_0(p^r M))$ *then* $q(\mathcal{V}_r)(f) = p^{n-r-1}(p-1)f$, *consequently,* $q(\mathcal{Y}_r)(f) = p^{n-r} f$.

Proof For $\Phi \in A_{2k}(N)$, using decomposition in Lemma 3.13 we have

$$\mathcal{T}_1(\Phi)(g) = \int_G X_{d(p)}(x)\Phi(gx)dx = \sum_{s=0}^{p-1} \Phi(gx(-s)d(p)).$$

Thus

$$q(\mathcal{T}_1)(f)(z) = f_{\mathcal{T}_1(\Phi_f)}(z) = \sum_{s=0}^{p-1} \Phi_f(g_\infty x(-s)d(p))j(g_\infty, i)^{2k}$$

where $g_\infty \in G_\infty$ such that $g_\infty i = z$. Since Φ_f is invariant under left multiplication by rational matrices, multiplying by $\gamma = d(p^{-1})x(s) \in \mathrm{GL}_2(\mathbb{Q})$ we obtain

$$\Phi_f(g_\infty x(-s)d(p)) = \Phi_f(d(p^{-1})x(s)g_\infty \cdot k_f) = \Phi_f(d(p^{-1})x(s)g_\infty)$$

where $k_f \in K_f(N)$. Thus we have

$$\begin{aligned} q(\mathcal{T}_1)(f)(z) &= \sum_{s=0}^{p-1} \Phi_f(d\left(p^{-1}\right)x(s)g_\infty)j(g_\infty, i)^{2k} \\ &= \sum_{s=0}^{p-1} f\left(d\left(p^{-1}\right)x(s)z\right) j\left(d\left(p^{-1}\right)x(s), z\right)^{-2k} = p^{-k}\sum_{s=0}^{p-1} f((z+s)/p). \end{aligned}$$

The proof of (2) is similar. Let $f \in S_{2k}(\Gamma_0(p^r M))$ where $1 \le r \le n$. Then using Lemma 3.13,

$$q(\mathcal{U}_r)(f)(z) = \sum_{s=0}^{p^{n-r}-1} \Phi_f(g_\infty x(s)w(p^r))j(g_\infty, i)^{2k}$$

where $z = g_\infty i$. Let $W_{p^r} = \begin{pmatrix} p^r\beta & 1 \\ p^r M\gamma & p^r \end{pmatrix}$ be an integer matrix of determinant p^r. Since $\Phi_f \in A_{2k}(p^r M)$ multiplying $g_\infty x(s)w(p^r)$ by the matrix $W_{p^r} z(p^{-r})x(-s) \in \mathrm{GL}_2(\mathbb{Q})$ we get that,

$$\Phi_f(g_\infty x(s)w(p^r)) = \Phi_f(h_\infty k_f) = \Phi_f(h_\infty)$$

where $h_\infty = z(p^{-r})W_{p^r}x(-s)g_\infty \in G_\infty$ and $k_f \in K_f(p^r M)$. Since $f|_{2k}W_{p^r} \in S_{2k}(\Gamma_0(p^r M))$,

$$q(\mathcal{U}_r)(f)(z) = \sum_{s=0}^{p^{n-r}-1} f|_{2k}W_{p^r}x(-s)(z) = p^{n-r} f|_{2k}W_{p^r}(z).$$

For (3), let $n \geq 2$. Using Lemmas 3.7 and 3.9 we have

$$\mathcal{V}_r(\Phi)(g) = \int_G X_{y(p^r)}\Phi(gh)dh = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \Phi(gd(s)y(p^r)).$$

Let $z \in \mathbb{H}$ be such that $z = g_\infty i$ for some $g_\infty \in \mathrm{G}_\infty$. Then,

$$q(\mathcal{V}_r)(f)(z) = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \Phi_f(g_\infty d(s)y(p^r))j(g_\infty, i)^{2k}.$$

By the strong approximation, $g_\infty d(s)y(p^r) = A_s^{-1}h_\infty k_f$ for some $A_s \in \mathrm{GL}_2(\mathbb{Q}), h_\infty \in \mathrm{G}_\infty$ and $k_f \in K_f(N)$. So we need $A_s \in \mathrm{GL}_2(\mathbb{Q})$ such that $A_s d(s)y(p^r)$ belongs to $K_0(p^n)$ and A_s belongs to K_q for $q \neq p$. So we must choose A_s with determinant 1. For any $s \in \mathbb{Z}_p^*$, we have $\gcd(p^r M, p^{n-r} - sM) = 1$, so there exists integers a_s, b_s such that $a_s(p^{n-r} - sM) - b_s p^r M = 1$. Take

$$A_s = \begin{pmatrix} a_s & b_s \\ p^r M & p^{n-r} - sM \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

then A_s belongs to K_q for $q \neq p$ and

$$A_s d(s)y(p^r) = \begin{pmatrix} a_s + b_s p^r & b_s \\ p^n & p^{n-r} - sM \end{pmatrix} \in K_0.$$

Thus

$$\Phi_f(g_\infty d(s)y(p^r)) = f(A_s z)j(A_s, z)^{-2k} j(g_\infty, i)^{-2k},$$

and so

$$q(\mathcal{V}_r)(f)(z) = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} f(A_s z)j(A_s, z)^{-2k} = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} f|_{A_s}(z).$$

Thus if $f \in S_{2k}(\Gamma_0(p^r M))$ then $q(\mathcal{V}_r)(f)(z) = p^{n-r-1}(p-1)f$. Further,

$$q(\mathcal{Y}_r)(f) = f + \sum_{j=r}^{n-1} \sum_{s\in\mathbb{Z}_p^*/1+p^{n-j}\mathbb{Z}_p} f|_{2k} \begin{pmatrix} a_{s,j} & b_{s,j} \\ p^j M & p^{n-j} - sM \end{pmatrix}$$

$$= f + \sum_{j=r}^{n-1}(p-1)p^{n-j-1} f = p^{n-r} f,$$

proving (4). □

Remark 4 The operator $q(\mathcal{U}_n)$ is the usual Atkin–Lehner operator W_{p^n} while the operator $q(\mathcal{T}_1)$ is the operator $\tilde{U}_p = p^{1-k}U_p$ where U_p is the usual Hecke operator. It is obvious that $q(\mathcal{Z})$ is the identity operator.

Let $N = pM$ where $p \nmid M$. Let $Q_p = q(\mathcal{U}_0)$ where $\mathcal{U}_0 \in H(G//K_0(p))$. Then using Lemma 3.5 we have

Corollary 4.2 *$Q_p = p^{1-k}U_p W_p$ and $(Q_p - p)(Q_p + 1) = 0$.*

Now consider $N = p^n M$ where $n \geq 2$. Let $Q_{p^m} = (\tilde{U}_p)^m W_{p^m}$ for $m \leq n$ where W_{p^m} is the Atkin–Lehner operator on $S_{2k}(\Gamma_0(p^m M))$. Using Lemma 3.13, Propositions 4.1 and 3.12 we have

Corollary 4.3 *For $\mathcal{U}_0 \in H(G//K_0(p^n))$, we have $Q_{p^n} = q(\mathcal{U}_0)$ and hence $Q_{p^n}(Q_{p^n} - p^n)(Q_{p^n} + p^{n-1}) = 0$. Further for $m \leq n$ we have $Q_{p^n} = (\tilde{U}_p)^m q(\mathcal{U}_m)$, hence if $f \in S_{2k}(\Gamma_0(p^m M)) \subseteq S_{2k}(\Gamma_0(N))$ then $Q_{p^n}(f) = p^{n-m} Q_{p^m}(f)$.*

Let $S_{p^n,r} = q(\mathcal{Y}_r)$ where $\mathcal{Y}_r \in H(G//K_0(p^n))$, $1 \leq r \leq n$. Using relations in Corollary 3.11, we have

Corollary 4.4 $S_{p^n,r}(S_{p^n,r} - p^{n-r}) = 0$ *for* $1 \leq r \leq n$.

5 Eigenspaces of classical operators and the characterization of the new space

Let N be a positive integer. In this section we shall look at the classical operators on $S_{2k}(\Gamma_0(N))$ that were introduced in Sect. 2 and Proposition 4.1 and study their eigenspaces. We shall prove the theorems stated in Sect. 2 including our main result Theorem 3.

5.1 N square-free

Let N be a square-free positive integer and S be the set of prime divisors of N. Let $p \in S$. Recall that translating $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{T}_1 \in H(G//K_0(p))$ we respectively obtained the classical operators Q_p, W_p and $\tilde{U}_p$ on $S_{2k}(\Gamma_0(N))$. For N, d any positive integers recall the shift operator $V(d) : S_{2k}(\Gamma_0(N)) \to S_{2k}(\Gamma_0(dN))$ given by $V(d)(f) = d^{-k} f|_{2k} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. It is well known [1] that the old space

$$\begin{aligned} S_{2k}^{\text{old}}(\Gamma_0(N)) &= \bigoplus_{dM|N,\ M \neq N} V(d) S_{2k}^{\text{new}}(\Gamma_0(M)) \\ &= \sum_{p_i \in S} S_{2k}(\Gamma_0(N/p_i)) + V(p_i) S_{2k}(\Gamma_0(N/p_i)). \end{aligned} \tag{1}$$

We will consider the action of Q_p on each of the above summands.

Lemma 5.1 *Let $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ be a newform. Then $Q_p(f) = -f$, that is, $S_{2k}^{\text{new}}(\Gamma_0(N))$ is contained in the -1 eigenspace of Q_p.*

Proof By [1, lemma 18], $S_{2k}^{\text{new}}(\Gamma_0(N))$ has a basis of primitive forms, so we can assume that f is primitive. By [1, Theorem 3], $W_p(f) = \lambda(p) f$ for some $\lambda(p) = \pm 1$ and $U_p(f) = -\lambda(p) p^{k-1} f$. Since $Q_p = p^{1-k} U_p W_p$ the result follows. □

Write $N = pM$ where M is a square-free integer coprime to p.

Lemma 5.2 *Let f be a form in $S_{2k}(\Gamma_0(M)) \subset S_{2k}(\Gamma_0(N))$. Then $Q_p(f) = pf$.*

Proof Since $\begin{pmatrix} \beta & 1 \\ M\gamma & p \end{pmatrix} \in \Gamma_0(M)$ we have

$$\begin{aligned} W_p(f)((z+s)/p) &= p^k (N\gamma(z+s)/p + p)^{-2k} f\left(\frac{p\beta(z+s)/p + 1}{N\gamma(z+s)/p + p}\right) \\ &= p^k (M\gamma(z+s) + p)^{-2k} f\left(\frac{\beta(z+s) + 1}{M\gamma(z+s) + p}\right) \\ &= p^k f|_{2k} \begin{pmatrix} \beta & 1 \\ M\gamma & p \end{pmatrix}(z+s) = p^k f(z+s) = p^k f(z). \end{aligned}$$

Hence

$$Q_p(f) = p^{-k} \sum_{s=0}^{p-1} W_p(f)((z+s)/p) = pf(z).$$

□

Next we look at the action of Q_p on the old subspace $V(p)(S_{2k}(\Gamma_0(M)))$.

Lemma 5.3 *Let $f \in S_{2k}(\Gamma_0(M))$ and $g(z) = f(pz) \in V(p)(S_{2k}(\Gamma_0(M)))$. Then*

$$Q_p(g) = p^{1-2k} T_p(f) - g,$$

where T_p is the usual Hecke operator on $S_{2k}(\Gamma_0(M))$.

Proof Note that from [1, Lemma 14],

$$p^{1-2k} T_p(f) = f(pz) + p^{-2k} \sum_{s=0}^{p-1} f((z+s)/p).$$

As before, we have

$$\begin{aligned} W_p(g)((z+s)/p) &= p^k (N\gamma(z+s)/p + p)^{-2k} g\left(\frac{p\beta(z+s)/p+1}{N\gamma(z+s)/p+p}\right) \\ &= p^k (M\gamma(z+s)+p)^{-2k} f\left(\frac{p\beta(z+s)+p}{M\gamma(z+s)+p}\right) \\ &= p^{-k} (M\gamma(z+s)/p+1)^{-2k} f\left(\frac{p\beta(z+s)/p+1}{M\gamma(z+s)/p+1}\right) \\ &= p^{-k} f|_{2k} \begin{pmatrix} p\beta & 1 \\ M\gamma & 1 \end{pmatrix} ((z+s)/p) = p^{-k} f((z+s)/p) \end{aligned}$$

since $\begin{pmatrix} p\beta & 1 \\ M\gamma & 1 \end{pmatrix} \in \Gamma_0(M)$. Thus

$$Q_p(g)(z) = p^{-2k} \sum_{s=0}^{p-1} f((z+s)/p) = p^{1-2k} T_p(f) - g.$$

□

We consider the subspace $X_p := S_{2k}(\Gamma_0(N/p)) \oplus V(p) S_{2k}(\Gamma_0(N/p))$ of $S_{2k}(\Gamma_0(N))$.

Corollary 5.4 *Q_p stabilizes X_p and the -1 eigenspace of Q_p inside X_p consists of forms $h(z) = -\frac{p^{1-2k}}{p+1} T_p(f)(z) + f(pz)$ where $f \in S_{2k}(\Gamma_0(M))$.*

Proof Let $h \in X_p$ be an old form. Then h can be uniquely written as $h(z) = f_1(z) + g(z)$ where $g(z) = f(pz)$ for some f, $f_1 \in S_{2k}(\Gamma_0(M))$. By Lemmas 5.2 and 5.3 we have $Q_p(h) = pf_1 + p^{1-2k} T_p(f) - g$ which is clearly in X_p.

Further since the above decomposition for $Q_p(h)$ is unique, if $f_1(z) + g(z)$ is an eigenfunction of Q_p with $g \neq 0$ then $Q_p(h) = -h$ and $f_1 = -\frac{p^{1-2k}}{p+1} T_p(f)$. Hence the -1 eigenspace of Q_p consists of forms $h(z) = -\frac{p^{1-2k}}{p+1} T_p(f)(z) + f(pz)$ for some $f \in S_{2k}(\Gamma_0(M))$. □

From Lemma 5.2 and Corollary 5.4 we obtain the following proposition.

Proposition 5.5 *The p eigenspace of Q_p in X_p is $S_{2k}(\Gamma_0(M))$.*

Next consider the operator $Q'_p = W_p Q_p W_p^{-1} = W_p Q_p W_p$. So, $Q'_p = W_p \tilde{U}_p$ and it satisfies the equation $(Q'_p - p)(Q'_p - 1) = 0$. Note that f is an eigenfunction of Q_p with eigenvalue λ if and only if $W_p(f)$ is an eigenfunction of Q'_p with eigenvalue λ. Since the action of Atkin–Lehner operator W_p on the space of new forms is surjective, Q'_p acts with the eigenvalue -1 on the space of new forms. We have the following lemma.

Lemma 5.6 *Let $f \in S_{2k}(\Gamma_0(M))$. Then $W_p(f)(z) = p^k f(pz)$. Further, if $g = f(pz)$ then $W_p(g)(z) = p^{-k} f(z)$. Consequently W_p maps $S_{2k}(\Gamma_0(M))$ onto $V(p)S_{2k}(\Gamma_0(M))$, so $V(p)S_{2k}(\Gamma_0(M))$ is contained in the p eigenspace of Q'_p. The operator Q'_p preserves X_p and the p eigenspace of Q'_p in X_p is the space $V(p)(S_{2k}(\Gamma_0(M))$.*

Proof Since $\begin{pmatrix} \beta & 1 \\ M\gamma & p \end{pmatrix} \in \Gamma_0(M)$ we get

$$\begin{aligned} W_p(f)(z) &= f|_{2k} \begin{pmatrix} p\beta & 1 \\ N\gamma & p \end{pmatrix}(z) = p^k (M\gamma(pz)+p)^{-2k} f\left(\frac{\beta(pz)+1}{M\gamma(pz)+p}\right) \\ &= p^k f|_{2k} \begin{pmatrix} \beta & 1 \\ M\gamma & p \end{pmatrix}(pz) = p^k f(pz). \end{aligned}$$

Further, since $\begin{pmatrix} p\beta & 1 \\ M\gamma & 1 \end{pmatrix} \in \Gamma_0(M)$ we get

$$\begin{aligned} W_p(g)(z) &= g|_{2k} \begin{pmatrix} p\beta & 1 \\ N\gamma & p \end{pmatrix}(z) = p^k (N\gamma z+p)^{-2k} f\left(\frac{p^2\beta z+p}{N\gamma z+p}\right) \\ &= p^{-k} (M\gamma z+1)^{-2k} f\left(\frac{p\beta z+1}{M\gamma z+1}\right) = p^{-k} f(z). \end{aligned}$$

Hence $W_p(X_p) = X_p$ and so Q'_p preserves X_p. It now follows from Proposition 5.5 that the p eigenspace of Q'_p in X_p is precisely $V(p)(S_{2k}(\Gamma_0(M))$. □

We shall need the following proposition.

Proposition 5.7 *The operators $Q_p = \tilde{U}_p W_p$ and $Q'_p = W_p Q_p W_p^{-1}$ are self-adjoint with respect to the Petersson inner product.*

Proof Recall that $\tilde{U}_p = p^{1-k} U_p = p^{1-k} T_p$. Following Miyake [12, Page 135]

$$T_p = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N)$$

and for $f \in S_{2k}(\Gamma_0(N))$,

$$T_p(f) = f|_{2k} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N) = p^{k-1} \sum_{m=0}^{p-1} f|_{2k} \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}.$$

Further,

$$T_p^* = \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N).$$

For $f,\ g \in S_{2k}(\Gamma_0(N))$, by [12, Theorem 4.5.4], $\langle T_p(f),\ g\rangle = \langle f,\ T_p^*(g)\rangle$.

The Atkin–Lehner operator W_p acts by a matrix $\begin{pmatrix} p\beta & 1 \\ N\gamma & p \end{pmatrix}$ such that $p^2\beta - N\gamma = p$. We want to show that the following diagram commutes:

$$\begin{array}{ccc} S_{2k}(\Gamma_0(N)) & \xrightarrow{T_p} & S_{2k}(\Gamma_0(N)) \\ {\scriptstyle W_p}\downarrow & & \downarrow{\scriptstyle W_p} \\ S_{2k}(\Gamma_0(N)) & \xrightarrow{T_p^*} & S_{2k}(\Gamma_0(N)) \end{array}$$

We have

$$\begin{aligned} f|_{2k} W_p^{-1} T_p W_p &= f|_{2k} W_p^{-1} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N) W_p \\ &= f|_{2k} \Gamma_0(N) W_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} W_p \Gamma_0(N) \end{aligned} \tag{2}$$

since $W_p \Gamma_0(N) W_p^{-1} = \Gamma_0(N)$.

We claim that $\Gamma_0(N) W_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} W_p \Gamma_0(N) = \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N)$. We note that

$$W_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} W_p = \begin{pmatrix} p\beta - N\gamma & 1 - p \\ -N\gamma\beta + pN\gamma\beta & -\frac{N\gamma}{p} + \beta p^2 \end{pmatrix}.$$

Choose $t \in \mathbb{Z}$ such that $t \equiv \beta M^{-1} \pmod{p}$ and consider the matrix $\begin{pmatrix} 1 & 0 \\ Nt & 1 \end{pmatrix}$ in $\Gamma_0(N)$. Then,

$$\begin{aligned} & W_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} W_p \cdot \begin{pmatrix} 1 & 0 \\ Nt & 1 \end{pmatrix} \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} p\beta - N\gamma & 1 - p \\ -N\gamma\beta + pN\gamma\beta & -\frac{N\gamma}{p} + \beta p^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{p} & 0 \\ \frac{Nt}{p} & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ -N\gamma\left(\frac{\beta - Mt}{p}\right) + N\gamma\beta + \beta pNt & * \end{pmatrix} \in \Gamma_0(N). \end{aligned}$$

Hence $W_p^{-1} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} W_p \in \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N)$ and our claim is proved.

Thus from (2), we have $f|_{2k} W_p^{-1} T_p W_p = f|_{2k} \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(N) = T_p^*(f)$. Using this, and that W_p is self-adjoint and it is an involution, we get that

$$\begin{aligned} \langle Q_p(f),\ g \rangle &= p^{1-k} \langle T_p W_p(f),\ g \rangle \\ &= p^{1-k} \langle W_p(f),\ T_p^*(g) \rangle \\ &= p^{1-k} \langle W_p(f),\ W_p T_p W_p^{-1}(g) \rangle \\ &= p^{1-k} \langle f,\ T_p W_p^{-1}(g) \rangle \\ &= p^{1-k} \langle f,\ T_p W_p(g) \rangle = \langle f,\ Q_p(g) \rangle. \end{aligned}$$

Hence Q_p and consequently Q'_p are self-adjoint. □

We now restate Theorem 1 and prove it below.

Theorem 8 *Let $N = p_1 p_2 \cdots p_r$ with p_i distinct primes. Then the space of new forms $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_{p_i} and Q'_{p_i} as $1 \le i \le r$. That is, $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ if and only if $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \le i \le r$.*

Proof We have already seen that if $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ then $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \le i \le r$.

Further it follows from Proposition 5.5 and Lemma 5.6 that for each p_i, the subspace $S_{2k}(\Gamma_0(N/p_i))$ is contained in the p_i eigenspace of Q_{p_i} and $V(p_i)S_{2k}(\Gamma_0(N/p_i))$ is contained in the p_i eigenspace of Q'_{p_i}.

Suppose $f \in S_{2k}(\Gamma_0(N))$ is such that $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \le i \le r$. Since Q_{p_i} and Q'_{p_i} are self-adjoint operators on $S_{2k}(\Gamma_0(N))$ we get that the p_i eigenspaces of Q_{p_i} and Q'_{p_i} are respectively orthogonal to the -1 eigenspaces of Q_{p_i} and Q'_{p_i}. Hence f is orthogonal to $S_{2k}(\Gamma_0(N/p_i))$ and $V(p_i)S_{2k}(\Gamma_0(N/p_i))$ for all $1 \le i \le r$. Thus f is orthogonal to the old space $S_{2k}^{\mathrm{old}}(\Gamma_0(N))$, that is, $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$. □

5.2 General case

Let N be a positive integer and p be a prime such that p^n strictly divides N, that is, $N = p^n M$ for some positive integer M coprime to p. Let $n \ge 2$. Recall that translating $\mathcal{U}_0$, $\mathcal{U}_n$, $\mathcal{T}_1$ and $\mathcal{Y}_r \in H(G//K_0(p^n))$ where $1 \le r \le n$, we respectively obtained the classical operators Q_{p^n}, W_{p^n}, $\tilde{U}_p$ and $S_{p^n,r}$ on $S_{2k}(\Gamma_0(N))$.

We have the following lemma.

Lemma 5.8 *For $1 \le r \le n$, a set of right coset representatives for $\Gamma_0(N)$ in $\Gamma_0(p^r M)$ consists of the identity element and elements of the form*

$$A_{s,j} = \begin{pmatrix} a_{s,j} & b_{s,j} \\ p^j M & p^{n-j} - sM \end{pmatrix} \text{ where } r \le j \le n-1 \text{ and } s \in \mathbb{Z}_p^*/1 + p^{n-j}\mathbb{Z}_p.$$

Proof First we check that the right cosets $\Gamma_0(N)$ and $\Gamma_0(N)A_{s,j}$ where j, s varies as above are mutually disjoint. For any such j and s clearly $A_{s,j} \in \Gamma_0(p^r M) \setminus \Gamma_0(N)$, hence $\Gamma_0(N)A_{s,j}$ and $\Gamma_0(N)$ are disjoint.

Now for any $r \le i,\ j \le n-1$ we have

$$\Gamma_0(N)A_{s,j} = \Gamma_0(N)A_{t,i} \iff p^j M(p^{n-i} - tM) - p^i M(p^{n-j} - sM) \in p^n M\mathbb{Z}_p.$$

Now if $i \ne j$, say $i > j$, then the equality of the above two cosets implies that $-tM \in p\mathbb{Z}_p$, leading to a contradiction.

Similarly, for $r \le j \le n-1$ we have

$$\begin{aligned} \Gamma_0(N)A_{s,j} &= \Gamma_0(N)A_{t,j} \iff p^j M(p^{n-j} - tM) - p^j M(p^{n-j} - sM) \in p^n M\mathbb{Z}_p \\ &\iff t \equiv s \pmod{p^{n-j}\mathbb{Z}_p} \iff t = s \in \mathbb{Z}_p^*/(1 + p^{n-j}\mathbb{Z}_p). \end{aligned}$$

Hence all the right cosets listed are mutually disjoint.

It is well known that $[\Gamma_0(p^r M) : \Gamma_0(N)] = p^{n-r}$ ([12, Theorem 4.2.5]). Since we have already checked that the right cosets $\Gamma_0(N)$, $\Gamma_0(N)A_{s,j}$ where $j,\ s$ varies as above are mutually disjoint and since there are exactly p^{n-r} of them the lemma follows. □

Lemma 5.9 *For $1 \le r \le n$, the operator $S_{p^n,r}$ takes the space $S_{2k}(\Gamma_0(N))$ to $S_{2k}(\Gamma_0(p^r M))$.*

Proof The proof follows using Lemma 5.8 and [1, Lemma 3]. □

Corollary 5.10 *For $1 \le r \le n$, the p^{n-r} eigenspace of $S_{p^n,r}$ is precisely the subspace $S_{2k}(\Gamma_0(p^r M))$.*

Proof It follows from Proposition 4.1 that $S_{2k}(\Gamma_0(p^r M))$ is contained in the p^{n-r} eigenspace of $S_{p^n,r}$. Let $f \in S_{2k}(\Gamma_0(N))$ be such that $S_{p^n,r}(f) = p^{n-r}(f)$. By Lemma 5.9, $S_{p^n,r}(f)$ belongs to $S_{2k}(\Gamma_0(p^r M))$. Thus $f \in S_{2k}(\Gamma_0(p^r M))$. □

Proposition 5.11 *Let $1 \le r \le n$. Then for each $r < \alpha \le n$, the space $S_{2k}^{\mathrm{new}}(\Gamma_0(p^\alpha M))$ is contained in the 0 eigenspace of $S_{p^n,r}$.*

Proof Let q be any prime that is coprime to N, then the Hecke operator T_q on $S_{2k}(\Gamma_0(N))$ corresponds to $\mathcal{T}_{(q)}$, the characteristic function of the double coset $\mathrm{GL}_2(\mathbb{Z}_q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_q)$ in the q-adic Hecke algebra $H(\mathrm{GL}_2(\mathbb{Z}_q))$. Since $\mathcal{Y}_r = \mathcal{Y}_{r(p)}$ belongs to the p-adic Hecke algebra $H(K_0(p^n))$, it follows from Remark 3 that the operators $\mathcal{T}_{(q)}$ and $\mathcal{Y}_{r(p)}$ commute and hence the operators $S_{p^n,r}$ and T_q on $S_{2k}(\Gamma_0(N))$ commute.

Let $r < \alpha \le n$ and $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(p^\alpha M))$ be a primitive form. Thus f is an eigenform with respect to T_q for any q coprime to N. Now since $S_{p^n,r}$ and T_q commute we get that $S_{p^n,r}(f)$ is also an eigenfunction with respect to all such T_q having the same eigenvalue as f.

By Corollary 5.9, $S_{p^n,r}(f) \in S_{2k}(\Gamma_0(p^r M))$ and as $r < \alpha$, it is an old form in the space $S_{2k}(\Gamma_0(p^\alpha M))$. It now follows from [1, Lemma 23] that $S_{p^n,r}(f) = 0$.

The proposition follows since $S_{2k}^{\mathrm{new}}(\Gamma_0(p^\alpha M))$ has a basis of primitive forms. □

Next consider the operator $S'_{p^n,r} = W_{p^n} S_{p^n,r} W_{p^n}^{-1} = W_{p^n} S_{p^n,r} W_{p^n}$. Then $S'_{p^n,r}$ clearly satisfies the equation $S'_{p^n,r}(S'_{p^n,r} - p^{n-r}) = 0$. Since the action of Atkin–Lehner operator W_{p^n} on the space of new forms is surjective, in particular we get that the space $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is contained in the 0 eigenspace of $S'_{p^n,n-1}$. We have the following lemma.

Lemma 5.12 *For $0 \le r \le n$, the operator W_{p^n} maps $S_{2k}(\Gamma_0(p^r M))$ onto $V(p^{n-r})$ $S_{2k}(\Gamma_0(p^r M))$ and takes the new space $S_{2k}^{\mathrm{new}}(\Gamma_0(p^r M))$ onto $V(p^{n-r})S_{2k}^{\mathrm{new}}(\Gamma_0(p^r M))$.*

Further, W_{p^n} maps the space $V(p^r)S_{2k}(\Gamma_0(M))$ onto $V(p^{n-r})S_{2k}(\Gamma_0(M))$.

Consequently for $1 \le r \le n$, the p^{n-r} eigenspace of $S'_{p^n,r}$ is precisely the space $V(p^{n-r})S_{2k}(\Gamma_0(p^r M))$.

Proof Let $r \ge 1$ be as above. Let $f \in S_{2k}(\Gamma_0(p^r M))$. Then,

$$\begin{aligned} W_{p^n}(f)(z) &= f\left(\frac{p^n\beta z + 1}{N\gamma z + p^n}\right)(N\gamma z + p^n)^{-2k} p^{nk} \\ &= f\left(\frac{p^r\beta(p^{n-r}z) + 1}{p^r M\gamma(p^{n-r}z) + p^n}\right)(N\gamma z + p^n)^{-2k} p^{nk} \\ &= p^{(n-r)k} f|_{2k}\begin{pmatrix} p^r\beta & 1 \\ p^r M\gamma & p^n\end{pmatrix}(p^{n-r}z) = p^{(n-r)k} f|_{2k} W_{p^r}(p^{n-r}z)\end{aligned}$$

which clearly belongs to $V(p^{n-r})S_{2k}(\Gamma_0(p^r M))$.

Note that since W_{p^r} is an involution on $S_{2k}(\Gamma_0(p^r M))$, it is a surjection, i.e, any $f \in S_{2k}(\Gamma_0(p^r M))$ is of the form $f'|_{2k} W_{p^r}$ for some $f' \in S_{2k}(\Gamma_0(p^r M))$. Let $g(z) = f(p^{n-r}z)$ where $f \in S_{2k}(\Gamma_0(p^r M))$. Then by above computation,

$$g(z) = f'|_{2k} W_{p^r}(p^{n-r}z) = p^{(r-n)k} W_{p^n}(f')(z).$$

Thus $W_{p^n}(g)(z) = p^{(r-n)k}(f')(z) = p^{(r-n)k} f|_{2k} W_{p^r}(z)$.

It is clear from the above that if $f \in S_{2k}(\Gamma_0(M))$ then $W_{p^n}(f)(z) = p^{nk} f(p^n z)$ and conversely if $g = f(p^n z)$ then $W_{p^n}(g) = p^{-nk} f$, proving the statement for $r = 0$. Moreover Atkin–Lehner involutions W_{p^r} are surjection on new spaces and hence takes $S_{2k}^{\text{new}}(\Gamma_0(p^r M))$ onto $V(p^{n-r}) S_{2k}^{\text{new}}(\Gamma_0(p^r M))$.

The proof of the second statement follows similarly. For the final statement let $1 \le r \le n$. Now h is in the p^{n-r} eigenspace of $S'_{p^n,r}$ if and only if $W_{p^n}(h)$ is in the p^{n-r} eigenspace of $S_{p^n,r}$. By Corollary 5.10, this is same as $W_{p^n}(h) \in S_{2k}(\Gamma_0(p^r M))$, that is, $h \in V(p^{n-r}) S_{2k}(\Gamma_0(p^r M))$. □

Applying the above results to the case $r = n - 1$ we have the following corollary.

Corollary 5.13 *The space $S_{2k}(\Gamma_0(p^{n-1}M))$ is the p eigenspace of $S_{p^n,n-1}$ and $V(p)S_{2k}(\Gamma_0(p^{n-1}M))$ is the p eigenspace of $S'_{p^n,n-1}$. Moreover, the space $S_{2k}^{\text{new}}(\Gamma_0(N))$ is contained in the intersection of the 0 eigenspaces of $S_{p^n,n-1}$ and $S'_{p^n,n-1}$.*

Next we have the following proposition.

Proposition 5.14 *The operators $S_{p^n,n-1}$ and $S'_{p^n,n-1}$ are self-adjoint with respect to the Petersson inner product.*

Proof Since $S_{p^n,n-1} = I + q(\mathcal{V}_{n-1})$, it is enough to prove that $q(\mathcal{V}_{n-1})$ on $S_{2k}(\Gamma_0(N))$ is self-adjoint. Recall that

$$q(\mathcal{V}_{n-1})(f) = \sum_{s=1}^{p-1} f|_{2k} A_s \quad \text{where } A_s = \begin{pmatrix} a_s & b_s \\ p^{n-1}M & p - sM \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

By [12, Theorem 2.8.2], $\langle q(\mathcal{V}_{n-1})(f), g\rangle = \langle f, \sum_{s=1}^{p-1} g|_{2k} A_s^{-1}\rangle$. We claim that for any $f \in S_{2k}(\Gamma_0(N))$ we have $\sum_{s=1}^{p-1} f|_{2k} A_s = \sum_{t=1}^{p-1} f|_{2k} A_t^{-1}$. Note that for each $1 \le t \le p-1$, the choice of a_t is unique mod p. Let $1 \le s \le p - 1$ be such that $s \equiv a_t M^{-1} \pmod{p}$. As t varies from 1 to $p - 1$, so does s. Now it is easy to see that

$$s \equiv a_t M^{-1} \pmod{p} \iff A_s A_t \in \Gamma_0(N) \iff f|_{2k} A_s = f|_{2k} A_t^{-1},$$

proving our claim. Thus

$$\langle q(\mathcal{V}_{n-1})(f), g\rangle = \langle f, q(\mathcal{V}_{n-1})g\rangle,$$

and so $S_{p^n,n-1}$ is self-adjoint. Since the Atkin–Lehner operator W_{p^n} is self-adjoint, it follows that $S'_{p^n,n-1}$ is also self-adjoint. □

Now we restate and give a proof of Theorem 3 of which Theorem 2' is a particular case.

Theorem 9 *Let $N = p_1 p_2 \cdots p_r q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ with p_i and q_j distinct primes and $\alpha_j \ge 2$ for all $1 \le j \le s$. Then the space of new forms $S_{2k}^{\text{new}}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_{p_i} and Q'_{p_i} as $1 \le i \le r$ and the 0 eigenspaces of $S_{{q_j}^{\alpha_j},\alpha_j-1}$ and $S'_{{q_j}^{\alpha_j},\alpha_j-1}$ for all $1 \le j \le s$. That is, $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \le i \le r$ and $S_{{q_j}^{\alpha_j},\alpha_j-1}(f) = 0 = S'_{{q_j}^{\alpha_j},\alpha_j-1}(f)$ for all $1 \le j \le s$.*

Proof We have already seen one side implication. Conversely suppose $f \in S_{2k}(\Gamma_0(N))$ is such that $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \le i \le r$ and $S_{{q_j}^{\alpha_j},\alpha_j-1}(f) = 0 = S'_{{q_j}^{\alpha_j},\alpha_j-1}(f)$ for all $1 \le j \le s$. It follows from the previous subsection that for each

$1 \le i \le r$, $S_{2k}(\Gamma_0(N/p_i))$ is contained in the p_i eigenspace of Q_{p_i} and $V(p_i)S_{2k}(\Gamma_0(N/p_i))$ is contained in the p_i eigenspace of Q'_{p_i}. Also from Corollary 5.13, for each $1 \le j \le s$, we get that $S_{2k}(\Gamma_0(N/q_j))$ is contained in the q_j eigenspace of $S_{{q_j}^{\alpha_j},\alpha_j-1}$ and $V(q_j)S_{2k}(\Gamma_0(N/q_j))$ is contained in the q_j eigenspace of $S'_{{q_j}^{\alpha_j},\alpha_j-1}$.

Since Q_{p_i}, Q'_{p_i} and $S_{{q_j}^{\alpha_j},\alpha_j-1}$, $S'_{{q_j}^{\alpha_j},\alpha_j-1}$ are self-adjoint operators we get that f is orthogonal to $S_{2k}(\Gamma_0(N/p))$ and $V(p)S_{2k}(\Gamma_0(N/p))$ for each prime divisor p of N. Thus f is orthogonal to the old space, that is, $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$. □

Next we consider N such that any prime divisor divides it with a power at most 2. Let p be a prime such that $N = p^2M$, so $(p, M) = 1$. Recall that $Q_{p^2} = (\tilde{U}_p)^2 W_{p^2}$ and $Q_{p^2}(Q_{p^2} - p^2)(Q_{p^2} + p) = 0$. It follows from Corollary 4.3 that if $f \in S_{2k}(\Gamma_0(pM))$ then $Q_{p^2}(f) = pQ_p(f)$, hence Q_{p^2} stabilizes $S_{2k}(\Gamma_0(pM))$ and acts with eigenvalues p^2 and $-p$ on this subspace. In particular if $f \in S_{2k}(\Gamma_0(M))$ then $Q_{p^2}(f) = p^2 f$ and if $f \in S_{2k}^{\text{new}}(\Gamma_0(pM))$ then $Q_{p^2}(f) = -pf$.

Further, if $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ is a primitive form then $\tilde{U}_p(f) = 0$ and so $Q_{p^2}(f) = 0$. Thus if $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ then $Q_{p^2}(f) = 0$.

Consider the operator $Q'_{p^2} = W_{p^2}Q_{p^2}W_{p^2} = W_{p^2}(\tilde{U}_p)^2$, then $Q'_{p^2}(Q'_{p^2} - p^2)(Q'_{p^2} + p) = 0$. We have the following lemma.

Lemma 5.15 *Let $N = p^2M$ with $(p, M) = 1$.*

(1) The operator Q'_{p^2} stabilizes the space $V(p)S_{2k}(\Gamma_0(pM))$ and its subspace $V(p)X_p$.
(2) If $g(z) = f(p^2z) \in V(p^2)S_{2k}(\Gamma_0(M))$ where $f \in S_{2k}(\Gamma_0(M))$, then $Q'_{p^2}(g) = p^2 g$. Consequently, Q'_{p^2} has eigenvalues p^2 and $-p$ on the space $V(p)X_p$.
(3) If $f \in S_{2k}^{\text{new}}(\Gamma_0(pM))$ and $g = f(pz) \in V(p)S_{2k}^{\text{new}}(\Gamma_0(pM))$. Then $Q'_{p^2}(g) = -pg$.
(4) Let q, M' be positive integers such that $(q, p) = 1$ and $qM' \mid M$. Then $V(pq)S_{2k}^{\text{new}}(\Gamma_0(pM'))$ is contained in the $-p$ eigenspace of Q'_{p^2}.

Thus Q'_{p^2} acts with eigenvalues p^2 and $-p$ on $V(p)S_{2k}(\Gamma_0(pM))$.

Proof Let $g = f(pz)$ where $f \in S_{2k}(\Gamma_0(pM))$. It follows from Lemma 5.12 that $W_{p^2}(g) = p^{-k}W_p(f)$ where W_p acts via $\begin{pmatrix} p^2\beta & 1 \\ pM\gamma & p \end{pmatrix}$. Since W_p is Atkin–Lehner operator on $S_{2k}(\Gamma_0(pM))$ and Q_{p^2} stabilizes $S_{2k}(\Gamma_0(pM))$ and W_{p^2} maps $S_{2k}(\Gamma_0(pM))$ onto $V(p)S_{2k}(\Gamma_0(pM))$ we get that $Q'_{p^2}(g)$ belongs to $V(p)S_{2k}(\Gamma_0(pM))$. Thus Q'_{p^2} stabilizes $V(p)S_{2k}(\Gamma_0(pM))$.

In particular, if $f \in S_{2k}^{\text{new}}(\Gamma_0(pM))$, since W_p preserves the space of newforms, we get that $W_{p^2}(g)$ belongs to $S_{2k}^{\text{new}}(\Gamma_0(pM))$. Thus

$$Q'_{p^2}(g) = W_{p^2}Q_{p^2}(W_{p^2}(g)) = -pW_{p^2}(W_{p^2}(g)) = -pg,$$

proving (3).

Recall that $V(p)X_p = V(p)S_{2k}(\Gamma_0(M)) \oplus V(p^2)S_{2k}(\Gamma_0(M))$. Let $g(z) = f(p^2z)$ where $f \in S_{2k}(\Gamma_0(M))$. Then by Lemma 5.12 we get that $W_{p^2}(g) = p^{-2k}f$ and thus

$$Q'_{p^2}(g) = W_{p^2}Q_{p^2}(p^{-2k}f) = p^2W_{p^2}(p^{-2k}f) = p^2 g,$$

proving a part of (2). Now we shall complete the proof of (1) and (2).

Let $g(z) = f(pz)$ where $f \in S_{2k}(\Gamma_0(M))$. By Lemma 5.12, $W_{p^2}(g) = g$ and using Lemma 5.3,

$$Q'_{p^2}(g) = W_{p^2} Q_{p^2}(g) = p W_{p^2}(p^{1-2k} T_p(f) - g) = p^2 T_p(f)(p^2 z) - pg,$$

which clearly belongs to $V(p)X_p$, showing (1). Now following arguments as in Corollary 5.4 and Proposition 5.5, we get that Q'_{p^2} acts with eigenvalues p^2 and $-p$ on $V(p)X_p$ and the p^2 eigenspace of Q'_{p^2} inside $V(p)X_p$ is $V(p^2)S_{2k}(\Gamma_0(M))$.

To prove (4), we check that the operators $V(q)$ and Q'_{p^2} on $S_{2k}(\Gamma_0(p^2 M'))$ commute. Since $(\tilde{U}_p)^2$ commutes with $V(q)$ [1, Lemma 15] it is enough to check that W_{p^2} commutes with $V(q)$. Let W_{p^2} acts via $\begin{pmatrix} p^2\beta & 1 \\ N\gamma & p^2 \end{pmatrix}$ of determinant p^2, then $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} W_{p^2} \left(W_{p^2} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1}$ belongs to $\Gamma_0(N/q)$. So for $f \in S_{2k}(\Gamma_0(N/q))$, $W_{p^2} V(q)(f) = V(q) W_{p^2}(f)$. Hence $Q'_{p^2} V(pq)(f) = V(q) Q'_{p^2} V(p)(f)$ for $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(pM'))$.

We note that $V(p)S_{2k}^{\mathrm{new}}(\Gamma_0(pM'))$ is contained in the $-p$ eigenspace of Q'_{p^2} and so, $Q'_{p^2} V(pq) S_{2k}^{\mathrm{new}}(\Gamma_0(pM')) = -pV(q)V(p)S_{2k}^{\mathrm{new}}(\Gamma_0(pM'))$ concluding the proof.

Finally, since

$$\begin{aligned} V(p)S_{2k}(\Gamma_0(pM)) &= V(p)S_{2k}^{\mathrm{new}}(\Gamma_0(pM)) \oplus V(p)X_p \oplus \\ &\quad \oplus_{qM'|M,(q,p)=1} V(pq) S_{2k}^{\mathrm{new}}(\Gamma_0(pM')), \end{aligned}$$

we get that Q'_{p^2} acts with eigenvalues p^2 and $-p$ on $V(p)S_{2k}(\Gamma_0(pM))$. □

Proposition 5.16 *The operators $Q_{p^2} = (\tilde{U}_p)^2 W_{p^2}$ and $Q'_{p^2} = W_{p^2} Q_{p^2} W_{p^2}$ are self-adjoint with respect to the Petersson inner product.*

Proof The proof is similar to that of Proposition 5.7. □

Now we restate and prove Theorem 2.

Theorem 10 *Let $N = M_1^2 M$ where M_1, M are square-free and coprime. Then $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p dividing M and $Q_{p^2}(f) = 0 = Q'_{p^2}(f)$ for all primes p dividing M_1.*

Proof The one side implication is clear.

Conversely if $f \in S_{2k}(\Gamma_0(N))$ is such that $Q_p(f) = -f = Q'_p(f)$ for all primes $p \mid M$, then as before f is orthogonal to both $S_{2k}(\Gamma_0(N/p))$ and $V(p)S_{2k}(\Gamma_0(N/p))$ for all $p \mid M$.

Let q be a prime dividing M_1 and $N = q^2 N'$, so $(q, N') = 1$. We have already checked that Q'_{q^2} stabilizes $S_{2k}(\Gamma_0(N/q)) = S_{2k}(\Gamma_0(qN'))$ and acts with eigenvalues q^2 and $-q$. Further it follows from Lemma 5.15 that Q'_{q^2} stabilizes $V(q)S_{2k}(\Gamma_0(N/q)) = V(q)S_{2k}(\Gamma_0(qN'))$ and acts with eigenvalues q^2 and $-q$. Thus if $Q_{q^2}(f) = 0 = Q'_{q^2}(f)$ for all primes q dividing M_1 then f is orthogonal to both $S_{2k}(\Gamma_0(N/q))$ and $V(q)S_{2k}(\Gamma_0(N/q))$ for all $q \mid M_1$. Hence f is in the new space at level N. □

Let p be an odd prime. Next we shall consider the action of twisting operators R_p and R_χ [1, Sect. 6] where R_p is the twist by the Dirichlet character given by Kronecker symbol $\left(\frac{\cdot}{p}\right)$

and R_χ is the twist by the Dirichlet character given by $\left(\frac{-1}{\cdot}\right)$. That is, if $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}(\Gamma_0(N))$ then

$$R_p(f)(z) = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) a_n q^n, \qquad R_\chi(f)(z) = \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) a_n q^n.$$

By [1, Lemma 33], R_p and R_χ are operators on $S_{2k}(\Gamma_0(N))$ provided $p^2 \mid N$ and $16 \mid N$ respectively.

It is well known that R_p and R_χ are self-adjoint operators with respect to the Petersson inner product.

Lemma 5.17 *Let $N = p^n M$ where p is odd and coprime to M and $n \geq 2$. If $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$, then $(R_p)^2(f) = f$. For $1 \leq \alpha \leq n$ the space $V(p^\alpha)(S_{2k}(\Gamma_0(p^{n-\alpha}M)))$ is contained in the 0 eigenspace of R_p^2.*

Proof If $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}^{\text{new}}(\Gamma_0(N))$ is a primitive form then since $p^2 \mid N$ we have $a_p = 0$ and consequently $a_m = 0$ for any m divisible by p. Thus $f(z) = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} a_n q^n$. Since $S_{2k}^{\text{new}}(\Gamma_0(N))$ has a basis of primitive forms, this holds for any $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$. It now follows that

$$R_p^2(f)(z) = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} \left(\frac{n^2}{p}\right) a_n q^n = \sum_{\substack{n=1\\(n,p)=1}}^{\infty} a_n q^n = f(z).$$

Let $g(z) = f(p^\alpha z)$ where $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}(\Gamma_0(p^{n-\alpha}M))$. Then $g(z) = \sum_{n=1}^{\infty} a_n q^{p^\alpha n}$. Since $\alpha \geq 1$ we have $R_p(g) = 0$. Hence the lemma follows. □

Following exactly similar arguments we also have the following lemma.

Lemma 5.18 *Let $N = 2^n M$ with M odd and $n \geq 4$. If $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$, then $(R_\chi)^2(f) = f$. For $1 \leq \alpha \leq n$ the space $V(p^\alpha)(S_{2k}(\Gamma_0(p^{n-\alpha}M)))$ is contained in the 0 eigenspace of R_χ^2.*

Since R_p^2 and R_χ^2 are self-adjoint operators, using Corollary 5.13 and Lemma 5.17 and Lemma 5.18, and following a similar argument as in Theorem 9 we obtain the following theorem (Theorem 4 of Sect. 2).

Theorem 11 *Let $N = 2^\beta p_1 p_2 \cdots p_r q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ where p_i, q_i are distinct odd primes and $\beta \geq 4$ and $\alpha_j \geq 2$ for all $1 \leq j \leq s$. Then $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$ if and only if $Q_{p_i}(f) = -f = Q'_{p_i}(f)$ for all $1 \leq i \leq r$, $(R_{q_j})^2(f) = f$ for all $1 \leq j \leq s$ and $(R_\chi)^2(f) = f$, and $S_{q^\gamma,\gamma-1}(f) = 0$ for all primes q such that $q^\gamma \| N$ with $\gamma \geq 2$.*

6 Characterization of old spaces

In the previous section we described the space of newforms in $S_{2k}(\Gamma_0(N))$ as a common eigenspace of certain Hecke operators. In this section we extend this description to the subspaces of old forms of type $V(d)S_{2k}^{\text{new}}(\Gamma_0(M))$ that appear in the direct sum decomposition of the old space $S_{2k}^{\text{old}}(\Gamma_0(N))$ in (1).

We first consider the case when N is square-free. In the theorem below we characterize the various summands in the old space as common eigenspaces of the operators Q_p, Q'_p as p varies over the prime divisors of N.

Theorem 12 *Let N be square-free and $f \in S_{2k}(\Gamma_0(N))$. Then*

(1) $f \in S_{2k}(\Gamma_0(1))$ if and only if $Q_p(f) = pf$ for all $p \mid N$.
(2) Let $1 \neq M \mid N$. Then $f \in S_{2k}^{\text{new}}(\Gamma_0(M))$ if and only if $Q_p(f) = -f = Q'_p f$ for all $p \mid M$ and $Q_q(f) = qf$ for all $q \mid (N/M)$.
(3) Let $1 \neq M' \mid N$. Then $f \in V(M')S_{2k}(\Gamma_0(1))$ if and only if $Q'_q(f) = qf$ for all $q \mid M'$ and $Q_q(f) = qf$ for all $q \mid (N/M')$.
(4) Let M and $M' > 1$ and $MM' \mid N$. Then $f \in V(M')S_{2k}^{\text{new}}(\Gamma_0(M))$ if and only if $Q_p(f) = -f = Q'_p f$ for all $p \mid M$, $Q'_q(f) = qf$ for all $q \mid M'$ and $Q_q(f) = qf$ for all $q \mid (N/MM')$.

The proof relies on the description of eigenspaces of Q_p and Q'_p in Sect. 5.1 and the following lemma.

Lemma 6.1 *Let $dM \mid N$ where $M \neq 1$ and d is coprime to M. If $f \in V(d)S_{2k}^{\text{new}}(\Gamma_0(M))$, then $Q_p(f) = -f = Q'_p f$ for all $p \mid M$.*

Proof Let $f = V(d)f_1$ where $f_1 \in S_{2k}^{\text{new}}(\Gamma_0(M))$ and p be a prime divisor of M. Then $Q_p(f) = \tilde{U}_p W_{p,N}(V(d)f_1)$ where $W_{p,N}$ is the Atkin–Lehner operator on $S_{2k}(\Gamma_0(N))$. Note that for $f \in S_{2k}(\Gamma_0(M))$, we have $W_{p,N}(f) = W_{p,M}(f)$. Further, $W_{p,N}$ commutes with $V(d)$ on $S_{2k}(\Gamma_0(M))$ as the matrix $W_{p,N}V(d)(V(d)W_{p,N})^{-1} \in \Gamma_0(M)$, and by [1, Lemma 15] $\tilde{U}_p$ commutes with $V(d)$ since $(d, p) = 1$. Hence by Theorem 8,

$$Q_p(f) = V(d)\tilde{U}_p W_{p,M} f_1 = V(d)Q_p(f_1) = -V(d)f_1 = -f.$$

The case of Q'_p follows similarly. □

Proof of Theorem 12 We shall give a proof of (4). The other parts follow similarly. Let M and $M' > 1$ and $N = MM't$ for some $t \in \mathbb{N}$.

If $f \in V(M')S_{2k}^{\text{new}}(\Gamma_0(M))$ then by Lemma 6.1, $Q_p(f) = -f = Q'_p f$ for all $p \mid M$. Further for each $q \mid M'$, $V(M')S_{2k}^{\text{new}}(\Gamma_0(M)) \subseteq V(q)S_{2k}(\Gamma_0(N/q))$, and so $Q'_q(f) = qf$ for all $q \mid M'$. Similarly for each $q \mid t$ we have $V(M')S_{2k}^{\text{new}}(\Gamma_0(M)) \subseteq S_{2k}(\Gamma_0(N/q))$ and so $Q_q(f) = qf$ for all $q \mid t$.

Conversely let $f \in S_{2k}(\Gamma_0(N))$ be such that $Q_p(f) = -f = Q'_p f$ for all $p \mid M$, $Q'_q(f) = qf$ for all $q \mid M'$ and $Q_q(f) = qf$ for all $q \mid (N/MM')$. Let q be any prime such that $q \mid M't$. Let $V := \oplus_{dr|N,q|r} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$ and $W := \oplus_{dr|N,(q,r)=1} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$. Then $S_{2k}(\Gamma_0(N)) = V \oplus W$ and since N is square-free we have $W = X_q$. By Lemma 6.1, V is contained in the intersection of the -1 eigenspace of Q_q and Q'_q. Now f can be uniquely written as $f = v + w$ with $v \in V$ and $w \in W$. If $q \mid t$ then $Q_q f = qf$ and so, $qv + qw = Q_q v + Q_q w = -v + Q_q w$ where $Q_q w \in W$. Thus $v = 0$ and $f \in W$. In the case $q \mid M'$ we get the same conclusion by using the operator Q'_q instead. Since the above argument works for all primes dividing $M't$, we get that f belongs to $\oplus_{dr|N,r|M} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$.

Now let $q \mid M'$ be any prime. Then $\oplus_{dr|N,r|M,(d,q)=1} V(d)S_{2k}^{\text{new}}(\Gamma_0(r)) \subseteq S_{2k}(\Gamma_0(N/q))$ and $\oplus_{dr|N,r|M,q|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r)) \subseteq V(q)S_{2k}(\Gamma_0(N/q))$. Thus $f \in X_q$. Since $Q'_q f = qf$ and the q eigenspace of Q'_q in X_q is precisely $V(q)S_{2k}(\Gamma_0(N/q))$, we get that f belongs to $\oplus_{dr|N,r|M,q|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$. Applying the same argument for all primes q dividing M' we get that f belongs to $\oplus_{dr|N,r|M,M'|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$.

Let q be a prime dividing t. Then $\oplus_{dr|N,r|M,M'|d,(d,q)=1} V(d)S_{2k}^{\text{new}}(\Gamma_0(r)) \subseteq S_{2k}(\Gamma_0(N/q))$ while $\oplus_{dr|N,r|M,M'|d,q|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r)) \subseteq V(q)S_{2k}(\Gamma_0(N/q))$. Thus $f \in X_q$. Since $Q_q f = qf$, $f \in \oplus_{dr|N,r|M,M'|d,(d,q)=1} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$. As before applying this argument for all primes q dividing t we get that f belongs to $\oplus_{dr|MM',r|M,M'|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r)) := Y$.

Finally, let p be a prime dividing M. Then $Y = Y_1 \oplus Y_2 \oplus Y_3$ where $Y_1 = \oplus_{dr|MM',r|M,M'|d,(dr,p)=1} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$, $Y_2 = \oplus_{dr|MM',r|M,M'|d,p|d} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$ and $Y_3 = \oplus_{dr|MM',r|M,M'|d,p|r} V(d)S_{2k}^{\text{new}}(\Gamma_0(r))$. Clearly, $Y_1 \oplus Y_2 \subseteq X_p$. We write f uniquely as $f = g + h$ where $g \in Y_1 \oplus Y_2$ and $h \in Y_3$. Since $Q_p(f) = -f = Q'_p f$ and $Q_p(h) = -h = Q'_p h$ we get that $Q_p(g) = -g = Q'_p g$. Thus g is orthogonal to X_p, but $g \in X_p$, hence $g = 0$. Applying the same argument for all primes p dividing M we get that $f \in \oplus_{dr|MM',r=M,M'|d}$ which is precisely $V(M')S_{2k}^{\text{new}}(\Gamma_0(M))$. □

We now consider the case $N = p^n$ where p is a prime. The characterization of the old space summands will be done inductively on n. The case $n = 1$ follows from Theorem 12. We assume that $n \geq 2$. It follows from (1) that

$$S_{2k}(\Gamma_0(p^n)) = S_{2k}(\Gamma_0(p^{n-1})) \oplus \bigoplus_{r=0}^{n} V(p^{n-r})S_{2k}^{\text{new}}(\Gamma_0(p^r)).$$

By Corollary 5.10, $S_{2k}(\Gamma_0(p^{n-1}))$ is precisely the p eigenspace of the operator $S_{p^n,p^{n-1}}$ and hence we can characterize the summands that appear inside the direct sum decomposition of $S_{2k}(\Gamma_0(p^{n-1}))$ using the induction hypothesis.

So we need to only deal with the spaces of type $V(p^{n-r})S_{2k}^{\text{new}}(\Gamma_0(p^r))$ for $0 \leq r \leq n$. Using Lemma 5.12, the operator W_{p^n} maps $S_{2k}^{\text{new}}(\Gamma_0(p^r))$ onto $V(p^{n-r})S_{2k}^{\text{new}}(\Gamma_0(p^r))$. Thus a form $f \in S_{2k}(\Gamma_0(p^n))$ belongs to the space $V(p^{n-r})S_{2k}^{\text{new}}(\Gamma_0(p^r))$ if and only if $W_{p^n}(f)$ belongs to $S_{2k}^{\text{new}}(\Gamma_0(p^r))$. By the previous section we already know how to characterize the forms in $S_{2k}^{\text{new}}(\Gamma_0(p^r))$, thus we can characterize $W_{p^n}(f)$ and hence f.

Using the above argument a similar statement as Theorem 12 can be made for general level N.

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