JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Soma PURKAIT

Hecke operators in half-integral weight

Tome 26, nº 1 (2014), p. 233-251.

<http://jtnb.cedram.org/item?id=JTNB_2014__26_1_233_0>

Journal de Théorie des Nombres
de Bordeaux **26** (2014), 233–251

Hecke operators in half-integral weight

par Soma PURKAIT

Résumé. Dans [6], Shimura a introduit la notion de formes modulaires de poids demi-entier et leurs algèbres de Hecke ; il a aussi établi leur lien avec les formes modulaires de poids entier via la correspondance de Shimura. Pour les formes modulaires de poids entier, les bornes de Sturm permettent de déterminer des générateurs de l'algèbre de Hecke comme module. L'on dispose également de formules de récurrence bien connues pour les opérateurs T_{p^ℓ} en les p premiers. Le but de cet article est d'établir des résultats analogues dans le cas de poids demi-entier. Nous donnons également une formule explicite sur la commutativité des opérateurs T_{p^ℓ} avec la correspondance de Shimura.

Abstract. In [6], Shimura introduced modular forms of half-integral weight, their Hecke algebras and their relation to integral weight modular forms via the Shimura correspondence. For modular forms of integral weight, Sturm's bounds give generators of the Hecke algebra as a module. We also have well-known recursion formulae for the operators T_{p^ℓ} with p prime. It is the purpose of this paper to prove analogous results in the half-integral weight setting. We also give an explicit formula for how operators T_{p^ℓ} commute with the Shimura correspondence.

1. Introduction

In [6], Shimura introduced modular forms of half-integral weight, their Hecke algebras and their relation to integral weight modular forms via the Shimura correspondence. For modular forms of integral weight, Sturm's bounds give generators of the Hecke algebra as a module. We also have well-known recursion formulae for the operators T_{p^ℓ} with p prime. It is the purpose of this paper to prove analogous results in the half-integral weight setting. We also give an explicit formula for how operators T_{p^ℓ} commute with the Shimura correspondence.

Let k, N be positive integers with k odd and $4 \mid N$. Let χ be a Dirichlet character modulo N. We shall denote by $M_{k/2}(N, \chi)$ the space of modular

Manuscrit reçu le 4 octobre 2012,
accepté le 22 octobre 2013..
Classification math. 11F37, 11F11.

forms of weight $k/2$, level N and character χ, and by $S_{k/2}(N,\chi)$ the subspace of cusp forms. We shall write $\mathbb{T}_{k/2}$ for the Hecke algebra acting on these spaces. For definitions we refer to Shimura's paper [6]. It is well-known [6, page 450] that $T_n = 0$ for n not square.

Theorem 1.1. *Let p be a prime and $\ell \geq 2$ be a positive integer. If $p \mid N$ then as Hecke operators in $\mathbb{T}_{k/2}$,*

$$T_{p^{2\ell}} = (T_{p^2})^{\ell}.$$

If $p \nmid N$ then

$$T_{p^{2\ell+2}} = T_{p^2}T_{p^{2\ell}} - \chi(p^2)p^{k-2}T_{p^{2\ell-2}}$$

as Hecke operators in $\mathbb{T}_{k/2}$.

Let $N' = N/2$. Let t be a square-free positive integer. For $k \geq 3$, Shimura [6] proved the so-called 'Shimura correspondence'

$$\mathrm{Sh}_t : S_{k/2}(N,\chi) \to M_{k-1}(N',\chi^2),$$

where $M_{k-1}(N',\chi^2)$ denotes the usual space of modular forms of integral weight $k-1$, level N' and character χ^2. Note that in Shimura's proof, N' was taken to be N and he conjectured that $N' = N/2$, which was later proven by Niwa [4].

It is well-known [2] and [5, page 53] that for $p \nmid tN$,

$$\mathrm{Sh}_t(T_{p^2}f) = T_p(\mathrm{Sh}_t(f))$$

where T_p is the usual integral weight Hecke operator. Our next theorem shows that the same identity holds even when $p \mid tN$, and also gives the precise relationship of how $T_{p^{2\ell}}$ commutes with Sh_t for all primes p, and for $\ell \geq 2$.

Theorem 1.2. *Let p be a prime and let $f \in S_{k/2}(N,\chi)$. Let t be a square-free positive integer. Then*

$$\mathrm{Sh}_t(T_{p^2}f) = T_p(\mathrm{Sh}_t(f)).$$

Let $\ell \geq 2$. Then

(a) *If $p \mid N$ then*

$$\mathrm{Sh}_t(T_{p^{2\ell}}f) = T_{p^\ell}(\mathrm{Sh}_t(f)).$$

(b) *If $p \nmid N$ then*

$$\mathrm{Sh}_t(T_{p^{2\ell}}f) = (T_{p^\ell} - \chi(p^2)p^{k-3}T_{p^{\ell-2}})(\mathrm{Sh}_t(f)).$$

We would like to point out that in the case $p \mid N$, it is easy to see that $T_{p^{2\ell}} = (T_{p^2})^{\ell}$ when viewed as the U-operator. This is well-known but is included in the Theorem 1.1 and Lemma 4.1 for completeness.

It is well-known [6, page 478] that the space $S_{3/2}(N,\chi)$ contains single-variable theta-series and we denote by $S_0(N,\chi)$ the subspace generated by these single-variable theta-series. The interesting part of the space $S_{3/2}(N,\chi)$ is the orthogonal complement of $S_0(N,\chi)$ with respect to the Petersson inner product. We will use the following notation:

$$S'_{k/2}(N,\chi) := \begin{cases} S_0(N,\chi)^{\perp} & k=3 \\ S_{k/2}(N,\chi) & \text{for } k\geq 5. \end{cases}$$

The Hecke algebra $\mathbb{T}_{k/2}$ preserves $S'_{k/2}(N,\chi)$. We denote

$$\mathbb{T}'_{k/2} = \left\{ T|_{S'_{k/2}(N,\chi)} : T \in \mathbb{T}_{k/2} \right\};$$

this is the restriction of the Hecke algebra to $S'_{k/2}(N,\chi)$.

Theorem 1.3. *Let k, N be positive integers with $k \geq 3$ odd, and $4 \mid N$. Let χ be a Dirichlet character modulo N. Let $N' = N/2$. Write*

$$m = N'^2 \prod_{p\mid N'} \left(1 - \frac{1}{p^2}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

Then T_{i^2} for $i \leq R$ generate $\mathbb{T}'_{k/2}$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. In particular the set of operators T_{p^2} for primes $p \leq R$ forms a generating set as an algebra. Moreover, if χ is a quadratic character, then the same result holds as above with

$$m = N' \prod_{p\mid N'} \left(1 + \frac{1}{p}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

2. Hecke operators

2.1. Integral weight Hecke operators. Let k, N be positive integers and χ be a Dirichlet character modulo N. Let $M_k(N,\chi)$ be the space of modular forms of weight k, level N and character χ and $S_k(N,\chi)$ be the subspace of cusp forms. Recall that given a positive integer n one can define Hecke operators T_n and $T_{(n,n)}$ (when $(n,N)=1$) acting on the space $M_k(N,\chi)$ that also preserve $S_k(N,\chi)$.

The following proposition lists the important properties of these Hecke operators.

Proposition 2.1. (a) *If $(m,n)=1$, then $T_{mn} = T_m T_n$.*

(b) *If p is a prime dividing N, then $T_{p^e} = T_p^e$ for any positive integer e.*

(c) *If p is a prime such that $(p,N)=1$, then for any positive integer e, $T_{p^{e+1}} = T_p T_{p^e} - pT_{(p,p)} T_{p^{e-1}}$ where for $f \in M_k(N,\chi)$ the action of $T_{(p,p)}$ can be explicitly expressed as $T_{(p,p)}(f) = p^{k-2}\chi(p) f$.*

Proof. See [3, Lemma 4.5.7] and [3, Pages 142-143]. □

Note that when $p \mid N$ then the operator T_p can be viewed as the $U(p)$-operator.

The Hecke algebra on $M_k(N,\chi)$, which we denote by $\mathbb{T}_k$ is an algebra over $\mathbb{Z}$ generated by T_p, $T_{(p,p)}$ and T_r where p, r varies over primes with $p \nmid N$ and $r \mid N$. We can write the action of Hecke operators in terms of q-expansions.

Proposition 2.2. *Let f be a modular form in $M_k(N,\chi)$ with q-expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$. Then $T_p(f)(z) = \sum_{n=0}^{\infty} b_n q^n$ where,*

$$b_n = a_{pn} + \chi(p)p^{k-1}a_{n/p}.$$

Here we take $a_{n/p} = 0$ if $p \nmid n$.

Proof. See [3, Lemma 4.5.14]. □

2.2. Half-integral weight forms and Hecke operators. Let G be the group consisting of all ordered pairs $(\alpha, \phi(z))$, where $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\phi(z)$ is a holomorphic function on $\mathbb{H}$ satisfying

$$\phi(z)^2 = t\frac{cz+d}{\sqrt{\det\alpha}}$$

for some $t \in \{\pm 1\}$, with the group law defined by

$$(\alpha, \phi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

Let $P : G \to \mathrm{GL}_2^+(\mathbb{Q})$ be the homomorphism given by the projection map onto the first coordinate. Let k be positive odd integer. The group G acts on the space of complex-valued functions on $\mathbb{H}$ by $f|[\xi]_{k/2}(z) := f(\alpha z)\phi(z)^{-k}$, where $\xi = (\alpha, \phi(z)) \in G$ and $f : \mathbb{H} \to \mathbb{C}$.

Let N be a positive integer with $4 \mid N$. Then for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ define

$$j(\gamma, z) := \left(\frac{c}{d}\right)\epsilon_d^{-1}\sqrt{cz+d}, \qquad \Delta_0(N) := \{\widetilde{\gamma} := (\gamma, j(\gamma, z)) | \gamma \in \Gamma_0(N)\},$$

where $\epsilon_d = 1$ or $\sqrt{-1}$ according as $d \equiv 1$ or $3 \pmod 4$.

Note that $\Delta_0(N)$ is a subgroup of G. The map $L : \Gamma_0(4) \to G$ given by $\gamma \mapsto \widetilde{\gamma}$ defines an isomorphism onto $\Delta_0(4)$. Thus $P|_{\Delta_0(4)} : \Delta_0(4) \to \Gamma_0(4)$ and $L : \Gamma_0(4) \to \Delta_0(4)$ are inverse of each other. Denote by $\Delta_1(N)$ and $\Delta(N)$ respectively the images of $\Gamma_1(N)$ and $\Gamma(N)$.

Let χ be a Dirichlet character modulo N and $M_{k/2}(N,\chi)$ and $S_{k/2}(N,\chi)$ be the spaces of modular forms and cusp forms of weight $k/2$, level N and character χ. The space $M_{k/2}(N,\chi)$ is $\{0\}$ unless χ is even, so henceforth we will be assuming χ to be even. As in the integral weight case one can define the Hecke operators on the spaces $M_{k/2}(N,\chi)$ and $S_{k/2}(N,\chi)$.

Let ξ be an element of G such that $\Delta_1(N)$ and $\xi^{-1}\Delta_1(N)\xi$ are commensurable. Define an operator $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ on $M_{k/2}(\Gamma_1(N))$ by

$$f|[\Delta_1(N)\xi\Delta_1(N)]_{k/2} = \det(\xi)^{k/4-1}\sum_{\nu} f|[\xi_\nu]_{k/2}$$

where $\Delta_1(N)\xi\Delta_1(N) = \bigcup_\nu \Delta_1(N)\xi_\nu$.

Now suppose m is a positive integer and $\alpha = \left[\begin{smallmatrix}1 & 0\\ 0 & m\end{smallmatrix}\right]$, $\xi = (\alpha, m^{1/4})$. Then the Hecke operator T_m is defined as the restriction of $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ to $M_{k/2}(N,\chi)$. It is to be noted that by [6, Proposition 1.0], if m is not a square and $(m,N)=1$ then $|[\Delta_1(N)\xi\Delta_1(N)]_{k/2}$ is the zero operator. So we assume that $m = n^2$ for a positive integer n. Shimura writes the Hecke operator T_{n^2} as

$$T_{n^2}(f) := n^{\frac{k}{2}-2}\sum_{\nu}\chi(a_\nu) f|[\xi_\nu]_{k/2},$$

where ξ_ν are the right coset representatives of $\Delta_0(N)$ in $\Delta_0(N)\xi\Delta_0(N)$ such that $P(\xi_\nu) = \left[\begin{smallmatrix}a_\nu & *\\ * & *\end{smallmatrix}\right]$. We have the following theorem.

Theorem 2.1. *(Shimura) Let* $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N,\chi)$. *Then* $T_{p^2}(f)(z) = \sum_{n=0}^{\infty} b_n q^n$ *where,*

$$b_n = a_{p^2n} + \chi(p)\left(\frac{-1}{p}\right)^{\lambda}\left(\frac{n}{p}\right)p^{\lambda-1}a_n + \chi(p^2)p^{k-2}a_{n/p^2},$$

and $\lambda = (k-1)/2$ *and* $a_{n/p^2} = 0$ *whenever* $p^2 \nmid n$.

Proof. See [6, Theorem 1.7]. □

3. Shimura Correspondence

For this section fix positive integers k, N with $k \geq 3$ odd and $4 \mid N$. Let χ be an even Dirichlet character of modulus N. Let $N' = N/2$. We recall Shimura's Theorem.

Theorem 3.1. *(Shimura) Let* $\lambda = (k-1)/2$. *Let* $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N,\chi)$. *Let t be a square-free integer and let ψ_t be the Dirichlet character modulo tN defined by*

$$\psi_t(m) = \chi(m)\left(\frac{-1}{m}\right)^{\lambda}\left(\frac{t}{m}\right).$$

Let $A_t(n)$ be the complex numbers defined by

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = \left(\sum_{i=1}^{\infty}\psi_t(i)i^{\lambda-1-s}\right)\left(\sum_{j=1}^{\infty} a_{tj^2}j^{-s}\right). \tag{3.1}$$

Let $\mathrm{Sh}_t(f)(z) = \sum_{n=1}^{\infty} A_t(n)q^n$. *Then*

(i) $\mathrm{Sh}_t(f) \in M_{k-1}(N',\chi^2)$.

(ii) *If* $k \geq 5$ *then* $\mathrm{Sh}_t(f)$ *is a cusp form.*

(iii) *If* $k = 3$ *and* $f \in S'_{3/2}(N, \chi)$ *then* $\mathrm{Sh}_t(f)$ *is a cusp form.*

Proof. For (i) and (ii) see [6, Section 3, Main Theorem], for the rest see [5, Theorem 3.14]. In particular, the fact that $N' = N/2$ was proved by Niwa [4, Section 3]. □

The form $\mathrm{Sh}_t(f)$ is called the *Shimura lift of* f *corresponding to* t. The following is clear from Equation(3.1).

Lemma 3.1. *The Shimura lift* Sh_t *is linear.*

Lemma 3.2. *If* $\mathrm{Sh}_t(f) = 0$ *for all positive square-free integers* t *then* $f = 0$.

Proof. By Equation (3.1) we know that $a_{tj^2} = 0$ for all positive square-free integers t and all positive integers j. Then $a_n = 0$ for all n. □

In Ono's book [5, Chapter 3, Corollary 3.16] and several other places [2] we find the following result stated without proof.

Proposition 3.1. *Suppose* $f \in S_{k/2}(N, \chi)$. *Let* t *be a square-free positive integer. If* $p \nmid tN$ *is a prime then*

$$\mathrm{Sh}_t(T_{p^2} f) = T_p \,\mathrm{Sh}_t(f).$$

Here T_{p^2} is the Hecke operator in $\mathbb{T}_{k/2}$ and T_p is the Hecke operator in $\mathbb{T}_{k-1}$. For what follows we shall need the following strengthening of this result.

Proposition 3.2. *Suppose* $f \in S_{k/2}(N, \chi)$ *and* t *a square-free positive integer. If* p *is a prime then*

$$\mathrm{Sh}_t(T_{p^2} f) = T_p \,\mathrm{Sh}_t(f).$$

We do not know why the above references impose the condition $p \nmid tN$. We shall give a careful proof that does not use this assumption.

Proof of Proposition 3.2. The proof uses the explicit formulae for Hecke operators in terms of q-expansions. As in Shimura's Theorem above, write $f(z) = \sum_{n=1}^{\infty} a_n q^n$. Fix t to be a positive square-free integer. To simplify notation, we shall write A_n for $A_t(n)$. Thus we have the relation

$$\sum_{n=1}^{\infty} A_n n^{-s} = \left(\sum_{i=1}^{\infty} \psi_t(i) i^{\lambda-1-s} \right) \left(\sum_{j=1}^{\infty} a_{tj^2} j^{-s} \right).$$

We may rewrite this as

$$A_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} a_{tj^2}. \tag{3.2}$$

Let

$$T_{p^2}(f)(z) = \sum_{n=1}^{\infty} b_n q^n.$$

Then using Theorem 2.1 we get,

$$b_n = a_{p^2 n} + \psi_1(p) \left(\frac{n}{p}\right) p^{\lambda-1} a_n + \chi^2(p) p^{k-2} a_{n/p^2}. \tag{3.3}$$

The reader will recall that if n/p^2 is not an integer then we take $a_{n/p^2} = 0$.

Let $g = \mathrm{Sh}_t(f)(z) = \sum_{n=1}^{\infty} A_n q^n$. Write

$$T_p(g)(z) = \sum_{n=1}^{\infty} B_n q^n.$$

Let

$$\mathrm{Sh}_t(T_{p^2} f)(z) = \sum_{n=1}^{\infty} C_n q^n.$$

To prove the proposition, it is enough to show that $B_n = C_n$ for all n. We shall do this by direct calculation, expressing both B_n and C_n in terms of the a_i.

Since $g(z) = \sum A_n q^n \in M_{k-1}(N', \chi^2)$ and $T_p(g)(z) = \sum B_n q^n$ we know by Proposition 2.2 that

$$B_n = A_{pn} + \chi^2(p) p^{k-2} A_{n/p}.$$

Substituting from (3.2) we have

$$B_n = \sum_{ij=pn} \psi_t(i) i^{\lambda-1} a_{tj^2} + \sum_{ij=n/p} \chi^2(p) \psi_t(i) p^{k-2} i^{\lambda-1} a_{tj^2}; \tag{3.4}$$

here the second sum is understood to vanish if $p \nmid n$.

Recall $T_{p^2} f(z) = \sum b_n q^n$ and $\mathrm{Sh}_t(T_{p^2} f)(z) = \sum C_n q^n$. Hence by (3.2) we have

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} b_{tj^2}.$$

Using (3.3) we obtain

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} \left(a_{p^2 tj^2} + \psi_1(p) \left(\frac{tj^2}{p}\right) p^{\lambda-1} a_{tj^2} + \chi^2(p) p^{k-2} a_{tj^2/p^2} \right).$$

Note that $\psi_1(p) \left(\frac{tj^2}{p}\right) = \psi_t(p) \left(\frac{j^2}{p}\right)$. So we can rewrite C_n as

$$C_n = \sum_{ij=n} \psi_t(i) i^{\lambda-1} \left(a_{p^2 tj^2} + \psi_t(p) \left(\frac{j^2}{p}\right) p^{\lambda-1} a_{tj^2} + \chi^2(p) p^{k-2} a_{tj^2/p^2} \right). \tag{3.5}$$

Note that the Legendre symbol here is 1 unless of course $p \mid j$ in which case it is 0. Moreover $a_{tj^2/p^2} = 0$ whenever $p \nmid j$; this is because t is square-free.

We consider the following two cases.

Case $p \nmid n$. In this case the formulae for B_n and C_n simplify as follows.

$$\begin{aligned}
B_n &= \sum_{ij=pn} \psi_t(i) i^{\lambda-1} a_{tj^2} \\
&= \sum_{ij=n} \psi_t(pi)(pi)^{\lambda-1} a_{tj^2} + \psi_t(i) i^{\lambda-1} a_{tp^2j^2} \\
&= \sum_{ij=n} \psi_t(i) i^{\lambda-1} (a_{tp^2j^2} + \psi_t(p) p^{\lambda-1} a_{tj^2}) \\
&= C_n.
\end{aligned}$$

Case $p \mid n$. Write $n = p^r m$ where $r \geq 1$ and $p \nmid m$. We rewrite (3.4) as follows.

$$\begin{aligned}
B_n = \sum_{j \mid p^{r+1}m} & \psi_t(p^{r+1}m/j)(p^{r+1}m/j)^{\lambda-1} a_{tj^2} \\
&+ \sum_{j \mid p^{r-1}m} \chi^2(p) \psi_t(p^{r-1}m/j) p^{k-2} (p^{r-1}m/j)^{\lambda-1} a_{tj^2}.
\end{aligned}$$

This may be re-expressed as $B_n = B_n^{(1)} + B_n^{(2)}$ where

$$B_n^{(1)} = \sum_{u=0}^{r+1} \sum_{k \mid m} \psi_t(p^{r+1-u}m/k)(p^{r+1-u}m/k)^{\lambda-1} a_{tp^{2u}k^2}$$

and

$$B_n^{(2)} = \sum_{u=0}^{r-1} \sum_{k \mid m} \chi^2(p) \psi_t(p^{r-1-u}m/k) p^{k-2} (p^{r-1-u}m/k)^{\lambda-1} a_{tp^{2u}k^2}.$$

Moreover, we can rewrite (3.5) as follows.

$$\begin{aligned}
C_n = \sum_{j \mid p^r m} & \psi_t(p^r m/j)(p^r m/j)^{\lambda-1} \\
& \left(a_{p^2tj^2} + \psi_t(p) \left(\frac{j^2}{p} \right) p^{\lambda-1} a_{tj^2} + \chi^2(p) p^{k-2} a_{tj^2/p^2} \right).
\end{aligned}$$

Thus we can write $C_n = C_n^{(1)} + C_n^{(2)} + C_n^{(3)}$ where

$$C_n^{(1)} = \sum_{u=0}^{r} \sum_{k \mid m} \psi_t(p^{r-u}m/k)(p^{r-u}m/k)^{\lambda-1} a_{tp^{2u+2}k^2},$$

and

$$C_n^{(2)} = \sum_{k \mid m} \psi_t(p^{r+1}m/k)(p^{r+1}m/k)^{\lambda-1} a_{tk^2},$$

and

$$C_n^{(3)} = \sum_{u=1}^{r} \sum_{k|m} \chi^2(p)\psi_t(p^{r-u}m/k)(p^{r-u}m/k)^{\lambda-1}p^{k-2}a_{tp^{2u-2}k^2}.$$

It is clear that $B_n^{(2)} = C_n^{(3)}$, and also that $B_n^{(1)} = C_n^{(1)} + C_n^{(2)}$; here $C_n^{(2)}$ corresponds to the $u = 0$ terms in $B_n^{(1)}$. Thus $B_n = C_n$ completing the proof. □

4. Recursion Formula for the Hecke Operators $T_{p^{2\ell}}$

We keep the notation as in the previous section. Let ℓ be a positive integer and p be a prime. In this section we are interested in the action of the Hecke operator $T_{p^{2\ell}}$ on the space $M_{k/2}(N,\chi)$. In the case $p \mid N$ we have the following easy lemma.

Lemma 4.1. *Let ℓ be a positive integer and p be a prime dividing N. Let t be a square-free positive integer. Then*

(i) $T_{p^{2\ell}} = (T_{p^2})^\ell$.

(ii) $\mathrm{Sh}_t(T_{p^{2\ell}}f) = T_{p^\ell}(\mathrm{Sh}_t(f))$ *for* $f \in S_{k/2}(N,\chi)$.

In the above statements $T_{p^{2\ell}} \in \mathbb{T}_{k/2}$ and $T_{p^\ell} \in \mathbb{T}_{k-1}$.

Proof. Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N,\chi)$. It follows using [6, Proposition 1.5] that $T_{p^{2\ell}}(f) = \sum_{n=1}^{\infty} a_{np^{2\ell}}q^n$. Now part (i) follows using Theorem 2.1. Part (ii) follows by using Proposition 3.2 and part (b) of Proposition 2.1 since $p \mid N'$. □

We will assume that $p \nmid N$ for the rest of this section. The main aim of this section is to prove the following result.

Theorem 4.1. *Let $p \nmid N$ be a prime and $\ell \geq 2$ be a positive integer. Then*

$$T_{p^{2\ell+2}} = T_{p^2}T_{p^{2\ell}} - \chi(p^2)p^{k-2}T_{p^{2\ell-2}}$$

as Hecke operators in $\mathbb{T}_{k/2}$.

It is to be noted that for $\ell = 1$ the above relation does not hold. One can check directly that in $\mathbb{T}_{k/2}$,

$$T_{p^4} = (T_{p^2})^2 - \chi(p^2)(p^{k-3} + p^{k-2}).$$

We need the following lemma on Gauss sums which can be easily deduced from [3, Lemma 3.1.3]:

Lemma 4.2. *Let p be an odd prime and n, α be a given positive integer. Then*

(i) $\sum_{m=0}^{p^\alpha-1} \left(\frac{m}{p}\right) e^{\frac{2\pi imn}{p^\alpha}} = \begin{cases} 0 & \text{if } p^{\alpha-1} \nmid n \\ p^{\alpha-1}\left(\frac{n'}{p}\right)\epsilon_p\sqrt{p} & \text{if } n = p^{\alpha-1}n'. \end{cases}$

(ii) $\sum_{m=0}^{p^\alpha-1} e^{\frac{2\pi imn}{p^\alpha}} = \begin{cases} 0 & p^\alpha \nmid n \\ p^\alpha & p^\alpha \mid n. \end{cases}$

Proof of Theorem 4.1. Let $f \in M_{k/2}(N,\chi)$. Let $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & p^{2\ell} \end{smallmatrix}\right]$, $\xi = (\alpha, p^{\ell/2})$. Using [3, Lemma 4.5.6] we know that

$$\Gamma_0(N)\alpha\Gamma_0(N) = \bigcup_{\nu,m} \Gamma_0(N)\alpha_{\nu,m}, \qquad \alpha_{\nu,m} = \begin{bmatrix} p^{2\ell-\nu} & m \\ 0 & p^\nu \end{bmatrix}$$

where $0 \le \nu \le 2\ell$, $0 \le m < p^\nu$ and $\gcd(m, p^\nu, p^{2\ell-\nu}) = 1$. Let G be the group defined as above. Let $\xi_{\nu,m} \in G$ be given by

$$\xi_{\nu,m} = \begin{cases} \left(\alpha_{\nu,m},\ p^{\frac{-2\ell+2\nu}{4}} \epsilon_p^{-1} \left(\frac{-m}{p}\right)\right) & \text{if } \nu \text{ is odd} \\ (\alpha_{\nu,m},\ p^{\frac{-2\ell+2\nu}{4}}) & \text{if } \nu \text{ is even.} \end{cases}$$

One can verify that $\xi_{\nu,m}$ with ν and m varying as above form a set of right coset representatives of $\Delta_0(N)$ in $\Delta_0(N)\xi\Delta_0(N)$ (see [6, Proposition 1.1]). Then we know by definition of $T_{p^{2\ell}}$ (see Subsection 2.2) that

$$T_{p^{2\ell}} f = (p^{2\ell})^{\frac{k}{4}-1} \left(A_0 + A_{2\ell} + \sum_{\nu=1}^{2\ell-1} A_\nu \right), \tag{4.1}$$

where

$$\begin{aligned} A_\nu &= \sum_{\substack{m=0 \\ (m,p)=1}}^{p^\nu-1} \chi(p^{2\ell-\nu}) f|[\xi_{\nu,m}]_{k/2}, \\ A_{2\ell} &= \sum_{m=0}^{p^{2\ell}-1} f|[\xi_{2\ell,m}]_{k/2}, \\ A_0 &= \chi(p^{2\ell}) f|[\xi_{0,0}]_{k/2}. \end{aligned}$$

Applying T_{p^2} to Equation (4.1) we obtain

$$\begin{aligned} T_{p^2} T_{p^{2\ell}} f &= (p^{2\ell})^{\frac{k}{4}-1} \left(\sum_{\nu=1}^{2\ell-1} T_{p^2} A_\nu + T_{p^2} A_{2\ell} + T_{p^2} A_0 \right) \\ &= (p^{2\ell+2})^{\frac{k}{4}-1} \left(\sum_{\nu=1}^{2\ell-1} B_\nu + B_{2\ell} + B_0 \right), \end{aligned} \tag{4.2}$$

where for ν with $0 \le \nu \le 2\ell - 2$ we have

$$B_\nu = \chi(p^{2\ell-\nu+2}) \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f|[(\left[\begin{smallmatrix} p^{2\ell-\nu+2} & m\\ 0 & p^\nu\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu-2}{4}} r_{\nu,m})]_{k/2}$$

$$+ \chi(p^{2\ell-\nu+1}) \sum_{m'=1}^{p-1} \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f|[(\left[\begin{smallmatrix} p^{2\ell-\nu+1} & p^{2\ell-\nu}m'+mp\\ 0 & p^{\nu+1}\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu}{4}} s_{\nu,m,m'})]_{k/2}$$

$$+ \chi(p^{2\ell-\nu}) \sum_{m'=0}^{p^2-1} \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f|[(\left[\begin{smallmatrix} p^{2\ell-\nu} & p^{2\ell-\nu}m'+mp^2\\ 0 & p^{\nu+2}\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu+2}{4}} r_{\nu,m})]_{k/2},$$

where

$$r_{\nu,m} = \begin{cases} \epsilon_p^{-1}\left(\frac{-m}{p}\right) & \nu \text{ odd}\\ 1 & \nu \text{ even}\end{cases}, \qquad s_{\nu,m,m'} = \begin{cases} \epsilon_p^{-2}\left(\frac{mm'}{p}\right) & \nu \text{ odd}\\ \epsilon_p^{-1}\left(\frac{-m'}{p}\right) & \nu \text{ even},\end{cases}$$

and $B_{2\ell}$ has the same expression as above with $\nu = 2\ell$ but without any coprimality condition on m, that is, we do not have $(m,p) = 1$ in the above terms while writing the expression for $B_{2\ell}$.

We express $T_{p^{2\ell+2}}f$ as in Equation (4.1) and compare it with Equation (4.2). Ruling out some of the terms using Euclidean algorithm and rewriting the action of matrices (we will give an example of the calculation later) we obtain

$$(4.3)\quad (T_{p^{2\ell+2}} - T_{p^2}T_{p^{2\ell}})(f) = -(p^{2\ell+2})^{\frac{k}{4}-1}\left(S_0 + S_{2\ell} + \sum_{\nu=1}^{2\ell-1}(D_\nu + E_\nu)\right)$$

where

$$S_0 = \sum_{m'=0}^{p^2-1} \chi(p^{2\ell}) f|[(\left[\begin{smallmatrix} p^{2\ell} & p^{2\ell}m'\\ 0 & p^2\end{smallmatrix}\right],\ p^{\frac{-\ell+1}{2}})]_{k/2}$$

$$S_{2\ell} = \sum_{\substack{m=0\\(m,p)\neq 1}}^{p^{2\ell}-1} \chi(p^2) f|[(\left[\begin{smallmatrix} p^2 & m\\ 0 & p^{2\ell}\end{smallmatrix}\right],\ p^{\frac{\ell-1}{2}})]_{k/2}$$

$$D_\nu = \chi(p^{2\ell-\nu}) \sum_{m'=0}^{p^2-1} \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f|[(\left[\begin{smallmatrix} p^{2\ell-\nu} & p^{2\ell-\nu}m'+mp^2\\ 0 & p^{\nu+2}\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu+2}{4}} r_{\nu,m})]_{k/2}$$

$$E_\nu = \chi(p^{2\ell-\nu+1}) \sum_{m'=1}^{p-1} \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f|[(\left[\begin{smallmatrix} p^{2\ell-\nu+1} & p^{2\ell-\nu}m'+mp\\ 0 & p^{\nu+1}\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu}{4}} s_{\nu,m,m'})]_{k/2}.$$

Further

$$\chi(p^2)p^{k-2}T_{p^{2\ell-2}}f = p^2(p^{2\ell+2})^{\frac{k}{4}-1}\left(\sum_{\nu=1}^{2\ell-3} C_\nu + C_{2\ell-2} + C_0\right), \tag{4.4}$$

where for ν with $0 \le \nu \le 2\ell-3$ we have

$$C_\nu = \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} \chi(p^{2\ell-\nu}) f|[([\begin{smallmatrix} p^{2\ell-\nu-2} & m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2\ell+2\nu+2}{4}} r_{\nu,m})]_{k/2}$$

and $C_{2\ell-2}$ has the same expression as above with $\nu = 2\ell-2$ but without the condition $(m,p)=1$ in the above sum. We first claim that the following relations hold:

(i) $D_\nu = p^2 C_\nu$ for $1 \le \nu \le 2\ell-3$, and $S_0 = p^2 C_0$.
(ii) $E_\nu = 0$ for $1 \le \nu \le 2\ell-2$.

We will only show the computation for part (ii) for case ν odd. The rest of the claim follows by similar method. Fix an odd ν with $1 \le \nu \le 2\ell-3$. Fix $1 \le m' \le p-1$. Then for each m with $0 \le m \le p^\nu - 1$ there exist unique a and b with $0 \le b \le p^\nu - 1$ such that $m + p^{2\ell-\nu-1}m' = ap^\nu + b$. Moreover $m \equiv b \pmod p$. Hence

$$(m,p) = 1 \iff (b,p) = 1, \qquad \left(\frac{-m}{p}\right) = \left(\frac{-b}{p}\right).$$

We can rewrite E_ν as

$$E_\nu = \chi(p^{2\ell-\nu+1}) \sum_{m'=1}^{p-1} \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f\left(\frac{p^{2\ell-\nu+1}z + p^{2\ell-\nu}m' + mp}{p^{\nu+1}}\right) \left(p^{\frac{-2\ell+2\nu}{4}} \epsilon_p^{-2}\left(\frac{mm'}{p}\right)\right)^{-k}$$

$$= \chi(p^{2\ell-\nu+1})\epsilon_p^k \sum_{m'=1}^{p-1}\left(\frac{-m'}{p}\right) \sum_{\substack{m=0\\(m,p)=1}}^{p^\nu-1} f\left|\left[\left([\begin{smallmatrix} p^{2\ell-\nu} & p^{2\ell-\nu-1}m'+m \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2\ell+2\nu}{4}}\epsilon_p^{-1}\left(\frac{-m}{p}\right)\right)\right]_{k/2}\right.$$

$$= \chi(p^{2\ell-\nu+1})\epsilon_p^k \sum_{m'=1}^{p-1}\left(\frac{-m'}{p}\right) \sum_{\substack{b=0\\(b,p)=1}}^{p^\nu-1} f\left|\left[\left([\begin{smallmatrix} p^{2\ell-\nu} & b \\ 0 & p^\nu \end{smallmatrix}],\ p^{\frac{-2\ell+2\nu}{4}}\epsilon_p^{-1}\left(\frac{-b}{p}\right)\right)\right]_{k/2}\right. = 0.$$

The second last equality follows since as elements of G we have

$$\left(\left[\begin{smallmatrix} p^{2\ell-\nu} & p^{2\ell-\nu-1}m'+m \\ 0 & p^{\nu}\end{smallmatrix}\right],\, p^{\frac{-2\ell+2\nu}{4}}\epsilon_p^{-1}\left(\frac{-m}{p}\right)\right)$$
$$=\left(\left[\begin{smallmatrix} 1 & a \\ 0 & 1\end{smallmatrix}\right], 1\right)\cdot\left(\left[\begin{smallmatrix} p^{2\ell-\nu} & b \\ 0 & p^{\nu}\end{smallmatrix}\right],\ p^{\frac{-2\ell+2\nu}{4}}\epsilon_p^{-1}\left(\frac{-b}{p}\right)\right).$$

By working out similarly as above one can further see that

$$p^2C_{2\ell-2}-D_{2\ell-2}=\chi(p^2)\sum_{m'=0}^{p^2-1}\sum_{\substack{m=0\\(m,p)\neq 1}}^{p^{2\ell-2}-1} f|[(\left[\begin{smallmatrix} p^2 & p^2m'+mp^2 \\ 0 & p^{2\ell}\end{smallmatrix}\right],\ p^{\frac{\ell-1}{2}})]_{k/2} =: F_{2\ell-2}.$$

Thus to prove the theorem we are left to show that

$$F_{2\ell-2}-S_{2\ell}-E_{2\ell-1}-D_{2\ell-1}=0.$$

We claim that $D_{2\ell-1}=0$ and $F_{2\ell-2}-S_{2\ell}-E_{2\ell-1}=0$ which proves the theorem.

We first show that $D_{2\ell-1}=0$. Let $f(z)=\sum_{n=0}^{\infty}a_ne(nz)$ where $e(nz)=e^{2\pi inz}$. Rewriting $D_{2\ell-1}$ in terms of coefficients a_n we obtain

$$\begin{aligned}
&D_{2\ell-1}\\
&=\chi(p)p^{\frac{-\ell k}{2}}\epsilon_p^k\left(\frac{-1}{p}\right)\sum_{m'=0}^{p^2-1}\sum_{\substack{m=0\\(m,p)=1}}^{p^{2\ell-1}-1}\sum_{n=0}^{\infty}a_ne\left(\frac{npz+npm'+nmp^2}{p^{2\ell+1}}\right)\left(\frac{m}{p}\right)\\
&=\chi(p)p^{\frac{-\ell k}{2}}\epsilon_p^k\left(\frac{-1}{p}\right)\sum_{n=0}^{\infty}a_ne\left(\frac{nz}{p^{2\ell}}\right)\sum_{m'=0}^{p^2-1}e\left(\frac{nm'}{p^{2\ell}}\right)\sum_{m=0}^{p^{2\ell-1}-1}e\left(\frac{nm}{p^{2\ell-1}}\right)\left(\frac{m}{p}\right)\\
&=\chi(p)p^{\frac{-\ell k+4\ell-3}{2}}\epsilon_p^{k+1}\left(\frac{-1}{p}\right)\sum_{\substack{n=0\\p^{2\ell-2}|n}}^{\infty}a_ne\left(\frac{nz}{p^{2\ell}}\right)\left(\frac{n/p^{2\ell-2}}{p}\right)\\
&\qquad\qquad\sum_{m'=0}^{p^2-1}e\left(\frac{nm'/p^{2\ell-2}}{p^2}\right)=0,
\end{aligned}$$

where last two equalities follows using Lemma 4.2 on Gauss sums. In order to prove the final claim we again use the coefficients method as above to obtain

$$F_{2\ell-2}-S_{2\ell}=\chi(p^2)p^{\frac{(-\ell+1)k+4\ell-2}{2}}\sum_{\substack{n=0\\p^{2\ell-2}\|n}}^{\infty}a_ne\left(\frac{nz}{p^{2\ell-2}}\right),$$

$$E_{2\ell-1}=\chi(p^2)p^{\frac{(-\ell+1)k+4\ell-2}{2}}\epsilon_p^{2k+2}\sum_{\substack{n=0\\p^{2\ell-2}\|n}}^{\infty}a_ne\left(\frac{nz}{p^{2\ell-2}}\right).$$

Now $\epsilon_p^{2k+2} = 1$ since $2k+2 \equiv 0 \pmod 4$. Hence we are done. $\square$

Corollary 4.1. *Let $p \nmid N$ be a prime and $\ell \geq 2$. Let $f \in S_{k/2}(N,\chi)$. Then*

$$\mathrm{Sh}_t(T_{p^{2\ell}} f) = (T_{p^\ell} - \chi(p^2)p^{k-3}T_{p^{\ell-2}})(\mathrm{Sh}_t\, f),$$

where as before $T_{p^{2\ell}} \in \mathbb{T}_{k/2}$ and T_{p^ℓ}, $T_{p^{\ell-2}} \in \mathbb{T}_{k-1}$.

Proof. We use induction on ℓ. Recall from part (c) of Proposition 2.1 that for prime $p \nmid N$, we have

$$T_{p^{e+1}}(\mathrm{Sh}_t\, f) = (T_pT_{p^e} - \chi(p^2)p^{k-2}T_{p^{e-1}})(\mathrm{Sh}_t\, f). \tag{4.5}$$

As we remarked earlier, for $l = 2$ we have the following relation in $\mathbb{T}_{k/2}$:

$$T_{p^4} = (T_{p^2})^2 - \chi(p^2)(p^{k-3} + p^{k-2}).$$

Hence we get

$$\begin{aligned}\mathrm{Sh}_t(T_{p^4} f) &= \mathrm{Sh}_t((T_{p^2})^2 f) - \chi(p^2)(p^{k-3}+p^{k-2})(\mathrm{Sh}_t\, f)\\ &= ((T_p)^2 - \chi(p^2)p^{k-2})(\mathrm{Sh}_t\, f) - \chi(p^2)p^{k-3}(\mathrm{Sh}_t\, f)\\ &= (T_{p^2} - \chi(p^2)p^{k-3})(\mathrm{Sh}_t\, f).\end{aligned}$$

Assume the statement holds for all $\ell \leq e$. Then

$$\begin{aligned}&\mathrm{Sh}_t(T_{p^{2e+2}} f) = \mathrm{Sh}_t(T_{p^2}T_{p^{2e}} f) - \chi(p^2)p^{k-2}\,\mathrm{Sh}_t(T_{p^{2e-2}} f)\\ &= T_p(\mathrm{Sh}_t(T_{p^{2e}} f)) - \chi(p^2)p^{k-2}\,\mathrm{Sh}_t(T_{p^{2e-2}} f)\\ &= (T_pT_{p^e} - \chi(p^2)(p^{k-3}T_pT_{p^{e-2}} + p^{k-2}T_{p^{e-1}}) + \chi(p^4)p^{2k-5}T_{p^{e-3}})(\mathrm{Sh}_t\, f)\\ &= (T_{p^{e+1}} - \chi(p^2)p^{k-3}(T_{p^{e-1}} + \chi(p^2)p^{k-2}T_{p^{e-3}}) + \chi(p^4)p^{2k-5}T_{p^{e-3}})(\mathrm{Sh}_t\, f)\\ &= (T_{p^{e+1}} - \chi(p^2)p^{k-3}T_{p^{e-1}})(\mathrm{Sh}_t\, f).\end{aligned}$$

The first equality uses Theorem 4.1, third equality follows by using inductive hypothesis for $\ell = e$ and $\ell = e-1$, the others follow by using Equation (4.5). $\square$

We also prove the following proposition, independently of the proof of Theorem 4.1.

Proposition 4.1. *Let $p \nmid N$ be a prime and ℓ be a positive integer. For positive integers r such that $1 \leq r \leq \lfloor \frac{\ell}{2} \rfloor$ we give the following recursive construction of sequences $A_{r,\ell}(m)$ and $B_{r,\ell}(m)$:*

$$A_{1,\ell}(m) = 1, \qquad A_{r,\ell}(m) = A_{r-1,\ell}(m) - \binom{\ell-2(r-1)}{m-(r-1)}A_{r-1,\ell}(r-1);$$

$$B_{1,\ell}(m) = \binom{\ell}{m} - 1,\ B_{r,\ell}(m) = B_{r-1,\ell}(m) - \binom{\ell-2(r-1)}{m-(r-1)}B_{r-1,\ell}(r-1).$$

Let $\alpha_{r,\ell} = A_{r,\ell}(r)$ and $\beta_{r,\ell} = B_{r,\ell}(r)$. Then

$$T_{p^{2\ell}} = (T_{p^2})^{\ell} - \sum_{r=1}^{\lfloor \frac{\ell}{2} \rfloor} \chi(p^{2r})(\alpha_{r,\ell} p^{r(k-2)-1} + \beta_{r,\ell} p^{r(k-2)})(T_{p^2})^{\ell-2r}$$

as Hecke operators in $\mathbb{T}_{k/2}$.

Proof. Let $f = \sum_{n=0}^{\infty} a(n) q^n \in M_{k/2}(N, \chi)$. Our strategy will be to compare the n-th coefficient of action of the above operators on f on both sides. Substituting the q-expansion of f in Equation (4.1) and using Lemma 4.2 on Gauss sums we obtain

$$T_{p^{2\ell}} f = I_0 + I_{2\ell} + \sum_{\substack{\nu=1 \\ \nu \text{odd}}}^{2\ell-1} I_{\nu}^{\text{odd}} + \sum_{\substack{\nu=1 \\ \nu \text{even}}}^{2\ell-1} I_{\nu}^{\text{even}}$$

where

$$I_0 = \chi(p^{2\ell}) p^{(k-2)\ell} \sum_{n=0}^{\infty} a(n/p^{2\ell}) q^n, \qquad I_{2\ell} = \sum_{n=0}^{\infty} a(np^{2\ell}) q^n,$$

$$\begin{aligned} I_{\nu}^{\text{odd}} &= \chi(p^{2\ell} - \nu) p^{(\frac{k}{2}-1)(2\ell-\nu)-\frac{1}{2}} \epsilon_p^{k+1} \left(\frac{-1}{p}\right) \\ &\qquad \sum_{\substack{n=0 \\ p^{2\ell-\nu-1} \mid n}}^{\infty} a(n/p^{2\ell-2\nu}) \left(\frac{n/p^{2\ell-\nu-1}}{p}\right) q^n, \\ I_{\nu}^{\text{even}} &= \chi(p^{2\ell} - \nu) p^{(\frac{k}{2}-1)(2\ell-\nu)-1} \\ &\qquad \Big(\sum_{\substack{n=0 \\ p^{2\ell-\nu} \mid n}}^{\infty} a(n/p^{2\ell-2\nu})(p-1) q^n - \sum_{\substack{n=0 \\ p^{2\ell-\nu-1} \| n}}^{\infty} a(n/p^{2\ell-2\nu}) q^n \Big). \end{aligned}$$

Let n be a positive integer with $p^{2(\ell-1)} \mid n$. We can write the n-th coefficient of $T_{p^2}^{\ell} f$ as

$$\begin{aligned} &a(np^{2\ell}) + \sum_{m=1}^{\ell-1} \binom{\ell}{m} \chi(p^{2m}) p^{(k-2)m} a(np^{2\ell-4m}) \\ &\qquad + \chi(p^{2\ell-1}) \left(\frac{-1}{p}\right)^{\frac{k-1}{2}} \left(\frac{n/p^{2\ell-2}}{p}\right) p^{\frac{k-3}{2}+(k-2)(\ell-1)} a(n/p^{2\ell-2}) \\ &\qquad + \chi(p^{2\ell}) p^{(k-2)\ell} a(n/p^{2\ell}). \end{aligned}$$

Thus the n-th coefficient of $T_{p^2}^{\ell} f - T_{p^{2\ell}} f$ is

$$\sum_{m=1}^{\ell-1} \left(\binom{\ell}{m} - 1 \right) \chi(p^{2m}) p^{(k-2)m} a(np^{2\ell-4m}) + \sum_{m=1}^{\ell-1} \chi(p^{2m}) p^{(k-2)m-1} a(np^{2\ell-4m}).$$

We want to subtract a suitable multiple of $T_{p^2}^{\ell-2} f$ from the above so as to remove the terms involving $a(np^{2\ell-4})$ and $a(np^{4-2\ell})$, thereby reducing the number of terms in the above sum. Indeed we obtain that the n-th coefficient of
$(T_{p^2}^{\ell} - T_{p^{2\ell}} - \chi(p^2)(p^{k-3} + (\ell-1)p^{k-2}) T_{p^2}^{\ell-2}) f$ is

$$\sum_{m=2}^{\ell-2} \left(1 - \binom{\ell-2}{m-1} \right) \chi(p^{2m}) p^{(k-2)m-1} a(np^{2\ell-4m}) + \sum_{m=2}^{\ell-2} \left(\binom{\ell}{m} - 1 - (\ell-1) \binom{\ell-2}{m-1} \right) \chi(p^{2m}) p^{(k-2)m} a(np^{2\ell-4m}).$$

We iterate this process of subtracting suitable multiples of $T_{p^2}^{\ell-2r} f$ which leads us to the recursive formulae for $\alpha_{r,\ell}$ and $\beta_{r,\ell}$. □

We obtain the following combinatorial result as a corollary of Theorem 4.1 and Proposition 4.1

Corollary 4.2. *Keeping the notation as in the previous proposition we get the following combinatorial identities for* $2 \le r \le \lfloor \frac{\ell}{2} \rfloor - 1$*:*

$$\alpha_{r-1,\ell-2} + \alpha_{r,\ell} - \alpha_{r,\ell-1} = 0, \qquad \beta_{r-1,\ell-2} + \beta_{r,\ell} - \beta_{r,\ell-1} = 0.$$

Proof. Let $p \nmid N$ be any prime. We substitute the formula for $T_{p^{2\ell}}$ given by Proposition 4.1 in the identity of Theorem 4.1,

$$T_{p^{2\ell+2}} - T_{p^2} T_{p^{2\ell}} + \chi(p^2) p^{k-2} T_{p^{2\ell-2}} = 0$$

to obtain

$$- \sum_{r=2}^{\lfloor \frac{\ell}{2} \rfloor} \chi(p^{2r}) (\alpha_{r,\ell} p^{r(k-2)-1} + \beta_{r,\ell} p^{r(k-2)}) (T_{p^2})^{\ell-2r}$$
$$+ \sum_{r=2}^{\lfloor \frac{\ell-1}{2} \rfloor} \chi(p^{2r}) (\alpha_{r,\ell-1} p^{r(k-2)-1} + \beta_{r,\ell-1} p^{r(k-2)}) (T_{p^2})^{\ell-2r}$$
$$- \sum_{r=2}^{\lfloor \frac{\ell-2}{2} \rfloor + 1} \chi(p^{2r}) (\alpha_{r-1,\ell-2} p^{r(k-2)-1} + \beta_{r-1,\ell-2} p^{r(k-2)}) (T_{p^2})^{\ell-2r} = 0.$$

It is clear, with fixed ℓ and varying r, that the operators $(T_{p^2})^{\ell-2r}$ are linearly independent elements of $\mathbb{T}_{k/2}$ and hence

$$-\alpha_{r,\ell}+\alpha_{r,\ell-1}-\alpha_{r-1,\ell-2}+(\beta_{r,\ell}+\beta_{r,\ell-1}-\beta_{r-1,\ell-2})p=0.$$

Since this holds for any prime p with $p\nmid N$ the above corollary follows.

□

5. Generators for the Hecke Action

Theorem 5.1. *Let k, N be positive integers with $k\geq 3$ odd, and $4\mid N$. Let χ be a Dirichlet character modulo N. Let $N'=N/2$. Let $\mathbb{T}$ be the restriction of Hecke algebra $\mathbb{T}_{k-1}$ to $S_{k-1}(N',\chi^2)$ and suppose $\mathbb{T}$ is generated as a $\mathbb{Z}$-module by the Hecke operators T_i for $i\leq r$. Then the Hecke operators T_{i^2} for $i\leq r$ generate $\mathbb{T}'_{k/2}$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. In particular, $f\in S'_{k/2}(N,\chi)$ is an eigenform for all Hecke operators if and only if it is an eigenform for T_{i^2} for $i\leq r$.*

Proof. Let n be a positive integer such that $n=p_1^{n_1}p_2^{n_2}\cdots p_s^{n_s}$, where p_i are distinct primes. Let $f\in S'_{k/2}(N,\chi)$. Let t be a square-free positive integer. Using Theorem 4.1 or Proposition 4.1, for any prime p and a positive integer ℓ we can express the action of $T_{p^{2\ell}}$ as

$$T_{p^{2\ell}}=\sum_{j=0}^{\ell}\gamma_j T_{p^2}^{j},\qquad \gamma_j\in\mathbb{Z}[\zeta_{\varphi(N)}]. \tag{5.1}$$

Note that in the above expression $\gamma_\ell=1$ and hence the Hecke operators $T_{p^2}^{j}$ with $1\leq j\leq \ell$ generates the same $\mathbb{Z}[\zeta_{\varphi(N)}]$-module as does the Hecke operators $T_{p^{2j}}$ with $1\leq j\leq\ell$. Thus we have

$$\begin{aligned}\mathrm{Sh}_t(T_{n^2}f)&=\mathrm{Sh}_t(T_{p_1^{2n_1}}T_{p_2^{2n_2}}\cdots T_{p_s^{2n_s}}f)\\&=\mathrm{Sh}_t\left(\left(\sum_{j_1=0}^{n_1}\gamma_{j_1}T_{p_1^2}^{j_1}\right)\cdots\left(\sum_{j_s=0}^{n_s}\gamma_{j_s}T_{p_s^2}^{j_s}\right)f\right)\\&=\left(\sum_{j_1=0}^{n_1}\gamma_{j_1}T_{p_1}^{j_1}\right)\cdots\left(\sum_{j_s=0}^{n_s}\gamma_{j_s}T_{p_s}^{j_s}\right)(\mathrm{Sh}_t f)\\&=\sum_{i=1}^{r}\delta_i T_i(\mathrm{Sh}_t f),\end{aligned}\tag{5.2}$$

where the last equality follows since the T_i, with $1\leq i\leq r$, generate $\mathbb{T}$ as a $\mathbb{Z}$-module, while the second last equality follows by Proposition 3.2.

Recall from Proposition 2.1, for any prime q and a positive integer ℓ, the action of Hecke operator T_{q^ℓ} on $S_{k-1}(N', \chi^2)$ can be expressed as

$$T_{q^\ell} = \sum_{j=0}^{\ell} \alpha_j T_q^j, \qquad \alpha_j \in \mathbb{Z}[\zeta_{\varphi(N')}] \subset \mathbb{Z}[\zeta_{\varphi(N)}].$$

Let $1 \le i \le r$ has prime factorization $i = q_1^{m_1} q_2^{m_2} \cdots q_v^{m_v}$. Then each term $T_i(\mathrm{Sh}_t f)$ in Equation (5.2) can be written as

$$\begin{aligned} T_i(\mathrm{Sh}_t f) &= T_{q_1^{m_1}} T_{q_2^{m_2}} \cdots T_{q_v^{m_v}} (\mathrm{Sh}_t f) \\ &= \left(\sum_{j_1=0}^{m_1} \alpha_{j_1} T_{q_1}^{j_1} \right) \cdots \left(\sum_{j_v=0}^{m_v} \alpha_{j_v} T_{q_v}^{j_v} \right) (\mathrm{Sh}_t f) \\ &= \mathrm{Sh}_t \left(\left(\sum_{j_1=0}^{m_1} \alpha_{j_1} T_{q_1^2}^{j_1} \right) \cdots \left(\sum_{j_v=0}^{m_v} \alpha_{j_v} T_{q_v^2}^{j_v} \right) f \right) \\ &= \mathrm{Sh}_t \left(\left(\sum_{j_1=0}^{m_1} \beta_{j_1} T_{q_1^{2j_1}} \right) \cdots \left(\sum_{j_v=0}^{m_v} \beta_{j_v} T_{q_v^{2j_v}} \right) f \right) \\ &= \mathrm{Sh}_t \left(\sum_{j=1}^{i} A_j T_{j^2} f \right), \end{aligned} \tag{5.3}$$

where $A_j \in \mathbb{Z}[\zeta_{\varphi(N)}]$. In the above equalities we repeatedly use Proposition 3.2 and Equation (5.1). For the second last equality we use the remark below Equation (5.1). Now using Equations (5.2) and (5.3) we get

$$\mathrm{Sh}_t(T_{n^2} f) = \mathrm{Sh}_t \left(\sum_{i=1}^{r} B_i T_{i^2} f \right), \qquad B_i \in \mathbb{Z}[\zeta_{\varphi(N)}].$$

Since this is true for all positive square-free integers t, using Lemma 3.2 we deduce that

$$T_{n^2} f = \sum_{i=1}^{r} B_i T_{i^2} f.$$

Hence T_{i^2} with $i \le r$ generate $\mathbb{T}'_{k/2}$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. □

We shall need the following theorem which is a consequence of Sturm's bound [8].

Theorem 5.2. *(Stein* [7, Theorem 9.23]*) Suppose* Γ *is a congruence subgroup that contains* $\Gamma_1(N)$*. Let*

$$r = \frac{km}{12} - \frac{m-1}{N}, \qquad m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma].$$

Then the Hecke algebra

$$\mathbb{T} = \mathbb{Z}[\dots, T_n, \dots] \subset \mathrm{End}(S_k(\Gamma))$$

is generated as a $\mathbb{Z}$-module by the Hecke operators T_n for $n \leq r$.

From Theorem 5.2 we deduce the following corollary.

Corollary 5.1. *Let k, N be positive integers with $k \geq 3$ odd, and $4 \mid N$. Let χ be a Dirichlet character modulo N. Let $N' = N/2$. Write*

$$m = N'^2 \prod_{p \mid N'} \left(1 - \frac{1}{p^2}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

Then T_{i^2} for $i \leq R$ generate $\mathbb{T}'_{k/2}$ as a $\mathbb{Z}[\zeta_{\varphi(N)}]$-module. In particular the set of operators T_{p^2} for primes $p \leq R$ forms a generating set as an algebra. Moreover, if χ is a quadratic character, then the same result holds as above with

$$m = N' \prod_{p \mid N'} \left(1 + \frac{1}{p}\right), \qquad R = \frac{(k-1)m}{12} - \frac{m-1}{N'}.$$

Proof. Note that $S_{k-1}(N', \chi^2) \subset S_{k-1}(\Gamma_1(N'))$. The first part of the corollary follows by applying Theorem 5.1 and Theorem 5.2 to the congruence subgroup $\Gamma_1(N')$ and using the formula for $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N')]$ that can be found for example in [1, Page 14].

Now suppose χ is a quadratic character. Then $S_{k-1}(N', \chi^2) = S_{k-1}(N')$. So we apply Theorem 5.2 to the group $\Gamma_0(N')$ and we now use the formula for $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N')]$. □

References

[1] F. Diamond and J. Shurman, *A First Course in Modular Forms*, GTM **228**, Springer-Verlag, 2005.

[2] W. Kohnen, *Newforms of half-integral weight*, J. Reine Angew. Math. **333** (1982), 32–72.

[3] T. Miyake, *Modular Forms*, Springer-Verlag, 1989.

[4] S. Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Mathematical Journal **56** (1975), 147–161.

[5] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS **102**, American Mathematical Society, 2004.

[6] G. Shimura, *On Modular Forms of Half Integral Weight*, Annals of Mathematics, Second Series, Vol. 97, **3** (1973), pp. 440–481.

[7] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics **79**, American Mathematical Society, 2007.

[8] J. Sturm, *On the Congruence of Modular Forms. Number theory (New York, 1984-1985)*, Lecture Notes in Math. **1240**, Springer, Berlin, (1987), 275–280.

Soma Purkait
Mathematics Institute
University of Warwick
Coventry
CV4 7AL
United Kingdom
E-mail: `Soma.Purkait@warwick.ac.uk`
URL: `http://homepages.warwick.ac.uk/~maslau/`