

# Number Theory in Tokyo

Date : March 20 (Monday) – March 24 (Friday), 2023

Venue : Tokyo Institute of Technology, Ookayama West Lecture Building 2, Room W631

Organizers : Soma Purkait (Tokyo Institute of Technology)

Shingo Sugiyama (Nihon University)

Shun'ichi Yokoyama (Tokyo Metropolitan University)

## Program

### March 20th (Mon)

9:30–9:45 Introduction

9:45–10:45 Takashi Hara (Tsuda University)

On  $p$ -adic Artin  $L$ -functions for CM fields

11:00–12:00 Ken Ono (University of Virginia) [**Online**]

Framework of Rogers-Ramanujan Identities

13:30–14:30 Samir Siksek (University of Warwick)

Differences of squares and perfect powers

15:00–16:00 Haluk Sengun (University of Sheffield)

Local theta correspondence via  $C^*$ -algebras of groups

### March 21st (Tue)

9:45–10:45 Masanobu Kaneko (Kyushu University)

On some formulas for quadratic class numbers

11:00–12:00 Hiro-aki Narita (Waseda University)

Automorphic forms generating quaternionic discrete series

13:30–14:30 Shingo Sugiyama (Nihon University)

Recent progress on the weighted density of low-lying zeros of  $L$ -functions in a family

14:45–15:45 Hirotaka Kodama (Tokoha University)

A note on the coefficients of vector valued Siegel modular forms of type  $(k, 2)$

16:00–17:00 Jerome Dimabayao (University of the Philippines)

An irrational variant of the congruent number problem

**March 22nd (Wed)**

- 9:45–10:45 Yukihiro Uchida (Tokyo Metropolitan University)  
Periodicity of Somos sequences related to curves of genus 2
- 11:00–12:00 Vaidehee Thatte (King's College, London)  
Understanding the Defect via Ramification Theory

**Free afternoon**

**March 23rd (Thu)**

- 9:45–10:45 Miyu Suzuki (Kyoto University)  
Towards the full epsilon dichotomy for linear periods
- 11:00–12:00 Vincent Sécherre (Université de Versailles Saint-Quentin)  
Congruence properties of the local functorial transfer from a classical group to  $GL(N)$  for cuspidal representations
- 13:30–14:30 Nadir Matringe (University Paris Cité)  
Symmetric periods for automorphic forms on unipotent groups
- 14:45–15:45 Ehud Moshe Baruch (Technion - Israel Institute of Technology)  
A 2-adic Hecke algebra and its applications
- 16:00–17:00 Ramla Abdellatif (Université de Picardie Jules Verne)  
On a question of Nigel Boston about Fontaine-Mazur Conjecture

**March 24th (Fri)**

- 9:45–10:45 Yoshinori Mishiba (University of the Ryukyus)  
Thakur's basis conjecture for multiple zeta values in positive characteristic
- 11:00–12:00 Masao Oi (Kyoto University)  
Swan conductors of local Galois representations and their functorial lifts
- 13:30–14:30 Filippo A. E. Nuccio (ICJ Université Jean Monnet Saint-Étienne)  
Explaining to a computer that the class group is finite
- 14:45–15:45 Narasimha Kumar (Indian Institute of Technology, Hyderabad) [**Online**]  
On the structure of Drinfeld modular forms of level  $\Gamma_0(T)$  and applications
- 16:00–17:00 Masao Tsuzuki (Sophia University)  
Weighted spectral equidistribution theorem for Siegel cusp forms of degree 2

## Abstracts

### March 20th (Mon)

9:45–10:45 Takashi Hara (Tsuda University)

On  $p$ -adic Artin  $L$ -functions for CM fields

Abstract: In 1983, Ralph Greenberg constructed  $p$ -adic Artin  $L$ -functions for totally real number fields, by appropriately patching  $p$ -adic  $L$ -functions associated to intermediate abelian extensions. Following Greenberg's strategy, we will construct  $p$ -adic Artin  $L$ -functions for CM fields assuming several technical conditions and the validity of (abelian) Iwasawa main conjecture for CM fields. We would especially like to focus on characteristic phenomena and difficulties observed in the CM field case. This is joint work with Tadashi Ochiai (Tokyo Institute of Technology).

11:00–12:00 Ken Ono (University of Virginia)

Framework of Rogers-Ramanujan Identities

Abstract: The two Rogers-Ramanujan identities, which express two modular units as strange  $q$ -series, are ubiquitous, arising in many different areas of mathematics: combinatorics, conformal field theory, Lie theory, number theory, representation theory, statistical mechanics. There are more than 1000 papers on these identities. In this talk the speaker will consider two fundamental questions. Are these two identities a glimpse of a large infinite framework of identities for modular units? We show that this is the case where infinitely many infinite families of modular functions are obtained as specialized characters of Lie type by making use of extensions of Hall-Littlewood polynomials. Secondly, the speaker will explain an infinite dimensional extension of a theorem of Bryden Cais and Brian Conrad on Ramanujan's conjectured construction of algebraic integral units.

13:30–14:30 Samir Siksek (University of Warwick)

Differences of squares and perfect powers

Abstract: Diophantine equations involving the difference between a square and an arbitrary perfect power have been studied by elementary means since 1850, and using Baker's theory of linear forms in logarithms since the 1960s. We illustrate how the proof of Fermat's Last Theorem, and in particular ideas of Kraus, can be made to work hand-in-hand with the elementary and Baker techniques to solve some difficult instances of these problems. This talk is based on joint work with Mike Bennett (University of British Columbia).

15:00–16:00 Haluk Sengun (University of Sheffield)

Local theta correspondence via  $C^*$ -algebras of groups

Abstract: Theta correspondence is a major theme in the theory of automorphic forms and in representation theory. The local version of the correspondence sets up a bijection between certain sets of admissible irreps of a pair of reductive groups  $G, H$  which sit as each others' centralizers in a larger symplectic group. The local correspondences then bundle up to set up a bijection between certain sets of automorphic representations.

There are two settings in which the local theta correspondence is known to enjoy important extra properties: if  $G$  and  $H$  are of 'equal size' then temperedness is preserved, and if one group is much larger than the other ('stable range'), then unitarity is preserved. In these two settings, Bram Mesland (Leiden) and I have used Rieffel's theory of induction for

representations of  $C^*$ -algebras to prove that local theta lifting is functorial and is continuous with respect to weak containment when restricted to tempered/unitary duals. In the talk, I will explain our approach and time permitting, will discuss further applications to transfer of characters and formal dimensions. Some of the results I will discuss can be found in the preprint arXiv:2207.13484.

### March 21st (Tue)

9:45–10:45 Masanobu Kaneko (Kyushu University)

On some formulas for quadratic class numbers

Abstract: A survey on two amusing formulas for the class number of an imaginary quadratic field is given: theorems of Hirzebruch-Zagier and of Girstmair. Then presented are complementary theorems for both, obtained by Mizuno and myself for the former, and by Mizuno for the latter. Also mentioned is a deduction of the classical class number formula via Gauss's formula for the values at rational arguments of the digamma function.

11:00–12:00 Hiro-aki Narita (Waseda University)

Automorphic forms generating quaternionic discrete series

Abstract: Holomorphic modular forms generate holomorphic discrete series representations. What are the next targets that come after the studies of holomorphic modular forms? One candidate would be to study the automorphic forms generating quaternionic discrete series representations, whose detailed studies were initiated by Gan-Gross-Savin and Arakawa and so on. The aim of this talk is to survey old and recent results including the interesting progress by Aaron Polack and speaker's current research.

13:30–14:30 Shingo Sugiyama (Nihon University)

Recent progress on the weighted density of low-lying zeros of  $L$ -functions in a family

Abstract: Katz and Sarnak suggested that low-lying zeros of  $L$ -functions in a family should be distributed like eigenvalues of random matrices in a compact matrix group. Inspired by Kowalski, Saha and Tsimerman, the density of low-lying zeros of  $L$ -functions in a family weighted by central  $L$ -values has been studied by several mathematicians. In this talk, the speaker suggests a conjecture on the density of low-lying zeros of  $L$ -functions weighted by general  $L$ -values, and gives results for a family of symmetric power  $L$ -functions and that of Dirichlet  $L$ -functions. The result on Dirichlet  $L$ -functions is a joint work with Ade Irma Suriajaya (Kyushu University).

14:45–15:45 Hirotaka Kodama (Tokoha University)

A note on the coefficients of vector valued Siegel modular forms of type  $(k, 2)$

Abstract: Let  $p \geq 5$  be a prime number. We introduce the generators of the module of vector valued Siegel modular forms of type  $(k, 2)$  over  $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$ . Moreover we introduce the examples of modular forms with Fourier coefficients in  $\mathbb{Z}$ .

16:00–17:00 Jerome Dimabayao (University of the Philippines)

An irrational variant of the congruent number problem

Abstract: We talk about a certain variant of the classical congruent number problem. We discuss integers which occur as areas of triangles with two rational sides and arbitrary fixed angle  $\psi$  with one adjacent side a rational multiple of a quadratic surd. We call such numbers  $\psi$ -congruent. We present a criterion that involves elliptic curves for deciding whether a given positive integer is  $\psi$ -congruent and talk about the density of  $\pi/4$ -congruent numbers. This is a joint work with Soma Purkait.

### March 22nd (Wed)

9:45–10:45 Yukihiro Uchida (Tokyo Metropolitan University)

Periodicity of Somos sequences related to curves of genus 2

Abstract: A Somos  $k$  sequence is a sequence defined by a bilinear recurrence relation of order  $k$ . For example, an elliptic divisibility sequence, which consists of the values of the division polynomials of an elliptic curve, is a Somos 4 sequence. It is known that Somos 4 sequences are periodic modulo all but finitely many primes. In this talk, we discuss periodicity of Somos sequences modulo primes when the sequences are related to curves of genus 2. In particular, we show that the Somos 8 sequence which consists of the values of Cantor's division polynomials of a curve of genus 2 is periodic modulo all but finitely many primes. This talk is based on joint work with Yasuhiro Ishitsuka, Tetsushi Ito, Tatsuya Ohshita, and Takashi Taniguchi.

11:00–12:00 Vaidehee Thatte (King's College, London)

Understanding the Defect via Ramification Theory

Abstract: Classical ramification theory deals with complete discrete valuation fields  $k((X))$  with perfect residue fields  $k$ . Invariants such as the Swan conductor capture important information about extensions of these fields. Many fascinating complications arise when we allow non-discrete valuations and imperfect residue fields  $k$ . Particularly in positive residue characteristic, we encounter the mysterious phenomenon of the *defect* (or ramification deficiency). The occurrence of a non-trivial defect is one of the main obstacles to long-standing problems, such as obtaining resolution of singularities in positive characteristic. Degree  $p$  extensions of valuation fields are building blocks of the general case. In this talk, we will present a generalization of ramification invariants for such extensions and discuss how this leads to a better understanding of the defect. If time permits, we will briefly discuss their connection with some recent work (joint with K. Kato) on upper ramification groups.

### March 23rd (Thu)

9:45–10:45 Miyu Suzuki (Kyoto University)

Towards the full epsilon dichotomy for linear periods

Abstract: Let  $G$  be an algebraic group over a local field and  $H$  be a closed subgroup. A representation of  $G$  is said to be  $H$ -distinguished if it has a nonzero  $H$ -invariant linear form. There are several conjectures/theorems that provide characterization of distinguished representations in generic  $L$ -packets by using epsilon factors and the characters of the  $S$ -groups. Such conjectures/theorems are called epsilon dichotomy. The most typical example is the local Gan-Gross-Prasad conjecture. Prasad and Takloo-Bighash formulated a conjecture

about necessary conditions for generic representations of general linear groups to be distinguished. Inspired by the Gan-Gross-Prasad conjecture, I will reformulate their conjecture so that it characterizes distinguished representations in generic  $L$ -packets in terms of epsilon factors and the characters of the  $S$ -groups. If time permits, I will also explain the strategy to prove this reformulated epsilon dichotomy using the conjectural multiplicity formula, which is proposed by Chen Wan for general spherical varieties.

14:45–15:45 Vincent Sécherre (Université de Versailles Saint-Quentin)

Congruence properties of the local functorial transfer from a classical group to  $\mathrm{GL}(N)$  for cuspidal representations

Abstract: Let  $G$  be the group of rational points of a quasi-split  $p$ -adic classical group for some odd prime number  $p$ . Following Arthur and Mok, there are a positive integer  $N$ , a  $p$ -adic field  $F$  and a local functorial transfer from the isomorphism classes of irreducible smooth complex representations of  $G$  to those of  $\mathrm{GL}(N, F)$ . Fixing a prime number  $l$  different from  $p$ , then an isomorphism between the field of complex numbers and an algebraic closure of the field of  $l$ -adic numbers, we get a transfer between representations with  $l$ -adic coefficients. Now consider a cuspidal irreducible  $l$ -adic representation  $V$  of  $G$ , and assume that its central character takes values in the ring of  $l$ -adic integers: one can thus define its reduction mod  $l$ , which is a semi-simple smooth representation of  $G$  of finite length, with coefficients in a finite field of characteristic  $l$ . Let  $V'$  be another cuspidal irreducible  $l$ -adic representation of  $G$  whose central character takes  $l$ -adic integer values and whose reduction mod  $l$  is isomorphic to that of  $V$ . We prove that the local functorial transfers  $W, W'$  of  $V, V'$  have reductions mod  $l$  which may not be isomorphic, but which share a unique common generic irreducible component. This is a joint work with Alberto Minguez.

13:30-14:30 Nadir Matringe (University Paris Cité)

Symmetric periods for automorphic forms on unipotent groups

Abstract: Let  $G$  be an algebraic group defined over a number field  $k$  with ring of adèles  $A$ , and let  $\sigma$  be a  $k$ -involuton of  $G$ . Studying the nonvanishing of (possible regularizations of) the period integral  $p : \phi \mapsto \int_{G^\sigma(k) \backslash G^\sigma(A)} \phi(h) dh$  on topologically irreducible submodules of  $L^2(G(k) \backslash G(A))$  is a very popular topic when  $G$  is reductive. Here I will focus on the case where  $G$  is unipotent, and explain that  $p$  does not vanish on smooth vectors of such a submodule  $\Pi$  if and only if  $\Pi^\vee = \Pi^\sigma$ .

14:45–15:45 Ehud Moshe Baruch (Technion - Israel Institute of Technology)

A 2-adic Hecke algebra and its applications

Abstract: We use the generators and relations of a particular 2-adic Hecke algebra to decompose the space of half integral weight forms of level  $8M$  where  $M$  is odd and square free and to compute Whittaker functions of new forms of level 4 and integer weight.

16:00–17:00 Ramla Abdellatif (Université de Picardie Jules Verne)

On a question of Nigel Boston about Fontaine-Mazur Conjecture

Abstract: In his seminal 1992 paper on Fontaine-Mazur Conjecture for prime degree extensions of number fields, Boston asked whether the method he develops could lead to a counter-example to Fontaine-Mazur Conjecture when applied to the biquadratic field  $\mathbb{Q}(\sqrt{-26}, \sqrt{229})$ .

In a joint work with S. Pisolkar (IISER Pune), we answer negatively to this question by making explicit the corresponding Galois group that was expected to provide the aforementioned counter-example. In particular, we prove that it is a finite group of order 6561 and that it is not a uniform group.

In this talk, which requires no specific pre-requisite beyond classical master courses, I will first explain the precise setting we are interested in, then I will introduce the various tools we use to answer Boston's original question.

## March 24th (Fri)

9:45–10:45 Yoshinori Mishiba (University of the Ryukyus)

Thakur's basis conjecture for multiple zeta values in positive characteristic

Abstract: In this talk, we will discuss multiple zeta values over function fields in positive characteristic. In particular, we will give a proof of Thakur's basis conjecture, which plays the analogue of Hoffman's basis conjecture for real multiple zeta values. This is a joint work with Chieh-Yu Chang and Yen-Tsung Chen.

11:00–12:00 Masao Oi (Kyoto University)

Swan conductors of local Galois representations and their functorial lifts

Abstract: Let us consider an  $N$ -dimensional Galois representation  $\rho$  of a  $p$ -adic field. For each finite-dimensional representation  $R$  of  $GL(N)$ , we can obtain another Galois representation  $\rho'$  by composing  $\rho$  with  $R$ . Given that the Swan conductor is one of the most fundamental invariants of Galois representations, it is natural to ask the relationship between the Swan conductor of  $\rho'$  and that of  $\rho$ . In this talk, I will explain several observations on this problem including a certain conjectural inequality proposed by Guy Henniart. I will also explain its application to the local Langlands correspondence for simple supercuspidal representations. This talk is based on my joint work with Guy Henniart.

13:30–14:30 Filippo A. E. Nuccio (ICJ Université Jean Monnet Saint-Étienne)

Explaining to a computer that the class group is finite

Abstract: In the first part of my talk I will explain what the Proof Assistant Lean is, and how it is able to "understand" mathematical definitions, statements and proofs. I will then discuss a joint work with A. Baanen, S. Daamen and Ashvni N., where we have formalised in Lean a proof of the finiteness of the ideal class group of a global field.

14:45–15:45 Narasimha Kumar (Indian Institute of Technology, Hyderabad)

On the structure of Drinfeld modular forms of level  $\Gamma_0(T)$  and applications

Abstract: In the literature, the structure of the graded  $\mathbb{C}$ -algebra  $M(\Gamma)$ , classical modular forms of level  $\Gamma$ , where  $\Gamma$  is a congruence subgroup of  $SL_2(\mathbb{Z})$ , is one of most important and well-studied question in modern number theory. It is well-known that the graded  $\mathbb{C}$ -algebra  $M(SL_2(\mathbb{Z}))$  is isomorphic to  $\mathbb{C}[X, Y]$ . Hence, to each  $f \in M(SL_2(\mathbb{Z}))$ , one can attach a unique polynomial. Using these polynomials, Serre, Swinnerton-Dyer studied the properties of the weight filtration for  $f \in M(SL_2(\mathbb{Z}))$ . For  $\Gamma_0(2)$ , the algebra  $M(\Gamma_0(2))$  is generated by two elements. There are many important works, by Rustom, Voight and Zureick-Brown, to understand the highest weight needed to generate  $M(\Gamma)$ .

Drinfeld modular forms are in many aspects analogues, over function fields of positive characteristic, of classical modular forms. It is of natural interest to understand the structure of the graded  $C$ -algebra  $M(\Gamma_0(\mathfrak{n}))$ , Drinfeld modular forms of level  $\Gamma_0(\mathfrak{n})$ , where  $\mathfrak{n}$  is an ideal of  $A := \mathbb{F}_q[T]$  and  $\Gamma_0(\mathfrak{n})$  is a congruent subgroup of  $\mathrm{GL}_2(A)$ . In 1988, Gekeler showed that the graded  $C$ -algebra  $M(\mathrm{GL}_2(A))$  is isomorphic to  $C[X, Y]$ . Hence, we can attach a unique polynomial to each  $f \in M(\mathrm{GL}_2(A))$ . Using these polynomials, in 2010, Vincent studied the properties of the weight filtration for  $f \in M(\mathrm{GL}_2(A))$ .

In joint work with Tarun Dalal, we show that the graded  $C$ -algebra  $M(\Gamma_0(T))$  is isomorphic to a quotient of the polynomial ring  $C[X, Y, Z]$ . Though we can attach a polynomial to each  $f \in M(\Gamma_0(T))$ , it may not be unique. The novelty in our work is to provide a way to attach a unique polynomial to each  $f \in M(\Gamma_0(T))$  and study the properties of the weight filtration for  $f \in M(\Gamma_0(T))$ . These properties are essential to provide an affirmative answer to a conjecture of Calegari and Stein in the context of Drinfeld modular forms.

16:00–17:00 Masao Tsuzuki (Sophia University)

Weighted spectral equidistribution theorem for Siegel cusp forms of degree 2

Abstract: We prove an asymptotic formula of spectral average of Satake parameters of Siegel cusp forms of degree 2 with growing even weight. Since our weight factors to form the average involve the norm square of Bessel periods and the central spinor  $L$ -values, the asymptotic formula has an application to non-vanishing of Bessel periods and central spinor  $L$ -values.