THE ALGEBRAIC p-ADIC L-FUNCTION AND ISOGENY BETWEEN FAMILIES OF GALOIS REPRESENTATIONS

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Abstract. In Theorem 1.6, we give a formula to compare the algebraic p-adic L-functions for two different lattices of a given family of Galois representations over a deformation ring $\mathcal{R}$. This generalize a classical comparison formula by Schneider [Sc] and Perrin-Riou [P], for which $\mathcal{R}$ is the cyclotomic Iwasawa algebra.

Recall that we studied the two-variable Iwasawa theory for residually irreducible nearly ordinary Hida deformations in [O1], [O2] and [O3] and we acquired sufficient understanding through these works. By applying our formula to Hida's nearly ordinary deformations, we understand better the two-variable Iwasawa theory for residually reducible cases, where the choice of lattices is not unique anymore.

Contents

1. Introduction

Let $\mathcal{R}$ be a local domain which is finitely generated and torsion-free over a power series algebra $\mathbb{Z}_p[[X_1, \cdots, X_n]]$ with p a fixed odd prime number and let $\mathcal{K}$ be the fraction field of $\mathcal{R}$. We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, where $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ are the algebraic closures of the rational number field $\mathbb{Q}$ and the p-adic field $\mathbb{Q}_p$ respectively. Throughout the paper, we denote by D_v the decomposition subgroup of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at a prime v.

Definition 1.1. Let $\mathcal{V}$ be a finite dimensional $\mathcal{K}$-vector space with $\mathcal{K}$-linear $G_{\mathbb{Q}}$-action. The representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{K}}(\mathcal{V})$ is called *continuous* if $\mathcal{V}$ has a finitely generated $\mathcal{R}$-submodule $\mathcal{T}$ with the following properties:

1. $\mathcal{T} \otimes_{\mathcal{R}} \mathcal{K}$ is isomorphic to $\mathcal{V}$.
2. The $\mathcal{R}$-submodule $\mathcal{T}$ is stable under $G_{\mathbb{Q}}$-action on $\mathcal{V}$.
3. The action of $G_{\mathbb{Q}}$ on $\mathcal{T}$ is continuous with respect to the $\mathfrak{M}$-adic topology on $\mathcal{T}$, where $\mathfrak{M}$ is the maximal ideal of $\mathcal{R}$.

An $\mathcal{R}$-submodule $\mathcal{T}$ of $\mathcal{V}$ satisfying the above properties is called *a lattice* (of $\mathcal{V}$). If $\mathcal{T}$ and $\mathcal{T}'$ are lattices of the same continuous representation $\mathcal{V}$, we say that $\mathcal{T}$ and $\mathcal{T}'$ are *isogenious* to each other. In general, a lattice $\mathcal{T}$ is not necessarily a free module over $\mathcal{R}$. When a lattice $\mathcal{T}$ is free $\mathcal{R}$-module, we call it *a free lattice*.

In this paper, we always assume that the action of $G_{\mathbb{Q}}$ on $\mathcal{T}$ is unramified outside a finite set of primes $\Sigma \supset \{p, \infty\}$. We introduce several notations. Let $\widetilde{\mathcal{R}}$ be the integral closure of $\mathcal{R}$ in $\mathcal{K}$. Note that $\widetilde{\mathcal{R}}$ is a finitely generated $\mathcal{R}$-module. Let $\widetilde{\mathcal{T}} = \mathcal{T} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}$ and let $\widetilde{\mathcal{A}} = \widetilde{\mathcal{T}} \otimes_{\widetilde{\mathcal{R}}} (\widetilde{\mathcal{R}})^\vee$ where $(\widetilde{\mathcal{R}})^\vee$ is the Pontrjagin dual of $\widetilde{\mathcal{R}}$. When $\widetilde{\mathcal{R}} = \mathcal{R}$, we denote $\widetilde{\mathcal{A}}$ by $\mathcal{A}$. We denote by $\mathrm{Hom}_{\mathbb{Z}_p}(\widetilde{\mathcal{R}}, \overline{\mathbb{Q}}_p)$ the set of non-trivial $\mathbb{Z}_p$-algebra homomorphisms, which is naturally endowed a p-adic topology and is regarded as a p-adic rigid analytic space. For $\phi \in \mathrm{Hom}_{\mathbb{Z}_p}(\widetilde{\mathcal{R}}, \overline{\mathbb{Q}}_p)$, we define $(\widetilde{\mathcal{T}})_\phi$ to be the p-adic Galois representation of $G_{\mathbb{Q}}$ in usual sense obtained by specializing $\widetilde{\mathcal{T}}$ via ϕ.

Definition 1.2. Suppose also that we have a D_p-stable $\mathcal{R}$-submodule $\mathrm{F}^+\widetilde{\mathcal{A}}$ of $\widetilde{\mathcal{A}}$. Then, we define the Selmer group $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ as follows:

$$\mathrm{Sel}_{\widetilde{\mathcal{T}}} = \mathrm{Ker}\left[H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \widetilde{\mathcal{A}}) \longrightarrow H^1(I_p, \mathrm{F}^-\widetilde{\mathcal{A}}) \times \prod_{l \in \Sigma \setminus \{p,\infty\}} H^1(I_l, \widetilde{\mathcal{A}}) \right], \tag{1}$$

where $\mathrm{F}^-\widetilde{\mathcal{A}}$ means $\widetilde{\mathcal{A}}/\mathrm{F}^+\widetilde{\mathcal{A}}$, $\mathbb{Q}_\Sigma$ is the maximal algebraic extension of $\mathbb{Q}$ unramified outside Σ and the map in the definition is the natural localization map. Though the Selmer group $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ depends on the fixed D_p-stable $\mathcal{R}$-submodule $\mathrm{F}^+\widetilde{\mathcal{A}}$ of $\widetilde{\mathcal{A}}$, we omit to note it in the notation $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ if there seems to be no confusion.

Remark 1.3. Since $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ is a discrete $\widetilde{\mathcal{R}}$-module, the Pontrjagin dual $(\mathrm{Sel}_{\widetilde{\mathcal{T}}})^\vee$ of $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ is naturally endowed with a structure of compact $\widetilde{\mathcal{R}}$-module. It is also not so hard to show that $(\mathrm{Sel}_{\widetilde{\mathcal{T}}})^\vee$ is a finitely generated $\widetilde{\mathcal{R}}$-module (cf. [Gr2, §4]).

We recall the following conjecture which we find in the paper [Gr2] (under slightly different assumptions):

Conjecture 1.4. *Let $\mathcal{T}$ be a lattice of an irreducible continuous $\mathcal{K}$-linear representation $\mathcal{V}$ of $G_{\mathbb{Q}}$. We denote by $d^\pm$ the $\mathcal{K}$-rank of the $\pm$-eigenspace of $\mathcal{V}$ under the action of the complex conjugation in $G_{\mathbb{Q}}$. Assume further that the following conditions are satisfied:*

1. *The action of $G_{\mathbb{Q}}$ on $\mathcal{T}$ is unramified outside a finite set of primes $\Sigma \supset \{p, \infty\}$.*
2. *$\mathcal{V}$ has a D_p-stable $\mathcal{K}$-subspace $\mathrm{F}^+\mathcal{V} \subset \mathcal{V}$ with $\mathrm{rank}_{\mathcal{K}} \mathrm{F}^+\mathcal{V} = d^+$ such that $\mathcal{V}/\mathrm{F}^+\mathcal{V}$ is free of rank d^-. (Note that $\mathrm{F}^+\mathcal{V}$ induces the filtration $\mathrm{F}^+\widetilde{\mathcal{T}} := \widetilde{\mathcal{T}} \cap \mathrm{F}^+\mathcal{V}$ in $\widetilde{\mathcal{T}}$ and the filtration $\mathrm{F}^+\widetilde{\mathcal{A}} := \mathrm{F}^+\widetilde{\mathcal{T}} \otimes_{\widetilde{\mathcal{R}}} (\widetilde{\mathcal{R}})^\vee$ in $\widetilde{\mathcal{A}}$).*
3. *There exists a dense subset $\mathcal{S} \subset \mathrm{Hom}_{\mathbb{Z}_p}(\widetilde{\mathcal{R}}, \overline{\mathbb{Q}}_p)$ such that every $\phi \in \mathcal{S}$ satisfies the following conditions:*
   (a) *For each $\phi \in \mathcal{S}$, there exists a pure motive M_ϕ over $\mathbb{Q}$ critical in the sense of* [De2] *such that $(\widetilde{\mathcal{T}})_\phi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to the p-adic étale realization $V_\phi = H_{ét}(M_\phi \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$.*
   (b) *The D_p-equivariant isomorphism $H_{\mathrm{dR}}(M_\phi) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} V_\phi \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ of p-adic Hodge theory induces an isomorphism $\mathrm{Fil}^0(H_{\mathrm{dR}}(M_\phi)) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{F}^+V_\phi \otimes_{\mathbb{Q}_p}$*

B_{dR}*, where* $H_{\mathrm{dR}}(M_\phi)$ *is the de Rham realization of* M_ϕ *and* $\{\mathrm{Fil}^i H_{\mathrm{dR}}(M_\phi)\}$ *is the de Rham filtration.*

(c) *The zero locus* $\mathcal{N} = \{\phi \in \mathcal{S} \mid L(M_\phi, 0) = 0\}$ *is contained in a certain Zariski closed subset of* $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \overline{\mathbb{Q}}_p)$.

Then, we conjecture that $(\mathrm{Sel}_{\widetilde{\mathcal{T}}})^\vee$ *is a finitely generated torsion* $\widetilde{\mathcal{R}}$*-module.*

For a noetherian integral domain R which is integrally closed in the fraction field, the localization $R_{\mathfrak{p}}$ is a discrete valuation ring for every height-one prime $\mathfrak{p}$ of R. Hence, for a finitely generated torsion R-module M, we have an invariant $\mathrm{length}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. We see that the invariant $\mathrm{length}_{\widetilde{\mathcal{R}}_{\mathfrak{p}}}((\mathrm{Sel}_{\widetilde{\mathcal{T}}})^\vee)_{\mathfrak{p}}$ makes sense if Conjecture 1.4 holds for $\mathcal{T}$. On the other hand, an isomorphism class of lattices $\mathcal{T}$ in $\mathcal{V}$ is not unique when the residual representation is reducible. Thus, it is important to compare the difference between $\mathrm{length}_{\widetilde{\mathcal{R}}_{\mathfrak{p}}}((\mathrm{Sel}_{\widetilde{\mathcal{T}}})^\vee)_{\mathfrak{p}}$ and $\mathrm{length}_{\widetilde{\mathcal{R}}_{\mathfrak{p}}}((\mathrm{Sel}_{\widetilde{\mathcal{T}'}})^\vee)_{\mathfrak{p}}$ when we take two lattices $\mathcal{T}$ and $\mathcal{T}'$. After preparing necessary notations, we will state our main result (Theorem 1.6) which calculate this difference.

Definition 1.5. Let us introduce the following conditions on a lattice $\mathcal{T}$:

($\mathbf{F}_{\mathbb{Q}}$) The coinvariant quotient $(\mathcal{T}^*)_{G_{\mathbb{Q}}}$ is a pseudo-null $\mathcal{R}$-module.

($\mathbf{F}_{\boldsymbol{p}}$) I_p acts non-trivially on every elements in $\mathrm{F}^-\mathcal{T}$ and D_p acts non-trivially on every elements in $\mathrm{F}^+\mathcal{T}(-1)$ and $\mathrm{F}^-\mathcal{T}(-1)$. The Pontrjagin dual of $(F^-\mathcal{A})^{D_p}$ is a pseudo-null $\mathcal{R}$-module.

(T) $\mathrm{Sel}_{\widetilde{\mathcal{T}}}$ is a cotorsion $\widetilde{\mathcal{R}}$-module.

(T*) $\mathrm{Sel}_{\widetilde{\mathcal{T}}^*(1)}$ is a cotorsion $\mathcal{R}$-module where $\widetilde{\mathcal{T}}^*(1) = \mathrm{Hom}_{\widetilde{\mathcal{R}}}(\widetilde{\mathcal{T}}, \widetilde{\mathcal{R}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$.

Further, we assume the following condition for each prime $v \in \Sigma \setminus \{\infty, p\}$:

($\mathbf{F}_{\boldsymbol{v}}$) D_v acts non-trivially on every elements in $\mathcal{T}(-1)$ and the modules $\mathcal{A}^{D_v}$ and $(\mathcal{A}^{I_v})_{D_v}$ are pseudo-null $\mathcal{R}$-modules.

Our main result is as follows:

Theorem 1.6. *Suppose that the ring* $\mathcal{R}$ *is isomorphic to* $\mathcal{O}[[X_1, \cdots, X_n]]$ *with the integer ring of integers* $\mathcal{O}$ *of a finite extension of* $\mathbb{Q}_p$ *and a certain natural number* n. *Suppose that* $\mathcal{T}$ *and* $\mathcal{T}'$ *are free lattices of a continuous representation* $\mathcal{V}$ *of* $G_{\mathbb{Q}}$ *over the fraction field* $\mathcal{K}$ *of* $\mathcal{R}$ *unramified outside a finite set of primes* $\Sigma \supset \{p, \infty\}$. *Let us assume the conditions* **($\mathbf{F}_{\mathbb{Q}}$), ($\mathbf{F}_{\boldsymbol{p}}$), (T), (T*)** *and* **($\mathbf{F}_{\boldsymbol{v}}$)** *for both of them.*

Then, we have

$$\begin{aligned}\mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}})^\vee)_{\mathfrak{p}} - \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}'})^\vee)_{\mathfrak{p}} \\ = \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathcal{T}/\mathcal{T}')_{G_{\mathbb{R}}})_{\mathfrak{p}} - \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{F}^+\mathcal{T}/\mathrm{F}^+\mathcal{T}'))_{\mathfrak{p}}\end{aligned}$$

for every height-one prime $\mathfrak{p}$ *of* $\mathcal{R}$ *(Note that* $\mathcal{R} = \widetilde{\mathcal{R}}$ *by assumption).*

As we will see in the following remark, the assumptions of this theorem are satisfied by a wide class of deformations. Hence, it ensures that we will have a plenty of applications of the theorem.

Remark 1.7. 1. It is not difficult to show that the conditions **(T)** and **(T*)** are *isogeny invariant* (That is, if the property holds for one lattice $\mathcal{T} \subset \mathcal{V}$, it holds for all lattices in $\mathcal{V}$). The conditions $(\mathbf{F}_{\mathbb{Q}}), (\mathbf{F}_{\boldsymbol{p}})$ and $(\mathbf{F}_{\boldsymbol{v}})$ also might be isogeny invariant under fairly general situation. But, we do not pursue the question of isogeny invariance of these conditions and we assume them for both of $\mathcal{T}$ and $\mathcal{T}'$.

2. Let $T \cong \mathbb{Z}_p^{\oplus d}$ be a p-adic Galois representation of $G_{\mathbb{Q}}$ unramified outside a finite set of primes $\Sigma \supset \{p, \infty\}$. We consider the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ with $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We denote by $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ a rank-one $\mathbb{Z}_p[[\Gamma]]$-module on which $G_{\mathbb{Q}}$ acts via the character $\widetilde{\chi}: \ G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p[[\Gamma]]^\times$ obtained by tautological injection $G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]^\times$. Then, we introduce the free $\mathbb{Z}_p[[\Gamma]]$-module $T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ on which $G_{\mathbb{Q}}$ acts diagonally and we call it *the cyclotomic deformation of* T. The Selmer group $\mathrm{Sel}_{\mathcal{T}}$ for $\mathcal{T} = T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ defined as in the definition (1) coincides with the cyclotomic Selmer group $\mathrm{Sel}_A(\mathbb{Q}_\infty) \subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$ given in [Gr1] where $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. When T is ordinary at p and $\mathrm{Sel}_A(\mathbb{Q}_\infty)^\vee$ is a torsion $\mathbb{Z}_p[[\Gamma]]$-module, Theorem 1.6 was obtained by Perrin-Riou in the paper [P].
3. We verify the conditions of the theorem in the following cases.
   (a) Let $T_p(E)$ be the p-Tate module of an elliptic curve E over $\mathbb{Q}$ with ordinary reduction at p and let $\mathcal{T} := T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. Note that a finitely generated $\mathbb{Z}_p[[\Gamma]]$-module is pseudo-null if and only if it is finite. In this situation, the conditions $(\mathbf{T})$ and $(\mathbf{T^*})$ always hold by results of Rubin [R1] (CM case) and Kato [Ka] (non CM case). Recall that the group $E(\mathbb{Q}_\infty)_{p\text{-tors}} := \underset{m \geq 1}{\cup} E(\mathbb{Q}_\infty)[p^m]$ is always finite as studied by Imai [I]. On the other hand, $(\mathcal{T}^*)_{G_{\mathbb{Q}}}$ is the Pontrjagin dual of $E(\mathbb{Q}_\infty)_{p\text{-tors}}$ by Shapiro's lemma. Hence, $\mathcal{T}$ always satisfies the condition $(\mathbf{F}_{\mathbb{Q}})$. Let $v \in \Sigma \setminus \{p, \infty\}$. We see easily that D_v acts non-trivially on every elements of $\mathcal{T}(-1) = T_p(E)(-1) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. The group $\mathcal{A}^{D_v}$ is isomorphic to a finite number of copies of $E(\mathbb{Q}_{v,\infty})_{p\text{-tors}} = \underset{m \geq 1}{\cup} E(\mathbb{Q}_{v,\infty})[p^m]$ where $\mathbb{Q}_v$ is the completion of $\mathbb{Q}$ at v and $\mathbb{Q}_{v,\infty} = \mathbb{Q}_\infty \mathbb{Q}_v$. Since it is easy to see that $E(\mathbb{Q}_{v,\infty})_{p\text{-tors}}$ is a finite group, the Pontrjagin dual of $\mathcal{A}^{D_v}$ is a pseudo-null $\mathcal{R}$-module. This implies immediately that the Pontrjagin dual of $(\mathcal{A}^{I_v})_{D_v}$ is a pseudo-null $\mathcal{R}$-module. Thus, $(\mathbf{F}_{\boldsymbol{v}})$ is satisfied for each $v \in \Sigma \setminus \{p, \infty\}$. Let us discuss about the condition $(\mathbf{F}_{\boldsymbol{p}})$. Since I_p acts non-trivially on any Tate-twist of $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$, I_p acts on every elements in $\mathrm{F}^-\mathcal{T}(r) = \mathrm{F}^- T_p(E)(r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ for any integer r. The group $(\mathrm{F}^-\mathcal{A})^{D_p}$ is isomorphic to $(\underset{m \geq 1}{\cup} \mathrm{F}^- E[p^m])^{G_{\mathbb{Q}_{p,\infty}}}$ where $\mathbb{Q}_{p,\infty} = \mathbb{Q}_\infty \mathbb{Q}_p$. This group is a finite group except when E has split-multiplicative reduction at p since the action of Frobenius element is non-trivial. Hence, the condition $(\mathbf{F}_{\boldsymbol{p}})$ is satisfied except when E has split-multiplicative reduction at p. More generally, let T be a lattice of the p-adic representation V associated to an ordinary eigen cuspform f of weight $k \geq 2$ in the sense of Deligne [De] and let $\mathcal{T} := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. The conditions $(\mathbf{T})$ and $(\mathbf{T^*})$ hold by the above mentioned results by Rubin and Kato. The conditions $(\mathbf{F}_{\mathbb{Q}}), (\mathbf{F}_{\boldsymbol{p}})$ and $(\mathbf{F}_{\boldsymbol{v}})$ for every $v \in \Sigma \setminus \{p\}$ hold in fairy general situations.

   Thus, the assumptions of Theorem 1.6 are checked to be true in these cases and we give another proof of results by Schneider [Sc] (for abelian varieties) and Perrin-Riou [P] (for ordinary p-adic representations).

(b) Suppose that $\mathcal{T}$ is a free lattice of the two-variable nearly ordinary Hida deformation $\mathcal{V} \cong \mathcal{K}^{\oplus 2}$ associated to a Λ-adic ordinary eigen cuspform $\mathcal{F}$. Here, $\mathcal{K}$ is isomorphic to a finite extension of the fraction field of $\mathbb{Z}_p[[X,Y]]$. Suppose that the ring of integers $\mathcal{R}$ of $\mathcal{K}$ is isomorphic to $\mathcal{O}[[X_1, X_2]]$ with the integer ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$ and let us take a height-one prime I of $\mathcal{R}$ such that $\mathcal{R}/I$ is also a regular local ring. Then, a finitely generated $\mathcal{R}$-module M is a torsion (resp. pseudo-null) $\mathcal{R}$-module if M/IM is a torsion (resp. pseudo-null) $\mathcal{R}/I$-module. The conditions **(T)** and **(T*)** are proved by the control theorem at a height-one prime I_f specializing to the cyclotomic deformation of a cusp form f in the Hida family $\mathcal{F}$ and the above mentioned results by Kato and Rubin for the cyclotomic deformation of f (The proof is the same as [O3, Proposition 4.9], where we prove it for $\mathcal{T}$ with irreducible residual representation). Similarly, $(\mathbf{F}_{\mathbb{Q}}), (\mathbf{F}_{\boldsymbol{p}})$ and $(\mathbf{F}_{\boldsymbol{v}})$ are proved by specializing method. For $(\mathbf{F}_{\boldsymbol{p}})$, if we have I_f corresponding to a cusp form f of weight two associated to an elliptic curve with split-multiplicative reduction at p, $(\mathbf{F}_{\boldsymbol{p}})$ is not true over $\mathcal{R}/I_f$ as we saw above. The specialization method does not work for this I_f. However, if we choose I_f so that the weight of f is greater than two, $(\mathbf{F}_{\boldsymbol{p}})$ is always true over $\mathcal{R}/I_f$. In this way, $(\mathbf{F}_{\mathbb{Q}}), (\mathbf{F}_{\boldsymbol{p}}), (\mathbf{T}), (\mathbf{T}^*)$ and $(\mathbf{F}_{\boldsymbol{v}})$ are always true for two-variable nearly ordinary deformations.

By applying Theorem 1.6, we obtain the difference of the algebraic p-adic L-function for Hida deformations in Corollary 4.4 after preparing notations in §4. Note that the method in [P] is not applicable anymore to such more general deformations. We also remark that previous results by Schneider [Sc] and Perrin-Riou [P] for the cyclotomic deformations of elliptic curves can be recovered from our result in a completely different manner.

We would like to study the Iwasawa Main conjecture for Hida deformations (equality between the algebraic p-adic L-function and the analytic p-adic L-function). When the residual representation for $\mathcal{F}$ is irreducible, we had much progress (We refer to [O3, §2] for a review of those as well as notations on Hida deformations which we will use in this paper). Thus, we are interested in the case where the reducible representation for $\mathcal{F}$ is reducible. In spite of the difficulty of ambiguity of the choice of lattice, it seems to be possible to say something around the main conjecture. First important ingredient for this is that we are able to calculate the difference of the algebraic p-adic L-functions for different choices of lattices by Corollary 4.4. Secondly, with help of [St] in which we find a conjectural answer on the minimal choice of lattice of the p-adic representation for elliptic modular forms of weight two, it seems reasonable to a canonical lattice $\mathcal{T}_{\mathcal{F}}$ associated to $\mathcal{F}$ constructed in §4 gives the minimal one in the sense that the algebraic p-adic L-function for $\mathcal{T}_{\mathcal{F}}$ is the minimal one among those for all lattices associated to $\mathcal{F}$. By combining these two points, we understand the Iwasawa Theory for $\mathcal{F}$ with reducible residual representation better than before.

For the proof of the main result (Theorem 1.6), the key tool is the Euler-Poincaré characteristic formula (Theorem 2.1) for a family of Galois representation proved in §2. The use of this generalized Euler-Poincaré characteristic formula is the most important difference from the proof of [P], which makes us generalize the result in [P]. We also remark that the of proof of Theorem 2.1 is based on the specialization principle developed in [O2] which allows us to recover the characteristic ideal of a given $\mathcal{R}$-module M from the information on various specializations of M. Though the proof of Theorem 2.1 occupies the whole of §2, the statement of the theorem is very natural and is understandable at a first glance.

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2. Euler-Poincaré characteristic with large coefficients

In this section, we prove the Euler-Poincaré characteristic formula for Galois cohomologies with coefficients in a power series algebra $\mathcal{R}$ generalizing the classical one with coefficients in the ring of integers of a p-adic field (cf. Theorem 2.1). The Euler-Poincaré characteristic formula for a family of Galois representation is studied a little bit in [Gr1, Proposition 3]. However, the formula in [Gr1, Proposition 3] concerns only the $\mathcal{R}$-rank of Galois cohomologies for only when $\mathcal{R} = \mathbb{Z}_p[[\Gamma]]$. In this section, we rather study the characteristic ideal of $\mathcal{R}$-cotorsion Galois cohomologies and we treat more general $\mathcal{R}$'s. The key tool for the proof of this section is "the specialization principle" established in the previous paper [O2]. The main result of this section (Theorem 2.1) will play an important role in the next section for the proof of Theorem 1.6.

Throughout the paper, we will denote the Galois group $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$ by G_Σ for short. For a finite abelian group A, we denote by $\sharp A$ the order of A. Our main result in this section is as follows:

Theorem 2.1. *Suppose that the ring $\mathcal{R}$ is isomorphic to $\mathcal{O}[[X_1, \cdots, X_n]]$ with the integer ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$ and with a certain natural number n. Let $\mathcal{C}$ be a discrete $\mathcal{R}$-module such that $\mathcal{C}^\vee$ is a finitely generated torsion $\mathcal{R}$-module with G_Σ-action. Then, we have the following formula for every height-one prime $\mathfrak{p}$ of $\mathcal{R}$:*

1. *We have the following Global Euler-Poincaré characteristic formula*:
$$\sum_{0 \le i \le 2} (-1)^i \mathrm{length}_{\mathcal{R}_\mathfrak{p}} (H^i(G_\Sigma, \mathcal{C})^\vee)_\mathfrak{p} = \mathrm{length}_{\mathcal{R}_\mathfrak{p}} (\mathcal{C}^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}} ((\mathcal{C}^{G_\mathbb{R}})^\vee)_\mathfrak{p}.$$
2. *We have the following local Euler-Poincaré characteristic formula*:
$$\begin{cases} \displaystyle\sum_{0 \le i \le 2} (-1)^i \mathrm{length}_{\mathcal{R}_\mathfrak{p}} (H^i(D_v, \mathcal{C})^\vee)_\mathfrak{p} = 0 & \text{for } v \neq p, \\ \displaystyle\sum_{0 \le i \le 2} (-1)^i \mathrm{length}_{\mathcal{R}_\mathfrak{p}} (H^i(D_p, \mathcal{C})^\vee)_\mathfrak{p} = \mathrm{length}_{\mathcal{R}_\mathfrak{p}} (\mathcal{C}^\vee)_\mathfrak{p} & \text{for } v = p. \end{cases}$$

To prove the above theorem, we recall necessary results from the paper [O2]. From now on throughout the section, we fix a power series $\mathcal{O}[[X_1, \cdots, X_n]]$ and denote it by $\Lambda^{(n)}_\mathcal{O}$ for short. We recall the following definition (cf. §3 of [O2]):

Definition 2.2. Let $n \geq 1$ be an integer.

1. *A linear element* l in an n-variable Iwasawa algebra $\Lambda_{\mathcal{O}}^{(n)} = \mathcal{O}[[X_1, \cdots, X_n]]$ is a polynomial $l = a_0 + a_1X_1 + \cdots a_nX_n \in \Lambda_{\mathcal{O}}^{(n)}$ with $a_i \in \mathcal{O}$ of degree at most one such that l is not divisible by a uniformizer π of $\mathcal{O}$ and is not invertible in $\Lambda_{\mathcal{O}}^{(n)}$. That is, l is a polynomial of degree at most one such that a_0 is divisible by π, but not all a_i are divisible by π.
2. We denote by $\mathcal{L}_{\mathcal{O}}^{(n)}$ the set of all linear ideals of $\Lambda_{\mathcal{O}}^{(n)}$. That is:
$$\mathcal{L}_{\mathcal{O}}^{(n)} = \left\{ (l) \subset \Lambda_{\mathcal{O}}^{(n)} \;\middle|\; l \text{ is a linear element in } \Lambda_{\mathcal{O}}^{(n)} \right\}.$$
3. Let $n \geq 2$. For a torsion $\Lambda_{\mathcal{O}}^{(n)}$-module M, we denote by $\mathcal{L}_{\mathcal{O}}^{(n)}(M)$ a subset of $\mathcal{L}_{\mathcal{O}}^{(n)}$ which consists of $(l) \subset \mathcal{L}_{\mathcal{O}}^{(n)}$ satisfying the following conditions:
   (a) The quotient $M/(l)M$ is a torsion $\Lambda_{\mathcal{O}}^{(n)}/(l)$-module.
   (b) The image of the characteristic ideal $\mathrm{char}_{\Lambda_{\mathcal{O}}^{(n)}}(M) \subset \Lambda_{\mathcal{O}}^{(n)}$ in $\Lambda_{\mathcal{O}}^{(n)}/(l)$ is equal to the characteristic ideal $\mathrm{char}_{\Lambda_{\mathcal{O}}^{(n)}/(l)}(M/(l)M) \subset \Lambda_{\mathcal{O}}^{(n)}/(l)$.

We will use the following theorem which was proved in [O2]:

Theorem 2.3 (Proposition 3.6 and Proposition 3.11 in [O2])**.** *Let M and N be finitely generated torsion $\mathcal{O}[[X]]$-modules. We have the following:*

(1) *The following conditions are equivalent:*
   (a) *There exists an integer $h \geq 0$ such that* $\mathrm{char}_{\Lambda_{\mathcal{O}}}(M) \supset (\pi^h)\mathrm{char}_{\Lambda_{\mathcal{O}}}(N)$.
   (b) *Let $\mathcal{O}'$ be arbitrary complete discrete valuation ring which is finite flat over $\mathcal{O}$. Then there exists a constant c depending only on $M_{\mathcal{O}'}$ and $N_{\mathcal{O}'}$ such that $\sharp(M_{\mathcal{O}'}/(l)M_{\mathcal{O}'})$ divides $c \cdot \sharp(N_{\mathcal{O}'}/(l)N_{\mathcal{O}'})$ for all but finitely many $(l) \in \mathcal{L}_{\mathcal{O}'}^{(1)}$.*

(2) *As for the difference by the power of π, we have the following equivalence:*
   (a) *Let $M_{(\pi)}$ (resp. $N_{(\pi)}$) be the localization of M (resp. N) at the prime ideal (π). Then we have* $\mathrm{length}_{(\Lambda_{\mathcal{O}})_{(\pi)}}(M_{(\pi)}) \leq \mathrm{length}_{(\Lambda_{\mathcal{O}})_{(\pi)}}(N_{(\pi)})$.
   (b) *There exists a set of principal ideals*
$$\{(E_m) \mid E_m \textit{ is an Eisenstein polynomial of degree } m\}_{m \in \mathbb{Z}_{\geq 1}}$$
   and a constant c depending only on M and N such that $\sharp(M/(E_m)M)$ divides $c \cdot \sharp(N/(E_m)N)$ for all but finitely many $m \in \mathbb{Z}_{\geq 1}$.

Let $n \geq 2$ be an integer and let M and N be a finitely generated torsion $\Lambda_{\mathcal{O}}^{(n)}$-modules. Then the following three statements are equivalent.

1. *We have* $\mathrm{char}_{\Lambda_{\mathcal{O}}^{(n)}}(M) \supset \mathrm{char}_{\Lambda_{\mathcal{O}}^{(n)}}(N)$.
2. *There exists a complete discrete valuation ring $\mathcal{O}'$ which is finite flat over $\mathcal{O}$ such that we have the inclusion*
$$\mathrm{char}_{\Lambda_{\mathcal{O}'}^{(n)}/(l)}(M_{\mathcal{O}'}/(l)M_{\mathcal{O}'}) \supset \mathrm{char}_{\Lambda_{\mathcal{O}'}^{(n)}/(l)}(N_{\mathcal{O}'}/(l)N_{\mathcal{O}'})$$
   for all but finitely many $(l) \in \mathcal{L}_{\mathcal{O}'}^{(n)}(M_{\mathcal{O}'}) \cap \mathcal{L}_{\mathcal{O}'}^{(n)}(N_{\mathcal{O}'})$.

Remark 2.4. 1. Not only the above theorem, but also a lot of basic properties on linear elements and linear specializations are developed in [O2, §3]. We refer to [O2, §3] for some properties possibly used implicitly later in this paper.

2. Concerning known other results on the technique of specialization, the referee informed the author that, for the case of modules of rank two, there are a similar version of Theorem 2.3 (1) proved by Plater and an equivalent version of Lemma 2.5 (1) by Nekovar and Plater (cf. [NP]).

Under the above preparation, we return to the proof of Theorem 2.1. For a finitely generated $\Lambda^{(n)}_{\mathcal{O}}$-module M, we denote by M_{null} the largest pseudo-null $\Lambda^{(n)}_{\mathcal{O}}$-submodule of M.

Lemma 2.5. *Let $\mathcal{C}$ be a discrete $\mathcal{R}$-module with continuous G_Σ-action such that $\mathcal{C}^\vee$ is a finitely generated torsion $\mathcal{R}$-module. Suppose that $\mathcal{R}$ is isomorphic to $\Lambda^{(n)}_{\mathcal{O}}$. For each complete discrete valuation ring $\mathcal{O}'$ finite flat over $\mathcal{O}$, the following statements follows:*

1. *Suppose that $n=1$ ($\mathcal{R}$ is isomorphic to $\mathcal{O}[[X]]$). The numerator and the denominator of the ratio $\dfrac{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f])}{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[f]}$ is bounded when $f \in \mathcal{O}'[[X]]$ runs over elements prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$. Similarly, for each $v \in \Sigma \setminus \{\infty\}$, the ratio $\dfrac{\sharp H^i(D_v, \mathcal{C}_{\mathcal{O}'}[f])}{\sharp H^i(D_v, \mathcal{C}_{\mathcal{O}'})[f]}$ is bounded when $f \in \mathcal{O}'[[X]]$ varies as above.*
2. *Suppose that $n \geq 2$. Then, there exist pseudo-null $\mathcal{R}$-modules $Y^i_\Sigma(\mathcal{O}')$ and $Z^i_\Sigma(\mathcal{O}')$ such that the Pontrjagin duals of the kernel and the cokernel of $H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[l]) \longrightarrow H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[l]$ are pseudo-null $\Lambda^{(n)}_{\mathcal{O}'}/(l)$-modules for every $l \in \mathcal{L}^{(n)}_{\mathcal{O}'}((\mathcal{C}^\vee_{\mathcal{O}'})_{\mathrm{null}} \oplus Y^i_\Sigma(\mathcal{O}') \oplus Z^i_\Sigma(\mathcal{O}'))$. Similarly, for each $v \in \Sigma \setminus \{\infty\}$, there exist pseudo-null $\mathcal{R}$-modules $Y^i_v(\mathcal{O}')$ and $Z^i_v(\mathcal{O}')$ such that the Pontrjagin duals of the kernel and the cokernel of $H^i(D_v, \mathcal{C}_{\mathcal{O}'}[l]) \longrightarrow H^i(D_v, \mathcal{C}_{\mathcal{O}'})[l]$ are pseudo-null $\Lambda^{(n)}_{\mathcal{O}'}/(l)$-modules for every $l \in \mathcal{L}^{(n)}_{\mathcal{O}'}((\mathcal{C}^\vee_{\mathcal{O}'})_{\mathrm{null}} \oplus Y^i_v(\mathcal{O}') \oplus Z^i_v(\mathcal{O}'))$.*

Proof. For both of the case $n=1$ and the case $n \geq 2$, we prove our statement only for the cohomology of the semi-global Galois group G_Σ. The case for the cohomology of D_v is done in the same way and will be omitted.

First, let us discuss the first assertion (the case with $n=1$). Let $\mathcal{C}^0$ be the $\mathcal{O}[[X]]$-submodule of $\mathcal{C}$ such that $(\mathcal{C}/\mathcal{C}^0)^\vee$ is the largest finite submodule of $\mathcal{C}^\vee$. Fix a complete discrete valuation ring $\mathcal{O}'$ which is finite flat over $\mathcal{O}$. Note that $\mathcal{C}^0_{\mathcal{O}'} := \mathcal{C}^0 \otimes_{\mathcal{O}} \mathcal{O}'$ is the $\mathcal{O}'[[X]]$-submodule of $\mathcal{C}_{\mathcal{O}'} := \mathcal{C} \otimes_{\mathcal{O}} \mathcal{O}'$ such that $(\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee$ is the largest finite submodule of $\mathcal{C}^\vee_{\mathcal{O}'}$. Since $H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})$ is a finite group for each i, we put:

$$a_i(\mathcal{O}') := \sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}).$$

For each $f \in \mathcal{O}'[[X]]$ prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$, we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}) & \longrightarrow & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}) & \longrightarrow & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}) & \longrightarrow & \cdots \\
 & & {\scriptstyle \times f}\downarrow & & \downarrow{\scriptstyle \times f} & & \downarrow{\scriptstyle \times f} & & \\
\cdots & \longrightarrow & H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}) & \longrightarrow & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}) & \longrightarrow & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}) & \longrightarrow & \cdots
\end{array}$$

By decomposing the two long exact sequences of the diagram into short exact sequences and by applying the snake lemma, we obtain the following inequality for each $i \geq 0$:

$$1 \leq \frac{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[f]}{\sharp H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})[f]} \leq a_i(\mathcal{O}'). \tag{2}$$

Since $(\mathcal{C}^0_{\mathcal{O}'})^\vee$ has no finite $\mathcal{O}'[[X]]$-submodule and f is prime to the characteristic ideal of $(\mathcal{C}^0_{\mathcal{O}'})^\vee$, we have:

$$0 \longrightarrow \mathcal{C}^0_{\mathcal{O}'}[f] \longrightarrow \mathcal{C}^0_{\mathcal{O}'} \xrightarrow{\times f} \mathcal{C}^0_{\mathcal{O}'} \longrightarrow 0.$$

We also put

$$b_i(\mathcal{O}') := \sharp\text{the largest finite } \mathcal{O}'[[X]]\text{-module of } H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})^\vee.$$

This gives us the following inequality for each $i \geq 0$:

$$1 \leq \frac{\sharp H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})[f]}{\sharp H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}[f])} \leq b_{i-1}(\mathcal{O}'), \tag{3}$$

where $b_{-1}(\mathcal{O}')$ is defined to be 1. We have the following short exact sequence for each $f \in \mathcal{O}'[[X]]$ prime to the characteristic ideal of $\mathcal{C}^\vee$:

$$0 \longrightarrow \mathcal{C}^0_{\mathcal{O}'}[f] \longrightarrow \mathcal{C}_{\mathcal{O}'}[f] \longrightarrow (\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})[f] \longrightarrow 0.$$

Since the number of $\mathcal{O}'$-submodules of $\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}$ is finite, we note that $\sharp H^i(G_\Sigma, (\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})[f])$ is bounded when f varies. We define:

$$c_i(\mathcal{O}') := \text{the maximum of } \sharp H^i(G_\Sigma, (\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})[f]) \text{ when } f \text{ varies.}$$

We have the following inequality for each $i \geq 0$:

$$\frac{c_{i-1}(\mathcal{O}')}{c_i(\mathcal{O}')} \leq \frac{\sharp H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}[f])}{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f])} \leq c_{i-1}(\mathcal{O}'), \tag{4}$$

where $c_{-1}(\mathcal{O}')$ is defined to be 1. By adding three inequalities $(2), (3)$ and (4), we obtain the following inequality :

$$\frac{c_{i-1}(\mathcal{O}')}{c_i(\mathcal{O}')} \leq \frac{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[f]}{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f])} \leq a_i(\mathcal{O}')b_{i-1}(\mathcal{O}')c_{i-1}(\mathcal{O}'). \tag{5}$$

Hence $\dfrac{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[f]}{\sharp H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f])}$ is bounded when $f \in \mathcal{O}'[[X]]$ runs elements prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$. This proves the first assertion of the lemma.

Next, we consider the case $n \geq 2$. Let $\mathcal{C}^0$ be a $\Lambda^{(n)}_{\mathcal{O}}$-submodule of $\mathcal{C}$ such that $(\mathcal{C}/\mathcal{C}^0)^\vee \cong (\mathcal{C}^\vee)_{\mathrm{null}}$. We consider the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}) \arrow[r] \arrow[d, "\times l"'] & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}) \arrow[r] \arrow[d, "\times l"] & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}) \arrow[r] \arrow[d, "\times l"] & \cdots \\
\cdots \arrow[r] & H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}) \arrow[r] & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}) \arrow[r] & H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'}) \arrow[r] & \cdots
\end{tikzcd}$$

For finitely generated torsion $\mathcal{R}$-modules M and N, having $\mathcal{R}$-linear map $M \longrightarrow N$ whose kernel and cokernel are pseudo-null $\mathcal{R}$-modules is an equivalence relation called pseudo-isomorphism. We write $M \sim N$ when M and N are pseudo-isomorphic. By decomposing the two long exact sequences of the above diagram into short exact sequences and by applying the snake lemma, we obtain the following statement for each $i \geq 0$:

(6) $$(H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[l])^\vee \sim (H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})[l])^\vee \qquad \text{if } l \in \mathcal{L}^{(n)}_{\mathcal{O}'}(Y^i_\Sigma(\mathcal{O}')),$$

where

$$Y^i_\Sigma(\mathcal{O}') = \begin{cases} H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee \oplus H^{i-1}(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee & \text{if } i \geq 1, \\ H^0(G_\Sigma, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee & \text{if } i = 0, \end{cases}$$

For a linear element $l \in \Lambda^{(n)}_{\mathcal{O}'}$ prime to the characteristic ideal of $\mathcal{C}^\vee$, we have the short exact sequence $0 \longrightarrow \mathcal{C}^0_{\mathcal{O}'}[l] \longrightarrow \mathcal{C}^0_{\mathcal{O}'} \xrightarrow{\times l} \mathcal{C}^0_{\mathcal{O}'} \longrightarrow 0$. This implies that

(7) $$H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}[l])^\vee \sim (H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})[l])^\vee \qquad \text{if } l \in \mathcal{L}^{(n)}_{\mathcal{O}'}(Z^i_\Sigma(\mathcal{O}')),$$

where

$$Z^i_\Sigma(\mathcal{O}') = \begin{cases} H^{i-1}(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'})^\vee)_{\mathrm{null}} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

Finally, the sequence $0 \longrightarrow \mathcal{C}^0_{\mathcal{O}'}[l] \longrightarrow \mathcal{C}_{\mathcal{O}'}[l] \longrightarrow (\mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})[l] \longrightarrow 0$ implies that we have the sequence for each $i \geq 0$:

(8) $$H^i(G_\Sigma, \mathcal{C}^0_{\mathcal{O}'}[l])^\vee \sim H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[l])^\vee \qquad \text{if } l \in \mathcal{L}^{(n)}_{\mathcal{O}'}((\mathcal{C}^\vee_{\mathcal{O}'})_{\mathrm{null}})$$

By combining (6), (7) and (8), we see that $(H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'})[l])^\vee$ and $H^i(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[l])^\vee$ is pseudo-isomorphic to each other for every $l \in \mathcal{L}^{(n)}_{\mathcal{O}'}((\mathcal{C}^\vee_{\mathcal{O}'})_{\mathrm{null}} \oplus Y^i_\Sigma(\mathcal{O}') \oplus Z^i_\Sigma(\mathcal{O}'))$. This completes the proof of the second assertion for the cohomologies of G_Σ. For the cohomologies of D_v, the proof is done exactly in the same way by putting :

$$Y^i_v(\mathcal{O}') = \begin{cases} H^i(D_v, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee \oplus H^{i-1}(D_v, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee & \text{if } i \geq 1, \\ H^0(D_v, \mathcal{C}_{\mathcal{O}'}/\mathcal{C}^0_{\mathcal{O}'})^\vee & \text{if } i = 0, \end{cases}$$

and

$$Z^i_v(\mathcal{O}') = \begin{cases} H^{i-1}(D_v, \mathcal{C}^0_{\mathcal{O}'})^\vee)_{\mathrm{null}} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

□

Let us return to the proof of Theorem 2.1.

Proof of Theorem 2.1. As in the proof of Lemma 2.5, we prove our statement only for the cohomology of the semi-global Galois group G_Σ. The case for the cohomology of D_v is done in the same way and will be omitted.

We apply Theorem 2.3 to the following modules:

$$M = H^0(G_\Sigma, \mathcal{C})^\vee \oplus H^2(G_\Sigma, \mathcal{C})^\vee \oplus (\mathcal{C}^{G_\mathbb{R}})^\vee,$$
$$N = H^1(G_\Sigma, \mathcal{C})^\vee \oplus \mathcal{C}^\vee$$

We will show that M is pseudo-isomorphic to N as a $\Lambda^{(n)}_{\mathcal{O}}$-module by induction with respect to n. Let us assume that $n = 1$ at first. For a complete discrete valuation ring $\mathcal{O}'$ finite flat over $\mathcal{O}$, $M_{\mathcal{O}'}$ (resp. $N_{\mathcal{O}'}$) is isomorphic to $H^0(G_\Sigma, \mathcal{C}_{\mathcal{O}'})^\vee \oplus H^2(G_\Sigma, \mathcal{C}_{\mathcal{O}'})^\vee \oplus ((\mathcal{C}_{\mathcal{O}})^{G_\mathbb{R}})^\vee$ (resp. $H^1(G_\Sigma, \mathcal{C}_{\mathcal{O}'})^\vee \oplus (\mathcal{C}_{\mathcal{O}'})^\vee$). Let f be any element in $\mathcal{O}'[[X]]$. The groups

$(M_{\mathcal{O}'}/(f)M_{\mathcal{O}'})$ and $(N_{\mathcal{O}'}/(f)N_{\mathcal{O}'})$ are finite when f is prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$. Lemma 2.5 immediately implies the following claim:

Claim 2.6. Let us introduce the following rational numbers:

$$r_f := \sharp(M_{\mathcal{O}'}/(f)M_{\mathcal{O}'})/\sharp(N_{\mathcal{O}'}/(f)N_{\mathcal{O}'}),$$
$$s_f := \sharp\left(H^0(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f]) \oplus H^2(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f]) \oplus (\mathcal{C}_{\mathcal{O}'}^{G_\mathbb{R}}[f])\right) \Big/ \sharp\left(H^1(G_\Sigma, \mathcal{C}_{\mathcal{O}'}[f]) \oplus \mathcal{C}_{\mathcal{O}'}[f]\right).$$

Then the numerator and the denominator of r_f/s_f is bounded when f runs elements prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$.

By the Global Euler-Poincaré characteristic formula for finite G_Σ-modules (cf. [NSW, Chap. VIII]), we have

$$\prod_{0\le i\le 2} \sharp H^i(G_\Sigma, A)^{(-1)^i} = \frac{\sharp A}{\sharp A^{G_\mathbb{R}}}. \tag{9}$$

This shows that $s_f = 1$ for every $f \in \mathcal{O}'[[X]]$ prime to the characteristic ideal of $(\mathcal{C}_{\mathcal{O}'})^\vee$. Thus, $r_f = \sharp(M_{\mathcal{O}'}/(f)M_{\mathcal{O}'})/\sharp(N_{\mathcal{O}'}/(f)N_{\mathcal{O}'})$ is bounded when f varies in $\mathcal{O}'[[X]]$. By the first half of Theorem 2.3, we conclude that M and N are pseudo-isomorphic to each other.

Next, let $n \ge 2$ and assume that Theorem 2.1 is proved for $n-1$. We define $M'_{\mathcal{O}'}$ and $N_{\mathcal{O}'}$ to be

$$M'_{\mathcal{O}'} = M_{\mathcal{O}'} \oplus Y^0_\Sigma(\mathcal{O}') \oplus Z^0_\Sigma(\mathcal{O}') \oplus Y^2_\Sigma(\mathcal{O}') \oplus Z^2_\Sigma(\mathcal{O}'),$$
$$N'_{\mathcal{O}'} = N_{\mathcal{O}'} \oplus Y^1_\Sigma(\mathcal{O}') \oplus Z^1_\Sigma(\mathcal{O}').$$

By Lemma 2.5 and inductive assumption, $M'_{\mathcal{O}'}/(l)M'_{\mathcal{O}'}$ is pseudo-isomorphic to $N'_{\mathcal{O}'}/(l)N'_{\mathcal{O}'}$ as $\Lambda^{(n)}_{\mathcal{O}'}/(l)$-module for every $l \in \mathcal{L}^{(n)}_{\mathcal{O}'}(M'_{\mathcal{O}'}) \cap \mathcal{L}^{(n)}_{\mathcal{O}'}(N'_{\mathcal{O}'})$. Hence, the last half of Theorem 2.3 implies that $M'_{\mathcal{O}}$ is pseudo-isomorphic to $N'_{\mathcal{O}}$ as $\Lambda^{(n)}_{\mathcal{O}}$-module. Since $M'_{\mathcal{O}}$ (resp. $N'_{\mathcal{O}}$) is pseudo-isomorphic to M (resp. N) over $\Lambda^{(n)}_{\mathcal{O}}$ by definition, this completes the proof for the cohomologies of G_Σ. For the cohomologies of D_v, we apply the local Euler-Poincaré characteristic formula for finite D_v-modules (cf. [NSW, Chap. VII]) in place of the Global Euler-Poincaré characteristic formula when $n = 1$. Since the proof for $n = 1$ and $n \ge 2$ is done in the same way, we omit it. $\square$

3. Proof of Theorem 1.6

In this section, we will complete the proof of Theorem 1.6 by using the results in §2. Throughout the section, we suppose that $\mathcal{R}$ is isomorphic to $\mathcal{O}[[X_1, \cdots, X_n]]$ with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$ and n a natural number. We fix an injection between two free lattices $\mathcal{T} \longrightarrow \mathcal{T}'$ and we denote by $\mathcal{C}$ the kernel of the induced surjective map $\mathcal{A} \longrightarrow \mathcal{A}'$ where $\mathcal{A}$ (resp. $\mathcal{A}'$) is $\mathcal{T} \otimes_\mathcal{R} \mathcal{R}^\vee$ (resp. $\mathcal{T}' \otimes_\mathcal{R} \mathcal{R}^\vee$). First, we prepare the following notation:

Definition 3.1. Let (M, F^+) be a discrete G_Σ-module with D_p-stable $\mathcal{R}$-submodule $\mathrm{F}^+M \subset M$.

1. Loc^i_M is defined to be $\mathrm{Loc}^i_M = H^i(D_p, \mathrm{F}^- M) \oplus \bigoplus_{v \in \Sigma \setminus \{\infty, p\}} H^i(D_v, M)$ (Though Loc^i_M depends on the choice of $\mathrm{F}^+ M$, we leave the notation as it is if there causes no confusion).
2. We denote by loc^i_M the natural localization map:

$$H^i(G_\Sigma, M) \longrightarrow \bigoplus_{v \in \Sigma \setminus \{\infty\}} H^i(D_v, M) \longrightarrow \mathrm{Loc}^i_M.$$

Our strategy is to consider the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i(G_\Sigma, \mathcal{C}) & \longrightarrow & H^i(G_\Sigma, \mathcal{A}) & \longrightarrow & H^i(G_\Sigma, \mathcal{A}') & \longrightarrow & \cdots \\ & & \Big\downarrow{\scriptstyle \mathrm{loc}^i_{\mathcal{C}}} & & \Big\downarrow{\scriptstyle \mathrm{loc}^i_{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \mathrm{loc}^i_{\mathcal{A}'}} & & \\ \cdots & \longrightarrow & \mathrm{Loc}^i_{\mathcal{C}} & \longrightarrow & \mathrm{Loc}^i_{\mathcal{A}} & \longrightarrow & \mathrm{Loc}^i_{\mathcal{A}'} & \longrightarrow & \cdots . \end{array} \tag{10}$$

We show that the kernels or cokernels of some of vertical homomorphisms in the above diagram vanish and this will give us a relation between the orders of the kernels and cokernels of the vertical homomorphisms. In the first half of this section, we will show that $\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}})$ (resp. $\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}'})$) is very “close to” $\mathrm{Sel}_{\mathcal{T}}$ (resp. $\mathrm{Sel}_{\mathcal{T}'}$). Then, in the latter half of the section, we calculate the difference between $\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}})$ and $\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}'})$ by the diagram (10) and by the Euler characteristic formula established in §2.

The goal of the first half of the section is the proposition as follows:

Proposition 3.2. *Let us assume the conditions* $(\boldsymbol{T^*})$, $(\mathbf{F}_{\boldsymbol{p}})$ *and* $(\mathbf{F}_{\boldsymbol{v}})$ *for each* $v \in \Sigma \setminus \{\infty, p\}$*. We have the following equality:*

$$\mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}})^\vee)_\mathfrak{p} = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{Sel}_{\mathcal{T}})^\vee)_\mathfrak{p}$$

for each height one prime $\mathfrak{p}$ *of* $\mathcal{R}$ *and* $\mathrm{Coker}(\mathrm{loc}^1_{\mathcal{A}})^\vee$ *is pseudo-null. The same result also holds for* $\mathcal{A}'$*.*

The proofs for $\mathcal{A}$ and $\mathcal{A}'$ are exactly the same. Hence we only treat the case of $\mathcal{A}$. Let us also recall the following notation:

$$H^1_{\mathrm{ur}}(D_v, M) = \mathrm{Ker}\left[H^1(D_v, M) \longrightarrow H^1(I_v, M)\right]$$

for any finite prime v of $\mathbb{Q}$ and for any D_v-module M. Before proving Proposition 3.2, we recall the following lemma:

Lemma 3.3. *Under the assumption* $(\boldsymbol{T^*})$*, the following localization map is surjective*:

$$H^1(G_\Sigma, \mathcal{A}) \longrightarrow \frac{H^1(D_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(D_p, \mathcal{A})} \oplus \bigoplus_{v \in \Sigma \setminus \{\infty, p\}} \frac{H^1(D_v, \mathcal{A})}{H^1_{\mathrm{ur}}(D_v, \mathcal{A})}, \tag{11}$$

where $H^1_{\mathrm{Gr}}(D_p, M) = \mathrm{Ker}\left[H^1(D_p, \mathcal{A}) \longrightarrow H^1(I_p, \mathrm{F}^- \mathcal{A})\right]$.

We omit the proof of the lemma, since it is done in the same way as [O3, Corollary 4.12] where the same statement is proved for two-variable Hida deformations.

Proof of Proposition 3.2. As we remarked earlier, it suffices to prove only in the case of $\mathcal{A}$. We start from the proof of the first assertion. By the definition of $\mathrm{loc}^1_{\mathcal{A}}$ (cf. Definition 3.1), we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^1(G_\Sigma,\mathcal{A}) & \longrightarrow & H^1(G_\Sigma,\mathcal{A}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{loc}^1_{\mathcal{A}} & & \downarrow \alpha & & \\
0 & \longrightarrow & H^1_{\mathrm{ur}}(D_p,\mathrm{F}^-\mathcal{A}) \oplus \bigoplus_{v\in\Sigma\setminus\{\infty,p\}} H^1_{\mathrm{ur}}(D_v,\mathcal{A}) & \longrightarrow & \mathrm{Loc}^1_{\mathcal{A}} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

where $X = \dfrac{H^1(D_p,\mathcal{A})}{H^1_{\mathrm{Gr}}(D_p,\mathcal{A})} \oplus \bigoplus_{v\in\Sigma\setminus\{\infty,p\}} \dfrac{H^1(D_v,\mathcal{A})}{H^1_{\mathrm{ur}}(D_v,\mathcal{A})}$. Note that $H^1_{\mathrm{ur}}(D_p,\mathrm{F}^-\mathcal{A}) = (\mathcal{A}^{I_v})_{D_v}$ and $H^1_{\mathrm{ur}}(D_v,\mathcal{A}) = (\mathcal{A}^{I_v})_{D_v}$ for each $v \in \Sigma\setminus\{\infty,p\}$ are pseudo-null by the assumptions $(\mathbf{F}_{\boldsymbol{p}})$ and $(\mathbf{F}_{\boldsymbol{v}})$. Since the module $\mathrm{Ker}(\alpha)$ is $\mathrm{Sel}_{\mathcal{T}}$ by definition, we complete the proof of the first assertion by the snake lemma. On the other hand, we have $\mathrm{Coker}(\alpha) = 0$ by Lemma 3.3. Hence we deduce that $\mathrm{Coker}(\mathrm{loc}^1_{\mathcal{A}})^\vee$ is a pseudo-null $\mathcal{R}$-module. $\square$

By a similar argument using the Poitou-Tate sequence, we obtain the following proposition:

Lemma 3.4. *Let us assume the conditions* $(\boldsymbol{T^*})$, $(\mathbf{F}_{\boldsymbol{p}})$ *and* $(\mathbf{F}_{\boldsymbol{v}})$ *for each* $v \in \Sigma\setminus\{\infty,p\}$ *on* $\mathcal{T}$. *Then,* $H^2(G_\Sigma,\mathcal{A})$ *and* $\mathrm{Loc}^2_{\mathcal{A}}$ *are trivial.*

Proof. We have $H^2(D_p,\mathrm{F}^-\mathcal{A})^\vee \cong ((\mathrm{F}^-\mathcal{T}(-1))^*)^{D_p}$ and $H^2(D_v,\mathcal{A})^\vee \cong (\mathcal{T}(-1)^*)^{D_v}$ by the local Tate duality. Hence $(\mathrm{Loc}^2_{\mathcal{A}})^\vee$ is trivial by the assumptions $(\mathbf{F}_{\boldsymbol{p}})$ and $(\mathbf{F}_{\boldsymbol{v}})$ for each $v \in \Sigma\setminus\{\infty,p\}$. This proves the second assertion.

Recall that $Ш^i_\Sigma(\mathcal{A})$ is defined by the following exact sequence for each i:

$$0 \longrightarrow Ш^i_\Sigma(\mathcal{A}) \longrightarrow H^i(G_\Sigma,\mathcal{A}) \longrightarrow \bigoplus_{v\in\Sigma\setminus\{\infty\}} H^i(D_v,\mathcal{A}). \tag{12}$$

For $i=2$, $H^2(D_p,\mathcal{A})$ is the Pontrjagin dual of $(\mathcal{T}^*(1))^{D_p}$ by the local Tate duality. Since $(\mathcal{T}^*(1))^{D_p}$ is zero by the assumption $(\mathbf{F}_{\boldsymbol{p}})$, the left terms in (12) is trivial. The group $Ш^2(\mathcal{A})$ is the Pontrjagin dual of $\varprojlim_n Ш^1(\mathcal{A}^*(1)[\mathfrak{M}^n])$ by the global duality theorem, where $\mathfrak{M}$ is the maximal ideal of $\mathcal{R}$. On the other hand, $\varprojlim_n Ш^1(\mathcal{A}^*(1)[\mathfrak{M}^n])$ is isomorphic to $\mathrm{Hom}_{\mathcal{R}}(Ш^1(\mathcal{A}^*(1))^\vee,\mathcal{R})$ (see [O1, Lemma 5.4] and its proof for the general result on such isomorphism exchanging the inverse limit, the Pontrjagin dual and the linear dual). Since $Ш^1(\mathcal{A}^*(1))^\vee$ is a torsion $\mathcal{R}$-module by the $(\boldsymbol{T^*})$, $Ш^2(\mathcal{A})$ is trivial. Since the first term and the third term of the equation (12) for $i=2$ is zero, this proves that $H^2(G_\Sigma,\mathcal{A}) = 0$. $\square$

Lemma 3.5. *Assume the conditions* $(\boldsymbol{T^*})$, $(\mathbf{F}_{\mathbb{Q}})$, $(\mathbf{F}_{\boldsymbol{p}})$ *and* $(\mathbf{F}_{\boldsymbol{v}})$ *for each* $v \in \Sigma\setminus\{\infty,p\}$. *We have the following equality for every height-one prime* $\mathfrak{p}$ *of* $\mathcal{R}$*:*

$$\begin{aligned}
&\mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}})^\vee)_{\mathfrak{p}} - \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}'})^\vee)_{\mathfrak{p}} \\
&\qquad = \sum_{1\le i\le 2}(-1)^i \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}(H^i(G_\Sigma,\mathcal{C})^\vee)_{\mathfrak{p}} - \sum_{1\le i\le 2}(-1)^i \mathrm{length}_{\mathcal{R}_{\mathfrak{p}}}((\mathrm{Loc}^i_{\mathcal{C}})^\vee)_{\mathfrak{p}}
\end{aligned}$$

Proof. The $\mathcal{R}$-modules $(\mathrm{Loc}^2_{\mathcal{A}})^\vee$ and $H^2(G_\Sigma,\mathcal{A})^\vee$ are trivial by Lemma 3.4. The $\mathcal{R}$-module $H^0(G_\Sigma,\mathcal{A})^\vee$ (resp. $(\mathrm{Loc}^0_{\mathcal{A}})^\vee$) is a pseudo-null $\mathcal{R}$-module by $(\mathbf{F}_{\mathbb{Q}})$ (resp. $(\mathbf{F}_{\boldsymbol{p}})$

and $(\mathbf{F}_v)$ for each $v \in \Sigma \setminus \{\infty, p\}$). Thus, we have the following commutative diagram for each height-one prime $\mathfrak{p}$ of $\mathcal{R}$:

(13)

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & (H^1(\mathcal{C})^\vee)_\mathfrak{p} & \longleftarrow & (H^1(\mathcal{A})^\vee)_\mathfrak{p} & \longleftarrow & (H^1(\mathcal{A}')^\vee)_\mathfrak{p} & \longleftarrow & (H^2(\mathcal{C})^\vee))_\mathfrak{p} & \longleftarrow & 0 \\ & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\ 0 & \longleftarrow & ((\mathrm{Loc}^1_\mathcal{C})^\vee)_\mathfrak{p} & \longleftarrow & ((\mathrm{Loc}^1_\mathcal{A})^\vee)_\mathfrak{p} & \longleftarrow & ((\mathrm{Loc}^1_{\mathcal{A}'})^\vee)_\mathfrak{p} & \longleftarrow & ((\mathrm{Loc}^2_\mathcal{C})^\vee)_\mathfrak{p} & \longleftarrow & 0, \end{array}$$

where $H^i(M)$ means $H^i(G_\Sigma, M)$ and the vertical maps are the Pontrjagin duals of loc^i. This implies the following equation:

$$\begin{aligned} (14)\quad & \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^1_\mathcal{C})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^1_\mathcal{A})^\vee)_\mathfrak{p} \\ & \qquad + \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^1_{\mathcal{A}'})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^2_\mathcal{C})^\vee)_\mathfrak{p} \\ & = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Coker}(\mathrm{loc}^1_\mathcal{C})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Coker}(\mathrm{loc}^1_\mathcal{A})^\vee)_\mathfrak{p} \\ & \qquad + \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Coker}(\mathrm{loc}^1_{\mathcal{A}'})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Coker}(\mathrm{loc}^2_\mathcal{C})^\vee)_\mathfrak{p} \end{aligned}$$

On the other hand, we easily prove the following equality:

$$\begin{aligned} (15)\quad & \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Ker}(\mathrm{loc}^i_\mathcal{C})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{Coker}(\mathrm{loc}^i_\mathcal{C})^\vee)_\mathfrak{p} \\ & \qquad = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(H^i(G_\Sigma, \mathcal{C})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{Loc}^i_\mathcal{C})^\vee)_\mathfrak{p} \end{aligned}$$

Thus, we complete the proof of the lemma by combining (14), (15) and Proposition 3.2. □

Let us return to the proof of our main theorem.

Proof of Theorem 1.6. Let us note that

(16)

$$\mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathcal{C})^\vee = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathcal{T}/\mathcal{T}') \ \ (\text{resp. } \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{F}^+\mathcal{C})^\vee = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathrm{F}^+\mathcal{T}/\mathrm{F}^+\mathcal{T}')).$$

On the other hand, we have the following equality by our Euler-Poincare characteristic formula (Theorem 2.1) and Lemma 3.5 :

$$\begin{aligned} & \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{Sel}_\mathcal{T})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{Sel}_{\mathcal{T}'})^\vee)_\mathfrak{p} \\ & = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{F}^-\mathcal{C})^\vee)_\mathfrak{p} - \Big(\mathrm{length}_{\mathcal{R}_\mathfrak{p}}(\mathcal{C}^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathcal{C}^{G_\mathbb{R}})^\vee)_\mathfrak{p}\Big) \\ & = \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathcal{C}^{G_\mathbb{R}})^\vee)_\mathfrak{p} - \mathrm{length}_{\mathcal{R}_\mathfrak{p}}((\mathrm{F}^+\mathcal{C})^\vee)_\mathfrak{p}. \end{aligned}$$

This completes the proof. □

4. Two-variable Iwasawa Main conjecture for Hida deformations

In this section, we will study the Iwasawa Main Conjecture for two-variable nearly ordinary deformations associated to Hida's ordinary Λ-adic forms. For general facts on Hida theory, we refer to papers [H1], [Wi] and a book [H2] by Hida, but we refer also to our previous papers [O2] and [O3] for the notations. Let $\mathbb{T}_N = \varprojlim_{r \geq 1} H^1_{\text{ét}}(X_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}}$, where $X_1(Np^r)_{\overline{\mathbb{Q}}}$ is a projective modular curve over $\overline{\mathbb{Q}}$ with level structure $\Gamma_1(Np^r)$ and $(\quad)^{\mathrm{ord}}$ means taking the unit-root subspace with respect to U_p-operator. Recall that

$\mathbb{T}_N$ is free of finite rank over $\Lambda = \mathbb{Z}_p[[\Gamma']]$ where $\Gamma' \cong 1 + p\mathbb{Z}_p$ is the group of diamond operators acting on the tower of modular curves $\{X_1(Np^r)\}_{r \geq 1}$. We denote by $\boldsymbol{h}_N^{\mathrm{ord}}$ Hida's ordinary Hecke algebra, which is a subalgebra of $\mathrm{End}_\Lambda(S_N^{\mathrm{ord}})$ with S_N^{ord} the space of whole ordinary Λ-adic cusp forms of tame conductor N. The algebra $\boldsymbol{h}_N^{\mathrm{ord}}$ naturally acts on $\mathbb{T}_N$. Let $\mathcal{F}$ be a Λ-adic eigen newform of tame conductor N and let $\mathbb{I}_{\mathcal{F}}$ be the subalgebra of the algebraic closure of $\mathrm{Frac}(\Lambda)$ generated by all Fourier coefficients of $\mathcal{F}$. We define the canonical ordinary Galois representation $\mathbb{T}_{\mathcal{F}}$ associated to $\mathcal{F}$ as follows:

$$\mathbb{T}_{\mathcal{F}} = \mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}}(\mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}}(\mathbb{T}_N[\lambda_{\mathcal{F}}], \mathbb{I}_{\mathcal{F}}), \mathbb{I}_{\mathcal{F}}),$$

where $\mathbb{T}_N[\lambda_{\mathcal{F}}]$ is defined to be the $\mathbb{I}_{\mathcal{F}}$-submodule $(\mathbb{T}_N \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{I}_{\mathcal{F}})[\lambda_{\mathcal{F}}]$ cut out by the condition that l-Hecke operator in $\boldsymbol{h}_N^{\mathrm{ord}}$ acts by l-th Fourier coefficient of $\mathcal{F}$ for almost all l. We have the following properties:

1. The module $\mathbb{T}_{\mathcal{F}}$ is a reflexive $\mathbb{I}_{\mathcal{F}}$-module since $\mathbb{T}_{\mathcal{F}}$ is the double $\mathbb{I}_{\mathcal{F}}$-linear dual of the $\mathbb{I}_{\mathcal{F}}$-module $\mathbb{T}_N \otimes_{\boldsymbol{h}_N^{\mathrm{ord}}} \mathbb{I}_{\mathcal{F}}$.
2. The module $\mathbb{V}_{\mathcal{F}} := \mathbb{T}_{\mathcal{F}} \otimes_{\mathbb{I}_{\mathcal{F}}} \mathbb{K}_{\mathcal{F}}$ is a $\mathbb{K}_{\mathcal{F}}$-vector space of rank two, where $\mathbb{K}_{\mathcal{F}} = \mathrm{Frac}(\mathbb{I}_{\mathcal{F}})$.
3. The module $\mathbb{T}_{\mathcal{F}}$ is equipped with continuous $G_{\mathbb{Q}}$-action.
4. $\mathbb{T}_{\mathcal{F}}$ is also equipped with a natural $G_{\mathbb{Q}}$-equivariant $\mathbb{I}_{\mathcal{F}}$-linear homomorphism $\mathbb{T}_N \otimes_{\boldsymbol{h}_N^{\mathrm{ord}}} \mathbb{I}_{\mathcal{F}} \longrightarrow \mathbb{T}_{\mathcal{F}}$.

We define the canonical nearly ordinary Galois representation $\mathcal{T}_{\mathcal{F}}$ associated to $\mathcal{F}$ to be $\mathcal{T}_{\mathcal{F}} = \mathbb{T}_{\mathcal{F}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. The action of $G_{\mathbb{Q}}$ on $\mathcal{T}_{\mathcal{F}}$ is the natural diagonal one. Thus, we have a lattice of the nearly ordinary representation $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{K}_{\mathcal{F}})$ associated to $\mathcal{F}$, where $\mathcal{K}_{\mathcal{F}} = \mathrm{Frac}(\mathbb{I}_{\mathcal{F}}[[\Gamma]])$.

Let $\mathcal{V}_{\mathcal{F}}$ be the underlying $\mathcal{K}_{\mathcal{F}}$-vector space of $\rho_{\mathcal{F}}$. As is well-known, two lattices of $\mathcal{V}_{\mathcal{F}}$ is not unique modulo $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-isomorphism when the residual representation for $\mathcal{F}$ is reducible (cf. [MW] for residual representation). A lattice of $\mathcal{V}_{\mathcal{F}}$ is not necessarily free over $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$ and a free lattice of $\mathcal{V}_{\mathcal{F}}$ is not necessarily isomorphic to $\mathcal{T}_{\mathcal{F}}$. Since it is not difficult to see, we introduce the following facts on lattices on $\mathcal{V}_{\mathcal{F}}$ without any proof:

Lemma 4.1. 1. *For any lattice $\mathcal{T}$ over $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$, we define $\mathcal{T}^{**}$ to be the double $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-linear dual as follows*:

$$\mathcal{T}^{**} = \mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}[[\Gamma]]}(\mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}[[\Gamma]]}(\mathcal{T}, \mathbb{I}_{\mathcal{F}}[[\Gamma]]), \mathbb{I}_{\mathcal{F}}[[\Gamma]]).$$

Then the natural $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-linear homomorphism $\mathcal{T} \longrightarrow \mathcal{T}^{**}$ is injective. Further, if $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$ is integrally closed in $\mathcal{K}_{\mathcal{F}}$, the cokernel of $\mathcal{T} \longrightarrow \mathcal{T}^{**}$ is a pseudo-null $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-module.

2. *Let $\mathcal{T}$ be a lattice of $\mathcal{V}_{\mathcal{F}}$ defined to be the cyclotomic deformation $\mathcal{T} = \mathbb{T} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ where $\mathbb{T}$ be a lattice of $\mathbb{V}_{\mathcal{F}}$. Then, $\mathcal{T}^{**}$ defined above is isomorphic to $\mathbb{T}^{**} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ where $\mathbb{T}^{**}$ is the $\mathbb{I}_{\mathcal{F}}$-linear double dual $\mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}}(\mathrm{Hom}_{\mathbb{I}_{\mathcal{F}}}(\mathbb{T}, \mathbb{I}_{\mathcal{F}}), \mathbb{I}_{\mathcal{F}})$ of $\mathbb{T}$.*
3. *If $\mathbb{T}$ is a reflexive lattice over $\mathbb{I}_{\mathcal{F}}$ and if $\mathbb{I}_{\mathcal{F}}$ is a regular local ring, then $\mathbb{T}$ (resp. $\mathcal{T} = \mathbb{T} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$) is a free lattice over $\mathbb{I}_{\mathcal{F}}$ (resp. $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$). Especially, the lattice $\mathbb{T}_{\mathcal{F}}$ (resp. $\mathcal{T}_{\mathcal{F}}$) constructed above is free over $\mathbb{I}_{\mathcal{F}}$ (resp. $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$) when $\mathbb{I}_{\mathcal{F}}$ is a regular local ring.*

Now we introduce the algebraic p-adic L-function for a lattice $\mathcal{T}$ under certain assumption.

Definition 4.2. Let $\mathbb{T}$ be a lattice of $\mathbb{V}_{\mathcal{F}}$ and let $\mathcal{T} = \mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. We assume that $\mathbb{I}_{\mathcal{F}}$ is a regular local ring (hence $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$ is also a regular local ring). Recall that $\mathbb{V}_{\mathcal{F}}$ has a unique D_p-stable one-dimensional subspace $\mathbb{V}^+_{\mathcal{F}}$ such that the quotient $\mathbb{V}^-_{\mathcal{F}} = \mathbb{V}_{\mathcal{F}}/\mathbb{V}^+_{\mathcal{F}}$ is an unramified representation of D_p. This induces a D_p-stable $\mathbb{I}_{\mathcal{F}}$-submodule $\mathrm{F}^+\mathbb{T} = \mathbb{T} \cap \mathbb{V}^+_{\mathcal{F}}$ on $\mathbb{T}$ and hence induces D_p-stable $\mathbb{I}_{\mathcal{F}}$-submodule $\mathrm{F}^+\mathcal{A}$ of $\mathcal{A} = \mathcal{T} \otimes_{\mathbb{I}_{\mathcal{F}}[[\Gamma]]} (\mathbb{I}_{\mathcal{F}}[[\Gamma]])^\vee$. We define the Selmer group $\mathrm{Sel}_{\mathcal{T}} \subset H^1(\mathbb{Q}, \mathcal{A})$ as in Definition 1.2. It is known that $(\mathrm{Sel}_{\mathcal{T}})^\vee$ is a finitely generated torsion $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-module. We define $L^{\mathrm{alg}}_p(\mathcal{T}) \in \mathbb{I}_{\mathcal{F}}[[\Gamma]]$ to be a generator of the characteristic ideal of $(\mathrm{Sel}_{\mathcal{T}})^\vee$, which is defined modulo multiplication by a unit of $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$.

We introduce the following lemma without any proof:

Lemma 4.3. *Let $\mathbb{T}$ be a lattice of $\mathbb{V}_{\mathcal{F}}$ and let $\mathcal{T} = \mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$. We assume that $\mathbb{I}_{\mathcal{F}}$ is a regular local ring.*

1. *We have* $\mathrm{length}_{\mathbb{I}_{\mathcal{F}}[[\Gamma]]_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}})^\vee)_{\mathfrak{p}} = \mathrm{length}_{\mathbb{I}_{\mathcal{F}}[[\Gamma]]_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}^{**}})^\vee)_{\mathfrak{p}}$ *for every height-one prime $\mathfrak{p}$ of $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$.*
2. *We have $L^{\mathrm{alg}}_p(\mathcal{T}) = L^{\mathrm{alg}}_p(\mathcal{T}^{**})$ and $\mathcal{T}^{**}$ is a free lattice.*

This lemma tells us that it suffices to treat only free lattices. Now, we state a corollary of Theorem 1.6 in the setting of Hida deformation.

Corollary 4.4. *Let $\mathcal{F}$ be a Λ-adic newform with reducible residual representation. Assume that $\mathbb{I}_{\mathcal{F}}$ is isomorphic to $\mathcal{O}[[X]]$ for a discrete valuation ring $\mathcal{O}$ finite flat over $\mathbb{Z}_p$. Let $\mathcal{T}$ and $\mathcal{T}'$ free $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$-lattices isogenious to $\mathcal{T}_{\mathcal{F}}$. Then we have*

$$\begin{aligned}\mathrm{length}_{(\mathbb{I}_{\mathcal{F}}[[\Gamma]])_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}})^\vee)_{\mathfrak{p}} - \mathrm{length}_{(\mathbb{I}_{\mathcal{F}}[[\Gamma]])_{\mathfrak{p}}}((\mathrm{Sel}_{\mathcal{T}'})^\vee)_{\mathfrak{p}} \\ = \mathrm{length}_{(\mathbb{I}_{\mathcal{F}}[[\Gamma]])_{\mathfrak{p}}}((\mathcal{T}/\mathcal{T}')_{G_{\mathbb{R}}})_{\mathfrak{p}} - \mathrm{length}_{(\mathbb{I}_{\mathcal{F}}[[\Gamma]])_{\mathfrak{p}}}((\mathrm{F}^+\mathcal{T}/\mathrm{F}^+\mathcal{T}'))_{\mathfrak{p}}\end{aligned}$$

for every height-one prime $\mathfrak{p}$ of $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$.

We finish the paper by the following question.

Question 4.5.
1. How much lattices exist for a given $\mathcal{F}$ (for a given $\mathbb{V}_{\mathcal{F}}$)?
2. Can we find and calculate examples of certain ordinary Λ-adic eigen cuspform $\mathcal{F}$ with reducible residual representation where we can calculate the difference between $(L^{\mathrm{alg}}_p(\mathcal{T}))$ and $(L^{\mathrm{alg}}_p(\mathcal{T}'))$ when $\mathcal{T}$ and $\mathcal{T}'$ vary in the set of lattices over $\mathbb{I}_{\mathcal{F}}[[\Gamma]]$ associated to $\mathcal{F}$?

All such questions naturally arise as continuation of the work in this paper and are related to a work by Emerton, Pollack and Weston [EPW] on the variation of Iwasawa λ-invariants in Hida families (Note that our Question 4.5 2 is related to the variation of Iwasawa μ-invariants). In this paper, we do not give an explicit example for explicit $\mathcal{F}$'s. Further results on residually reducible Hida deformations will be discussed in a joint project [OP] with Robert Pollack.

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