LMS J. Comput. Math. 16 (2013) 216–245
doi:10.1112/S1461157013000144

Explicit application of Waldspurger's theorem

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Abstract

For a given cusp form ϕ of even integral weight satisfying certain hypotheses, Waldspurger's theorem relates the critical value of the L-function of the nth quadratic twist of ϕ to the nth coefficient of a certain modular form of half-integral weight. Waldspurger's recipes for these modular forms of half-integral weight are far from being explicit. In particular, they are expressed in the language of automorphic representations and Hecke characters. We translate these recipes into congruence conditions involving easily computable values of Dirichlet characters. We illustrate the practicality of our 'simplified Waldspurger' by giving several examples.

1. *Introduction*

In 1983 Tunnell [28] gave a remarkable solution to the congruent number problem, assuming that the celebrated Birch and Swinnerton-Dyer conjecture holds. This ancient Diophantine question asks for the classification of congruent numbers, those positive integers which are the areas of right-angled triangles whose sides are rational numbers. For positive n, write $E_n : y^2 = x^3 - n^2x$; note that E_n is the nth quadratic twist of E_1. It is straightforward to show that n is a congruent number if and only if the elliptic curve $E_n/\mathbb{Q}$ has positive rank. Tunnell expresses the critical value of the L-function of E_n in terms of coefficients of certain modular forms of weight $3/2$. These modular forms are in turn written in terms of theta series of ternary quadratic forms. Applying the conjecture of Birch and Swinnerton-Dyer, Tunnell is then able to give a simple and elegant criterion for n to be a congruent number.

Tunnell's theorem is a highly non-trivial consequence of a theorem of Waldspurger [30]. For a given cusp form ϕ of even integral weight satisfying certain hypotheses, Waldspurger's theorem relates the critical value of the L-function of the nth quadratic twist of ϕ to the nth coefficient of a certain modular form of half-integral weight. Waldspurger's recipes for these modular forms of half-integral weight are far from being explicit. In particular, they are expressed in the language of automorphic representations and Hecke characters. We translate these recipes into congruence conditions involving easily computable values of Dirichlet characters. We illustrate the practicality of our 'simplified Waldspurger' by giving several Tunnell-like examples, of which the following is the simplest.

Proposition 1.1. *Let E be the elliptic curve of conductor* 50 *given by*

$$E : Y^2 + XY + Y = X^3 + X^2 - 3X + 1. \tag{1}$$

Let Q_1, Q_2, Q_3, Q_4 be the following positive-definite ternary quadratic forms,

$$Q_1 = 25x^2 + 25y^2 + z^2, \quad Q_2 = 14x^2 + 9y^2 + 6z^2 + 4yz + 6xz + 2xy,$$
$$Q_3 = 25x^2 + 13y^2 + 2z^2 + 2yz, \quad Q_4 = 17x^2 + 17y^2 + 3z^2 - 2yz - 2xz + 16xy.$$

Received 4 October 2012; revised 10 April 2013.

2010 Mathematics Subject Classification 11F37 (primary), 11F70, 11F11 (secondary).

The author is supported by a Warwick Postgraduate Research Scholarship and an EPSRC Delivering Impact Award.

Let n be a positive square-free number such that $5 \nmid n$. Then,

$$\mathrm{L}(E_{-n}, 1) = \frac{\mathrm{L}(E_{-1}, 1)}{\sqrt{n}} \cdot c_n^2$$

where E_{-n} is the $-n$th quadratic twist of E and

$$c_n = \sum_{i=1}^{4} \frac{(-1)^{i-1}}{2} \cdot \#\{(x, y, z) : Q_i(x, y, z) = n\}.$$

For the elliptic curve E in (1) and n a positive square-free integer such that $5 \nmid n$, we give a similar formula for $\mathrm{L}(E_n, 1)$ that involves 38 quadratic forms.

We would like to note here that all of the examples we consider deal with newforms whose levels are neither odd nor square-free. In fact, for newforms ϕ of weight $k-1$ and odd and square-free level N with $\mathrm{L}(\phi, 1) \neq 0$ and $k \equiv 3 \pmod 4$ there is an explicit recipe by Böcherer and Schulze-Pillot [**4**] for constructing a modular form of weight $k/2$, level $4N$ that is Shimura equivalent to ϕ. Their method uses generalized theta series and the Eichler correspondence with automorphic forms on quaternion algebras. In particular, they show that given a rational elliptic curve E of odd and square-free conductor, an inverse Shimura lift of ϕ_E (the newform corresponding to E) comes from ternary quadratic forms if and only if $\mathrm{L}(E, 1) \neq 0$. We note that the form they construct belongs to the Kohnen subspace [**15**]. It follows from Waldspurger [**30**] that in these cases the space of Shimura equivalent forms at level $4N$ is two-dimensional. In a recent paper Hamieh [**12**] used [**4**] and Waldspurger's recipe to compute a basis for this two-dimensional space.

We would also like to mention the work of Shin-ichi Yoshida [**31**] in which he considers $2\pi/3$- and $\pi/3$-congruent number problems and uses Waldspurger's result (Corollary 5.2 below) to give a Tunnell-like criterion for a square-free number in certain congruence classes to be $2\pi/3$- and $\pi/3$-congruent.

The paper is arranged as follows. In § 2 we review Shimura's decomposition of the space of cusp forms of a certain level and half-integral weight into certain subspaces appearing in Waldspurger's theorem. In § 3 we review the correspondence between Dirichlet characters and Hecke characters and we prove a result that allows us to evaluate components of a Hecke character corresponding to a given Dirichlet character. Next, in § 4 we review the correspondence between modular forms of even integral weight and automorphic representations and prove a result needed for simplifying the hypotheses of Waldspurger's theorem. In § 5 we state Waldspurger's theorem in simplified form. To apply Waldspurger's theorem in conjunction with the Birch and Swinnerton-Dyer conjectures it is convenient to express the period of the nth twist of a given elliptic curve in terms of the period of the elliptic curve itself. We do this in § 6. To apply Waldspurger's recipes we need to be able to answer questions of the following form: for a given cusp form of half-integral weight $f = \sum a_n q^n$, and positive integers a, M, is $a_n = 0$ for all $n \equiv a \pmod M$? We give an algorithm for answering this question in § 7. Finally, § 8 is devoted to extensive examples which combine our algorithm [**18**] for computing Shimura's decomposition with Waldspurger's theorem as made explicit in this paper.

2. *Shimura decomposition*

Let $k \geqslant 3$ be an odd integer and N a positive integer such that $4 \mid N$. Let χ be an even Dirichlet character modulo N. We denote by $S_{k/2}(N, \chi)$ the space of cusp forms of weight $k/2$, level N and character χ. Let $S^0_{k/2}(N, \chi)$ be the subspace of $S_{k/2}(N, \chi)$ spanned by single-variable[†]

[†]The term 'single-variable theta series' refers to the theta series of weights 1/2 and 3/2 that come from a quadratic form of one variable and are of the form $\sum_{n=-\infty}^{\infty} \psi(n) n^{\nu} q^{n^2}$, where $\nu \in \{0, 1\}$ and ψ is a Dirichlet character such that $\psi(-1) = (-1)^{\nu}$.

theta series when $k=3$; for $k \geqslant 5$, we define $S^0_{k/2}(N,\chi)=0$. More precisely, a generating set for $S^0_{3/2}(N,\chi)$ is given by

$$S=\left\{\sum_{m=1}^{\infty}\psi(m)mq^{tm^2} : 4r_\psi^2 t \mid N \text{ and } \psi \text{ is a primitive odd character of conductor } r_\psi \text{ such that } \chi=\left(\frac{-4t}{\cdot}\right)\psi\right\},$$

which in fact constitutes a basis for $S^0_{3/2}(N,\chi)$, as shown in [**18**]. The interesting part (from the point of view of Waldspurger's theorem) of the space $S_{k/2}(N,\chi)$ is the orthogonal complement of $S^0_{k/2}(N,\chi)$ with respect to the Petersson inner product, denoted by $S'_{k/2}(N,\chi)$.

In his thesis Basmaji [**3**] gave an algorithm for computing a basis for the space of half-integral weight modular forms of level divisible by 16. The main idea of the algorithm is to use theta series $\Theta=\sum_{n=-\infty}^{\infty}q^{n^2}$, $\Theta_1=(\Theta-V(4)\Theta)/2$ and the following embedding,

$$\varphi: S_{k/2}(N,\chi)\to S\times S, \quad f\mapsto (f\Theta, f\Theta_1),$$

where $S=S_{(k+1)/2}(N,\chi\cdot\chi_{-1}^{(k+1)/2})$ and V is the usual shift operator. This idea has been generalized by Steve Donnelly for level divisible by four and is implemented in `MAGMA`.

Let $N'=N/2$. For $M\mid N'$ such that $\mathrm{Cond}(\chi^2)\mid M$ and a newform $\phi\in S^{\mathrm{new}}_{k-1}(M,\chi^2)$ Shimura defines

$$S_{k/2}(N,\chi,\phi)=\{f\in S'_{k/2}(N,\chi): T_{p^2}(f)=\lambda_p(\phi)f \text{ for almost all } p\nmid N\},$$

where $T_p(\phi)=\lambda_p(\phi)\phi$, and gives the following decomposition theorem [**22**].

THEOREM 1 (Shimura [**22**]). *We have* $S'_{k/2}(N,\chi)=\bigoplus_\phi S_{k/2}(N,\chi,\phi)$ *where* ϕ *runs through all newforms* $\phi\in S^{\mathrm{new}}_{k-1}(M,\chi^2)$ *with* $M\mid N'$ *and* $\mathrm{Cond}(\chi^2)\mid M$.

We point out that the summands $S_{k/2}(N,\chi,\phi)$ occur in Waldspurger's theorem and their computation is necessary for explicit applications of that theorem. However, the above theorem is not suitable for computation since for any particular prime $p\nmid N$, we do not know if it is included or excluded in the 'almost all' condition. In [**18**] we proved the above theorem with a more precise definition for the spaces $S_{k/2}(N,\chi,\phi)$:

$$S_{k/2}(N,\chi,\phi)=\{f\in S'_{k/2}(N,\chi): T_{p^2}(f)=\lambda_p(\phi)f \text{ for all } p\nmid N\},$$

whilst showing that our definition is equivalent to Shimura's definition. We also proved the following theorem that gives an algorithm for computing the Shimura decomposition.

THEOREM 2 (Purkait [**18**]). *Let* ϕ *be a newform of weight* $k-1$*, level* M *dividing* N'*, and character* χ^2*. Let* $p_1,\ldots,p_n$ *be primes not dividing* N *satisfying the following: for every newform* $\phi'\neq\phi$ *of weight* $k-1$*, level dividing* N' *and character* χ^2*, there is some* p_i *such that* $\lambda_{p_i}(\phi')\neq\lambda_{p_i}(\phi)$*, where* $T_{p_i}(\phi)=\lambda_{p_i}(\phi)\cdot\phi$*. Then*

$$S_{k/2}(N,\chi,\phi)=\{f\in S_{k/2}(N,\chi): T_{p_i^2}(f)=\lambda_{p_i}(\phi)f \text{ for } i=1,\ldots,n\}.$$

3. *Correspondence between Dirichlet characters and Hecke characters on* $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ *of finite order*

We shall need the correspondence between Dirichlet characters and Hecke characters on $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ of finite order. This material is in Tate's thesis [**26**], but we found the presentation in [**6**, § 3.1] more useful.

PROPOSITION 3.1. *Let $\boldsymbol{\chi} = (\chi_p)$ be a character on $\mathbb{A}_{\mathbb{Q}}^{\times}$. Then there exists a finite set S of places, including the Archimedean one, such that if $p \notin S$, then χ_p is trivial on the unit group $\mathbb{Z}_p^{\times}$.*

Recall that if χ_p is trivial on the unit group $\mathbb{Z}_p^{\times}$, then χ_p is unramified. Thus by the above proposition, χ_p is unramified for all but finitely many p.

THEOREM 3 (Bump [**6**, Proposition 3.1.2]). *Suppose that $\boldsymbol{\chi} = (\chi_p)$ is a character of finite order on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$. There exists an integer N whose prime divisors are precisely the non-Archimedean primes p such that χ_p is ramified, and a primitive Dirichlet character χ modulo N such that if $p \nmid N$ is non-Archimedean, then $\chi(p) = \chi_p(p)$. This correspondence $\boldsymbol{\chi} \mapsto \chi$ is a bijection between characters of finite order of $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ and the primitive Dirichlet characters.*

In our work, we shall need to start with a Dirichlet character χ of modulus N and then perform computations with the corresponding Hecke character $\boldsymbol{\chi}$. We collect here some facts that will help us with these computations.

LEMMA 3.2. *We keep the notation of Theorem 3.*

(i) *For any $\alpha \in \mathbb{Q}^{\times}$, $\prod \chi_p(\alpha) = 1$ where the product is taken over all places.*
(ii) *Suppose that $p = \infty$ and $\alpha \in \mathbb{Q}_{\infty}^{\times} = \mathbb{R}^{\times}$. Then $\chi_{\infty}(\alpha) = 1$ if $\alpha > 0$ or if χ has odd order.*
(iii) *Let p be a non-Archimedean prime such that $p \mid N$ and α, $\beta \in \mathbb{Z}_p$ be non-zero. Suppose that $\beta \equiv \alpha \pmod{\alpha N \mathbb{Z}_p}$. Then $\chi_p(\beta) = \chi_p(\alpha)$.*
(iv) *Let p be non-Archimedean such that $p \nmid N$. Then χ_p is unramified.*

PROPOSITION 3.3. *Let χ be a Dirichlet character modulo N (not necessarily primitive) and let $\boldsymbol{\chi} = (\chi_p)$ be the corresponding character on $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$. Let $a \in \mathbb{Z}$ be non-zero. For a prime q, let $\nu_q(a)$ denote the exponent of the highest power of q that divides a.*

(a) *If $q \nmid N$, then $\chi_q(a) = \chi(q)^r$ where $r = \nu_q(a)$.*

(b) *Suppose that q divides N and let $q_1, \ldots, q_r$ be the other primes dividing N. Let b be a* positive *integer satisfying*

$$b \equiv \begin{cases} a \pmod{aN\mathbb{Z}_q} \\ 1 \pmod{N\mathbb{Z}_{q_i}} \quad i = 1, \ldots, r; \end{cases}$$

such b can easily be constructed by the Chinese remainder theorem. Write

$$b = q^{\nu_q(a)} \prod_{j=1}^{s} \ell_j^{\beta_j}$$

where the ℓ_j are distinct primes. Then

$$\chi_q(a) = \prod_{j=1}^{s} \chi(\ell_j)^{-\beta_j}.$$

Proof. Let N' be the conductor of χ and note that $N' \mid N$. Now if $q \nmid N$, then χ_q is unramified. Write $a = q^r a'$ where $q \nmid a'$. Then $a' \in \mathbb{Z}_q^{\times}$. Thus, by definition of unramified, $\chi_q(a') = 1$. Moreover, from Theorem 3, $\chi_q(q) = \chi(q)$. This proves part (a).

Now suppose that $q \mid N$ and let $q_1, \ldots, q_r$ be the other primes dividing N. Let b be as in the proposition. Since $N' \mid N$, we have

$$b \equiv \begin{cases} a \pmod{aN'\mathbb{Z}_q} \\ 1 \pmod{N'\mathbb{Z}_{q_i}} \quad i = 1, \ldots, r. \end{cases}$$

By Lemma 3.2, $\chi_q(b) = \chi_q(a)$, and $\chi_{q_i}(b) = 1$ for $i = 1, \ldots, r$. Now

$$\begin{aligned} \chi_q(a) &= \chi_q(b) \\ &= \prod_{p \neq q} \chi_p(b)^{-1} \quad \text{by part (i) of Lemma 3.2,} \\ &= \prod_{p \nmid N} \chi_p(b)^{-1} \quad \text{since } \chi_{q_i}(b) = 1, \\ &= \prod_{j=1}^{s} \chi(\ell_j)^{-\beta_j} \quad \text{using part (a).} \end{aligned}$$

This completes the proof. □

4. *Local components of the automorphic representations associated to modular forms of even integer weight*

Let k be a positive odd integer with $k \geqslant 3$. Let $\phi = \sum_{n=1}^{\infty} a_n q^n \in S_{k-1}^{\mathrm{new}}(N, \chi)$ be a newform of weight $k-1$, level N and character χ.

We can associate to ϕ an automorphic representation ρ. Let ρ_p be the local component of ρ at a prime p.

If $\phi = \sum_{n=1}^{\infty} a_n q^n$ is an eigenform, then we define its twist by a character μ to be the modular form $\phi_\mu = \sum_{n=1}^{\infty} a_n \mu(n) q^n$.

Waldspurger works with the following different definition of twist: let ϕ be a newform of weight $k-1$ and character χ. Let μ a Dirichlet character. We denote by $\phi \otimes \mu$ the (unique) newform of weight $k-1$ with character $\chi\mu^2$ satisfying $\lambda_p(\phi \otimes \mu) = \mu(p)\lambda_p(\phi)$ for almost all primes p, where λ_p is the eigenvalue under T_p.

Now fix a prime number p. Let ξ_p be the set of primitive Dirichlet characters with p-power conductor. The following hold (see [**30**, § III]):

(i) ρ_p is supercuspidal if and only if for all $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p and $\lambda_p(\phi \otimes \mu) = 0$;

(ii) ρ_p is an irreducible principal series if and only if either:
  - (a) there exists a character μ in ξ_p such that p does not divide the level of $\phi \otimes \mu$; or
  - (b) there exist two distinct characters μ_1, μ_2 in ξ_p such that $\lambda_p(\phi \otimes \mu_1) \neq 0$, $\lambda_p(\phi \otimes \mu_2) \neq 0$;

(iii) ρ_p is a special representation if and only if the following conditions hold:
  - (a) for all $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p; and
  - (b) there exists a unique μ in ξ_p such that $\lambda_p(\phi \otimes \mu) \neq 0$.

We shall need the following theorem which is extracted from the paper of Atkin and Li [**1**].

THEOREM 4 (Atkin and Li [**1**]). *Let $\phi = \sum_{n=1}^{\infty} a_n q^n$ be a newform of weight $k-1$, character χ and level N. Let μ be a primitive character of conductor m. Then the following hold.*

(a) *If $\gcd(m, N) = 1$, then $\phi \otimes \mu = \phi_\mu$, and it is a newform of weight $k-1$, character $\chi\mu^2$ and level Nm^2 (see [**1**, Introduction]).*

(b) *Suppose that μ is of q-power conductor where $q \mid N$ and write $N = q^s M$ where $q \nmid M$. Then $\phi \otimes \mu$ is a newform of weight $k-1$, character $\chi\mu^2$ and level $q^{s'} M$ for some $s' \geqslant 0$. Moreover, $\lambda_p(\phi \otimes \mu) = \mu(p)\lambda_p(\phi)$ for all primes $p \nmid N$ (see [**1**, Theorem 3.2]). In particular, if $s = 1$ and χ is trivial, then for μ with conductor q^r, $r \geqslant 1$, it turns out that $\phi \otimes \mu = \phi_\mu$ is a newform of level $q^{2r} M$ and character μ^2 (see [**1**, Corollary 4.1]).*

(c) *Let $q \mid N$. Suppose that ϕ is q-primitive and $a_q = 0$. Then for all characters μ of q-power conductor, $\phi \otimes \mu = \phi_\mu$ is a newform of level divisible by N. (Recall that ϕ is q-primitive if ϕ*

is not a twist of any newform of level lower than N by a character of conductor equal to some power of q.) See [**1**, *Proposition* 4.1].

(d) *Let $N = q^s M$ where $q \nmid M$; let $Q = q^s$. Let χ_Q be the Q-part*† *of the character χ. If s is odd and* cond $\chi_Q \leqslant \sqrt{Q}$, *then ϕ is q-primitive.*
Now suppose that $q = 2$. Then, if $s = 2$, then ϕ is always 2-primitive; if s is odd, then ϕ is 2-primitive if and only if cond $\chi_Q < \sqrt{Q}$; *if s is even and $s \geqslant 4$, then ϕ is 2-primitive if and only if* cond $\chi_Q = \sqrt{Q}$ (*see* [**1**, *Theorem* 4.4]).

We deduce the following corollaries which we use later.

Corollary 4.1. *Let $\phi = \sum_{n=1}^{\infty} a_n q^n \in S_{k-1}^{\text{new}}(N)$ be a newform with trivial character. Let ρ_2 be the local component at 2 of the corresponding automorphic representation. Suppose that either N is odd or $\nu_2(N) = 1$. Then ρ_2 is not supercuspidal. Further, if $\nu_2(N) \geqslant 2$ and ϕ is 2-primitive, then ρ_2 is supercuspidal, hence if either $\nu_2(N) = 2$ or $\nu_2(N) > 1$ is odd then ρ_2 is supercuspidal.*

Proof. If N is odd, take μ to be the identity character. Thus, $\mu \in \xi_2$ and the level of $\phi \otimes \mu$ is odd and, hence, ρ_2 is not supercuspidal. If $N = 2M$ such that M is odd, then $a_2 \neq 0$, so taking μ as the identity character we get that $\lambda_2(\phi \otimes \mu) = a_2 \neq 0$ and, thus, ρ_2 is not supercuspidal.

Let $\nu_2(N) \geqslant 2$. Then $a_2 = 0$. If ϕ is 2-primitive, then it follows using part (c) of Theorem 4 that for any $\mu \in \xi_2$, $\phi \otimes \mu = \phi_\mu$ is newform of level divisible by 2. Write $T_2(\phi_\mu) = \sum_{n=1}^{\infty} b_n q^n$. Then, $b_n = a_{2n}\mu(2n) + \mu^2(2)2^{k-2}a_{n/2}\mu(n/2)$ for all n. Thus, $T_2(\phi_\mu) = 0$. Therefore, $\lambda_2(\phi \otimes \mu) = \lambda_2(\phi_\mu) = 0$ and ρ_2 is supercuspidal. The final statement is a direct application of part (d) of Theorem 4. $\square$

Corollary 4.2. *Let ϕ be as in the above corollary.*
(i) *If $N = pM$ with M coprime to p and $a_p \neq 0$, then ρ_p is a special representation.*
(ii) *If $p \nmid N$, then ρ_p is an irreducible principal series.*

Proof. We first prove part (i). By part (b) of Theorem 4, for any $\mu \in \xi_p$, the level of $\phi \otimes \mu$ is divisible by p. Further if μ is the identity character then $\lambda_p(\phi \otimes \mu) = a_p \neq 0$; we claim that this is unique such character in ξ_p. Let $\mu \in \xi_p$ be such that μ is a character of conductor p^r, $r \geqslant 1$. Then $\phi \otimes \mu = \phi_\mu$ is a newform in $S_{k-1}(p^{2r}M, \mu^2)$ such that $\lambda_p(\phi_\mu) = a_p\mu(p) = 0$ and, hence, $\lambda_p(\phi \otimes \mu) = 0$.

The proof of part (ii) is obvious and does not require the condition that newform ϕ has trivial character. $\square$

5. *Waldspurger's theorem and notation*

In this section we present Waldspurger's theorem. We introduce and simplify the notation used in the theorem. This is needed in the following section where we will discuss how to use the theorem for elliptic curves and compute critical values of L-functions in terms of coefficients of corresponding half-integral weight forms. An important application is the computation of orders of the Tate–Shafarevich groups assuming the Birch and Swinnerton-Dyer conjecture.

Let k be positive integers with $k \geqslant 3$ odd. Let χ be an even Dirichlet character with modulus divisible by 4. Fix a newform ϕ of level M_ϕ in $S_{k-1}^{\text{new}}(M_\phi, \chi^2)$. Let p be a prime number. Let ν_p be the p-adic valuation on $\mathbb{Q}$ and $\mathbb{Q}_p^\times$. Let $m_p = \nu_p(M_\phi)$ and λ_p be the Hecke eigenvalue of ϕ corresponding to the Hecke operator T_p.

†Let χ be a Dirichlet character with modulus $p_1^{r_1} \ldots p_n^{r_n}$ where the p_i are distinct primes. Then χ can be written uniquely as a product $\prod \chi_{p_i^{r_i}}$ where $\chi_{p_i^{r_i}}$ has modulus $p_i^{r_i}$. See [**1**].

Let ρ be the automorphic representation associated to ϕ and ρ_p be the local component of ρ at p. Let S be the (finite) set of primes p such that ρ_p is not irreducible principal series. If $p \notin S$, ρ_p is equivalent to $\pi(\mu_{1,p}, \mu_{2,p})$ where $\mu_{1,p}$ and $\mu_{2,p}$ are two continuous characters on $\mathbb{Q}_p$ such that $\mu_{1,p}\mu_{2,p} \neq |\cdot|^{\pm 1}$. Let (H1) be the following hypothesis:

$$\text{(H1)} \quad \text{For all } p \notin S, \quad \mu_{1,p}(-1) = \mu_{2,p}(-1) = 1.$$

THEOREM 5 (Flicker [11]). *There exists N such that $S_{k/2}(N, \chi, \phi) \neq \{0\}$ if and only if the hypothesis* (H1) *holds.*

THEOREM 6 (Vigneras [29]). *Flicker's condition* (H1) *always holds whenever ϕ is a newform of even weight with trivial character.*

Proof. For the proof refer to [29]. □

From the theorems of Flicker and Vigneras we have the following easy corollary.

COROLLARY 5.1. *Let ϕ be a newform of weight $k-1$, level M_ϕ and trivial character χ_{triv}. Let χ be a Dirichlet character satisfying $\chi^2 = \chi_{\text{triv}}$. Then there exists some N such that $S_{k/2}(N, \chi, \phi) \neq \{0\}$.*

Henceforth, we will always assume that ϕ has trivial character and χ is quadratic, thus the conclusion of the corollary holds. We will now introduce several pieces of notation used by Waldspurger [30, § VIII] before stating his main theorem.

Let χ_0 be the Dirichlet character associated to χ given by

$$\chi_0(n) := \chi(n)\left(\frac{-1}{n}\right)^{(k-1)/2}.$$

Let $\chi_{0,p}$ be the local component of χ_0 at a prime p. For each prime p we will define (page 225) a non-negative integer $\widetilde{n_p}$ that depends only on the local components ρ_p and $\chi_{0,p}$. Let $\widetilde{N_\phi}$ be given by

$$\widetilde{N_\phi} := \prod_p p^{\widetilde{n_p}}.$$

For prime p and natural number e, we will also define a set $\mathrm{U}_p(e, \phi)$ which consists of some finite number of complex-valued functions on $\mathbb{Q}_p^\times$ having support in $\mathbb{Z}_p \cap \mathbb{Q}_p^\times$.

Let $\mathbb{N}^{\text{sc}}$ be the set of positive square-free numbers and for $n \in \mathbb{N}$, let n^{sc} be the square-free part of n. Let A be a function on the set $\mathbb{N}^{\text{sc}}$ having values in $\mathbb{C}$ and E be an integer such that $\widetilde{N_\phi} \mid E$. We use the notation $e_p = \nu_p(E)$ for all prime numbers p and let $\underline{c} = (c_p)$ be any element of $\prod_p \mathrm{U}_p(e_p, \phi)$. Define

$$f(\underline{c}, A)(z) := \sum_{n=1}^{\infty} A(n^{\text{sc}}) n^{(k-2)/4} \prod_p c_p(n)\, q^n, \quad z \in \mathbb{H}$$

and let $\overline{\mathrm{U}}(E, \phi, A)$ be the space generated by these functions $f(\underline{c}, A)$ on $\mathbb{H}$ where $\underline{c} \in \prod_p \mathrm{U}_p(e_p, \phi)$.

With the above notation, we are now ready to state the main theorem of Waldspurger [30, Théorème 1].

THEOREM 7 (Waldspurger [30]). *Let* (H2) *be the hypothesis that one of the following holds:*
(a) *the local component ρ_2 is not supercuspidal;*
(b) *the conductor of χ_0 is divisible by 16;*
(c) $16 \mid M_\phi$.

Let χ be a Dirichlet character and ϕ be a newform of weight $k-1$ and character χ^2 such that (H1) *and* (H2) *hold. Then there exists a function A_ϕ on $\mathbb{N}^{\mathrm{sc}}$ such that for $t \in \mathbb{N}^{\mathrm{sc}}$:*

$$A_\phi(t)^2 := \mathrm{L}\left(\phi \otimes \chi_0^{-1}\chi_t, \frac{k-1}{2}\right) \cdot \epsilon\left(\chi_0^{-1}\chi_t, \frac{1}{2}\right). \tag{2}$$

Moreover, for $N \geqslant 1$,

$$S_{k/2}(N, \chi, \phi) = \bigoplus \overline{\mathrm{U}}(E, \phi, A_\phi)$$

where the sum is over all $E \geqslant 1$ such that $\widetilde{N_\phi} \mid E \mid N$.

Here $\chi_t = (\frac{t}{\cdot})$ is a quadratic character with conductor $|t|$ if $t \equiv 1 \pmod 4$, otherwise with conductor $|4t|$ if $t \equiv 2, 3 \pmod 4$.

Remark. Note that the function A_ϕ depends only on χ and ϕ. However, A_ϕ is not determined by (2), so we cannot use this theorem for computing a basis for the space $S_{k/2}(N, \chi, \phi)$. However, in Theorem 2 we have already given an algorithm to compute this space, and if $f(z) = \sum_{n=1}^{\infty} a_n q^n$ is one of the basis elements, then we can express the critical value of the L-function of the twist of the newform ϕ by the character $\chi_0^{-1}\chi_t$, in terms of the square of the Fourier coefficient a_t and the factor $\epsilon(\chi_0^{-1}\chi_t, 1/2)$ which depends on the local components of ϕ and χ_0.

It is to be noted that $\epsilon(\chi, 1/2)$ for any Hecke character χ can be computed as shown in Tate's article [27]. In particular, when χ is quadratic, $\epsilon(\chi, 1/2)$=1. Since we will be dealing only with quadratic characters, we can ignore the ϵ-factor. Moreover, note that if χ is quadratic, then the conductor of χ_0 is at most divisible by 8, so we do not need to consider possibility (b) of the hypothesis (H2).

Further by Corollary 4.1, possibilities (a) and (c) of the hypothesis (H2) can be simply stated in terms of the level M_ϕ. Assuming χ to be quadratic, Waldspurger's theorem is applicable whenever one of the following holds: M_ϕ is odd; $\nu_2(M_\phi) = 1$ and $\lambda_2 \neq 0$; or $\nu_2(M_\phi) \geqslant 4$. The last condition is the same as possibility (c) of (H2).

We also state the following corollary of Waldspurger [30, p. 483].

Corollary 5.2 (Waldspurger [30]). *Let $\phi \in S_{k-1}^{\mathrm{new}}(M_\phi, \chi^2)$ be a newform such that ϕ satisfies* (H1). *Suppose*[†] *that $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi, \phi)$ for some $N \geqslant 1$ such that M_ϕ divides $N/2$. Suppose that $n_1, n_2 \in \mathbb{N}^{\mathrm{sc}}$ such that $n_1/n_2 \in {\mathbb{Q}_p^\times}^2$ for all $p \mid N$. Then we have the following relation:*

$$a_{n_1}^2 \mathrm{L}(\phi\chi_0^{-1}\chi_{n_2}, 1)\chi(n_2/n_1) n_2^{k/2-1} = a_{n_2}^2 \mathrm{L}(\phi\chi_0^{-1}\chi_{n_1}, 1) n_1^{k/2-1}.$$

In what follows $(\cdot,\cdot)_p$ denotes the Hilbert symbol defined on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Recall that (see, for example, [8]) if $p = 2$ and a, b are odd, then

$$(2^s a, 2^t b)_2 = \left(\frac{2}{|a|}\right)^t \left(\frac{2}{|b|}\right)^s (-1)^{(a-1)(b-1)/4}.$$

For an odd prime p and a, b coprime to p,

$$(p^s a, p^t b)_p = \left(\frac{-1}{p}\right)^{st} \left(\frac{a}{p}\right)^t \left(\frac{b}{p}\right)^s.$$

In particular, for an odd n, $(n, -1)_2 = (-1)^{(n-1)/2}$ and $(2, n)_2 = (-1)^{(n^2-1)/8}$. Also, if $\nu_p(n) = 0$, then $(p, n)_p = (\frac{n}{p})$, and if $\nu_p(n) = 1$ and $n = pn'$, then $(p, n)_p = (\frac{-n'}{p})$.

[†] In this corollary we do not require f to be of the form $f(\underline{c}, A_\phi)$.

We now write down explicitly the definitions of the integers $\widetilde{n_p}$ and the local factors $U(e,\phi)$ used in Waldspurger's theorem. It should be noted that for Waldspurger's theorem, we require the values of the functions in $U_p(e,\phi)$ only at square-free positive integers. We will first define a certain set of functions.

Case 1: *p odd*. Waldspurger considered the following set of functions.

$$\Lambda_p := \{c^{(0)}_{p,\delta}, c^{(1)}_{p,\delta}, c^{(2)}_{p,\delta}, c^{(3)}_{p,\delta}, c^{(4)}_{p,\delta}, c^{(5)}_{p,\delta}, c^{(6)}_{p,\delta} : \delta \in \mathbb{C}\}.$$

We are interested only in values of the functions in Λ_p at square-free numbers in $\mathbb{Z}_p \setminus \{0\}$. Let $n \in \mathbb{Z}_p \setminus \{0\}$ be square-free, that is $\nu_p(n)=0$ or $\nu_p(n)=1$. We get the following after simplification:

$$c^{(0)}_{p,\delta}(n) = 1,$$

$$c^{(1)}_{p,\delta}(n) = \begin{cases} 1 & \text{if } \nu_p(n)=0;\\ \delta & \text{if } \nu_p(n)=1,\end{cases}$$

$$c^{(2)}_{p,\delta}(n) = \begin{cases} 1-(p,n)_p\chi_{0,p}(p)p^{-1/2}\delta^{-1} & \text{if } \nu_p(n)=0;\\ 1 & \text{if } \nu_p(n)=1,\end{cases}$$

$$c^{(3)}_{p,\delta}(n) = \begin{cases} 1 & \text{if } \nu_p(n)=0;\\ \delta-(p,n)_p\chi_{0,p}(p)p^{-1/2} & \text{if } \nu_p(n)=1,\end{cases}$$

$$c^{(4)}_{p,\delta}(n) = \begin{cases} 0 & \text{if } \nu_p(n)=0;\\ \delta(p-1)^{-1} & \text{if } \nu_p(n)=1,\end{cases}$$

$$c^{(5)}_{p,\delta}(n) = \begin{cases} 2^{1/2} & \text{if } \nu_p(n)=0 \text{ and } (p,n)_p = -p^{1/2}\chi_{0,p}(p^{-1})\delta;\\ 0 & \text{if } \nu_p(n)=0 \text{ and } (p,n)_p = p^{1/2}\chi_{0,p}(p^{-1})\delta;\\ 1 & \text{if } \nu_p(n)=1,\end{cases}$$

$$c^{(6)}_{p,\delta}(n) = \begin{cases} 1 & \text{if } \nu_p(n)=0;\\ 2^{1/2}\delta & \text{if } \nu_p(n)=1 \text{ and } (p,n)_p = -p^{1/2}\chi_{0,p}(p^{-1})\delta;\\ 0 & \text{if } \nu_p(n)=1 \text{ and } (p,n)_p = p^{1/2}\chi_{0,p}(p^{-1})\delta.\end{cases}$$

Case 2: $p=2$. In this case Waldspurger considered the following set of functions:

$$\Lambda_2 := \{c^{(0)}_{2,\delta}, c^{(1)}_{2,\delta}, c^{(2)}_{2,\delta}, c^{(3)}_{2,\delta}, c^{(4)}_{2,\delta}, c^{(5)}_{2,\delta}, c^{(6)}_{2,\delta} : \delta \in \mathbb{C}\}.$$

Let $n \in \mathbb{Z}_2 \setminus \{0\}$ be square-free so that either $\nu_2(n)=0$ or $\nu_2(n)=1$. We have

$$c^{(0)}_{2,\delta}(n) = \begin{cases} 1 & \text{if } \nu_2(n)=0;\\ \delta & \text{if } \nu_2(n)=1,\end{cases}$$

$$c^{(1)}_{2,\delta}(n) = \begin{cases} \delta-(2,n)_2\chi_{0,2}(2)2^{-1/2} & \text{if } \nu_2(n)=0 \text{ and } (n,-1)_2=\chi_{0,2}(-1);\\ 1 & \text{if } \nu_2(n)=0 \text{ and } (n,-1)_2=-\chi_{0,2}(-1);\\ 1 & \text{if } \nu_2(n)=1,\end{cases}$$

$$c^{(2)}_{2,\delta}(n) = \begin{cases} \delta & \text{if } \nu_2(n)=0 \text{ and } (n,-1)_2=\chi_{0,2}(-1);\\ 0 & \text{if } \nu_2(n)=0 \text{ and } (n,-1)_2=-\chi_{0,2}(-1);\\ 0 & \text{if } \nu_2(n)=1,\end{cases}$$

$$c^{(3)}_{2,\delta}(n) = \begin{cases} \delta^{-1} & \text{if } \nu_2(n)=0;\\ \delta-(2,n)_2\chi_{0,2}(2)2^{-1/2} & \text{if } \nu_2(n)=1 \text{ and } (n,-1)_2=\chi_{0,2}(-1);\\ 1 & \text{if } \nu_2(n)=1 \text{ and } (n,-1)_2=-\chi_{0,2}(-1),\end{cases}$$

$$c_{2,\delta}^{(4)}(n) = \begin{cases} 0 & \text{if } \nu_2(n) = 0; \\ 2\delta - (2,n)_2\chi_{0,2}(2)2^{-1/2} & \text{if } \nu_2(n) = 1 \text{ and } (n,-1)_2 = \chi_{0,2}(-1); \\ 1 & \text{if } \nu_2(n) = 1 \text{ and } (n,-1)_2 = -\chi_{0,2}(-1), \end{cases}$$

$$c_{2,\delta}^{(5)}(n) = \begin{cases} 0 & \text{if } \nu_2(n) = 0,\ (n,-1)_2 = \chi_{0,2}(-1) \text{ and } (2,n)_2 = 2^{1/2}\chi_{0,2}(2^{-1})\delta; \\ 2^{1/2}\delta & \text{if } \nu_2(n) = 0,\ (n,-1)_2 = \chi_{0,2}(-1) \text{ and } (2,n)_2 = -2^{1/2}\chi_{0,2}(2^{-1})\delta; \\ 1 & \text{if } \nu_2(n) = 0 \text{ and } (n,-1)_2 = -\chi_{0,2}(-1); \\ 1 & \text{if } \nu_2(n) = 1, \end{cases}$$

$$c_{2,\delta}^{(6)}(n) = \begin{cases} \delta^{-1} & \text{if } \nu_2(n) = 0; \\ 0 & \text{if } \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1) \text{ and } (2,n)_2 = 2^{1/2}\chi_{0,2}(2^{-1})\delta; \\ 2^{1/2}\delta & \text{if } \nu_2(n) = 1,\ (n,-1)_2 = \chi_{0,2}(-1) \text{ and } (2,n)_2 = -2^{1/2}\chi_{0,2}(2^{-1})\delta; \\ 1 & \text{if } \nu_2(n) = 1 \text{ and } (n,-1)_2 = -\chi_{0,2}(-1). \end{cases}$$

We will be interested in the above functions only for particular values of δ. We will specify and further simplify them later.

Recall that λ_p is the Hecke eigenvalue of ϕ corresponding to the Hecke operator T_p for any prime p, and $m_p = \nu_p(M_\phi)$. Let $\lambda'_p = p^{1-k/2}\lambda_p$. For $p \nmid M_\phi$ let α_p and α'_p be such that

$$\alpha_p + \alpha'_p = \lambda'_p,$$
$$\alpha_p \cdot \alpha'_p = 1.$$

It should be noted that if ϕ is rational newform of weight 2, then $\alpha_p \neq \alpha'_p$, since otherwise $\lambda_p^2 = 4$, which is a contradiction as λ_p is rational (pth Fourier coefficient of ϕ).

Next, we need to consider a subset of $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$, denoted by $\Omega_p(\phi)$, which is defined as

$$\Omega_p(\phi) = \{\omega \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} :\ \exists f \in S_{k/2}(N,\chi,\phi) \text{ for some } N \text{ and } \exists n \geqslant 1 \text{ such that:}$$
$$\text{(i) the image of } n \text{ in } \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \text{ is } \omega; \text{(ii) the } n\text{th coefficient of } f \neq 0\}. \tag{3}$$

Note that the set $\Omega_p(\phi)$ depends on the newform ϕ and character χ that we started with. Computation of this set is important in our applications and we will see that we need this set only in the case when $m_p \geqslant 1$ and $\lambda_p = 0$. We use the results of § 7 and our algorithm in Theorem 2 to compute most of the elements of this set.

Waldspurger defined another set of local functions on $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ taking values in $\mathbb{Z}/2\mathbb{Z}$:

$$\Gamma_p := \{\gamma_{e,\upsilon} : e \in \mathbb{Z}, \upsilon \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \text{such that } \nu_p(\upsilon) \equiv e \pmod 2\},$$

where

$$\gamma_{e,\upsilon}(u) = \begin{cases} 1 & \text{if } u \in \upsilon\mathbb{Q}_p^{\times 2} \text{ and } \nu_p(u) = e; \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 2$, define

$$\gamma'_{e,\upsilon} = \tfrac{1}{2}(\gamma_{e,\upsilon} + \gamma_{e,5\upsilon}),$$
$$\gamma''_e(u) = \begin{cases} 1 & \text{if } \nu_2(u) = e; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_e^0(u) = \begin{cases} 1 & \text{if } \nu_2(u) = e \text{ and } (u,-1)_2 = -\chi_{0,2}(-1) \text{ or } \nu_2(u) = e+1; \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to define the local factors $\widetilde{n_p}$ and the set $U_p(e,\phi)$ for $e = \widetilde{n_p}$. We will be dealing with several cases and subcases and in each of them we will be simplifying Waldspurger's formulae and making them more explicit for our use.

Case 1: *p odd and $m_p \geqslant 1$.* We consider the following subcases.

(a) $\lambda_p = 0$.

In this case we need to compute $\Omega_p(\phi)$. We know that $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} = \{1, p, u, \ pu\}$ where u is unit in $\mathbb{Z}_p$ which is a non-square mod p. If there exists a $\omega \in \Omega_p(\phi)$ such that $\nu_p(\omega) = 0$, then $\widetilde{n_p} = m_p$, and $U_p(\widetilde{n_p}, \phi) = \{\gamma_{0,\omega} : \omega \in \Omega_p(\phi)$ and $\nu_p(\omega) = 0\}$. In this case, the set $U_p(\widetilde{n_p}, \phi)$ consists of at most the functions $\gamma_{0,1}$ and $\gamma_{0,u}$. Otherwise, for all $\omega \in \Omega_p(\phi)$, $\nu_p(\omega) = 1$. In this case $\widetilde{n_p} = m_p + 1$, and $U_p(\widetilde{n_p}, \phi) = \{\gamma_{1,\omega} : \omega \in \Omega_p(\phi)$ and $\nu_p(\omega) = 1\}$, hence $U_p(\widetilde{n_p}, \phi)$ consists of at most $\gamma_{1,p}$ and $\gamma_{1,pu}$. Note that $\gamma_{0,1}, \gamma_{0,u}, \gamma_{1,p}, \gamma_{1,pu}$ are characteristic functions of $1, u, p, pu$ modulo $\mathbb{Q}_p^{\times 2}$, respectively.

(b) $\lambda_p \neq 0$.

In this case we must have $m_p = 1$, since $m_p \geqslant 2$ implies that $\lambda_p = 0$. Note that $p \in S$ since by Corollary 4.2 ρ_p is a special representation and, hence, not irreducible principal series. We have further subcases.

(i) $\chi_{0,p}$ is unramified.

Here again $\widetilde{n_p} = m_p = 1$ and $U_p(1, \phi) = \{c^{(5)}_{p,\lambda'_p}\}$. We use the theory of newforms to simplify the function $c^{(5)}_{p,\lambda'_p}$. Since $m_p = 1$ we get that $\lambda_p = -\omega_p p^{(k-3)/2}$ and $\lambda'_p = -\omega_p p^{-1/2}$. Here $\omega_p \in \{\pm 1\}$ is the eigenvalue under the Atkin–Lehner involution corresponding to the prime p. Hence, we have in this case,

$$c^{(5)}_{p,\lambda'_p}(n) = \begin{cases} 2^{1/2} & \text{if } \nu_p(n) = 0 \text{ and } \left(\dfrac{n}{p}\right) = \omega_p \chi_{0,p}(p^{-1}); \\ 0 & \text{if } \nu_p(n) = 0 \text{ and } \left(\dfrac{n}{p}\right) = -\omega_p \chi_{0,p}(p^{-1}); \\ 1 & \text{if } \nu_p(n) = 1. \end{cases}$$

(ii) $\chi_{0,p}$ is ramified.

We have $\widetilde{n_p} = m_p = 1$ and $U_p(1, \phi) = \{c^{(6)}_{p,\lambda'_p}\}$. As in the above subcase, we get the following simplification:

$$c^{(6)}_{p,\lambda'_p}(n) = \begin{cases} 1 & \text{if } \nu_p(n) = 0; \\ -\omega_p 2^{1/2} p^{-1/2} & \text{if } \nu_p(n) = 1 \text{ and } (p, n)_p = \omega_p \chi_{0,p}(p^{-1}); \\ 0 & \text{if } \nu_p(n) = 1 \text{ and } (p, n)_p = -\omega_p \chi_{0,p}(p^{-1}). \end{cases}$$

Case 2: *p odd and $m_p = 0$.* We have the following subcases.

(a) $\chi_{0,p}$ is unramified.

Here, $\widetilde{n_p} = m_p = 0$ and $U_p(0, \phi) = \{c^{(0)}_{p,\lambda'_p}\}$. It should be noted that $c^{(0)}_{p,\lambda'_p}$ takes the value 1 at any square-free n.

(b) $\chi_{0,p}$ is ramified.

We have $\widetilde{n_p} = 1$ and $U_p(1, \phi) = \{c^{(3)}_{p,\alpha_p}, c^{(3)}_{p,\alpha'_p}\}$ if $\alpha_p \neq \alpha'_p$, otherwise $U_p(1, \phi) = \{c^{(3)}_{p,\alpha_p}, c^{(4)}_{p,\alpha_p}\}$. We note that if p does not divide the modulus of χ, then we do not need to consider this subcase because in this case $\chi_{0,p}$ is unramified by Lemma 3.2.

Case 3: *$p = 2$ and $m_2 \geqslant 1$.* Consider the following subcases.

(a) $\lambda_2 = 0$.

We compute $\Omega_2(\phi)$. Note that $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} = \{\pm 1, \pm 2, \pm 5, \pm 10\}$. If there exists a $\omega \in \Omega_2(\phi)$ such that $\nu_2(\omega) = 0$, then $\widetilde{n_2} = m_2 + 2$, and $U_2(\widetilde{n_2}, \phi) = \{\gamma_{0,\omega} : \omega \in \Omega_2(\phi)$ and $\nu_2(\omega) = 0\}$. In this case, the set $U_2(\widetilde{n_2}, \phi)$ consists of at most $\gamma_{0,1}$, $\gamma_{0,3}$, $\gamma_{0,5}$ and $\gamma_{0,7}$. Otherwise, for all $\omega \in \Omega_2(\phi)$, $\nu_2(\omega) = 1$ and then $\widetilde{n_2} = m_2 + 3$, and $U_2(\widetilde{n_2}, \phi) = \{\gamma_{1,\omega} : \omega \in \Omega_2(\phi)$ and $\nu_2(\omega) = 1\}$, hence $U_2(\widetilde{n_2}, \phi)$ consists of at most $\gamma_{1,2}$, $\gamma_{1,6}$, $\gamma_{1,10}$ and $\gamma_{1,14}$. As above, $\gamma_{0,i}$ for $i \in \{1, 3, 5, 7\}$ are the characteristic functions of an odd residue class modulo 8 and $\gamma_{1,j}$ for $j \in \{2, 6, 10, 14\}$ are the characteristic functions of even residue class modulo $\mathbb{Q}_2^{\times 2}$.

(b) $\lambda_2 \neq 0$.
We must have $m_2 = 1$. As before Corollary 4.2 implies that ρ_2 is a special representation and hence $p \in S$. We have the following subcases.
(i) $\chi_{0,2}$ is trivial on $1 + 4\mathbb{Z}_2$.
Here $\widetilde{n_2} = 2$ and $U_2(2, \phi) = \{c^{(5)}_{2,\lambda'_2}\}$. Since $m_2 = 1$ we get that $\lambda_2 = -\omega_2 2^{(k-3)/2}$ and $\lambda'_2 = -\omega_2 2^{-1/2}$; $\omega_2 \in \{\pm 1\}$ is the eigenvalue under the Atkin–Lehner involution corresponding to 2. Hence, we have

$$c^{(5)}_{2,\lambda'_2}(n) = \begin{cases} 0 & \text{if } \nu_2(n) = 0,\, (-1)^{(n-1)/2} = \chi_{0,2}(-1) \text{ and} \\ & \quad (-1)^{(n^2-1)/8} = -\omega_2 \chi_{0,2}(2^{-1}); \\ -\omega_2 & \text{if } \nu_2(n) = 0,\, (-1)^{(n-1)/2} = \chi_{0,2}(-1) \text{ and} \\ & \quad (-1)^{(n^2-1)/8} = \omega_2 \chi_{0,2}(2^{-1}); \\ 1 & \text{if } \nu_2(n) = 0,\, (-1)^{(n-1)/2} = -\chi_{0,2}(-1); \\ 1 & \text{if } \nu_2(n) = 1. \end{cases}$$

(ii) $\chi_{0,2}$ is non-trivial on $1 + 4\mathbb{Z}_2$.
Here $\widetilde{n_2} = 3$ and $U_2(3, \phi) = \{c^{(6)}_{p,\lambda'_2}, \gamma''_0\}$ and we get the following simplification:

$$c^{(6)}_{2,\lambda'_2}(n) = \begin{cases} -\omega_2 2^{1/2} & \text{if } \nu_2(n) = 0; \\ 0 & \text{if } \nu_2(n) = 1,\, (n, -1)_2 = \chi_{0,2}(-1) \text{ and} \\ & \quad (2, n)_2 = -\omega_2 \chi_{0,2}(2^{-1}); \\ -\omega_2 & \text{if } \nu_2(n) = 1,\, (n, -1)_2 = \chi_{0,2}(-1) \text{ and} \\ & \quad (2, n)_2 = \omega_2 \chi_{0,2}(2^{-1}); \\ 1 & \text{if } \nu_2(n) = 1,\, (n, -1)_2 = -\chi_{0,2}(-1). \end{cases}$$

Case 4: $p = 2$ *and* $m_2 = 0$. We have the following subcases.
(a) $\chi_{0,2}$ is trivial on $1 + 4\mathbb{Z}_2$.
We have $\widetilde{n_2} = 2$ and $U_2(2, \phi) = \{c^{(1)}_{2,\alpha_2}, c^{(1)}_{2,\alpha'_2}\}$ if $\alpha_2 \neq \alpha'_2$, otherwise $U_2(2, \phi) = \{c^{(1)}_{2,\alpha_2}, c^{(2)}_{2,\alpha_2}\}$.
(b) $\chi_{0,2}$ is non-trivial on $1 + 4\mathbb{Z}_2$.
Here $\widetilde{n_2} = 3$ and $U_2(3, \phi) = \{c^{(3)}_{2,\alpha_2}, c^{(3)}_{2,\alpha'_2}, \gamma''_0\}$ if $\alpha_2 \neq \alpha'_2$, otherwise $U_2(3, \phi) = \{c^{(3)}_{2,\alpha_2}, c^{(4)}_{2,\alpha_2}, \gamma''_0\}$.
We would like to point out the following useful lemma.

LEMMA 5.3. *Let χ be a quadratic character modulo N such that $\nu_2(N)$ is at most 2. Then, $\chi_{0,2}$ is trivial on $1 + 4\mathbb{Z}_2$.*

Proof. Since χ is a quadratic, χ_0 is also quadratic with modulus $\operatorname{lcm}(4, N) = 4N'$ where $2 \nmid N'$. Now the lemma follows from part (iii) of Lemma 3.2. □

REMARK. These simplifications along with our method to compute a basis for $S_{k/2}(N, \chi, \phi)$ for suitable N and χ lead to an algorithm for computing critical values of the L-functions of certain quadratic twists of ϕ. For example, if $M_\phi = p^\alpha$ for some odd prime p, then the possible values for $\widetilde{N_\phi}$ are either $4p^\alpha$ or $4p^{\alpha+1}$, hence we compute bases for spaces $S_{k/2}(4p^\alpha, \chi_{\text{triv}}, \phi)$ and $S_{k/2}(4p^{\alpha+1}, \chi_{\text{triv}}, \phi)$ and the sets $U_2(2, \phi)$, $U_p(\alpha, \phi)$, $U_p(\alpha + 1, \phi)$ to apply Theorem 7 in order to obtain the desired results.

Note that in the above we have discussed computation of $U_p(e, \phi)$ only for $e = \widetilde{n_p}$. But in certain cases working with the level $\widetilde{N_\phi}$ is not sufficient to obtain the complete information and one might need to go to higher levels.

6. *Periods*

LEMMA 6.1. *Let $E/\mathbb{Q}$ be an elliptic curve, given by a minimal Weierstrass model, and let E_n be the minimal model of its twist by a square-free positive integer n. Then there is a computable non-zero rational number α_n such that*

$$\Omega(E_n) = \frac{\alpha_n \Omega(E)}{\sqrt{n}}.$$

The proof we give also explains how to compute α_n.

Proof. Let $\omega = dx/(2y + a_1x + a_3)$ be the invariant differential for the model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

By definition, the period

$$\Omega(E) = \int_{E(\mathbb{R})} |\omega|.$$

Recall [**23**, p. 49] that a change of variable

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$

leads to a model E' with invariant differential $\omega' = u\omega$; thus, the periods are related by $\Omega(E') = |u|\Omega(E)$. Completing the square in y we obtain the model

$$E' : {y'}^2 = {x'}^3 + A{x'}^2 + Bx' + C$$

where

$$A = \frac{b_2}{4}, \quad B = \frac{b_4}{2}, \quad C = \frac{b_6}{4}.$$

Since $u = 1$ in this change of variable, $\omega' = \omega$ and $\Omega(E') = \Omega(E)$. Now let the model E'' be the twist of E' by n:

$$E'' : {y''}^2 = {x''}^3 + An{x''}^2 + Bn^2x'' + Cn^3.$$

Note that these are related by the change of variable

$$y'' = n^{3/2}y', \quad x'' = nx'.$$

Thus, the invariant differentials satisfy

$$\omega'' = \frac{dx''}{2y''} = \frac{\omega'}{\sqrt{n}}.$$

Thus,

$$\Omega(E'') = \frac{\Omega(E')}{\sqrt{n}} = \frac{\Omega(E)}{\sqrt{n}}.$$

Now the model E'' is not necessarily minimal (nor even integral at 2), but by Tate's algorithm there is a change of variables

$$x'' = u^2X + r, \quad y'' = u^3Y + u^2sX + t$$

with rational u, s, t (and $u \neq 0$) such that the resulting model E_n is minimal. By the above

$$\Omega(E_n) = u\Omega(E'') = \frac{|u|\Omega(E)}{\sqrt{n}}. \qquad \square$$

LEMMA 6.2. *Let $E : Y^2 = X^3 + AX^2 + BX + C$ be an elliptic curve with A, B, $C \in \mathbb{Z}$. Suppose that the discriminant of this model is sixth-power free. Let n be a square-free* *positive integer. Then a minimal model for the nth twist is $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$. Moreover, the periods are related by the formula*

$$\Omega(E_n) = \frac{\Omega(E_1)}{\sqrt{n}}.$$

Proof. Let Δ be the discriminant of the model $E : Y^2 = X^3 + AX^2 + BX + C$. We are assuming that Δ is sixth-power free. Thus, it is 12th-power free, and so E is minimal. Now the model $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$ has discriminant $\Delta_n = \Delta \cdot n^6$. Since n is square-free this is 12th-power free. Thus, the model for E_n is minimal. The argument in the proof of Lemma 6.1 completes the proof. □

7. *Modular forms are determined by coefficients modulo n*

As usual N is a positive integer divisible by 4 and χ is a Dirichlet character modulo N. Let k be an odd integer. Let ϕ be a newform of weight $k-1$, level dividing $N/2$ and character χ^2. To apply Waldspurger's theorem, we need to know (see (3)) for certain primes p, certain $\omega \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ and certain forms $f = \sum a_n q^n \in S_{k/2}(N, \chi, \phi)$, whether there is some n such that the image of n in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is ω and $a_n \neq 0$. Given such p, f and ω we can write down the first few coefficients of f and test whether the image of n in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is ω and $a_n \neq 0$. If there is such an n, then we should be able to find it by writing down enough coefficients. However, sometimes it appears that $a_n = 0$ for all n that are equivalent in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ to ω. To be able to prove that, we have developed the results in this section.

THEOREM 8. *Let N be a positive integer such that $4 \mid N$ and χ be a Dirichlet character modulo N. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{k/2}(N, \chi)$. Let a, M be integers such that $(a, M) = 1$. Let $R = (k/24)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(NM^2)]$. Suppose that $a_n = 0$ whenever $n \not\equiv a \pmod{M}$ for all integers n up to $R+1$. Then $a_n = 0$ whenever $n \not\equiv a \pmod{M}$ for all n. Moreover, if $M^2 \mid N$, then the above statement holds with*

$$R = \begin{cases} \dfrac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] & \text{if } \dfrac{N}{M} \equiv 0 \pmod{4}; \\[2mm] \dfrac{k}{24}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(2N)] & \text{if } \dfrac{N}{M} \equiv 2 \pmod{4}. \end{cases}$$

We will be requiring the analogue, in the case of half-integral weight forms, of the following theorem of Sturm.

THEOREM 9 (Sturm [25, p. 276]). *Let Γ be a congruence subgroup and k be a positive integer. Let f, $g \in M_k(\Gamma)$ such that f and g have coefficients in $\mathcal{O}_F$, the ring of integers of a number field F. Let λ be a prime ideal of $\mathcal{O}_F$. If*

$$\mathrm{ord}_\lambda(f-g) > \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma],$$

then $\mathrm{ord}_\lambda(f-g) = \infty$, that is, $f \equiv g \pmod{\lambda}$.

In the above statement if $f(z) = \sum_{n \geqslant 0} a_n q^n$, then $\mathrm{ord}_\lambda(f) := \inf\{n : a_n \notin \lambda\}$. If $a_n \in \lambda$ for all n, then we let $\mathrm{ord}_\lambda(f) := \infty$.

LEMMA 7.1. *Let Γ' be a congruence subgroup such that $\Gamma' \subseteq \Gamma_0(4)$ and k' be a positive odd integer. Then the statement of Theorem 9 is valid for $\Gamma = \Gamma'$ and $k = k'/2$.*

Proof. Let $h := f - g \in S_{k'/2}(\Gamma')$. By assumption, $\mathrm{ord}_\lambda(h) > (k'/24)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma']$. Let $h' = h^4$. Then $h' \in M_{2k'}(\Gamma')$. This is because for any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma'$ and $z \in \mathbb{H}$,

$$\begin{aligned} h'(\gamma z) &= h^4(\gamma z) \\ &= j(\gamma, z)^{4k'} h^4(z) \\ &= (cz+d)^{2k'} h'(z). \end{aligned}$$

Also, $\mathrm{ord}_\lambda(h') = 4 \cdot \mathrm{ord}_\lambda(h) > (2k'/12)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma']$. So we apply Theorem 9 to h' to get that $\mathrm{ord}_\lambda(h') = \infty$. Hence, $\mathrm{ord}_\lambda(h) = \infty$. □

We note that the above lemma still holds if $f, g \in M_{k'/2}(\Gamma_0(N), \chi)$; the above proof works by taking $h' = h^{4n}$ where n is the order of Dirichlet character χ.

We will need the following lemmas for the proof of Theorem 8.

LEMMA 7.2. *Let M be a positive integer and $a \in \mathbb{Z}$ such that $(a, M) = 1$. Define*

$$\mathrm{I_a}(n) := \begin{cases} 1 & \text{if } n \equiv a \pmod{M}; \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathrm{I_a}(n) = \sum_{\psi \in \mathrm{X}(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n)$$

where $\mathrm{X}(M)$ denotes the group of Dirichlet characters of modulus M and φ is Euler's phi function.

Proof. See [**20**, p. 63, Chapter 6]. □

LEMMA 7.3. *Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N)$ and $m^2 \mid N$. Let $0 \leqslant \nu' < m$ and $c\nu'/m \equiv 0 \pmod 4$. Then, $\left(\frac{c}{d + c\nu'/m}\right) = \left(\frac{c}{d}\right)$.*

The proof of the above lemma requires the following reciprocity law as stated in Cassels and Fröhlich [**26**, p. 350].

PROPOSITION 7.4. *Let P, Q be positive odd integers and a be any non-zero integer with $a = 2^\alpha a_0$, a_0 odd. Then,*

$$\left(\frac{a}{P}\right) = \left(\frac{a}{Q}\right) \quad \text{if } P \equiv Q \pmod{8a_0}.$$

PROPOSITION 7.5. *Let k be a positive odd integer, χ be a Dirichlet character modulo N where $4 \mid N$ and $f(z) = \sum_{n=0}^\infty a_n q^n \in M_{k/2}(N, \chi)$. Suppose that ψ is a Dirichlet character of conductor m and $f_\psi(z) = \sum_{n=0}^\infty \psi(n) a_n q^n$. Then:*

(i) *$f_\psi \in M_{k/2}(Nm^2, \chi\psi^2)$;*
(ii) *if $m^2 \mid N$ and $N/m \equiv 0 \pmod 4$, then $f_\psi \in M_{k/2}(N, \chi\psi^2)$;*
(iii) *if $m^2 \mid N$ and $N/m \equiv 2 \pmod 4$, then $f_\psi \in M_{k/2}(2N, \chi\psi^2)$.*

Moreover, if f is a cusp form, then so is f_ψ.

Proof. The proof essentially follows that of Proposition 17 of [**14**, Chapter III], which is the integral weight case, with some necessary changes. We use Lemma 7.3 to obtain (ii) and (iii). □

LEMMA 7.6. *Let k, N be positive integers such that k is odd and $4 \mid N$. Suppose that $f(z) = \sum_{n=1}^\infty a_n q^n \in S_{k/2}(N, \chi)$. Let a, M be positive integers such that $(a, M) = 1$.*

Define

$$g(z) := \sum_{n=1}^{\infty} \mathrm{I_a}(n) a_n q^n.$$

Then $g \in S_{k/2}(\Gamma_1(NM^2))$.

Proof. We have

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} \mathrm{I_a}(n) a_n q^n \\ &= \sum_{n=1}^{\infty} \sum_{\psi \in \mathrm{X}(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n) a_n q^n \\ &= \sum_{\psi \in \mathrm{X}(M)} \alpha_\psi \sum_{n=1}^{\infty} \psi(n) a_n q^n \\ &= \sum_{\psi \in \mathrm{X}(M)} \alpha_\psi f_\psi, \end{aligned}$$

where $\alpha_\psi = \psi(a)^{-1}/\varphi(M)$. Using Proposition 7.5, for all $\psi \in \mathrm{X}(M)$ we have $f_\psi \in S_{k/2}(\Gamma_1(NM^2))$. Hence, $g \in S_{k/2}(\Gamma_1(NM^2))$. □

Now we are ready to prove Theorem 8.

Proof of Theorem 8. Let $h = f - g$ where g is as in the above lemma. Since $f \in S_{k/2}(\Gamma_1(NM^2))$, so does h. It is clear that

$$\text{coefficient of } q^n \text{ in } h = \begin{cases} a_n & \text{if } n \not\equiv a \pmod{M}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $h(z) = \sum_{n \not\equiv a \pmod{M}} a_n q^n \in S_{k/2}(\Gamma_1(NM^2))$. Since we have assumed $a_n = 0$ whenever $n \not\equiv a \pmod{M}$ for all integers n up to $R+1$, we apply Lemma 7.1 to get $h = 0$. If $M^2 \mid N$ we apply parts (ii) and (iii) of Proposition 7.5 to Lemma 7.6. □

REMARK. Note that in Lemma 7.6 if all of the Dirichlet characters modulo M are quadratic then by Proposition 7.5, in fact, $g \in S_{k/2}(\Gamma_0(NM^2, \chi))$. Hence, in this case Theorem 8 holds with $R = (k/24)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(NM^2)]$. For example, if $N = 1984$, $k = 3$ and $M = 8$, since all Dirichlet characters modulo 8 are quadratic we have $R = (3/24)[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(1984)] = 384$.

8. *Applications of Waldspurger's theorem*

In this section we will present a few examples explaining how to use Waldspurger's theorem. The idea of using Waldspurger's theorem for an elliptic curve is motivated by Tunnell's famous work on the congruent number problem. We will see, however, that our case needs many more computations to get any desired result. In the examples that follow we will first use our algorithm in Theorem 2 to compute the space of cusp forms that are Shimura equivalent to the given elliptic curve and then use Waldspurger's theorem to investigate some L-values. We will follow the notation adopted in the previous section.

8.1. *A first example*

Our first example will be the elliptic curve E over $\mathbb{Q}$ given by

$$E : Y^2 = X^3 + X + 1.$$

The conductor of E is $496 = 16 \times 31$ and E does not have complex multiplication. Let $\phi \in S_2^{\mathrm{new}}(496, \chi_{\mathrm{triv}})$ be the corresponding newform given by the Modularity theorem; ϕ has the following q-expansion:

$$\phi = q - 3q^5 + 3q^7 - 3q^9 - 2q^{11} - 4q^{13} - q^{19} + O(q^{20}).$$

It is to be noted that ϕ satisfies the hypothesis (H1): this follows by Theorem 6, and since $16 \mid M_\phi$, ϕ satisfies (H2). Let χ be a Dirichlet character with $\chi^2 = \chi_{\mathrm{triv}}$. By Theorem 5 there exists N such that $S_{3/2}(N, \chi, \phi) \neq \{0\}$. Note that we must have $496 \mid (N/2)$.

In order to apply Waldspurger's theorem we would like to compute an eigenbasis for the summand $S_{3/2}(N, \chi, \phi)$ for a suitable N and χ. We will assume χ to be the trivial character χ_{triv}. We use Theorem 2 to find out that $S_{3/2}(992, \chi, \phi) = \{0\}$. However, at level 1984 we get that the space $S_{3/2}(1984, \chi, \phi)$ has a basis $\{f_1, f_2, f_3\}$ where f_1, f_2 and f_3 have the following q-expansions:

$$f_1 = q^3 + q^{43} - 2q^{75} + 2q^{83} + q^{91} + 3q^{115} - 3q^{123} + O(q^{145}) := \sum_{n=1}^{\infty} a_n q^n,$$

$$f_2 = q^{15} + q^{23} - q^{31} + 2q^{55} + q^{79} - 3q^{119} + O(q^{145}) := \sum_{n=1}^{\infty} b_n q^n,$$

$$f_3 = q^{17} + q^{57} + q^{65} + 2q^{73} - q^{89} - q^{105} + q^{137} + O(q^{145}) := \sum_{n=1}^{\infty} c_n q^n.$$

We note that the space $S_{3/2}(1984, \chi)$ is 119-dimensional.

By Waldspurger's theorem (Theorem 7) there exists a function A_ϕ on square-free positive integers n such that

$$A_\phi(n)^2 = \mathrm{L}(E_{-n}, 1)$$

and

$$S_{3/2}(1984, \chi, \phi) = \bigoplus \overline{\mathrm{U}}(E, \phi, A_\phi),$$

where the sum is over all $E \geqslant 1$ such that $\widetilde{N_\phi} \mid E \mid 1984$. We already know the left-hand side of the above identity. Henceforth, we will be interested in computing the right-hand side. We will first compute $\widetilde{N_\phi}$ and then $\overline{\mathrm{U}}(E, \phi, A_\phi)$ for $\widetilde{N_\phi} \mid E \mid 1984$.

We need to compute local components $\widetilde{n_p}$ for each prime p. We consider the following cases.

Case 1: *p odd and $p \neq 31$*. In this case $m_p = 0$ and since $p \nmid N$ the local character $\chi_{0,p}$ is unramified. Hence we get that $\widetilde{n_p} = 0$.

Case 2: $p = 31$. Here $m_{31} = 1$. Since $\lambda_{31} \neq 0$, using Corollary 4.2 it follows that the local component ρ_{31} is a special representation of $\mathrm{GL}_2(\mathbb{Q}_{31})$ and so $31 \in S$. Also, note that $\mathbb{Z}_{31}^\times/{\mathbb{Z}_{31}^\times}^2$ is generated by 11 mod ${\mathbb{Z}_{31}^\times}^2$ and using Proposition 3.3 we can show that $\chi_{0,31}(11) = 1$. Thus $\chi_{0,31}$ is unramified and so, $\widetilde{n_{31}} = 1$.

Case 3: $p = 2$. In this case $m_2 = 4$ and it is clear from the q-expansion of ϕ that $\lambda_2 = 0$. We need some information about the set $\Omega_2(\phi)$ (see (3)). In our case, looking at f_1, f_2 and f_3, we get that $\{1, 3, 7\} \subseteq \Omega_2(\phi)$. Since $\nu_2(1) = \nu_2(3) = \nu_2(7) = 0$, we get $\widetilde{n_2} = m_2 + 2 = 6$.

Hence,

$$\widetilde{N_\phi} = 31 \times 2^6 = 1984.$$

Thus, we have $E = \widetilde{N_\phi} = 1984$ and we would like to know how the space $\overline{\mathrm{U}}(1984, \phi, A_\phi)$ looks. For that the next immediate task will be to compute $\mathrm{U}_p(e_p, \phi)$ where $e_p = \nu_p(1984)$. We consider the following cases.

Case 1: *p odd and $p \neq 31$*. Here, $e_p = 0$ and $\mathrm{U}_p(0, \phi)$ consists of only one function $c^{(0)}_{p,\lambda'_p}$ defined on $\mathbb{Q}_p^\times$. Recall that $c^{(0)}_{p,\lambda'_p}(n) = 1$ for n square-free.

Case 2: $p = 31$. In this case $e_{31} = 1$ and as already seen, $31 \in S$ and $\chi_{0,31}$ is unramified. So, $\mathrm{U}_{31}(1, \phi) = \{c^{(5)}_{31,\lambda'_{31}}\}$. Note that $\lambda_{31} = -1$ and, hence, $\lambda'_{31} = (31)^{-1/2}\lambda_{31} = -(31)^{-1/2}$. Again using Proposition 3.3 we can show that $\chi_{0,31}(31^{-1}) = -1$. Also note that $(31, n)_{31} = (\frac{n}{31})$. So for n square-free, we have

$$c^{(5)}_{31,\lambda'_p}(n) = \begin{cases} 2^{1/2} & \text{if } \nu_{31}(n) = 0 \text{ and } \left(\dfrac{n}{31}\right) = -1; \\ 0 & \text{if } \nu_{31}(n) = 0 \text{ and } \left(\dfrac{n}{31}\right) = 1; \\ 1 & \text{if } \nu_{31}(n) = 1. \end{cases}$$

Case 3: $p = 2$. Here $e_2 = 6$. Since $\lambda_2 = 0$ and $\{1, 3, 7\} \subseteq \Omega_2(\phi)$, we see that $\mathrm{U}_2(6, \phi)$ consists of $\gamma_{0,1}, \gamma_{0,3}, \gamma_{0,7}$ which are the characteristic functions of residue classes of $1, 3, 7$ modulo 8, respectively. By our methods so far we do not know whether 5 belongs to $\Omega_2(\phi)$ or not.

Recall that $\overline{\mathrm{U}}(E, \phi, A_\phi)$ is the space generated by the functions $f(\underline{c}, A_\phi)$ where $\underline{c} \in \prod_p \mathrm{U}_p(e_p, \phi)$. Thus, in our case $\underline{c} = (c_p)_p$ where, for odd primes $p \neq 31$ we have $c_p = c^{(0)}_{p,\lambda'_p}$, $c_{31} = c^{(5)}_{31,\lambda'_{31}}$ and for c_2 the possible choices are $\gamma_{0,1}$, $\gamma_{0,3}$, $\gamma_{0,5}$ and $\gamma_{0,7}$. By using Waldspurger's theorem (Theorem 7) we have

$$S_{3/2}(1984, \chi, \phi) = \overline{\mathrm{U}}(1984, \phi, A_\phi)$$

and so every cusp form in the space on the left-hand side can be written in terms of

$$f(\underline{c}, A_\phi)(z) := \sum_{n=1}^{\infty} A_\phi(n^{\mathrm{sc}}) n^{1/4} \prod_p c_p(n) q^n$$

for some $\underline{c} = (c_p) \in \prod U_p(e_p, \phi)$.

We use Theorem 8 to conclude that f_1 has non-zero nth coefficients only for $n \equiv 3 \pmod 8$, f_2 has non-zero coefficients only for $n \equiv 7 \pmod 8$ and f_3 has non-zero coefficients only for $n \equiv 1 \pmod 8$.

Since f_1 has non-zero a_n only for $n \equiv 3 \pmod 8$, taking $\underline{c}$ as above with $c_2 = \gamma_{0,3}$ we get that for n square-free,

$$\begin{aligned} a_n &= \beta_1 A_\phi(n) n^{1/4} c_2(n) c_{31}(n) \\ &= \begin{cases} 2^{1/2}\beta_1 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n) = 0, \left(\dfrac{n}{31}\right) = -1 \text{ and } n \equiv 3 \pmod 8; \\ \beta_1 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n) = 1 \text{ and } n \equiv 3 \pmod 8; \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4}$$

for some complex constant β_1. Similarly, taking $c_2 = \gamma_{0,7}$ for f_2 and $c_2 = \gamma_{0,1}$ for f_3 respectively we get that

$$\begin{aligned} b_n &= \beta_2 A_\phi(n) n^{1/4} c_2(n) c_{31}(n) \\ &= \begin{cases} 2^{1/2}\beta_2 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n) = 0, \left(\dfrac{n}{31}\right) = -1 \text{ and } n \equiv 7 \pmod 8; \\ \beta_2 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n) = 1 \text{ and } n \equiv 7 \pmod 8; \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{5}$$

for some complex constant β_2 and

$$c_n = \beta_3 A_\phi(n) n^{1/4} c_2(n) c_{31}(n)$$
$$= \begin{cases} 2^{1/2}\beta_3 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n)=0, \left(\dfrac{n}{31}\right) = -1 \text{ and } n \equiv 1 \pmod 8; \\ \beta_3 A_\phi(n) n^{1/4} & \text{if } \nu_{31}(n)=1 \text{ and } n \equiv 1 \pmod 8; \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for some complex constant β_3.

We have the following proposition which allows us to calculate the critical values of the L-functions of E_{-n}, the $(-n)$th quadratic twists of E.

Proposition 8.1. *Let E be as above and n be a positive square-free integer.*

(i) *If $\nu_{31}(n)=0$, $n \equiv 3 \pmod 8$ and $(\frac{n}{31}) = -1$, then*

$$\mathrm{L}(E_{-n},1) = \frac{a_n^2}{2{\beta_1}^2\sqrt{n}}.$$

(ii) *If $\nu_{31}(n)=1$, $n \equiv 3 \pmod 8$, then*

$$\mathrm{L}(E_{-n},1) = \frac{a_n^2}{{\beta_1}^2\sqrt{n}}.$$

(iii) *If $\nu_{31}(n)=0$, $n \equiv 7 \pmod 8$ and $(\frac{n}{31}) = -1$, then*

$$\mathrm{L}(E_{-n},1) = \frac{b_n^2}{2{\beta_2}^2\sqrt{n}}.$$

(iv) *If $\nu_{31}(n)=1$, $n \equiv 7 \pmod 8$, then*

$$\mathrm{L}(E_{-n},1) = \frac{b_n^2}{{\beta_2}^2\sqrt{n}}.$$

(v) *If $\nu_{31}(n)=0$, $n \equiv 1 \pmod 8$ and $(\frac{n}{31}) = -1$, then*

$$\mathrm{L}(E_{-n},1) = \frac{c_n^2}{2{\beta_3}^2\sqrt{n}}.$$

(vi) *If $\nu_{31}(n)=1$, $n \equiv 1 \pmod 8$, then*

$$\mathrm{L}(E_{-n},1) = \frac{c_n^2}{{\beta_3}^2\sqrt{n}}.$$

Proof. Using Waldspurger's theorem (Theorem 7) we know the existence of a function A_ϕ on square-free numbers such that $A_\phi(n)^2 = \mathrm{L}(E_{-n},1)$. The proof now follows using (4)–(6). □

We will show now how we use the above to calculate the order of the Tate–Shafarevich group $Ш(E_{-n}/\mathbb{Q})$. We will be assuming the Birch and Swinnerton-Dyer conjecture for rank zero elliptic curves:

$$\mathrm{L}(E_{-n},1) = \frac{|Ш(E_{-n}/\mathbb{Q})| \cdot \Omega_{E_{-n}} \cdot \prod_p c_p}{|E_{-n,\mathrm{tor}}|^2} \tag{7}$$

where $\Omega_{E_{-n}}$ stands for the real period of E_{-n} (since $E_{-n}(\mathbb{R})$ is connected), c_p for the pth Tamagawa number of E_{-n} and $E_{-n,\mathrm{tor}}$ stands for the torsion group of E_{-n}, all of which are easily computable.

We have the following lemma.

Lemma 8.2. *Let $E : Y^2 = X^3 + X + 1$. Then $E_{n,\mathrm{tor}} = 0$ for all square-free integers n.*

Proof. Let $K = \mathbb{Q}(\sqrt{n})$. It is well-known that the map

$$E_n(\mathbb{Q}) \to E(K)$$

given by

$$O \mapsto O, \quad (X, Y) \mapsto \left(\frac{X}{n}, \frac{Y}{n\sqrt{n}}\right)$$

is an injective group homomorphism†. Thus, it is sufficient to show that $E(K)$ has trivial torsion subgroup. Recall that the discriminant of E is $-496 = -16 \times 31$. Let $p \neq 2,\ 31$ be a rational prime and let $\mathfrak{P}$ be a prime ideal of K dividing p. Then E has good reduction at $\mathfrak{P}$. Moreover, if $e_{\mathfrak{P}} < p - 1$, then the reduction map $E(K)_{\mathrm{tor}} \to E(\mathbb{F}_{\mathfrak{P}})$ is injective [**13**, p. 501], where $e_{\mathfrak{P}}$ is the ramification index for $\mathfrak{P}$ and $\mathbb{F}_{\mathfrak{P}}$ denotes the residue field of $\mathfrak{P}$. Thus, if $p \geqslant 5$ and $p \neq 31$, then this map is injective. Now we take $p = 5,\ 7$, so $E(\mathbb{F}_{\mathfrak{P}})$ is a subgroup of $E(\mathbb{F}_{25})$ and $E(\mathbb{F}_{49})$ respectively. Using `MAGMA` we find

$$E(\mathbb{F}_{25}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}, \quad E(\mathbb{F}_{49}) \cong \mathbb{Z}/55\mathbb{Z}.$$

Since these two groups have coprime orders, it follows that $E(K)_{\mathrm{tor}} = 0$ and so $E_{n,\mathrm{tor}} = 0$. $\square$

Further, since the discriminant of E_{-1} is $-496 = 2^4 \times 31$, by Lemma 6.2 we know that $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$.

It is clear that the quantity $\mathrm{L}(E_{-n}, 1)/\Omega_{E_{-n}}$ is an integer, according to the Birch and Swinnerton-Dyer conjecture. Using `MAGMA` we compute this integer for $n \in \{3, 15, 17\}$. In particular for $n = 3$ we get that $\mathrm{L}(E_{-3}, 1)/\Omega_{E_{-3}} = 2$. Substituting this in part (i) of Proposition 8.1 and using Lemma 6.2, it follows that $\Omega_{E_{-1}} = 1/4{\beta_1}^2$. Doing similar calculations with $n = 15,\ 17$ we get

$$\Omega_{E_{-1}} = \frac{1}{4{\beta_1}^2} = \frac{1}{4{\beta_2}^2} = \frac{1}{8{\beta_3}^2}. \tag{8}$$

Now recall that $W(E_{-n}/\mathbb{Q})$ denotes the root number for elliptic curve E_{-n} over rational numbers. We have the following proposition. The methods used here to compute the root numbers are well known and we refer to [**9**].

PROPOSITION 8.3. *For E as above and n positive square-free the following hold.*

(i) *If $\nu_{31}(n) = 0$, then*

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & \text{if } n \equiv 1, 3, 7 \pmod 8,\ \left(\frac{n}{31}\right) = 1 \text{ or} \\ & n \equiv 5 \pmod 8,\ \left(\frac{n}{31}\right) = -1 \text{ or} \\ & n \text{ even},\ \left(\frac{n}{31}\right) = -1; \\ 1 & \text{if } n \equiv 1, 3, 7 \pmod 8,\ \left(\frac{n}{31}\right) = -1 \text{ or} \\ & n \equiv 5 \pmod 8,\ \left(\frac{n}{31}\right) = 1 \text{ or} \\ & n \text{ even},\ \left(\frac{n}{31}\right) = 1. \end{cases}$$

†As the map simply scales the variables, it takes lines to lines and so must define a homomorphism of Mordell–Weil groups.

(ii) *If* $\nu_{31}(n) = 1$, *then*

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & \text{if } n \equiv 5 \pmod 8 \text{ or} \\ & \quad n \text{ even;} \\ 1 & \text{if } n \equiv 1, 3, 7 \pmod 8. \end{cases}$$

Before computing the order of the Tate–Shafarevich group $Ш(E_{-n}/\mathbb{Q})$, we have the following refinement of Theorem 8.1.

THEOREM 10. *Let* $E : Y^2 = X^3 + X + 1$ *and* $f = f_1 + f_2 + \sqrt{2} f_3 = \sum d_n q^n$. *Then, for positive square-free* $n \equiv 1,\ 3,\ 7 \pmod 8$,

$$\mathrm{L}(E_{-n}, 1) = \frac{2^{(\nu_{31}(n)+1)}\Omega_{E_{-1}}}{\sqrt{n}} \cdot d_n^2.$$

Proof. Note that $d_n = a_n + b_n + \sqrt{2} c_n$. It is important for the proof to note that $a_n = 0$ for $n \not\equiv 3 \pmod 8$, $b_n = 0$ for $n \not\equiv 7 \pmod 8$ and $c_n = 0$ for $n \not\equiv 1 \pmod 8$; we proved this by applying Theorem 8. It follows from (4)–(6) that $d_n = 0$ whenever $n \equiv 1, 3, 7 \pmod 8$ and the Kronecker symbol $(\frac{n}{31}) = 1$. Further by Proposition 8.3 if $n \equiv 1, 3, 7 \pmod 8$ and $(\frac{n}{31}) = 1$, then $W(E_{-n}, \mathbb{Q}) = -1$ and so $\mathrm{L}(E_{-n}, 1) = 0$. Thus, the theorem follows when $(\frac{n}{31}) = 1$.

In the case when $(\frac{n}{31}) = -1$, the refinement follows by using (8) in Proposition 8.1. □

We have now the following corollary which computes the order of the Tate–Shafarevich group $Ш(E_{-n}/\mathbb{Q})$.

COROLLARY 8.4. *Let* $E : Y^2 = X^3 + X + 1$ *and* $f = f_1 + f_2 + \sqrt{2} f_3 = \sum d_n q^n$. *Let* n *be positive square-free number such that* $n \equiv 1,\ 3,\ 7 \pmod 8$ *and* E_{-n} *has rank zero. Then, assuming the Birch and Swinnerton-Dyer conjecture,*

$$|Ш(E_{-n}/\mathbb{Q})| = \frac{2^{(\nu_{31}(n)+1)}}{\prod_p c_p} \cdot d_n^2$$

where the Tamagawa numbers c_p *of* E_{-n} *are given by*

$$c_2 = \begin{cases} 1 & \text{if } n \equiv 3, 7 \pmod 8; \\ 2 & \text{if } n \equiv 1, 5 \pmod 8, \end{cases} \qquad c_{31} = \begin{cases} 1 & \text{if } 31 \nmid n; \\ 4 & \text{if } 31 \mid n \text{ and } \left(\dfrac{n/31}{31}\right) = 1; \\ 2 & \text{if } 31 \mid n \text{ and } \left(\dfrac{n/31}{31}\right) = -1, \end{cases}$$

$c_p = \# E_{-1}(\mathbb{F}_p)[2]$ *for* $p \mid n$, $p \neq 31$, *and* $c_p = 1$ *for all other primes* p.

Proof. From Lemma 8.2 we have $E_{-n,\mathrm{tor}} = 0$ for all square-free integers n. Substituting this and $\Omega(E_{-n}) = \Omega(E_{-1})/\sqrt{n}$ in (7) we get that

$$|Ш(E_{-n}/\mathbb{Q})| = \frac{\mathrm{L}(E_{-n}, 1) \cdot \sqrt{n}}{\Omega_{E_{-1}} \cdot \prod_p c_p} = \frac{2^{(\nu_{31}(n)+1)}}{\prod_p c_p} \cdot d_n^2;$$

the last equality follows by Theorem 10.

We use Tate's algorithm (see [**24**, pp. 364–368]) to compute the Tamagawa numbers c_p. □

We have the following easy corollary to Theorem 10.

COROLLARY 8.5. *Suppose* $n \equiv 1, 3, 7 \pmod 8$ *and* $(\frac{n}{31}) = -1$. *Then assuming the Birch and Swinnerton-Dyer conjecture,*

$$\mathrm{Rank}(E_{-n}) \geqslant 2 \Leftrightarrow d_n = 0.$$

Proof. By Proposition 8.3, if $n \equiv 1, 3, 7 \pmod 8$ and $(\frac{n}{31}) = -1$ then $W(E_{-n}/\mathbb{Q}) = 1$. Thus, the analytic rank is even, and so by the Birch and Swinnerton-Dyer conjecture, the rank is even. The corollary now follows using Theorem 10. $\square$

REMARK. By Proposition 8.3 if n is a square-free integer such that $n \equiv 5 \pmod 8$, then $\mathrm{L}(E_{-n}, 1) = 0$ whenever either $\nu_{31}(n) = 1$ or $\nu_{31}(n) = 0$ and $(\frac{n}{31}) = -1$. One can also obtain this by using Waldspurger's Theorem. In fact, since f_1, f_2, f_3 span $S_{3/2}(1984, \chi, \phi)$ and none of them have a non-zero coefficient for $n \equiv 5 \pmod 8$ we obtain

$$A_\phi(n) c_{31}(n) = 0 \quad \text{whenever } n \equiv 5 \pmod 8.$$

The statement now follows since $c_{31}(n) \neq 0$ if either $\nu_{31}(n) = 1$ or $\nu_{31}(n) = 0$ and $(\frac{n}{31}) = -1$. However, these methods fail to provide any information[†] about $\mathrm{L}(E_{-n}, 1)$ if $n \equiv 5 \pmod 8$ and $(\frac{n}{31}) = 1$. We hope to predict what happens in these cases by either going to higher levels, by suitably twisting E or by allowing non-trivial characters.

We note here that for newforms ϕ of weight $k-1$ and odd and square-free level Baruch and Mao [**2**, Theorem 10.1] obtain Waldspurger-type results for $\mathrm{L}(\phi \otimes \chi_D, (k-1)/2)$ for all fundamental discriminants D. In a subsequent paper Mao [**17**, Theorem 1.3] removes the square-free condition using the generalized Shimura correspondence.

8.2. *Second example*

Our second example will be the rational elliptic curve E of conductor 144 given by

$$E : Y^2 = X^3 - 1.$$

The corresponding newform ϕ is given by

$$\phi = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} - 10q^{37} - 8q^{43} + 9q^{49} + O(q^{50}).$$

Here $M_\phi = 144$. Using Theorem 2 for computing Shimura's decomposition, we find that at the level 576, the space $S_{3/2}(576, \chi_{\mathrm{triv}}, \phi) \neq \{0\}$; and this space has a basis $\{f_1, f_2, f_3, f_4\}$ where f_1, f_2, f_3 and f_4 have the following q-expansions:

$$f_1 = q - q^{25} + 5q^{49} - 6q^{73} - 6q^{97} + O(q^{100}) := \sum_{n=1}^{\infty} a_n q^n,$$

$$f_2 = q^5 + q^{29} - q^{53} - 2q^{77} + O(q^{100}) := \sum_{n=1}^{\infty} b_n q^n,$$

$$f_3 = q^{13} - 2q^{61} + q^{85} + O(q^{100}) := \sum_{n=1}^{\infty} c_n q^n,$$

$$f_4 = q^{17} - q^{41} - q^{89} + O(q^{100}) := \sum_{n=1}^{\infty} d_n q^n.$$

Doing similar calculations as in the previous example we have the following result.

THEOREM 11. *Let* $E : Y^2 = X^3 - 1$. *Let*

$$f = f_1/\sqrt{6} + f_2 + \sqrt{2} f_3 + \sqrt{3} f_4 := \sum_{n=1}^{\infty} e_n q^n.$$

[†]In fact, performing computations using MAGMA we get, for example, that $\mathrm{L}(E_{-n}, 1) \neq 0$ for $n = 5, 69, 101, 109, 133, 157, 165$; these n satisfy the conditions $n \equiv 5 \pmod 8$ and $(\frac{n}{31}) = 1$. However, for $n = 149, 173$, which also satisfy the same two conditions, we get that $\mathrm{L}(E_{-n}, 1) = 0$ (using the root number argument $\mathrm{Rank}(E_{-n}) \geqslant 2$ for $n = 149, 173$). We do not detect a general pattern.

Let $n \neq 1$[†] *be a positive square-free integer such that $n \equiv 1$ or $2 \pmod 3$. Then,*

$$\mathrm{L}(E_{-n},1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot e_n^2. \tag{9}$$

Further, assuming that the Birch and Swinnerton-Dyer conjecture holds, if E_{-n} has rank zero, then

$$|\text{Ш}(E_{-n}/\mathbb{Q})| = \frac{4}{\prod_p c_p} \cdot e_n^2$$

where the Tamagawa numbers $c_2 = 3$ if $n \equiv 1 \pmod 8$, $c_2 = 1$ if $n \equiv 3, 5, 7 \pmod 8$; $c_3 = 2$; $c_p = \#E_{-1}(\mathbb{F}_p)[2]$ for $p \mid n$, $p \neq 3$; and $c_p = 1$ for all other primes p.

Remark. To consider the case when $3 \mid N$, we try to instead work with elliptic curve E_3. The curve E_3 has conductor 36 and is isogenous to E_{-1}. Hence, $\mathrm{L}(E_{3n},1) = \mathrm{L}(E_{-n},1)$ and $\mathrm{L}(E_n,1) = \mathrm{L}(E_{-3n},1)$ for all positive square-free n coprime to 3. Thus, computation of $\mathrm{L}(E_{-3n},1)$ for all such n will lead to a formula for $\mathrm{L}(E_n,1)$ for all n square-free. Since the hypothesis (H2) is not satisfied we cannot apply Theorem 7 to E_3. Let $F := E_3$ and ϕ' be the corresponding newform. Using Theorem 2 we find that $S_{3/2}(72, \chi_{\mathrm{triv}}, \phi')$ is two-dimensional spanned by g_1 and g_2 where

$$g_1 = q - 2q^{10} - 2q^{13} + 4q^{22} - q^{25} + 2q^{34} + 4q^{37} + O(q^{40}),$$
$$g_2 = q^2 - q^5 - 2q^{14} + q^{17} + 3q^{29} + O(q^{40}).$$

Let $g = g_1 + g_2 = \sum_{n=1}^{\infty} a_n q^n$. We try to instead apply Corollary 5.2. Let $I = \{1, 5, 13, 17\}$, then for each i in I we obtain

$$\mathrm{L}(F_{-n},1) = \frac{a_n^2 \cdot \mathrm{L}(F_{-i},1)}{a_i^2} \sqrt{\frac{i}{n}} \quad \text{for } n \equiv i \pmod{24}.$$

Also by root number calculations $\mathrm{L}(F_{-n},1) = 0$ for $n \equiv 7, 11, 19, 23 \pmod{24}$. So the cases we are left with are $n \equiv j \pmod{24}$ for $j \in J = \{2, 10, 14, 22\}$. We make the following observation. Using `MAGMA` for positive square-free $n \leqslant 1000$ we check that up to 30 decimal places

$$\mathrm{L}(F_{-n},1) = \frac{a_n^2 \cdot \mathrm{L}(F_{-j},1)}{a_j^2} \sqrt{\frac{j}{n}} \quad \text{for } n \equiv j \pmod{24}.$$

This observation does not follow from Corollary 5.2, for example $\mathrm{L}(F_{-74},1) = 4 \cdot (\mathrm{L}(F_{-2},1)/\sqrt{37})$ but $74/2 \notin \mathbb{Q}_2^{\times 2}$.

8.3. *Example with a non-rational newform*

In this example we start with a non-rational newform ψ and we show that we can get similar formulae as before for the critical values of L-functions of $\psi \otimes \chi_{-n}$.

Let $\psi \in S_2^{\mathrm{new}}(62, \chi_{\mathrm{triv}})$ be a newform of weight 2, level 62 and trivial character given by the following q-expansion,

$$\psi = q - q^2 + aq^3 + q^4 + (-2a+2)q^5 - aq^6 + 2q^7 - q^8 + (2a-1)q^9 + O(q^{10})$$

where a has minimal polynomial $x^2 - 2x - 2$.

As before using Theorem 2 we get that the space $S_{3/2}(124, \chi_{\mathrm{triv}}, \psi) = \langle f \rangle$ where f has the following q-expansion,

$$f = q + (a+1)q^2 - q^4 - 2aq^5 - aq^7 + (-a-1)q^8 + (a+1)q^9 - 2q^{10} + O(q^{12}).$$

[†]In the case $n = 1$ we still have $\mathrm{L}(E_{-n},1) = (\Omega_{E_{-1}}/\sqrt{n}) \cdot e_n^2$, but since $|E_{-1,\mathrm{tor}}| = 6$ we get that $|\text{Ш}(E_{-n}/\mathbb{Q})| = (36/\prod_p c_p) \cdot e_n^2$.

Note that Waldspurger's theorem is applicable for the newform ψ since ρ_2, the local automorphic representation of ψ at 2, is not supercuspidal; this follows since $\nu_2(62) = 1$ (see Corollary 4.1).

We have the following proposition.

PROPOSITION 8.6. *Let ψ and $f := \sum_{n=1}^{\infty} a_n q^n$ be as above. Let n be square-free such that $n \not\equiv 3 \pmod 8$ and $(\frac{n}{31}) \neq -1$. Then*

$$\mathrm{L}(\psi \otimes \chi_{-n}, 1) = \begin{cases} \dfrac{\beta}{\sqrt{n}} \cdot a_n^2 & \text{if } \nu_{31}(n) = 1; \\ \dfrac{\beta}{2\sqrt{n}} \cdot a_n^2 & \text{if } \nu_{31}(n) = 0, \end{cases}$$

where $\beta = 2 \cdot \mathrm{L}(\psi \otimes \chi_{-1}, 1)$.

Proof. The proof follows by calculations similar to those in the previous examples. □

8.4. *Ternary quadratic forms and Tunnell-like formulae*

For a positive-definite integral quadratic form $Q(x_1, \ldots, x_m)$ we define its theta series by

$$\theta_Q(z) = \sum_{n=0}^{\infty} \#\{\mathbf{a} \in \mathbb{Z}^m : Q(\mathbf{a}) = n\} \cdot q^n; \quad q = \exp(2\pi i z).$$

Let $A_Q = (\partial^2 Q / \partial x_i \partial x_j)$ be the matrix of the quadratic form Q. The level of Q is defined to be the smallest positive integer N_Q such that $N_Q {A_Q}^{-1}$ is a matrix with integer entries that has even integers on the main diagonal. Let $d_Q = \det(A_Q)$ if $m \equiv 0 \pmod 4$, $d_Q = -\det(A_Q)$ if $m \equiv 2 \pmod 4$ and $d_Q = \det(A_Q)/2$ if m is odd. Then the character of Q is defined to be $\chi_{d_Q} = (\frac{d_Q}{\cdot})$.

Shimura [21] showed that $\theta_Q \in M_{m/2}(N_Q, \chi_{d_Q})$. Further Siegel [19] showed that if Q_1 and Q_2 are positive-definite integral ternary quadratic forms both having level N, character χ_d and belonging to the same genus, then $\theta_{Q_1} - \theta_{Q_2} \in S_{3/2}(N, \chi_d)$. Denote by $S_q(N, \chi_d)$ the subspace of $S_{3/2}(N, \chi_d)$ generated by all such differences of theta series.

It is interesting, when applying Waldspurger's theorem to a weight 2 cuspform ϕ, to ask whether the relevant modular form of weight 3/2 belongs to $S_q(N, \chi_d)$; in this case we would obtain a Tunnell-like formula expressing the critical values of the L-functions of twists of ϕ in terms of ternary quadratic forms. We will illustrate this below by presenting several examples. We point out, however, that this is not always possible. In particular, for the elliptic curve in our first example, $E : Y^2 = X^3 + X + 1$, the space $S_{3/2}(1984, \chi_{\mathrm{triv}}, \phi_E)$ has trivial intersection with the subspace $S_q(1984, \chi_{\mathrm{triv}})$. Note that $\mathrm{L}(E, 1) = 0$. As we mentioned in the introduction, for elliptic curves of odd and square-free conductor, Böcherer and Schulze-Pillot [4] showed that an inverse Shimura lift comes from ternary quadratic forms if and only if the curve has analytic rank zero. In the examples below we consider levels that are neither odd and square-free but the result of Böcherer and Schulze-Pillot still seems to hold.

We do not give details of how to compute $S_q(N, \chi_d)$ or the intersection $S_q(N, \chi_d) \cap S_{3/2}(N, \chi_d, \phi)$. We merely point out that it is straightforward to compute a basis for the space $S_q(N, \chi_d)$ with the help of an algorithm of Dickson [10, 16] for computing quadratic forms of a given level and character up to equivalence. Computing the intersection with $S_{3/2}(N, \chi_d, \phi)$ is easy using a suitable adaptation of our Theorem 2, and a result of Bungert [7, Proposition 4] for computing the Hecke action on theta series.

We note here that expressing the forms in $S_{3/2}(N, \chi, \phi)$ in terms of ternary quadratic forms has a big advantage in running time for the computation of coefficients of such modular forms for large values of n and, hence, for the computation of critical values of the L-functions for large twists of ϕ. In Example 2 below the run time for computing the first 10^5 coefficients of the theta series is just 304.200 s on a modest laptop while the same computation takes over 36 CPU hours if we do not use the representation in terms of ternary quadratic forms. Similarly in Example 1 the run time for computing the first 10^5 coefficients of the theta series is 358.820 s.

Notation. We will denote by $[a, b, c, r, s, t]$ the ternary quadratic form given by $ax^2 + by^2 + cz^2 + ryz + sxz + txy$.

Example 1. Let E be an elliptic curve of conductor 50 as in Proposition 1.1. Let ϕ be the newform corresponding to E,

$$\phi = q + q^2 - q^3 + q^4 - q^6 - 2q^7 + q^8 - 2q^9 - 3q^{11} + O(q^{12}).$$

Note that $\nu_2(50) = 1$, hence ρ_2 is not supercuspidal and we can apply Waldspurger's theorem.

We get that $\widetilde{N_\phi} = 100$ and $S_{3/2}(100, \chi_{\mathrm{triv}}, \phi)$ has a basis consisting of f_1 and f_2 where

$$f_1 = q + q^4 - q^6 - q^{11} - 2q^{14} + O(q^{15}) := \sum_{n=1}^{\infty} a_n q^n,$$
$$f_2 = q^2 - q^3 + q^8 - q^{12} + 2q^{13} + O(q^{15}) := \sum_{n=1}^{\infty} b_n q^n.$$

In fact, it turns out that $f_1 = (\theta_{Q_1} - \theta_{Q_2})/2$ and $f_2 = (\theta_{Q_3} - \theta_{Q_4})/2$ where Q_i are the ternary quadratic forms in Proposition 1.1. This can now be proved along similar lines to Theorem 10.

Again we can compute the order of $\text{Ш}(E_{-n}/\mathbb{Q})$ assuming that the Birch and Swinnerton-Dyer conjecture holds. For example, we get that

$$|\text{Ш}(E_{-9318}/\mathbb{Q})| = 33^2 = 1089.$$

We can further consider the real quadratic twists E_n. For this we work with the elliptic curve E_{-1} of conductor 400,

$$E_{-1} : Y^2 = X^3 + X^2 - 48X - 172.$$

We can show that if $5 \nmid n$, then

$$\mathrm{L}(E_n, 1) = \begin{cases} \dfrac{\mathrm{L}(E_1, 1)}{\sqrt{n}} \cdot c_n^2 & \text{if } \left(\dfrac{n}{5}\right) = 1; \\ \mathrm{L}(E_{17}, 1) \cdot \sqrt{\dfrac{17}{n}} \cdot c_n^2 & \text{if } \left(\dfrac{n}{5}\right) = -1, \end{cases}$$

where c_n is the nth coefficient of the following linear combination of theta series of weight 3/2 and level 1600 coming from the ternary quadratic forms:

$$\begin{aligned} &-\tfrac{1}{5} \cdot \theta_{[5,5,17,-2,-4,0]} + \tfrac{1}{5} \cdot \theta_{[5,9,10,2,2,4]} + \tfrac{1}{10} \cdot \theta_{[1,4,400,0,0,0]} - \tfrac{1}{10} \cdot \theta_{[5,17,20,-8,0,-2]} \\ &- \tfrac{1}{10} \cdot \theta_{[5,17,20,4,4,2]} + \tfrac{1}{10} \cdot \theta_{[8,13,20,12,8,4]} - \tfrac{1}{5} \cdot \theta_{[1,32,52,-16,0,0]} + \tfrac{1}{5} \cdot \theta_{[8,13,17,6,4,4]} \\ &+ \tfrac{1}{10} \cdot \theta_{[4,5,400,0,0,-4]} - \tfrac{1}{10} \cdot \theta_{[4,16,101,0,-4,0]} + \tfrac{1}{10} \cdot \theta_{[400,100,1,0,0,0]} \\ &- \tfrac{1}{10} \cdot \theta_{[125,100,4,0,0,100]} + \tfrac{1}{5} \cdot \theta_{[89,56,9,-4,-2,-44]} - \tfrac{1}{5} \cdot \theta_{[49,36,29,24,22,16]} \\ &- \tfrac{1}{2} \cdot \theta_{[400,13,8,4,0,0]} - \tfrac{1}{10} \cdot \theta_{[100,25,17,10,0,0]} + \tfrac{1}{10} \cdot \theta_{[52,32,25,0,0,16]} \\ &+ \tfrac{1}{2} \cdot \theta_{[53,33,25,-10,-10,-14]} + \tfrac{1}{2} \cdot \theta_{[400,400,1,0,0,0]} + \tfrac{9}{10} \cdot \theta_{[400,25,16,0,0,0]} \\ &- \tfrac{1}{2} \cdot \theta_{[201,201,4,4,4,2]} + \tfrac{1}{10} \cdot \theta_{[224,89,9,-2,-8,-88]} - \tfrac{1}{10} \cdot \theta_{[209,36,25,20,10,36]} \end{aligned}$$

$$
\begin{aligned}
&- \tfrac{9}{10} \cdot \theta_{[129,100,16,0,-16,-100]} - \tfrac{4}{5} \cdot \theta_{[84,81,25,10,20,4]} + \tfrac{4}{5} \cdot \theta_{[89,49,41,-6,-14,-38]} \\
&- \tfrac{1}{5} \cdot \theta_{[400,29,16,16,0,0]} + \tfrac{1}{5} \cdot \theta_{[125,100,16,0,0,100]} - \tfrac{2}{5} \cdot \theta_{[100,96,21,8,20,80]} \\
&+ \tfrac{2}{5} \cdot \theta_{[84,69,29,2,12,28]} - \tfrac{2}{5} \cdot \theta_{[400,32,13,8,0,0]} + \tfrac{2}{5} \cdot \theta_{[117,52,32,-16,-24,-44]} \\
&+ \tfrac{1}{5} \cdot \theta_{[400,25,17,10,0,0]} + \tfrac{1}{5} \cdot \theta_{[212,48,17,8,4,48]} + \tfrac{1}{10} \cdot \theta_{[208,32,25,0,0,32]} \\
&- \tfrac{1}{5} \cdot \theta_{[212,33,25,-10,-20,-28]} - \tfrac{1}{10} \cdot \theta_{[208,33,32,32,32,16]} - \tfrac{1}{5} \cdot \theta_{[113,52,32,16,8,52]}.
\end{aligned}
$$

Further, using root number arguments, we get that $\mathrm{L}(E_{-5n},1)=0$ whenever $n \not\equiv 3 \pmod 8$ and $\mathrm{L}(E_{5n},1)=0$ whenever $n \equiv 5 \pmod 8$. We have been unable to derive similar formulae for $\mathrm{L}(E_{-5n},1)$ when $n \equiv 3 \pmod 8$ and for $\mathrm{L}(E_{5n},1)$ when $n \not\equiv 5 \pmod 8$. We consider the twist E_5 whose conductor is again 50 and for which the minimum level to obtain non-zero Shimura equivalent forms is 500, however no new information can be obtained from these forms.

Example 2. This example formulates Theorem 11 in terms of ternary quadratic forms. Let $E: Y^2 = X^3 - 1$. Let n be a positive square-free integer such that $n \equiv 1$ or $2 \pmod 3$. Then

$$
\mathrm{L}(E_{-n},1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot a_n^2
$$

where a_n is the nth coefficient of the cusp form f of weight $3/2$ and level 576 that can be written as follows as a linear combination of theta series:

$$
\begin{aligned}
f &= \sum_{n=1}^{\infty} a_n q^n \\
&= \frac{1}{\sqrt{6}} \cdot \Bigg(\frac{1}{2} \cdot \theta_{[1,36,45,-36,0,0]} - \frac{1}{2} \cdot \theta_{[4,9,37,0,-4,0]} + \frac{1}{2} \cdot \theta_{[144,9,4,0,0,0]} - \frac{1}{2} \cdot \theta_{[45,36,4,0,0,36]} \\
&\quad - \frac{1}{2} \cdot \theta_{[144,16,9,0,0,0]} + \frac{1}{2} \cdot \theta_{[49,36,16,0,-16,-36]} \Bigg) + \frac{1}{2} \cdot \theta_{[144,29,5,2,0,0]} \\
&\quad - \frac{1}{2} \cdot \theta_{[45,32,20,-16,-12,-24]} + \sqrt{2} \cdot \left(\frac{1}{4} \cdot \theta_{[144,13,13,10,0,0]} - \frac{1}{4} \cdot \theta_{[45,36,16,0,0,36]} \right) \\
&\quad + \sqrt{3} \cdot \Bigg(\frac{1}{6} \cdot \theta_{[1,4,144,0,0,0]} - \frac{1}{6} \cdot \theta_{[4,4,37,0,-4,0]} + \frac{1}{6} \cdot \theta_{[4,5,36,0,0,-4]} - \frac{1}{6} \cdot \theta_{[4,13,13,-10,0,0]} \\
&\quad + \frac{1}{3} \cdot \theta_{[1,20,32,-16,0,0]} + \frac{1}{6} \cdot \theta_{[4,5,29,-2,0,0]} - \frac{1}{2} \cdot \theta_{[4,9,17,-6,0,0]} + \frac{1}{6} \cdot \theta_{[144,16,1,0,0,0]} \\
&\quad - \frac{1}{6} \cdot \theta_{[16,16,9,0,0,0]} - \frac{1}{3} \cdot \theta_{[144,5,4,4,0,0]} + \frac{1}{6} \cdot \theta_{[37,16,4,0,4,0]} + \frac{1}{6} \cdot \theta_{[16,13,13,10,0,0]} \\
&\quad + \frac{1}{6} \cdot \theta_{[32,21,4,-4,0,-16]} - \frac{1}{6} \cdot \theta_{[29,16,5,0,2,0]} - \frac{1}{2} \cdot \theta_{[144,36,1,0,0,0]} + \frac{1}{2} \cdot \theta_{[144,9,4,0,0,0]} \\
&\quad - \frac{1}{6} \cdot \theta_{[144,144,1,0,0,0]} + \frac{1}{6} \cdot \theta_{[49,36,16,0,-16,-36]} + \frac{1}{2} \cdot \theta_{[45,32,20,-16,-12,-24]} \\
&\quad - \frac{1}{2} \cdot \theta_{[32,29,29,22,16,16]} - \frac{1}{6} \cdot \theta_{[80,32,9,0,0,32]} + \frac{1}{2} \cdot \theta_{[80,17,17,-2,-16,-16]} \\
&\quad - \frac{1}{3} \cdot \theta_{[41,32,20,16,20,8]} \Bigg).
\end{aligned}
$$

Example 3. Let $E: Y^2 + Y = X^3 - 7$ be an elliptic curve of conductor 27 and let ϕ be the corresponding newform. Using Corollary 4.1, we get that the local component of ϕ at 2 is not supercuspidal and hence we can apply Waldspurger's theorem. We have the following proposition.

Proposition 8.7. *With E as above let n be a square-free integer.*

(i) *Suppose that $n \equiv 1 \pmod 3$. Let f be given by*

$$f = \sum_{n=1}^{\infty} a_n q^n = -\frac{1}{2}\cdot\theta_{[1,6,15,-6,0,0]} + \frac{1}{2}\cdot\theta_{[4,4,7,4,4,2]} + \theta_{[27,27,1,0,0,0]} - \theta_{[28,27,4,0,4,0]} - \frac{1}{2}\cdot\theta_{[27,7,4,2,0,0]} - \frac{1}{2}\cdot\theta_{[16,9,7,-6,-4,-6]} + \theta_{[31,16,7,4,2,16]}.$$

If either $\nu_2(n) = 1$, or $\nu_2(n) = 0$ and $n \equiv 1, 5 \pmod 8$, then

$$\mathrm{L}(E_{-n}, 1) = \frac{\mathrm{L}(E_{-1}, 1)}{\sqrt{n}}\cdot a_n^2.$$

Otherwise,

$$\mathrm{L}(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}}\cdot a_n^2$$

where $\kappa = \sqrt{19}\cdot \mathrm{L}(E_{-19}, 1)$ if $n \equiv 3 \pmod 8$ and $\kappa = \sqrt{7}\cdot\mathrm{L}(E_{-7}, 1)$ if $n \equiv 7 \pmod 8$.

(ii) *Suppose that $n \equiv 0 \pmod 3$ and let $n = 3m$. Let $h \in S_{3/2}(324, \chi_{\mathrm{triv}}, \phi)$ be the cusp form having the following q-expansion*

$$h = q^3 - q^{21} + 2q^{30} - q^{39} - 2q^{48} - q^{57} - 2q^{66} + q^{75} + O(q^{80}) := \sum_{n=1}^{\infty} b_n q^n.$$

Further suppose that $(\frac{m}{3}) = 1$. If either $\nu_2(n) = 1$, or $\nu_2(n) = 0$ and $n \equiv 1, 5 \pmod 8$, then

$$\mathrm{L}(E_{-n}, 1) = \mathrm{L}(E_{-21}, 1)\cdot\sqrt{\frac{21}{n}}\cdot b_n^2.$$

If $n \equiv 3, 7 \pmod 8$, then

$$\mathrm{L}(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}}\cdot b_n^2$$

where $\kappa = \sqrt{3}\cdot\mathrm{L}(E_{-3}, 1)$ if $n \equiv 3 \pmod 8$ and $\kappa = \sqrt{39}\cdot\mathrm{L}(E_{-39}, 1)$ if $n \equiv 7 \pmod 8$.

(iii) *If $n = 3m$ and $(\frac{m}{3}) = -1$, then $\mathrm{L}(E_{-n}, 1) = 0$.*

(iv) *If $n \equiv 2 \pmod 3$, then $\mathrm{L}(E_{-n}, 1) = 0$.*

The proof of (i) and (ii) follows as in the previous examples, while for (iii) and (iv) one can use root number arguments. We point out that the cusp form h which appears in (ii) does not come from ternary quadratic forms. Moreover, since E is isogenous to E_{-3}, for n positive square-free $\mathrm{L}(E_n, 1) = \mathrm{L}(E_{-3n}, 1)$. Thus, using the above proposition we are able to compute the critical values $\mathrm{L}(E_n, 1)$ for all n square-free.

9. *Tables*

In this section we present tables of orders of Tate–Shafarevich groups for twists of some of the elliptic curves considered in the previous section. We are assuming that the Birch and Swinnerton-Dyer conjecture holds.

We first consider the elliptic curve $E : Y^2 = X^3 + X + 1$ and use the formula in Corollary 8.4 to obtain the following table of orders of $Ш(E_{-n}/\mathbb{Q})$ for positive square-free $n \leqslant 2000$ with $n \equiv 1, 3, 7 \pmod 8$ and $\mathrm{L}(E_{-n}, 1) \neq 0$.

$|Ш(E_{-n}/\mathbb{Q})| = 1$ for $n =$ 15, 17, 23, 31, 43, 57, 65, 79, 89, 91, 105, 137, 145, 151, 155, 161, 179, 201, 215, 217, 239, 251, 263, 303, 305, 313, 321, 323, 337, 339, 393, 395, 399, 401, 403, 409, 465, 471, 527, 551, 571, 595, 601, 611, 619, 633, 651, 663, 673, 681, 697, 699, 705, 755, 759, 767, 787, 843, 849, 871, 879, 895, 921, 953, 959, 991, 1015, 1019, 1057, 1119, 1129, 1153, 1159, 1171, 1193, 1201, 1209, 1235, 1255, 1257, 1271, 1329, 1339, 1355, 1367, 1385, 1401, 1441, 1479, 1481, 1545, 1553, 1615, 1633, 1641, 1649, 1673, 1689, 1691, 1729, 1731, 1735, 1751, 1759, 1767, 1779, 1841, 1851, 1887, 1891, 1921, 1939, 1951, 1959

$|Ш(E_{-n}/\mathbb{Q})| = 4$ for $n =$ 55, 73, 83, 167, 209, 223, 241, 259, 265, 331, 371, 385, 415, 449, 457, 491, 499, 587, 649, 695, 703, 761, 881, 983, 1023, 1047, 1049, 1067, 1115, 1139, 1145, 1199, 1295, 1297, 1379, 1407, 1439, 1463, 1483, 1577, 1579, 1603, 1655, 1687, 1703, 1705, 1793, 1811, 1889, 1903, 1913, 1915, 1937, 1979, 1999

$|Ш(E_{-n}/\mathbb{Q})| = 9$ for $n =$ 115, 119, 123, 177, 203, 247, 271, 291, 347, 427, 433, 447, 455, 489, 523, 579, 615, 713, 719, 739, 743, 771, 817, 823, 827, 863, 899, 905, 911, 943, 951, 1003, 1065, 1191, 1195, 1231, 1239, 1267, 1313, 1319, 1391, 1417, 1491, 1505, 1511, 1515, 1531, 1635, 1695, 1711, 1819, 1897, 1977, 1983

$|Ш(E_{-n}/\mathbb{Q})| = 16$ for $n =$ 353, 463, 643, 647, 859, 947, 1097, 1111, 1147, 1243, 1345, 1363, 1387, 1393, 1419, 1447, 1487, 1571, 1643, 1667, 1697, 1835, 1855, 1943, 1945, 1987

$|Ш(E_{-n}/\mathbb{Q})| = 25$ for $n =$ 327, 487, 553, 623, 923, 1207, 1263, 1315, 1455, 1543, 1607, 1627, 1747, 1763, 1995

The following is data for higher orders of $Ш(E_{-n}/\mathbb{Q})$ which are attained for n as above with $n \leqslant 10\,000$:

$|Ш(E_{-n}/\mathbb{Q})| = 36$ for $n =$ 383, 635, 967, 2347, 2351, 2383, 2411, 2563, 3155, 3391, 3743, 4091, 4367, 4487, 4519, 4591, 4609, 5323, 5327, 5393, 5423, 5467, 5479, 5555, 5657, 5659, 5803, 5883, 5963, 6691, 6863, 7159, 7215, 7297, 7307, 7343, 7559, 7567, 7607, 7639, 7895, 7963, 8159, 8283, 8515, 8635, 8683, 9047, 9385, 9631, 9665, 9667, 9787, 9791

$|Ш(E_{-n}/\mathbb{Q})| = 49$ for $n =$ 1623, 1753, 2337, 2603, 2927, 2999, 3153, 3279, 3347, 3563, 4043, 4115, 4331, 4507, 4555, 4955, 4971, 5199, 5347, 5595, 5795, 5955, 6131, 6227, 6447, 6593, 6663, 6695, 7123, 7283, 7545, 7591, 7687, 7951, 8071, 8135, 8259, 8407, 8431, 8455, 8567, 8755, 8835, 8897, 8923, 9609, 9771, 9827, 9839

$|Ш(E_{-n}/\mathbb{Q})| = 64$ for $n =$ 1007, 1727, 2183, 2243, 2455, 2555, 2723, 3763, 3905, 4963, 5137, 5417, 6587, 6935, 7467, 7483, 7811, 8273, 8737, 9343, 9923

$|Ш(E_{-n}/\mathbb{Q})| = 81$ for $n =$ 1567, 2683, 2931, 3247, 3323, 3547, 3587, 3855, 3867, 6087, 6305, 6403, 7153, 7223, 7339, 7833, 7993, 8227, 8447, 8779, 8887, 8895, 9327, 9393, 9931

$|Ш(E_{-n}/\mathbb{Q})| = 100$ for $n =$ 2827, 3463, 4103, 4543, 5207, 5663, 6847, 7415, 8011, 8015, 8335, 8393, 9143, 9323, 9379

$|Ш(E_{-n}/\mathbb{Q})| = 121$ for $n =$ 2743, 5703, 7451, 7703, 7873, 7903, 8795, 8983, 9755, 9763

$|Ш(E_{-n}/\mathbb{Q})| = 144$ for $n =$ 3307, 4643, 9497, 9995

$|Ш(E_{-n}/\mathbb{Q})| = 169$ for $n =$ 3687, 6527

$|Ш(E_{-n}/\mathbb{Q})| = 196$ for $n =$ 7867, 9355

$|Ш(E_{-n}/\mathbb{Q})| = 225$ for $n =$ 7143

$|Ш(E_{-n}/\mathbb{Q})| = 289$ for $n =$ 5003, 6823, 8903

Moreover, using Corollary 8.5 we obtain the following list of positive square-free $n \leqslant 10\,000$ with $n \equiv 1, 3, 7 \pmod 8$ and $(\frac{n}{31}) = -1$ such that $d_n = 0$. Hence, for these values of n, $\mathrm{Rank}(E_{-n}) \geqslant 2$, assuming that the Birch and Swinnerton-Dyer conjecture holds.

11, 127, 139, 185, 199, 367, 451, 511, 519, 561, 569, 631, 641, 737, 799, 809, 835, 883, 889, 897, 929, 985, 987, 995, 1009, 1081, 1091, 1131, 1137, 1169, 1177, 1283, 1443, 1499, 1561, 1563, 1639, 1739, 1801, 1871, 1873, 1883, 2207, 2409, 2441, 2479, 2495, 2571, 2627, 2785, 2905, 2935, 3081, 3121, 3143, 3289, 3343, 3377, 3431, 3487, 3499, 3551, 3561, 3799, 3927, 3929, 3959, 4145, 4177, 4209, 4339, 4355, 4395, 4415, 4463, 4481, 4663, 4735, 4811, 4921, 5017, 5169, 5335, 5345, 5449, 5561, 5579, 5665, 5671, 5779, 5793, 5849, 5889, 5919, 5951, 5969, 5979, 5995, 6007, 6031, 6153, 6193, 6211, 6289, 6409, 6465, 6491, 6505, 6719, 6739, 6761, 6857, 6895, 6911, 6959, 6967, 6999, 7023, 7195, 7207, 7265, 7315, 7331, 7359, 7513, 7601, 7643, 7711, 7777, 7815, 8139, 8201, 8241, 8249, 8363, 8369, 8507, 8691, 8769, 8807, 8889, 9127, 9129, 9281, 9311, 9313, 9415, 9417, 9515, 9543, 9551, 9591, 9647, 9795, 9851, 9895

Next we consider the curve $E : Y^2 = X^3 - 1$. We use Theorem 11 to obtain the orders of $Ш(E_{-n}/\mathbb{Q})$ for $n \leqslant 100\,000$ positive square-free with $n \equiv 1, 2 \pmod 3$ and $\mathrm{L}(E_{-n}, 1) \neq 0$. Here we present a table for values of such n with $|Ш(E_{-n}/\mathbb{Q})| \geqslant 256$.

$|Ш(E_{-n}/\mathbb{Q})| = 256$ for $n =$ 33 997, 35 341, 38 821, 48 109, 50 893, 62 261, 62 821, 65 285, 70 573, 71 501, 73 309, 75 493, 77 773, 77 797, 84 157, 85 277, 85 333, 89 045, 90 037, 94 813, 96 613, 97 205

$|Ш(E_{-n}/\mathbb{Q})| = 289$ for $n =$ 12 893, 14 717, 14 845, 27 893, 28 661, 30 029, 37 589, 37 621, 39 821, 41 189, 44 789, 45 293, 45 677, 45 869, 53 149, 53 437, 55 061, 55 313, 58 757, 62 989, 68 141, 68 501, 72 077, 72 301, 72 341, 73 421, 80 317, 80 533, 80 813, 82 141, 85 165, 86 357, 87 485, 87 797, 89 501, 89 909, 93 497, 93 565, 95 021, 95 717, 96 221, 96 989, 97 397

$|Ш(E_{-n}/\mathbb{Q})| = 324$ for $n =$ 34 501, 64 237, 79 693, 82 549

$|Ш(E_{-n}/\mathbb{Q})| = 361$ for $n =$ 18 773, 30 341, 31 541, 31 765, 40 949, 43 517, 43 853, 48 341, 49 789, 58 733, 59 021, 61 949, 63 773, 69 541, 71 693, 75 269, 75 949, 76 957, 78 893, 83 093, 83 597, 86 077, 863 41, 86 813, 86 981, 88 045, 92 357, 93 629, 95 429, 95 957, 96 157, 98 269

$|Ш(E_{-n}/\mathbb{Q})| = 400$ for $n =$ 52 261, 64 693, 66 373, 80 029

$|Ш(E_{-n}/\mathbb{Q})| = 441$ for $n =$ 15 629, 23 957, 24 533, 49 157, 53 549, 66 029, 68 813, 70 853, 71 893, 82 333, 82 781, 86 837

$|Ш(E_{-n}/\mathbb{Q})| = 484$ for $n =$ 83 677, 92 797

$|Ш(E_{-n}/\mathbb{Q})| = 529$ for $n =$ 40 829, 51 869, 70 157, 70 877, 73 517, 76 541, 77 213, 79 901, 83 117, 86 117

$|Ш(E_{-n}/\mathbb{Q})| = 576$ for $n =$ 60 037, 85 669, 99 109, 99 469

$|Ш(E_{-n}/\mathbb{Q})| = 625$ for $n =$ 56 605, 57 221, 60 101, 61 757, 85 853, 92 237, 95 653

$|Ш(E_{-n}/\mathbb{Q})| = 729$ for $n =$ 57 557, 65 309, 69 221, 71 741, 71 837, 82 613, 88 661, 98 573

$|Ш(E_{-n}/\mathbb{Q})| = 841$ for $n =$ 76 733

$|Ш(E_{-n}/\mathbb{Q})| = 1089$ for $n =$ 74 933

$|Ш(E_{-n}/\mathbb{Q})| = 1225$ for $n =$ 78 797

Acknowledgements. This article is part of my thesis and I would like to thank my advisor Professor Samir Siksek for suggesting this problem and for several helpful discussions. I am grateful to Professor John Cremona and Dr Neil Dummigan for careful reading of my thesis and for providing valuable insights and suggestions. I am grateful to the referee for providing many helpful remarks and suggestions and pointing out many useful references.

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