THE COLEMAN MAP FOR HIDA FAMILIES OF GSp_4

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ABSTRACT. In this paper, we construct a Coleman map for ordinary Hida deformations associated to the symplectic group GSp_4, which interpolates the dual exponential maps when arithmetic specializations of this family vary. The main result (Theorem 3.2) will play an important role in our forthcoming work ([LO]) discussing a conjecture on the existence of an Euler system which would give rise via our map to a three variable p-adic L-function.

CONTENTS

1. INTRODUCTION

Let p be an odd prime number and let $\mathcal{T}$ be a p-adic family of Galois representations of the absolute Galois group $G_F = \mathrm{Gal}(\overline{F}/F)$ of a number field F. That is, $\mathcal{T}$ is a free module of finite rank over a complete local Noetherian ring $\mathcal{R}$ with finite residue field of characteristic p which has an $\mathcal{R}$-linear continuous action of G_F. As is indicated in [Gr] and [O4], we believe that we can develop the Iwasawa theory for such families of Galois representations $\mathcal{T}$ satisfying several reasonable conditions.

In the spirit of Iwasawa theory, our interest is in constructing and studying p-adic analytic L-functions $L_p^{\mathrm{an}}(\mathcal{T})$ for $\mathcal{T}$ and in proving the Iwasawa main conjecture which relates the principal ideal $(L_p^{\mathrm{an}}(\mathcal{T}))$ to the characteristic ideal of the Pontryagin dual of the Selmer group for $\mathcal{T}$.

Iwasawa Main Conjecture is proved by Mazur-Wiles [MW] for $\mathcal{T}$ which are the cyclotomic deformations of ideal class groups of abelian extensions of $\mathbb{Q}$. Iwasawa Main Conjecture for $\mathcal{T}$ which are obtained as the cyclotomic deformations of modular forms has progressed a lot by Kato [Ka] and Skinner-Urban. For these cyclotomic deformations, $\mathcal{R}$ is the cyclotomic Iwasawa algebra $\mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$ where $\mathcal{O}$ is the ring of integers of a finite extension of

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$\mathbb{Q}_p$ and $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The deformation $\mathcal{T}$ is defined to be $T \otimes_{\mathcal{O}} \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]](\widetilde{\chi})$ where T is a continuous representation of $G_{\mathbb{Q}}$ of finite rank over $\mathcal{O}$ and $\mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]](\widetilde{\chi})$ is a free $\mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$-module of rank one on which $G_{\mathbb{Q}}$ acts via $\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \hookrightarrow \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]^\times$.

If $\mathcal{T}$ is any deformation of rank equal to or greater than two over $\mathcal{R}$, then $\mathcal{T}$ is not necessarily obtained as the cyclotomic deformation of a continuous representation T of $G_{\mathbb{Q}}$ of finite rank over $\mathcal{O}$. However, among such general non-cyclotomic deformations $\mathcal{T}$, there are very few examples where we establish some positive results on the Iwasawa theory for $\mathcal{T}$. The first example of $\mathcal{T}$ where successful results on the Iwasawa theory are obtained is the case of Hida deformation for $\mathrm{GL}_{2/\mathbb{Q}}$ where $\mathcal{R}$ is a local component of Hida's nearly ordinary Hecke algebra and $\mathcal{T}$ is the universal Galois representation, which is a finitely generated and generically of rank two $\mathcal{R}$-module on which $G_{\mathbb{Q}}$-acts continuously. We do not recall the definition and the construction of $\mathcal{R}$ and $\mathcal{T}$. However, the basic properties are summarized as follows.

- (i) The ring $\mathcal{R}$ is a local domain of Krull dimension three with finite residue field of characteristic p which is finite flat over a power series algebra of two variables $\mathbb{Z}_p[[X, Y]]$.
- (ii) The module $\mathcal{T}$ is equipped with continuous $G_{\mathbb{Q}}$-action and $\mathcal{T}$ is a free module of rank two over $\mathcal{R}$ under some technical conditions.
- (iii) Recall that $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \overline{\mathbb{Q}}_p)$ is regarded as a finite cover of $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[X, Y]], \overline{\mathbb{Q}}_p)$ which is isomorphic to an open p-adic ball in $\overline{\mathbb{Z}}_p^{\oplus 2}$. There is a dense discrete subset S of $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \overline{\mathbb{Q}}_p)$ which consists of arithmetic specializations such that, for any $\lambda \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \overline{\mathbb{Q}}_p)$, the specialization $V_\lambda := \mathcal{T} \otimes_{\mathcal{R}} \overline{\mathbb{Q}}_p$ is a Tate twist of a Galois representation associated to a certain ordinary cuspform f_λ of some weight ≥ 2.

In this case, there are the ideal of algebraic p-adic L-function $(L_p^{\mathrm{alg}}(\mathcal{T})) \subset \mathcal{R}$ and the ideal of analytic p-adic L-function $(L_p^{\mathrm{an}}(\mathcal{T})) \subset \mathcal{R}$. The ideal $(L_p^{\mathrm{alg}}(\mathcal{T})) \subset \mathcal{R}$ is defined to be the characteristic ideal of the Pontryagin dual of the Selmer group for $\mathcal{T}$. The ideal $(L_p^{\mathrm{an}}(\mathcal{T}))$ is the principal ideal generated by the element whose specialization $\lambda(L_p^{\mathrm{an}}(\mathcal{T}))$ interpolates the special values $L(V_\lambda, 0)$ of the Hasse-Weil L-function $L(V_\lambda, s)$ modified with certain factors like complex periods, p-adic periods, Gauss sums, Euler factors etc at every arithmetic specialization $\lambda \in S$.

By the two-variable Iwasawa Main Conjecture, we expect the equality

$$(L_p^{\mathrm{alg}}(\mathcal{T})) = (L_p^{\mathrm{an}}(\mathcal{T})).$$

The result we obtained in [O1], [O2] and [O3] is the half of expected equality:

$$(L_p^{\mathrm{alg}}(\mathcal{T})) \supset (L_p^{\mathrm{an}}(\mathcal{T})) \tag{1}$$

under some technical conditions. Our long-term project for $\mathrm{GSp}_{4/\mathbb{Q}}$ is to pursue the analogue of this study for $\mathrm{GL}_{2/\mathbb{Q}}$. In order to sketch our study in the case of $\mathrm{GSp}_{4/\mathbb{Q}}$, we have to review briefly the main ingredients of the proof of (1) in the case of $\mathrm{GL}_{2/\mathbb{Q}}$. As we will see later, for each p-adic representation W of $G_{\mathbb{Q}_p}$, we have a subspace $H^1_f(\mathbb{Q}_p, W)$ of

$H^1(\mathbb{Q}_p, W)$ called the finite part and we also have an important map

$$\exp^* : \frac{H^1(\mathbb{Q}_p, W)}{H^1_f(\mathbb{Q}_p, W)} \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(W)$$

called the dual exponential map where D_{dR} is a functor defined by Fontaine. Let us denote by $\overline{\mathcal{T}}$ the Kummer dual $\mathrm{Hom}_{\mathcal{R}}(\mathcal{T}, \mathcal{R}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. We denote also by $\overline{V}_\lambda$ the Kummer dual $\mathrm{Hom}_{\overline{\mathbb{Q}}_p}(V_\lambda, \overline{\mathbb{Q}}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. Note that $\overline{V}_\lambda$ is nothing but the specialization of $\overline{\mathcal{T}}$ at λ.

(A) We show that the dual exponential maps for $\lambda \in S$ are interpolated. That is,
  (i) There is a finitely generated $\mathcal{R}$-module $H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ which interpolates $\frac{H^1(\mathbb{Q}_p, \overline{T}_\lambda)}{H^1_f(\mathbb{Q}_p, \overline{T}_\lambda)}$ for $\lambda \in S$.
  (ii) There is a free $\mathcal{R}$-module $\overline{\mathcal{D}}$ of rank one such that $\overline{\mathcal{D}} \otimes_{\mathcal{R}} \lambda(\mathcal{R}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is canonically isomorphic to $\mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}_\lambda)$ for every $\lambda \in S$.
  (iii) There is an $\mathcal{R}$-linear map $\overline{\Xi} : H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow \overline{\mathcal{D}}$ interpolating the dual exponential maps in the sense that, for every $\lambda \in S$, we have the following commutative diagram:

$$\begin{array}{ccc} H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}}) & \xrightarrow{\overline{\Xi}} & \overline{\mathcal{D}} \\ \downarrow & & \downarrow \\ \dfrac{H^1(\mathbb{Q}_p, \overline{V}_\lambda)}{H^1_f(\mathbb{Q}_p, \overline{V}_\lambda)} & \longrightarrow & \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}_\lambda), \end{array}$$

  where the bottom map is equal to the dual exponential map modified by a Euler-like factor and a Gamma factor at λ.

(B) We establish a generalized Euler system theory for such a deformation. That is, if $\mathcal{Z} \in H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ is not an $\mathcal{R}$-torsion element and if $\mathcal{Z}$ is a part of a certain norm compatible system satisfying well-known Euler system condition, the characteristic ideal of $H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})/\mathcal{Z}\mathcal{R}$ is contained in the characteristic ideal of the Pontryagin dual of the Selmer group for $\mathcal{T}$.

(C) There exists an element $\mathcal{Z}^{\mathrm{opt}} \in H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ which is extended to a part of Euler system. That is, there is an Euler system $\{\mathcal{Z}(r) \in H^1(\mathbb{Q}(\zeta_r), \overline{\mathcal{T}})\}_r$ for square-free natural numbers r such that the image of $\mathcal{Z}(1)$ via

$$H^1(\mathbb{Q}, \overline{\mathcal{T}}) \to H^1(\mathbb{Q}_p, \overline{\mathcal{T}}) \twoheadrightarrow H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$$

is equal to $\mathcal{Z}^{\mathrm{opt}}$. Further, for any specialization $\lambda : \mathcal{R} \longrightarrow \overline{\mathbb{Q}}_p$ such that $\overline{V}_\lambda$ is associated to a j-th Tate twist of a modular form f, the image of $\mathcal{Z}^{\mathrm{opt}}_\lambda \in H^1(\mathbb{Q}_p, \overline{V}_\lambda)$ via the dual exponential map:

$$\exp^* : \frac{H^1(\mathbb{Q}_p, \overline{V}_\lambda)}{H^1_f(\mathbb{Q}_p, \overline{V}_\lambda)} \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}_\lambda) \cong \overline{\mathbb{Q}}_p$$

is equal to the special value $L(f_\lambda, j)$ divided by an optimal complex period, where the last isomorphism is obtained by (the dual of) the modular form f_λ which is regarded as an element of $\mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}_\lambda)$. ([Ka], [O3])

The ingredient (A) will provide us an equality

$$\mathrm{char}_{\mathcal{R}}\left(H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})\Big/\mathcal{Z}^{\mathrm{opt}}\mathcal{R}\right)=(L_p^{\mathrm{an}}(\mathcal{T})). \tag{2}$$

The ingredient (B) will provide us an inequality

$$(L_p^{\mathrm{alg}}(\mathcal{T}))\supset\mathrm{char}_{\mathcal{R}}\left(H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})\Big/\mathcal{Z}^{\mathrm{opt}}\mathcal{R}\right). \tag{3}$$

We obtain the inequality (1) by combining these two equations.

In our project, we are interested in extending this work to Hida deformations associated to $\mathrm{GSp}_{4/\mathbb{Q}}$. Roughly speaking, in this case, $\mathcal{R}$ is of Krull dimension 4 (of three variable) and $\mathcal{T}$ is generically of rank 4 over $\mathcal{R}$. Among three main ingredients, we devote ourselves to the part (A) of the above plan. Since both the Krull dimension of the deformation ring $\mathcal{R}$ and the rank of the Galois representation $\mathcal{T}$ are larger than the case of $\mathrm{GL}_{2/\mathbb{Q}}$, the interpolation of the dual exponential maps is more difficult in the case of $\mathrm{GSp}_{4/\mathbb{Q}}$.

We do not discuss the ingredients (B) and (C). However, the ingredient (B) will be done by the same principle as [O2]. The ingredient (C) to construct an Euler system will be more difficult. The first named author has established partial results to construct elements related to L-value [Le]. However, it is still far from the best possible result we want. In any case, this work is the first step towards for our project and we plan to proceed on other ingredients (B) and (C) with help of our insight obtained through this work. In a forthcoming paper [LO], we will discuss a more detailed framework of the theory. The plan of the paper is as follows:

Plan: In Section 2, we recall basic facts on Hida families of GSp_4. We will fix the notations and list basic assumptions which we suppose in the article. In Section 3, we will state the main theorems of the article. In Section 4, we recall the translation of classical Coleman power series into the language of Galois cohomology (Prop. 4.2) and we give a variant of Prop. 4.2 (Prop. 4.3). In Section 5, we give some technical calculations related to comparisons between local conditions and treatments on inverse limits and duals. In Section 6, we combine all results obtained in earlier sections to prove the essential part of our main theorem.

Since the strategy for the proof of our Main Theorem is complicated and involves a lot of statements, we would like to visualize the relations between several key propositions and theorems below. First, our main results are Theorem 3.1 (freeness) and Theorem 3.2 (construction of the Coleman map). Theorem 3.1 is proved in Section 5. For Theorem 3.2 which is the most essential result, the steps for the proof are organized as follows:

Thm. 6.4

$\nearrow$ $\searrow$

Prop. 4.2 $\rightarrow$ Prop. 4.3 Thm. 6.1 $\rightarrow$ Thm. 3.2

$\searrow$ $\nearrow$

Thm. 6.10

Notation: throughout the article, we fix an odd prime number p. We will also fix the complex embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and the p-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ of the rationals $\mathbb{Q}$. By this, we will identify the decomposition group of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ with $G_{\mathbb{Q}_p}$. For a commutative ring R, we denote by $R^\times$ the group of invertible elements in R. When a character $\rho : G \longrightarrow R^\times$ of a group G is given, we denote by $R(\rho)$ the free R-module of rank one with the action of G via ρ.

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2. Review of Hida families of GSp(4)

Let us define $\mathrm{GSp}_{4/\mathbb{Z}}$ by:

$$\mathrm{GSp}_{4/\mathbb{Z}} = \{g \in \mathrm{GL}_{4/\mathbb{Z}} \,|\, \exists\, \nu(g) \in \mathbb{G}_m,\ {}^{\mathrm{t}}gJg = \nu(g)J\},$$

where $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$, with derived group $\mathrm{Sp}_{4/\mathbb{Z}} = \ker \nu$. Then $\mathrm{GSp}_{4/\mathbb{Z}}$ is a reductive group over $\mathbb{Z}$. We denote by T the maximal torus of $\mathrm{GSp}_{4/\mathbb{Z}}$ and by B the standard Borel. We have

$$\mathrm{T} \;=\; \left\{ \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2^{-1}\nu, \alpha_1^{-1}\nu) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2^{-1}\nu & 0 \\ 0 & 0 & 0 & \alpha_1^{-1}\nu \end{pmatrix},\ (\alpha_1, \alpha_2, \nu) \in \mathbb{G}_m^3 \right\}.$$

Hence we have a canonical isomorphism $\mathrm{T} \simeq (\mathbb{G}_m)_1 \times (\mathbb{G}_m)_2 \times (\mathbb{G}_m)_3$ where each $(\mathbb{G}_m)_i$ is isomorphic to $\mathbb{G}_m$ and where $(\mathbb{G}_m)_1 \times (\mathbb{G}_m)_2$ is the maximal torus of $\mathrm{Sp}_{4/\mathbb{Z}}$. The group of characters $X^*(\mathrm{T})$ is identified with the subgroup of $\mathbb{Z}^2 \oplus \mathbb{Z}$ consisting of triples (k, k', t) such that $k + k' \equiv t \bmod 2$ via

$$\lambda(k, k', t) : \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2^{-1}\nu, \alpha_1^{-1}\nu) \mapsto \alpha_1^k \alpha_2^{k'} \nu^{\frac{t-k-k'}{2}}.$$

Write $\rho_1 = \lambda(1, -1, 0)$ and $\rho_2 = \lambda(0, 2, 0)$. Then the roots of T in $\mathrm{GSp}_{4/\mathbb{Z}}$ are $R = \{\pm\rho_1, \pm\rho_2, \pm(\rho_1 + \rho_2), \pm(2\rho_1 + \rho_2)\}$ and the positive roots with respect to B are $R^+ = \{\rho_1, \rho_2, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}$. Then the dominant weights with respect to B are the $\lambda(k, k', t)$ such that $k \geq k' \geq 0$.

Let G_i be the p-Sylow subgroup of $(\mathbb{G}_m)_i(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$ for $i = 1, 2, 3$. Then $G_1 \times G_2 \times G_3$ is the p-Sylow subgroup of the maximal torus of $\mathrm{GSp}_4(\mathbb{Z}_p)$ such that $G_1 \times G_2$ is the p-Sylow subgroup of the maximal torus of $\mathrm{Sp}_4(\mathbb{Z}_p) \subset \mathrm{GSp}_4(\mathbb{Z}_p)$.

Note that each G_i has a canonical isomorphism $\chi_i : G_i \xrightarrow{\sim} 1+p\mathbb{Z}_p$ such that $g_1 \times g_2 \times g_3 \in G_1 \times G_2 \times G_3$ is identified with elements of the maximal subtorus of $\mathrm{GSp}_4(\mathbb{Z}_p)$ via

$$\begin{pmatrix} \chi_1(g_1) & & & \\ & \chi_2(g_2) & & \\ & & \chi_2(g_2)^{-1}\chi_3(g_3) & \\ & & & \chi_1(g_1)^{-1}\chi_3(g_3) \end{pmatrix}$$

Let $\Lambda^{\mathrm{ord}} = \mathbb{Z}_p[[G_1 \times G_2]]$ and let $\Lambda^{\mathrm{n.ord}} = \mathbb{Z}_p[[G_1 \times G_2 \times G_3]]$. As is well-known, Λ^{ord} (resp. $\Lambda^{\mathrm{n.ord}}$) is non-canonically isomorphic to $\mathbb{Z}_p[[X_1, X_2]]$ (resp. $\mathbb{Z}_p[[X_1, X_2, X_3]]$).

Definition 2.1. For each pair of integers (a_1, a_2) and for each pair of non-negative integers (l, m), we denote by $I^{(a_1,a_2)}_{l,m}$ the ideal of height two in Λ^{ord} generated by

$$\left\{ (g_1^{p^l} - \chi_1(g_1)^{a_1 p^l}, g_2^{p^m} - \chi_2(g_2)^{a_2 p^m}) \mid g_i \in G_i \ (i = 1, 2) \right\}.$$

Similarly, for each triple of integers (a_1, a_2, a_3)] and for each triple of non-negative integers (l, m, n), we denote by $I^{(a_1,a_2,a_3)}_{l,m,n}$ the ideal of height three in $\Lambda^{\mathrm{n.ord}}$ generated by

$$\left\{ (g_1^{p^l} - \chi_1(g_1)^{a_1 p^l}, g_2^{p^m} - \chi_2(g_2)^{a_2 p^m}, g_3^{p^n} - \chi_3(g_3)^{a_3 p^n}) \ |g_i \in G_i \ (i = 1, 2, 3) \right\}.$$

Definition 2.2.
(1) Let (a_1, a_2) be a pair of integers. $\kappa \in \mathrm{Hom}(\Lambda^{\mathrm{ord}}, \overline{\mathbb{Q}}_p)$ is called an arithmetic specialization of weight (a_1, a_2) on Λ^{ord} if $\mathrm{Ker}(\kappa)$ contains $I^{(a_1,a_2)}_{l,m}$ for some l, m. For an algebra R which is finite and torsion-free over Λ^{ord}, $\kappa \in \mathrm{Hom}(R, \overline{\mathbb{Q}}_p)$ is called an arithmetic specialization of weight (a_1, a_2) on R if $\kappa|_{\Lambda^{\mathrm{ord}}}$ is an arithmetic specialization of weight (a_1, a_2) on Λ^{ord}.
(2) Let (a_1, a_2, a_3) be a triple of integers. Similarly, $\lambda \in \mathrm{Hom}(\Lambda^{\mathrm{n.ord}}, \overline{\mathbb{Q}}_p)$ is called an arithmetic specialization of weight (a_1, a_2, a_3) on $\Lambda^{\mathrm{n.ord}}$ if $\mathrm{Ker}(\lambda)$ contains $I^{(a_1,a_2,a_3)}_{l,m,n}$ for some l, m, n. For an algebra R' which is finite and torsion-free over $\Lambda^{\mathrm{n.ord}}$, $\lambda \in \mathrm{Hom}(R', \overline{\mathbb{Q}}_p)$ is called an arithmetic specialization of weight (a_1, a_2, a_3) on $R^{\mathrm{n.ord}}$ if $\lambda|_{\Lambda^{\mathrm{n.ord}}}$ is an arithmetic specialization of weight (a_1, a_2, a_3) on $\Lambda^{\mathrm{n.ord}}$.

We call a character of $G_1 \times G_2$ (resp. $G_1 \times G_2 \times G_3$) which is equal to $\chi_1^{a_1} \times \chi_2^{a_2}$ (resp. $\chi_1^{a_1} \times \chi_2^{a_2} \times \chi_3^{a_3}$) modulo a finite character an arithmetic character of weight (a_1, a_2) (resp. (a_1, a_2, a_3)). In fact, there is a natural one-to-one correspondence between arithmetic characters of weight (a_1, a_2) (resp. (a_1, a_2, a_3)) and arithmetic specializations of weight (a_1, a_2) (resp. (a_1, a_2, a_3)) on Λ^{ord} (resp. $\Lambda^{\mathrm{n.ord}}$).

Up to now, we are discussing arbitrary algebras R and R' which are finite and torsion-free over Λ^{ord} and $\Lambda^{\mathrm{n.ord}}$ respectively. From now on, in place of general R and R', we take only R^{ord} and $R^{\mathrm{n.ord}}$ called "a branch" of the (nearly) ordinary Hecke algebra of GSp_4 of tame conductor N which will be defined below.

Let us fix a positive integer N which is prime to p. For each positive integer r, we denote by $U_1(Np^r) \subset \mathrm{Sp}_4(\widehat{\mathbb{Z}})$ be a congruence subgroup of level Np^r which is defined as follows:

$$U_1(Np^r) := \left\{ g \in \mathrm{Sp}_4(\widehat{\mathbb{Z}}) | g \bmod Np^r \in (B \cap \mathrm{Sp}_4)(\mathbb{Z}/Np^r\mathbb{Z}) \right\}.$$

For each r, we have a Siegel three fold S_{Np^r} over $\mathbb{C}$ whose underlying space has an identification:

$$S_{Np^r} = \mathrm{GSp}_4(\mathbb{Q})\backslash \mathfrak{H}^{(2)} \times \mathrm{GSp}_4(\mathbb{A}_f)/U_1(Np^r)$$

where $\mathfrak{H}^{(2)}$ is a disjoint union of upper and lower Siegel half planes of genus 2 and $\mathbb{A}_f$ is the ring of finite adèles of $\mathbb{Q}$. For each weight (a_1, a_2) satisfying the condition $a_1 > a_2 > 0$ and for any $\mathbb{Z}$ algebra A, we have a standard local system $\mathcal{L}_{a_1,a_2}(A)$ over S_{Np^r} which is locally free over A for which we refer to [TU, §1.2]. We define the interior cohomology $H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))$ at the middle degree to be the image of the natural map from the compact support cohomology $H^3_c(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))$ to the singular cohomology $H^3(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))$. For each prime number l, the double coset:

$$U_1(Np^r)\begin{pmatrix}1&&&\\&1&&\\&&l&\\&&&l\end{pmatrix}U_1(Np^r),\ U_1(Np^r)\begin{pmatrix}1&&&\\&l&&\\&&l&\\&&&l^2\end{pmatrix}U_1(Np^r),\ U_1(Np^r)\begin{pmatrix}l&&&\\&l&&\\&&l&\\&&&l\end{pmatrix}U_1(Np^r),$$

induces $\mathbb{Z}_p$-linear endomorphisms on the cohomology $H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))$, which will be denoted by T_l, T'_l and T''_l respectively. By a standard argument on the $\mathbb{Z}_p$-linear endomorphism of $\mathbb{Z}/p^s\mathbb{Z}$-modules, $e = \lim_{n\to\infty}(T_pT'_p)^{n!}$ defines a well-defined $\mathbb{Z}_p$-linear endmorphism on $H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))$. We will denote $e(H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z})))$ by $H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))^{\mathrm{ord}}$.

The Pontrjagin dual $M_N^{(a_1,a_2)} = \left(\varinjlim_{r,s} H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))^{\mathrm{ord}}\right)^{\vee}$ of the injective limit $\varinjlim_{r,s} H^3_!(S_{Np^r}, \mathcal{L}_{a_1,a_2}(\mathbb{Z}/p^s\mathbb{Z}))^{\mathrm{ord}}$ is naturally regarded as Λ^{ord}-module and is known to be finitely generated over Λ^{ord}. We denote by $\mathbb{H}_N^{\mathrm{ord}}$ the sub-algebra of $\mathrm{End}_{\Lambda^{\mathrm{ord}}}(M_N^{(a_1,a_2)})$ generated by T_l, T'_l, T''_l for primes $l \nmid Np$ as well as the endomorphism induced by the cosets $U_1(Np^r)\begin{pmatrix}x_p&&&\\&x_p&&\\&&x_p^{-1}&\\&&&x_p^{-1}\end{pmatrix}U_1(Np^r)$ with $x_p \in \mathbb{Z}_p^{\times}$ embedded in the p-component of $\widehat{\mathbb{Z}}^{\times} = \prod_{l:\text{primes}} \mathbb{Z}_l^{\times}$ in an obvious way. We call $\mathbb{H}_N^{\mathrm{ord}}$ the ordinary Hecke algebra for GSp_4 of tame conductor N. Note that the module $M_N^{(a_1,a_2)}$ and $M_N^{(a'_1,a'_2)}$ are isomorphic as Λ^{ord}-module when we take another (a'_1, a'_2) with $a'_1 > a'_2 > 0$. Thus, $\mathbb{H}_N^{\mathrm{ord}}$ which a priori depends on (a_1, a_2) is isomorphic to each other when we change the auxiliary weight (a_1, a_2) (cf. [TU, Thm. 6.1]).

Since we identify Λ^{ord} as $\mathbb{Z}_p[[X_1, X_2]]$, Λ^{ord}-algebra $\mathbb{H}_N^{\mathrm{ord}}$ is also finite and torsion-free over $\mathbb{Z}_p[[X_1, X_2]]$. We have only finitely many ideals of height zero in $\mathbb{H}_N$ and we call the algebra $R_{\mathfrak{I}}^{\mathrm{ord}} := \mathbb{H}_N^{\mathrm{ord}}/\mathfrak{I}$ a branch of $\mathbb{H}_N^{\mathrm{ord}}$ (corresponding to the ideal $\mathfrak{I}$ of height zero). Note that each branch $R_{\mathfrak{I}}^{\mathrm{ord}}$ is a local domain which is finite and torsion-free over $\mathbb{Z}_p[[X_1, X_2]]$. From now on, when we do not have to specify the ideal $\mathfrak{I}$ of $\mathbb{H}_N^{\mathrm{ord}}$, we denote a branch of $\mathbb{H}_N^{\mathrm{ord}}$ by R^{ord}.

Similarly, we define the nearly ordinary Hecke algebra of GSp_4 of tame conductor N to be $\mathbb{H}_N^{\mathrm{ord}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_3]]$ and denote it by $\mathbb{H}_N^{\mathrm{n.ord}}$. By the same way as above, a branch $R^{\mathrm{n.ord}}$ of

$\mathbb{H}_N^{\text{n.ord}}$ is defined and each branch $R^{\text{n.ord}}$ is a local domain which is finite and torsion-free over $\mathbb{Z}_p[[X_1, X_2, X_3]]$.

A branch R^{ord} of $\mathbb{H}_N^{\text{ord}}$ gives us a branch $R^{\text{n.ord}} = R^{\text{ord}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_3]]$ of $\mathbb{H}_N^{\text{n.ord}}$ and a branch $R^{\text{n.ord}}$ of $\mathbb{H}_N^{\text{n.ord}}$ gives us a branch $R^{\text{ord}} = R^{\text{n.ord}}\otimes_{\mathbb{H}_N^{\text{n.ord}}}\mathbb{H}_N^{\text{ord}}$ of $\mathbb{H}_N^{\text{ord}}$. Thus there is a one-to-one correspondence between the set of branches of $\mathbb{H}_N^{\text{ord}}$ and the set of branches of $\mathbb{H}_N^{\text{n.ord}}$.

Definition 2.3. Let R^{ord} be a branch of $\mathbb{H}_N^{\text{ord}}$ and let $\mathfrak{M}^{\text{ord}}$ be the maximal ideal of R^{ord}. A module $\mathcal{V}$ of rank 4 over $\mathbb{F} = R^{\text{ord}}/\mathfrak{M}^{\text{ord}}$ with continuous action of the absolute Galois group $G_{\mathbb{Q}}$ is called the residual representation for R^{ord} if the following conditions hold:

(i) The action $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{F}}(\mathcal{V})$ over $\mathbb{F}$ is semi-simple and unramified outside Np.

(ii) For every prime $l \nmid Np$, we have $\mathrm{Tr}(\overline{\rho}(\mathrm{Frob_l})) \equiv T_l \mod \mathfrak{M}^{\text{ord}}$.

Thanks to Laumon, Taylor and Weissauer [L], [Tay], [W], the residual representation for R^{ord} always exists. The residual representation is unique modulo isomorphism thanks to Chebotarev density theorem.

From now on throughout the article, we will fix a branch R^{ord} of $\mathbb{H}_N^{\text{ord}}$ and we discuss the following conditions for a fixed branch R^{ord} of $\mathbb{H}_N^{\text{ord}}$:

I. The residual representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{F}}(\mathcal{V})$ for R^{ord} is absolutely irreducible.

II. R^{ord} is a local domain which is Gorenstein.

Let Γ_{cyc} be the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$. By the p-adic cyclotomic character χ_{cyc}, we have a canonical isomorphism

$$\chi_{\text{cyc}} : \Gamma_{\text{cyc}} \xrightarrow{\sim} 1 + p\mathbb{Z}_p.$$

For each $1 \leq i \leq 3$, we denote by $\widetilde{\chi}_i$, the character of $G_{\mathbb{Q}}$:

$$G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\text{cyc}} \xrightarrow{\sim} G_i \hookrightarrow (\Lambda^{\text{n.ord}})^\times.$$

Definition 2.4. Let $\pi = \pi_\infty \otimes \pi_f$ an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{R} \times \mathbb{A}_f)$ whose archimedean component is a discrete series. We say that π is stable at infinity if for every discrete series π'_∞ in the discrete series L-packet of π_∞ (i.e. the set of discrete series with the same Harish-Chandra parameter as π_∞), $\pi'_\infty \otimes \pi_f$ is automorphic, cuspidal.

The following theorem is due to [TU, Theorem 7.1] and [U] which proves the conjecture 3 which appears in the statement of [TU, Theorem 7.1] as well as [Pi].

Theorem 2.5. *Under the assumption* **I**, *we have a module $\mathbb{T}^{\text{ord}}$ with continuous $G_{\mathbb{Q}}$-action which is free of rank 4 over R^{ord} and which satisfies the following properties:*

(1) *For any arithmetic specialization κ of weight (a_1, a_2) such that $a_1 > a_2 > 0$, we have a cuspidal automorphic representation π_κ cohomological of weight (a_1, a_2) such that $V_\kappa := \mathbb{T}^{\text{ord}} \otimes_{R^{\text{ord}}} \kappa(R^{\text{ord}})$ is the Galois representation associated to π_κ constructed by Laumon, Taylor and Weissauer* [L], [Tay], [W].

(2) *The subset* $\{\mathrm{Ker}(\kappa)\}$ *of* $\mathrm{Spec}(R^{\text{ord}})$ *for arithmetic specializations κ on R^{ord} of weight (a_1, a_2) with $a_1 > a_2 > 0$ such that V_κ is crystalline at p is dense.*

(3) *Let us denote by* $\mathbb{V}^{\mathrm{ord}}$ *the base extension* $\mathbb{T}^{\mathrm{ord}} \otimes_{R^{\mathrm{ord}}} \mathrm{Frac}(R^{\mathrm{ord}})$ *of* $\mathbb{T}^{\mathrm{ord}}$*, where* $\mathrm{Frac}(R^{\mathrm{ord}})$ *is the fraction field of* R^{ord}*. Under the assumption that one of the specializations* π_κ *is stable at infinity and that* π_κ *is ordinary at* p *for the Borel subgroup, the representation on* $\mathbb{V}^{\mathrm{ord}}$ *restricted to* $G_{\mathbb{Q}_p}$ *is conjugate to:*

$$\begin{pmatrix} \widetilde{\alpha}^{(1)}\omega^{b_1} & * & * & * \\ 0 & \widetilde{\alpha}^{(2)}\chi^{-1}\widetilde{\chi}_2^{-1}\omega^{b_2} & * & * \\ 0 & 0 & \widetilde{\alpha}^{(3)}\chi^{-2}\widetilde{\chi}_1^{-1}\omega^{b_3} & * \\ 0 & 0 & 0 & \widetilde{\alpha}^{(4)}\chi^{-3}\widetilde{\chi}_1^{-1}\widetilde{\chi}_2^{-1}\omega^{b_4} \end{pmatrix}$$

where $\widetilde{\alpha}^{(i)} : G_{\mathbb{Q}_p} \longrightarrow (R^{\mathrm{ord}})^\times$ *is an unramified character for* $i = 1, 2, 3, 4$ *such that, for each arithmetic specialization* κ *of weight* $a_1 > a_2 > 0$ *whose* V_κ *is crystalline at* p*,*

$$\{\widetilde{\alpha}^{(1)}_\kappa(\mathrm{Frob}_p), \widetilde{\alpha}^{(2)}_\kappa(\mathrm{Frob}_p)p^{a_2+1}, \widetilde{\alpha}^{(3)}_\kappa(\mathrm{Frob}_p)p^{a_1+2}, \widetilde{\alpha}^{(4)}_\kappa(\mathrm{Frob}_p)p^{a_1+a_2+3}\}$$

are equal to the set of Satake parameters of π_κ *at* p *where* Frob_p *means a geometric Frobenius element at* p*. Here,* ω *be the Teichmuller character which is canonically identified with a character of* $G_{\mathbb{Q}}$ *and* b_1, b_2, b_3, b_4 *are integers satisfying* $0 \le b_1, b_2, b_3, b_4 \le p-2$*.*

Remark 2.6. (1) In general, it is not clear if we have a free lattice for the Galois representation over the fraction field of an algebra like R^{ord}. Thanks to the comment at the end of §7 of [TU], we have the free Galois representation $\mathbb{T}^{\mathrm{ord}} \cong (R^{\mathrm{ord}})^{\oplus 4}$ equipped with a continuous Galois action of $G_{\mathbb{Q}}$ under our assumption **I**.

(2) In the above theorem, only specializations with regular weight $a_1 > a_2 > 0$ are controlled because of a technical restriction in the work [TU] to assure certain vanishing of cohomology. However, the same statements are expected to be true for non-regular cohomological weights $a_1 \ge a_2 \ge 0$. In the statement (1), the existence of a cuspidal automorphic representation π_κ (or equivalently a classical Siegel modular form corresponding to π_κ) follows applying [Pi, Cor 1.1] with the relation $(k_1, k_2) = (a_1+3, a_2+3)$ for his modular weight (k_1, k_2).

Throughout the article, we suppose another condition as follows:

III. For $i = 1, 2, 3, 4$, the characters $\widetilde{\alpha}^{(i)} : G_{\mathbb{Q}_p} \longrightarrow (R^{\mathrm{ord}})^\times$ are non-trivial modulo the maximal ideal $\mathfrak{M}^{\mathrm{n.ord}}$ of $R^{\mathrm{n.ord}}$.

For each a satisfying $1 \le a \le p-1$, let us define $\mathbb{T}^{\mathrm{n.ord},(a)}$ to be $\mathbb{T}^{\mathrm{ord}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[G_3]](\widetilde{\chi}_3\omega^a)$. Similarly as above, we denote by $\mathbb{V}^{\mathrm{n.ord}}$ the base extension $\mathbb{T}^{\mathrm{n.ord}} \otimes_{R^{\mathrm{n.ord}}} \mathrm{Frac}(R^{\mathrm{n.ord}})$ of $\mathbb{T}^{\mathrm{n.ord}}$.

We have an immediate consequence of the statement (3) of Theorem 2.5 as follows:

Corollary 2.7. *The action of* $G_{\mathbb{Q}_p}$ *on* $\mathbb{V}^{\mathrm{n.ord},(a)}$ *is conjugate to:*

$$\begin{pmatrix} \widetilde{\alpha}^{(1)}\widetilde{\chi}_3\omega^{b_1+a} & * & * & * \\ 0 & \widetilde{\alpha}^{(2)}\chi^{-1}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^{b_2+a} & * & * \\ 0 & 0 & \widetilde{\alpha}^{(3)}\chi^{-2}\widetilde{\chi}_1^{-1}\widetilde{\chi}_3\omega^{b_3+a} & * \\ 0 & 0 & 0 & \widetilde{\alpha}^{(4)}\chi^{-3}\widetilde{\chi}_1^{-1}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^{b_4+a} \end{pmatrix}.$$

In general, we do not know if the filtration given by (2.7) is realized without the base extension $\otimes_{R^{\mathrm{n.ord}}}\mathrm{Frac}(R^{\mathrm{n.ord}})$. Thus, we consider the following condition:

IV. For a suitable choice of R^{ord}-basis (resp. $R^{\mathrm{n.ord}}$-basis) of $\mathbb{T}^{\mathrm{ord}}$ (resp. $\mathbb{T}^{\mathrm{n.ord}}$), the action of $G_{\mathbb{Q}_p}$ is represented by the equation (3) (resp. the equation (2.7)) without taking the base extension $\otimes_{R^{\mathrm{n.ord}}}\mathrm{Frac}(R^{\mathrm{ord}})$ (resp. $\otimes_{R^{\mathrm{n.ord}}}\mathrm{Frac}(R^{\mathrm{n.ord}})$).

Remark 2.8. (1) If the condition **IV** is true for $\mathbb{T}^{\mathrm{ord}}$, it is true for $\mathbb{T}^{\mathrm{n.ord}}$ since $\mathbb{T}^{\mathrm{n.ord},(a)}$ is defined to be $\mathbb{T}^{\mathrm{ord}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_3]](\widetilde{\chi}_3\omega^a)$.

(2) Let us consider the following condition:
(Reg) All the Jordan-Hölder components of the representation $\mathbb{T}^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}\mathbb{T}^{\mathrm{ord}} \cong \mathbb{T}^{\mathrm{n.ord}}/\mathfrak{M}^{\mathrm{n.ord}}\mathbb{T}^{\mathrm{n.ord}}$ as $G_{\mathbb{Q}_p}$-modules are of multiplicity free.
We prove that the condition **(Reg)** implies the condition **IV**. In fact, let us denote the three dimensional $\mathrm{Frac}(R^{\mathrm{ord}})$-vector space $(\mathbb{V}^{\mathrm{ord}})'$ of $\mathbb{V}^{\mathrm{ord}}$ stable under the action (3). and we define the one-dimensional quotient $(\mathbb{V}^{\mathrm{ord}})''$ of $\mathbb{V}^{\mathrm{ord}}$ to be the quotient $\mathbb{V}^{\mathrm{ord}}/(\mathbb{V}^{\mathrm{ord}})'$. Then, we define $(\mathbb{T}^{\mathrm{ord}})'$ (resp. $(\mathbb{T}^{\mathrm{ord}})''$) to be the kernel (resp. the image) of the composite $\mathbb{T}^{\mathrm{ord}} \hookrightarrow \mathbb{V}^{\mathrm{ord}} \twoheadrightarrow (\mathbb{V}^{\mathrm{ord}})''$. We thus have the following exact sequence:

$$0 \longrightarrow (\mathbb{T}^{\mathrm{ord}})' \longrightarrow \mathbb{T}^{\mathrm{ord}} \longrightarrow (\mathbb{T}^{\mathrm{ord}})'' \longrightarrow 0.$$

By applying the functor $\otimes_{R^{\mathrm{ord}}}R^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}$ to this sequence, we obtain the exact sequence as follows:

$$(\mathbb{T}^{\mathrm{ord}})'/\mathfrak{M}^{\mathrm{ord}}(\mathbb{T}^{\mathrm{ord}})' \longrightarrow \mathbb{T}^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}\mathbb{T}^{\mathrm{ord}} \longrightarrow (\mathbb{T}^{\mathrm{ord}})''/\mathfrak{M}^{\mathrm{ord}}(\mathbb{T}^{\mathrm{ord}})'' \longrightarrow 0.$$

A priori, we only know that the dimension of $R^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}$-vector space $\mathbb{T}^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}\mathbb{T}^{\mathrm{ord}}$ is 4 thanks to the condition **I**. However, the condition **(Reg)** implies that the module $(\mathbb{T}^{\mathrm{ord}})''/\mathfrak{M}^{\mathrm{ord}}(\mathbb{T}^{\mathrm{ord}})''$ is of dimension 1 over $R^{\mathrm{ord}}/\mathfrak{M}^{\mathrm{ord}}$, which implies that $(\mathbb{T}^{\mathrm{ord}})''$ is a cyclic R^{ord}-module. Since $(\mathbb{T}^{\mathrm{ord}})''$ is torsion-free over R^{ord} by construction, $(\mathbb{T}^{\mathrm{ord}})''$ is free of rank 1 over R^{ord}. Hence $(\mathbb{T}^{\mathrm{ord}})'$ is free of rank 3 over R^{ord}. By repeating the same argument for $(\mathbb{T}^{\mathrm{ord}})'$, we will complete the proof of **(Reg)** $\Rightarrow$ **IV**.

In the below, we denote by $F^+\mathbb{T}^{\mathrm{n.ord},(a)}$ the $R^{\mathrm{n.ord}}[G_{\mathbb{Q}_p}]$-submodule of rank two in $\mathbb{T}^{\mathrm{n.ord},(a)}$ which is an extension of $R^{\mathrm{n.ord}}(\widetilde{\alpha}^{(2)}\chi^{-1}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^a)$ by $R^{\mathrm{n.ord}}(\widetilde{\alpha}^{(1)}\widetilde{\chi}_3\omega^a)$ and $F^-\mathbb{T}^{\mathrm{n.ord},(a)}$ the quotient $\mathbb{T}^{\mathrm{n.ord},(a)}/F^+\mathbb{T}^{\mathrm{n.ord},(a)}$. We denote the Kummer dual $\mathrm{Hom}_{R^{\mathrm{n.ord}}}(\mathbb{T}^{\mathrm{n.ord},(a)}, R^{\mathrm{n.ord}})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$ of $\mathbb{T}^{\mathrm{n.ord},(a)}$ by $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$. Similarly $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$ has a submodule of rank two $F^+\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$ given by $F^+\overline{\mathbb{T}}^{\mathrm{n.ord},(a)} = \mathrm{Hom}_{R^{\mathrm{n.ord}}}(F^-\mathbb{T}^{\mathrm{n.ord},(a)}, R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ stable by the action of $G_{\mathbb{Q}_p}$ and we denote by $F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$ the quotient $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}/F^+\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$. Let us define free $R^{\mathrm{n.ord}}$-modules of rank one $\overline{\mathbb{S}}^{(a)}$, $\overline{\mathbb{U}}^{(a)}$ on which $G_{\mathbb{Q}_p}$ acts continuously:

$$\overline{\mathbb{S}}^{(a)} = R^{\mathrm{n.ord}}((\widetilde{\alpha}^{(2)})^{-1}\chi^2\widetilde{\chi}_2\widetilde{\chi}_3^{-1}\omega^{1-a}), \quad \overline{\mathbb{U}}^{(a)} = R^{\mathrm{n.ord}}((\widetilde{\alpha}^{(1)})^{-1}\chi\widetilde{\chi}_3^{-1}\omega^{1-a}). \tag{4}$$

By definition, we have an exact sequence of Galois modules

$$0 \longrightarrow \overline{\mathbb{S}}^{(a)} \longrightarrow F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)} \longrightarrow \overline{\mathbb{U}}^{(a)} \longrightarrow 0. \tag{5}$$

3. Statement of the main theorems

Denote by $\overline{T}^{(a_1,a_2,a_3)}_{l,m,n}$ the module $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}/I^{(a_1,a_2,a_3)}_{l,m,n}$. To avoid a complicated notation, we will not show the dependence on a in the symbol $\overline{T}^{(a_1,a_2,a_3)}_{l,m,n}$. By Proposition 5.8 which will be proved later, the inverse limit $\varprojlim_{l,m,n} \dfrac{H^1(\mathbb{Q}_p, \overline{T}^{(a_1,a_2,a_3)}_{l,m,n})}{H^1_f(\mathbb{Q}_p, \overline{T}^{(a_1,a_2,a_3)}_{l,m,n})}$ is independent of the choice of (a_1, a_2, a_3) whenever we have $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$. We denote this limit by $H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$.

Theorem 3.1. *Let us assume the conditions* **I**, **III** *and* **IV**. *Then, for each natural number a satisfying $1 \leq a \leq p-1$, $H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ is free of rank two over $R^{\mathrm{n.ord}}$.*

The proof of this theorem is given in Section 5. For every arithmetic specialization λ of $R^{\mathrm{n.ord}}$, we denote by $\overline{\mathbb{U}}^{(a)}_\lambda$ (resp. $\overline{\mathbb{S}}^{(a)}_\lambda$) the p-adic representation $\overline{\mathbb{U}}^{(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $\overline{\mathbb{S}}^{(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$). In order to state the main result of this paper, let us mention that we will construct two $R^{\mathrm{n.ord}}$-modules of rank one $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}}$ and $\mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$ (see the equations (10) and (11)) which interpolate lattices of the filtered modules $D_{\mathrm{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda)$ and $D_{\mathrm{dR}}(\overline{\mathbb{S}}^{(a)}_\lambda)$ respectively when λ varies, in the sense that $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $\mathcal{D}_{\overline{\mathbb{S}}^{(a)}} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) is canonically isomorphic to $D_{\mathrm{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda)$ (resp. $D_{\mathrm{dR}}(\overline{\mathbb{S}}^{(a)}_\lambda)$). We will see that $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \otimes_{R^{\mathrm{n.ord}}} \mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$ interpolates lattices of $\bigwedge^2 \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda)$ when λ runs over arithmetic specializations of $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) with $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$ where the specializations $\overline{V}^{(a)}_\lambda$ corresponds to motives which are critical in the sense of Deligne [De].

Theorem 3.2. *Assume the conditions* **I**, **II**, **III** *and* **IV** *for the fixed branch R^{ord} of $\mathbb{H}^{\mathrm{odd}}_N$. Let a be a natural number satisfying $1 \leq a \leq p-1$ and put $\overline{\mathcal{D}}^{(a)} = \mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \otimes_{R^{\mathrm{n.ord}}} \mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$. Then, the following statements hold:*

(1) *For every arithmetic specialization λ on $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) satisfying $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$, $\overline{\mathcal{D}}^{(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is canonically isomorphic to $\bigwedge^2 \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda)$.*

(2) *There exists an $R^{\mathrm{n.ord}}$-linear isomorphism*

$$\Xi^{(a)} : \bigwedge^2 H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \longrightarrow \overline{\mathcal{D}}^{(a)}$$

such that, for every arithmetic specialization λ on $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) with $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$, we have the following commutative diagram:

$$\begin{array}{ccc} \bigwedge^2 H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) & \xrightarrow{\Xi^{(a)}} & \overline{\mathcal{D}}^{(a)} \\ \lambda \big\downarrow & & \big\downarrow \lambda \\ \bigwedge^2 \dfrac{H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)}{H^1_f(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)} & \xrightarrow[\overline{m}_\lambda]{} & \bigwedge^2 \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda), \end{array}$$

where the map $\overline{m}_\lambda$ is equal to:

$$a_3!(a_3 - a_2 + 1)! \left(\frac{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a_3}} \right)^{-s} \left(1 - \frac{p^{a_3}\phi(p)}{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)} \right) \left(1 - \frac{\phi(p)\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a_3+1}} \right)^{-1}$$

$$\times \left(\frac{\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)}{p^{a_3-a_2+1}} \right)^{-s'} \left(1 - \frac{p^{a_3-a_2+1}\phi'(p)}{\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)} \right) \left(1 - \frac{\phi'(p)\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)}{p^{a_3-a_2+2}} \right)^{-1} \times \bigwedge^2 \exp^*_\lambda$$

with the dual exponential map $\exp^*_\lambda : \dfrac{H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)}{H^1_f(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)} \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}_\lambda)$ *and where ϕ (resp. ϕ') is the finite order character $\lambda|_{G_3}\chi_3^{-a_3}$ (resp. $\lambda|_{G'_3}\chi_2^{a_2}\chi_3^{-a_3}$) on G_3 (resp. $G'_3 :=$ the image of $G_2 \longrightarrow G_2 \times G_3$, $g \mapsto g^{-1} \times g$) and s (resp. s') is the p-order of the conductor of ϕ (resp. ϕ').*

4. The Coleman map for a power of the universal cyclotomic character

In this section, we study the local Galois cohomology and the Coleman map for a power of the universal cyclotomic character as follows: let G (resp. Δ) be a group which is equipped with a fixed isomorphism $G \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ (resp. $\Delta \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\times$). This allows us to identify G (resp. Δ) with Γ_{cyc} (resp. $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$).

Definition 4.1.

(1) Let G be a group with the above fixed identification $G \xrightarrow{\sim} 1+p\mathbb{Z}_p$. Let η be a $(\overline{\mathbb{Q}}_p)^\times$-valued character on G. Then we say that η is an arithmetic character of weight $w(\eta) \in \mathbb{Z}$ if there exists an open subgroup $U \subset G$ such that $\eta|_U$ coincides with $\chi_{\mathrm{cyc}}^{w(\eta)}$.

(2) For an arithmetic character η of G, we denote by K_η a finite extension of $\mathbb{Q}_p$ obtained by adjoining values of η.

Let us consider the universal character

$$\widetilde{\chi} : \ G_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathrm{cyc}} \xrightarrow{\sim} G \longrightarrow \mathbb{Z}_p[[G]]^\times$$
$$\widetilde{\omega} : \ G_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \xrightarrow{\sim} \Delta \longrightarrow \mathbb{Z}_p[\Delta]^\times$$

Perrin-Riou [P2] interpolates the Bloch-Kato exponential maps for crystalline representations of the Galois group of an absolute unramified complete discrete valuation field of mixed characteristic in the cyclotomic tower. As a consequence of her work for the trivial representation, we have the following proposition proved in [O1] (The corresponding statement Proposition 5.10 in [O1] contains typos, but we see the right statement by comparing with Proposition 5.7 in [O1]):

Proposition 4.2 ([O1] Proposition 5.10)**.** *We have a $\mathbb{Z}_p[[G\times\Delta]]$-linear map:*

$$\Xi^{\mathrm{ur}} : \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G\times\Delta]] \longrightarrow \frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G\times\Delta]](\widetilde{\chi}\widetilde{\omega}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p)}$$

with the following commutative diagram for each arithmetic character η of G with $w(\eta)\geq 1$:

$$\begin{array}{ccc}
\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G\times\Delta]] & \xrightarrow{\Xi^{\mathrm{ur}}} & \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G\times\Delta]](\widetilde{\chi}\widetilde{\omega}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p)} \\
{\scriptstyle \mathrm{Sp}_\eta}\downarrow & & \downarrow{\scriptstyle \mathrm{Sp}_\eta} \\
D_{\mathrm{dR}}^{\mathrm{ur}}(K_\eta(\eta)) & \xrightarrow[m_\eta]{} & H^1(\mathbb{Q}_p^{\mathrm{ur}}, K_\eta(\eta)).
\end{array}$$

In the above diagram m_η is the map

$$(-1)^{w-1}(w-1)!\exp\left(\frac{\sigma}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{\sigma}\right)\left(1-\frac{\sigma\phi(p)}{p^w}\right)^{-1}$$

where ϕ is the finite order character $\eta(\chi\omega)^{-w(\eta)}$ of G and s is the p-order of the conductor of ϕ. Further, the $\mathbb{Z}_p[[G\times\Delta]]$-linear map Ξ^{ur} is an injective map whose cokernel is isomorphic to $\mathbb{Z}_p(\omega^{-1}\chi^{-1})$.

Proposition 4.3. *Let b be an integer and let c be a natural number satisfying $(c,p)=1$. For each integer a with $1\leq a\leq p-1$, we have a $\mathbb{Z}_p[[G]]$-linear map:*

$$\Xi^{\mathrm{ur},(a,b,c)} : \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]] \longrightarrow \frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G]](\omega^a\chi^b\widetilde{\chi}^c))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p)(\omega^a\chi^b)}$$

with the following commutative diagram for each arithmetic character η of G with $w = w(\eta)\geq 1-b$:

$$\begin{array}{ccc}
\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]] & \xrightarrow{\Xi^{\mathrm{ur},(a,b,c)}} & \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G]](\omega^a\chi^b\widetilde{\chi}^c))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p)(\omega^a\chi^b)} \\
{\scriptstyle \mathrm{Sp}_\eta}\downarrow & & \downarrow{\scriptstyle \mathrm{Sp}_\eta} \\
D_{\mathrm{dR}}^{\mathrm{ur}}(K_\eta(\omega^a\chi^b\eta^c)) & \xrightarrow[m_\eta]{} & H^1(\mathbb{Q}_p^{\mathrm{ur}}, K_\eta(\omega^a\chi^b\eta^c)).
\end{array}$$

In the above diagram m_η is the map

$$(-1)^{w+b-1}(w+b-1)!\exp\left(\frac{\sigma}{p^{w+b-1}}\right)^{-s}\left(1-\frac{p^{w+b-1}\phi^c(p)}{\sigma}\right)\left(1-\frac{\sigma\phi^c(p)}{p^{w+b}}\right)^{-1}$$

where ϕ is the finite order character $\eta(\chi\omega)^{-w(\eta)}$ of G and s is the p-order of the conductor of ϕ. Further, the $\mathbb{Z}_p[[G]]$-linear map $\Xi^{\mathrm{ur},(a,b,c)}$ is an injective map whose cokernel is isomorphic to $\mathbb{Z}_p(\omega^{a-1}\chi^{b-1})$.

Proposition 4.3 is a rather immediate consequence of Proposition 4.2. We will explain how to deduce Proposition 4.3 from Proposition 4.2. First, we project the whole commutative diagram of Proposition 4.2 to ω^a-component with respect to the action of Δ. Then, we twist the G-action of the map Ξ^{ur} by the character χ^b of G. We note that we have a unique ring automorphism $[c] : \mathbb{Z}_p[[G]] \longrightarrow \mathbb{Z}_p[[G]]$ which extends the automorphism $G \overset{\sim}{\longrightarrow} G,\ g \mapsto g^c$. Finally, we obtain Proposition 4.3 by taking the base change of the $\mathbb{Z}_p[[G]]$-module structure via $[c]$.

The following result is an important corollary of Proposition 4.3.

Corollary 4.4. *Let R be a complete local domain which is free of finite rank over $\mathbb{Z}_p[[G]]$ and let $\widetilde{\alpha} : G_{\mathbb{Q}_p} \longrightarrow R^\times$ be a non-trivial continuous unramified character. Let b be an integer and let c be a natural numbers satisfying $(c,p)=1$. Then, if $(R(\widetilde{\alpha}) \otimes_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is trivial, the Galois cohomology group $H^1(\mathbb{Q}_p, R(\widetilde{\alpha}\omega^a\chi^b\widetilde{\chi}^c))$ is a free R-module of rank one for each integer a with $1 \leq a \leq p-1$.*

Proof. By Proposition 4.3, we have

$$0 \longrightarrow \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]] \longrightarrow \frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G]](\omega^a\chi^b\widetilde{\chi}^c))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p)(\omega^a\chi^b)} \longrightarrow \mathbb{Z}_p(\omega^{a-1}\chi^{b-1}) \longrightarrow 0.$$

We apply $\otimes_{\mathbb{Z}_p[[G]]} R(\widetilde{\alpha})$ to the above exact sequence and take the $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-invariant. Let us recall the following lemma:

Lemma 4.5 ([O1] Lemma 3.3)**.** *Let R be a complete local domain which is finite and torsion-free over $\mathbb{Z}_p[[G]]$ and let M be a free R-module of finite rank e endowed with an unramified action of $G_{\mathbb{Q}_p}$. Then $(M \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ is a free R-module of rank e.*

By the above lemma, $\left(\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G]] \otimes_{\mathbb{Z}_p[[G]]} R(\widetilde{\alpha})\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is free of rank one over R. On the other hand, $\left(\mathbb{Z}_p(\omega^{a-1}\chi^{b-1}) \otimes_{\mathbb{Z}_p[[G]]} R(\widetilde{\alpha})\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is trivial by the assumption. Since the functor of taking the $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-invariant part is left exact, we complete the proof. □

5. Calculation of local Iwasawa modules

Definition 5.1. For a p-adic representation V of $G_{\mathbb{Q}_p}$ the subspaces $H^1_f(\mathbb{Q}_p, V)$ and $H^1_g(\mathbb{Q}_p, V)$ of $H^1(\mathbb{Q}_p, V)$ are defined as follows (see [BK, §3]):

$$\begin{aligned} H^1_f(\mathbb{Q}_p, V) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})] \\ H^1_g(\mathbb{Q}_p, V) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}})] \end{aligned}$$

We have

$$H^1_f(\mathbb{Q}_p, V) \subset H^1_g(\mathbb{Q}_p, V) \subset H^1(\mathbb{Q}_p, V)$$

by definition. Let T (resp. A) be a $G_{\mathbb{Q}_p}$-stable lattice of V (resp. the discrete Galois module $T \otimes \mathbb{Q}_p/\mathbb{Z}_p$). They fit into the following exact sequence:

$$0 \longrightarrow T \overset{i}{\longrightarrow} V \overset{p}{\longrightarrow} A \longrightarrow 0.$$

This induces the following exact sequence:

$$H^1(\mathbb{Q}_p, T) \xrightarrow{i_*} H^1(\mathbb{Q}_p, V) \xrightarrow{p_*} H^1(\mathbb{Q}_p, A).$$

We define $H^1_f(\mathbb{Q}_p, T) \subset H^1(\mathbb{Q}_p, T)$ (resp. $H^1_f(\mathbb{Q}_p, A) \subset H^1(\mathbb{Q}_p, A)$) to be the pull-back $(i_*)^{-1} H^1_f(\mathbb{Q}_p, V)$ (resp. the push-forward $p_* H^1_f(\mathbb{Q}_p, V)$).

Definition 5.2. Suppose that V is a p-adic representation of $G_{\mathbb{Q}_p}$ which is admissible in the sense that V has a $G_{\mathbb{Q}_p}$-stable subrepresentation $F^+V \subset V$ such that every Hodge-Tate weight of $F^+V \otimes \mathbb{C}_p$ is positive and every Hodge-Tate weight of $(V/F^+V) \otimes \mathbb{C}_p$ is non-positive. Then, we introduce the following notations:

(1) We define subspaces $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V)$ and $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, V)$ of $H^1(\mathbb{Q}_p, V)$ as follows:

$$\begin{aligned} H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, V) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V/F^+V)], \\ H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p^{ur}, V/F^+V)]. \end{aligned}$$

(2) Let A be the discrete Galois representation $T \otimes \mathbb{Q}_p/\mathbb{Z}_p$ where T is a $G_{\mathbb{Q}_p}$-stable lattice of V. As above, we define subspaces $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A)$ and $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A)$ of $H^1(\mathbb{Q}_p, A)$ as follows:

$$\begin{aligned} H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, A) \longrightarrow H^1(\mathbb{Q}_p, A/F^+A)] \\ H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A) &= \mathrm{Ker}[H^1(\mathbb{Q}_p, A) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, A/F^+A)] \end{aligned}$$

By a result of Flach [Fl], we have the following lemma:

Lemma 5.3. *Suppose that V is an admissible p-adic representation of $G_{\mathbb{Q}_p}$. Then, the subspace $H^1_g(\mathbb{Q}_p, V)$ is equal to $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V)$.*

Though Flach [Fl] states the theorem only for ordinary p-adic representations, exactly the same proof works for admissible representations. Hence we omit the proof of the above lemma.

By Proposition 3.8 and Corollary 3.8.4 of [BK], we have the following lemma:

Lemma 5.4. *Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. Then, we have*

$$\frac{H^1_g(\mathbb{Q}_p, V)}{H^1_f(\mathbb{Q}_p, V)} = \left(D_{\mathrm{crys}}(\overline{V})/(1-\varphi) D_{\mathrm{crys}}(\overline{V})\right)^*,$$

where $\overline{V}$ is the Kummer dual of V and φ is the Frobenius operator acting on D_{crys} and $(\ \ )^*$ means $\mathbb{Q}_p$-linear dual.

We consider the specialization of $\mathbb{T}^{\mathrm{n.ord},(a)}$ and $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}$ at each arithmetic specialization λ on $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) which satisfies $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$. We denote by $V^{(a)}_\lambda$ (resp. $\overline{V}^{(a)}_\lambda$) the p-adic representation which has a lattice isomorphic to the specialization $\mathbb{T}^{\mathrm{n.ord},(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})$ (resp. $\overline{\mathbb{T}}^{\mathrm{n.ord},(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})$).

Lemma 5.5. *Let us assume the conditions* **III** *and* **IV**. *Assume that λ is an arithmetic specialization on $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) satisfying $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$. Then, we have $H^1_f(\mathbb{Q}_p, V^{(a)}_\lambda) = H^1_g(\mathbb{Q}_p, V^{(a)}_\lambda)$ for each integer a with $1 \leq a \leq p-1$.*

Proof. By Lemma 5.4, the difference between H^1_g and H^1_f for $V^{(a)}_\lambda$ is bounded by the φ-coinvariant quotient of D_{crys} of the Kummer dual $\overline{V}^{(a)}_\lambda$. By the condition **IV**, $\overline{V}^{(a)}_\lambda$ is a successive extension of one-dimensional representations $\overline{V}_1, \dots, \overline{V}_4$ of $G_{\mathbb{Q}_p}$ for which $D_{\mathrm{crys}}(\overline{V}_i)/(1-\varphi)D_{\mathrm{crys}}(\overline{V}_i)$ is trivial. This completes the proof. □

Note that under the assumption $a_1 \geq a_3 - 2 \geq a_2 \geq 0$, the Galois representation $V^{(a)}_\lambda$ (resp. $\overline{V}^{(a)}_\lambda$) has an admissible filtration $F^+V^{(a)}_\lambda$ (resp. $F^+\overline{V}^{(a)}_\lambda$) which has a lattice isomorphic to $F^+\mathbb{T}^{\mathrm{n.ord},(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})$ (resp. $\mathrm{Hom}_{R^{\mathrm{n.ord}}}(\mathbb{T}^{\mathrm{n.ord},(a)}/F^+\mathbb{T}^{\mathrm{n.ord},(a)}, R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}})$). Hence we define the local cohomologies H^1_{Gr} and $H^1_{\mathrm{Gr}'}$.

Lemma 5.6. *Let λ be an arithmetic specialization on $R^{\mathrm{n.ord}}$. Under the assumptions* **I**, **III** *and* **IV**, *we have $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) = H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$ for each integer a with $1 \leq a \leq p-1$.*

Proof. Let us denote the quotient $\overline{V}^{(a)}_\lambda / F^+\overline{V}^{(a)}_\lambda$ by $F^-\overline{V}^{(a)}_\lambda$. We have a commutative diagram with exact lines

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) & \longrightarrow & H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) & \longrightarrow & H^1(\mathbb{Q}_p, F^-\overline{V}^{(a)}_\lambda) \\
& \downarrow & & \| & & \downarrow \\
0 \longrightarrow & H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) & \longrightarrow & H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) & \longrightarrow & H^1(\mathbb{Q}_p^{ur}, F^-\overline{V}^{(a)}_\lambda).
\end{array}$$

By the snake lemma, the injectivity of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) \longrightarrow H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$ is obvious and the cokernel of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) \longrightarrow H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$ is a subquotient of

$$\mathrm{Ker}[H^1(\mathbb{Q}_p, F^-\overline{V}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, F^-\overline{V}^{(a)}_\lambda)] \cong H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (F^-\overline{V}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{ur}}}).$$

Note that the representation $F^-\overline{V}^{(a)}_\lambda$ has an extension as follows:

$$0 \longrightarrow \overline{\mathbb{S}}^{(a)}_\lambda \longrightarrow F^-\overline{V}^{(a)}_\lambda \longrightarrow \overline{\mathbb{U}}^{(a)}_\lambda \longrightarrow 0, \tag{6}$$

where $\overline{\mathbb{S}}^{(a)}_\lambda$ (resp. $\overline{\mathbb{U}}^{(a)}_\lambda$) is the representation of rank one over $\mathrm{Frac}(\lambda(R^{\mathrm{n.ord}}))$ on which $G_{\mathbb{Q}_p}$ acts via the character $\omega^{-a}\chi^{a_2+2-a_3}(\alpha^{(2)}_\lambda)^{-1}$ (resp. $\omega^{-a}\chi^{1-a_3}(\alpha^{(1)}_\lambda)^{-1}$), where (a_1, a_2, a_3) is the weight of λ and $\alpha^{(i)}_\lambda$ is the unramified character obtained by the specialization of $\widetilde{\alpha}^{(i)}$. Since the functor $(\ )^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}$ is left-exact, we have:

$$0 \longrightarrow (\overline{\mathbb{S}}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{\mathrm{ur}}}} \longrightarrow (F^-\overline{V}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{\mathrm{ur}}}} \longrightarrow (\overline{\mathbb{U}}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}.$$

Both $(\overline{\mathbb{S}}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}$ and $(\overline{\mathbb{U}}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}$ are of dimension 1 or 0 over the fraction field of $\lambda(R^{\mathrm{n.ord}})$ and $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ acts on them non-trivially by the assumption **III**. This shows that the group $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\overline{V}^{(a)}_\lambda/F^+\overline{V}^{(a)}_\lambda)^{G_{\mathbb{Q}_p^{ur}}})$ is trivial and we complete the proof. □

For a finitely generated $R^{\mathrm{n.ord}}$-module M we denote by $M^{(a_1,a_2,a_3)}_{l,m,n}$ the specialization $M/I^{(a_1,a_2,a_3)}_{l,m,n}$.

If we denote the discrete representation $\mathbb{T}^{\text{n.ord},(a)}/I_{l,m,n}^{(a_1,a_2,a_3)}\mathbb{T}^{\text{n.ord},(a)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ by $A_{l,m,n}^{(a_1,a_2,a_3)}$ omitting the dependence on the power a, we have

$$\varinjlim_{l,m,n} H^1_{\text{Gr}}(\mathbb{Q}_p, A_{l,m,n}^{(a_1,a_2,a_3)}) = \text{Ker}[H^1(\mathbb{Q}_p,\mathcal{A}^{(a)}) \longrightarrow H^1(\mathbb{Q}_p,\mathcal{A}^{(a)}/F^+\mathcal{A}^{(a)})].$$

Since it is independent of the choice of (a_1,a_2,a_3) we denote it by $H^1_{\text{Gr}}(\mathbb{Q}_p,\mathcal{A}^{(a)})$. Now we define

$$H^1_f(\mathbb{Q}_p,\mathcal{A}^{(a)})^{(a_1,a_2,a_3)} = \varinjlim_{l,m,n} H^1_f(\mathbb{Q}_p, A_{l,m,n}^{(a_1,a_2,a_3)})$$

By [BK] Proposition 3.8, the group $\varprojlim_{l,m,n} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{l,m,n}^{(a_1,a_2,a_3)})$ is the Pontryagin dual of $H^1_f(\mathbb{Q}_p,\mathcal{A})^{(a_1,a_2,a_3)}$.

Lemma 5.7. *Let us assume the conditions* **III** *and* **IV**. *For any $l,m,n \geq 0$, the group:*

$$(\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)} \quad \left(\text{resp. } (F^+\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}\right)$$

is trivial.

Proof. The proof of the lemma is exactly the same for $(\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$ and for $(F^+\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$. Thus, we only prove it for $(F^+\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$. By definition, we have an exact sequence of Galois modules:

$$0 \longrightarrow R^{\text{n.ord}}(\widetilde{\alpha}^{(1)}\chi^{-1}\widetilde{\chi}_3\omega^{a-1}) \longrightarrow F^+\mathbb{T}^{\text{n.ord},(a)}(-1) \longrightarrow R^{\text{n.ord}}(\widetilde{\alpha}^{(2)}\chi^{-2}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^{a-1}) \longrightarrow 0$$

inducing an exact sequence

$$(7) \quad (R^{\text{n.ord}}(\widetilde{\alpha}^{(1)}\chi^{-1}\widetilde{\chi}_3\omega^{a-1})_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)} \longrightarrow (F^+\mathbb{T}^{\text{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)} \longrightarrow (R^{\text{n.ord}}(\widetilde{\alpha}^{(2)}\chi^{-2}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^{a-1})_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)} \longrightarrow 0.$$

The first term $(R^{\text{n.ord}}(\widetilde{\alpha}^{(1)}\chi^{-1}\widetilde{\chi}_3\omega^{a-1})_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$ modulo the maximal ideal $\mathfrak{M}$ of $R^{\text{n.ord}}$ is $(R^{\text{n.ord}}/\mathfrak{M})(\widetilde{\alpha}^{(1)}\omega^{a-1})_{G_{\mathbb{Q}_p}}$, which is trivial by the condition **III**. By Nakayama's lemma, $(R^{\text{n.ord}}(\widetilde{\alpha}^{(1)}\chi^{-1}\widetilde{\chi}_3\omega^{a-1})_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$ is also trivial. By the same reason using the condition **III** and Nakayama's lemma, the third term $(R^{\text{n.ord}}(\widetilde{\alpha}^{(2)}\chi^{-2}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^{a-1})_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}$ is also a trivial $R^{\text{n.ord}}$-module. We complete the proof by the sequence (7). □

Proposition 5.8. *Let (a_1,a_2,a_3) be a triple of integers satisfying $a_1 \geq a_3-2 \geq a_2 > 0$ and $a_1 > a_2$. Then, under the assumptions* **III** *and* **IV**, *$H^1_f(\mathbb{Q}_p,\mathcal{A}^{(a)})^{(a_1,a_2,a_3)}$ is equal to $H^1_{\text{Gr}}(\mathbb{Q}_p,\mathcal{A}^{(a)})$ for each integer a with $1 \leq a \leq p-1$.*

Proof. We have the following commutative diagram with exact lines:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1_{\text{Gr}'}(\mathbb{Q}_p, V_{l,m,n}^{(a_1,a_2,a_3)}) & \longrightarrow & H^1(\mathbb{Q}_p, V_{l,m,n}^{(a_1,a_2,a_3)}) & \longrightarrow & H^1(\mathbb{Q}_p, F^-V_{l,m,n}^{(a_1,a_2,a_3)}) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^1_{\text{Gr}'}(\mathbb{Q}_p, A_{l,m,n}^{(a_1,a_2,a_3)}) & \longrightarrow & H^1(\mathbb{Q}_p, A_{l,m,n}^{(a_1,a_2,a_3)}) & \longrightarrow & H^1(\mathbb{Q}_p, F^-A_{l,m,n}^{(a_1,a_2,a_3)}), &
\end{array}$$

where we denote $V_{l,m,n}^{(a_1,a_2,a_3)}/F^+V_{l,m,n}^{(a_1,a_2,a_3)}$ (resp. $A_{l,m,n}^{(a_1,a_2,a_3)}/F^+A_{l,m,n}^{(a_1,a_2,a_3)}$) by $F^-V_{l,m,n}^{(a_1,a_2,a_3)}$ (resp. $F^-A_{l,m,n}^{(a_1,a_2,a_3)}$). Note that $H^1(\mathbb{Q}_p, V_{l,m,n}^{(a_1,a_2,a_3)}) \longrightarrow H^1(\mathbb{Q}_p, F^-V_{l,m,n}^{(a_1,a_2,a_3)})$ is surjective since we have $H^2(\mathbb{Q}_p, F^+V_{l,m,n}^{(a_1,a_2,a_3)}) \simeq H^0(\mathbb{Q}_p, F^-\overline{V}_{l,m,n}^{(a_1,a_2,a_3)})^* = 0$ by the assumption **III** and by the same argument as the proof of Lemma 5.6.

By Lemma 5.3, Lemma 5.5 and Lemma 5.6, the space $H^1_f(\mathbb{Q}_p, A_{l,m,n}^{(a_1,a_2,a_3)})$ is equal to the image of the left hand vertical arrow. We have the following exact sequence by applying the snake lemma to the above commutative diagram:

$$\varinjlim_{l,m,n} \frac{H^1(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)})}{H^1(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)})_{\mathrm{tor}}} \longrightarrow \varinjlim_{l,m,n} \frac{H^1(\mathbb{Q}_p, F^-T_{l,m,n}^{(a_1,a_2,a_3)})}{H^1(\mathbb{Q}_p, F^-T_{l,m,n}^{(a_1,a_2,a_3)})_{\mathrm{tor}}}$$
$$\longrightarrow \frac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}^{(a)})}{H^1_f(\mathbb{Q}_p, \mathcal{A}^{(a)})^{(a_1,a_2,a_3)}} \longrightarrow \varinjlim_{l,m,n} H^2(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)})_{\mathrm{tor}},$$

where "tor" means the $\mathbb{Z}_p$-torsion part. Now we have:

$$\varinjlim_{l,m,n} H^2(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)}) \simeq \varinjlim_{l,m,n} H^0(\mathbb{Q}_p, \mathrm{Hom}(T_{l,m,n}^{(a_1,a_2,a_3)}, \mathbb{Q}_p/\mathbb{Z}_p(1)))^\vee$$
$$\simeq \varinjlim_{l,m,n} (\mathbb{T}^{\mathrm{n.ord},(a)}(-1)_{G_{\mathbb{Q}_p}})_{l,m,n}^{(a_1,a_2,a_3)}.$$

Then it follows from Lemma 5.7 that $\varinjlim_{l,m,n} (\mathbb{T}^{\mathrm{n.ord},(a)}_{G_{\mathbb{Q}_p}}(-1))_{l,m,n}^{(a_1,a_2,a_3)} = 0$. On the other hand, the group

$$\varinjlim_{l,m,n} \mathrm{Coker}\left[\frac{H^1(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)})}{H^1(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)})_{\mathrm{tor}}} \longrightarrow \frac{H^1(\mathbb{Q}_p, F^-T_{l,m,n}^{(a_1,a_2,a_3)})}{H^1(\mathbb{Q}_p, F^-T_{l,m,n}^{(a_1,a_2,a_3)})_{\mathrm{tor}}}\right]$$

is a quotient of

$$\varinjlim_{l,m,n} \mathrm{Coker}\left[H^1(\mathbb{Q}_p, T_{l,m,n}^{(a_1,a_2,a_3)}) \longrightarrow H^1(\mathbb{Q}_p, F^-T_{l,m,n}^{(a_1,a_2,a_3)})\right]$$

which is a subgroup of $\varinjlim_{l,m,n} H^2(\mathbb{Q}_p, F^+T_{l,m,n}^{(a_1,a_2,a_3)})$. By the same argument as above, we prove $\varinjlim_{l,m,n} H^2(\mathbb{Q}_p, F^+T_{l,m,n}^{(a_1,a_2,a_3)}) = 0$. This completes the proof. □

Since $\varprojlim_{l,m,n} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{l,m,n}^{(a_1,a_2,a_3)})$ is the Pontryagin dual of $H^1_f(\mathbb{Q}_p, \mathcal{A}^{(a)})^{(a_1,a_2,a_3)}$, Proposition 5.8 implies the following fact:

Corollary 5.9. *Let a be an integer with $1 \leq a \leq p-1$ and let (a_1, a_2, a_3) be a triple of integers satisfying $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$. We denote by $\overline{T}_{l,m,n}^{(a_1,a_2,a_3)}$ the representation $\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbb{T}^{\mathrm{n.ord},(a)}/I_{l,m,n}^{(a_1,a_2,a_3)}\mathbb{T}^{\mathrm{n.ord},(a)}, \mathbb{Z}_p(1)\big)$. Then, the following statements hold:*

1. *We have an isomorphism $H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \cong H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ for each integer a with $1 \leq a \leq p-1$.*
2. *The group $\varprojlim_{l,m,n} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{l,m,n}^{(a_1,a_2,a_3)})$ is independent of the choice of (a_1, a_2, a_3).*

Proposition 5.10. *Let a be an integer with $1 \leq a \leq p-1$ and let (a_1, a_2, a_3) be a triple of integers satisfying $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$ Then, $\varprojlim_{l,m,n} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(a_1,a_2,a_3)}_{l,m,n})$ is a free $R^{\mathrm{n.ord}}$-module of rank two.*

Proof. By Proposition 5.8, we have to calculate the Pontryagin dual of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}^{(a)})$. By definition, we have the following exact sequence:

$$H^0(\mathbb{Q}_p, F^-\mathcal{A}^{(a)}) \longrightarrow H^1(\mathbb{Q}_p, F^+\mathcal{A}^{(a)}) \longrightarrow H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}^{(a)}) \longrightarrow 0.$$

As $H^0(\mathbb{Q}_p, F^-\mathcal{A}^{(a)})$ is a torsion $R^{\mathrm{n.ord}}$-module it suffices to show that the Pontryagin dual $H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ of $H^1(\mathbb{Q}_p, F^+\mathcal{A}^{(a)})$ is free of rank two over $R^{\mathrm{n.ord}}$. Taking the Galois cohomology of (5), we have:

$$H^0(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) \longrightarrow H^1(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)}) \longrightarrow H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \longrightarrow H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) \longrightarrow H^2(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)})$$

Since the action of $G_{\mathbb{Q}_p}$ on $\overline{\mathbb{U}}^{(a)}$ is non-trivial, we have $H^0(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) = 0$. By the local Tate duality and by the assumption **II** saying that $R^{\mathrm{n.ord}}$ is Gorenstein, we have

$$H^2(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)}) \cong H^0(\mathbb{Q}_p, \mathrm{Hom}_{R^{\mathrm{n.ord}}}(\overline{\mathbb{S}}^{(a)}, R^{\mathrm{n.ord}})(1)) = 0.$$

Hence, the above exact sequence becomes

$$0 \longrightarrow H^1(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)}) \longrightarrow H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \longrightarrow H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) \longrightarrow 0. \tag{8}$$

Hence it is enough to show that the left hand term and the right hand term are free $R^{\mathrm{n.ord}}$-modules of rank one. By applying Corollary 4.4, $H^1(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)})$ and $H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)})$ are both free of rank one over $R^{\mathrm{n.ord}}$. This completes the proof of the proposition. □

We have the following lemma:

Lemma 5.11. *Under the assumptions* **I**, **III** *and* **IV**, *we have*

$$H^2(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) = H^2(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) = 0.$$

Proof. Since the proof of $H^2(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ and the proof of $H^2(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ consist of the same argument, we will prove the lemma for $H^2(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$. By using the Tate local duality, we have $H^2(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \cong (\mathbb{T}^{\mathrm{n.ord},(a)}(-1))_{G_{\mathbb{Q}_p}}$. We complete the proof, since $(\mathbb{T}^{\mathrm{n.ord},(a)}(-1))_{G_{\mathbb{Q}_p}}$ is trivial by Lemma 5.7. □

6. Proof of the main result

The main result of this section which will be the main step for the main result of the paper (Theorem 3.1) is as follows (see the end of §1 for the visual plan of the proof).

Theorem 6.1. *Let the assumptions and the notations be as in Theorem 3.2 and put $\overline{\mathcal{D}}^{(a)} = \mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \otimes_{R^{\mathrm{n.ord}}} \mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$ by using the modules $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}}$ and $\mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$ which will be introduced at* (10) *and* (12) *respectively. Then, the following statements hold.*

(1) *For every arithmetic specialization* λ *on* $R^{\mathrm{n.ord}}$ *of weight* (a_1, a_2, a_3) *with* $a_1 \geq a_3 - 2 \geq a_2 > 0$ *and* $a_1 > a_2$, $\overline{\mathcal{D}}^{(a)} \otimes_{R^{\mathrm{n.ord}}} \lambda(R^{\mathrm{n.ord}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is canonically isomorphic to* $\bigwedge^2 D_{\mathrm{dR}}(F^- \overline{V}^{(a)}_\lambda)$.

(2) *We have an* $R^{\mathrm{n.ord}}$*-linear isomorphism*

$$\Xi^{(a)}_- : \bigwedge^2 H^1(\mathbb{Q}_p, F^- \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \longrightarrow \overline{\mathcal{D}}^{(a)}$$

such that, for every arithmetic specialization λ *on* $R^{\mathrm{n.ord}}$ *of weight* (a_1, a_2, a_3) *with* $a_1 \geq a_3 - 2 \geq a_2 \geq 0$ *and* $a_1 > a_2 > 0$, *we have the following commutative diagram:*

$$\begin{array}{ccc}
\bigwedge^2 H^1(\mathbb{Q}_p, F^- \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) & \xrightarrow{\Xi^{(a)}_-} & \overline{\mathcal{D}}^{(a)} \\
\lambda \downarrow & & \downarrow \lambda \\
\bigwedge^2 H^1(\mathbb{Q}_p, F^- \overline{V}^{(a)}_\lambda) & \xrightarrow[\overline{m}_\lambda]{} & \bigwedge^2 D_{\mathrm{dR}}(F^- \overline{V}^{(a)}_\lambda),
\end{array}$$

where the map $\overline{m}_\lambda$ *is equal to:*

$$a_3!(a_3 - a_2 + 1)! \left(\frac{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a_3}} \right)^{-s} \left(1 - \frac{p^{a_3}\phi(p)}{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)} \right) \left(1 - \frac{\phi(p)\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a_3+1}} \right)^{-1}$$
$$\times \left(\frac{\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)}{p^{a_3-a_2+1}} \right)^{-s'} \left(1 - \frac{p^{a_3-a_2+1}\phi'(p)}{\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)} \right) \left(1 - \frac{\phi'(p)\widetilde{\alpha}^{(2)}_\lambda(\mathrm{Frob}_p)}{p^{a_3-a_2+2}} \right)^{-1} \times \bigwedge^2 \exp^*_\lambda$$

with the dual exponential map $\exp^*_\lambda : H^1(\mathbb{Q}_p, F^- \overline{V}^{(a)}_\lambda) \longrightarrow D_{\mathrm{dR}}(F^- \overline{V}_\lambda)$ *and where* ϕ *(resp.* ϕ'*) is the finite order character* $\lambda|_{G_3} \chi_3^{-a_3}$ *(resp.* $\lambda|_{G'_3} \chi_2^{a_2} \chi_3^{-a_3}$*) on* G_3 *(resp.* $G'_3 :=$ *the image of* $G_2 \longrightarrow G_2 \times G_3$, $g \mapsto g^{-1} \times g$*) and* s *(resp.* s'*) is the* p*-order of the conductor of* ϕ *(resp.* ϕ'*).*

We will prove this theorem at the end of this section.

Proposition 6.2. *Let* λ *be an arithmetic specialization of weight* (a_1, a_2, a_3) *satisfying* $a_1 \geq a_3 - 2 \geq a_2 \geq 0$ *and* $a_1 > a_2$. *Then the filtered module* $D_{\mathrm{dR}}(F^+ V_\lambda)$ *is canonically isomorphic to* $D_{\mathrm{dR}}(V_\lambda)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V_\lambda)$ *and the filtered module* $D_{\mathrm{dR}}(F^- \overline{V}^{(a)}_\lambda)$ *is canonically isomorphic to* $\mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda)$.

Proof. By Corollary 2.7, the action of the inetia subgroup of $G_{\mathbb{Q}_p}$ to the specialization of $\mathbb{T}^{\mathrm{n.ord},(a)}$ at λ has a decreasing filtration represented by:

$$\begin{pmatrix}
\chi_{\mathrm{cyc}}^{a_3}\phi_1 & * & * & * \\
0 & \chi_{\mathrm{cyc}}^{-a_2+a_3-1}\phi_2 & * & * \\
0 & 0 & \chi_{\mathrm{cyc}}^{-a_1+a_3-2}\phi_3 & * \\
0 & 0 & 0 & \chi_{\mathrm{cyc}}^{-a_1-a_2+a_3-3}\phi_4
\end{pmatrix}.$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are finite characters of $G_{\mathbb{Q}_p}$ which depend on λ. Hence, there exists a finite extension K of $\mathbb{Q}_p$ such that V_λ is an ordinary representation of G_K (recall that an ordinary representation is defined to be a p-adic representation of a p-adic field whose restriction to the inertia subgroup has a filtration with i-th graded piece having the inertia action through i-th power of the cyclotomic character). By a result of Perrin-Riou [P3], an ordinary representation is semi-stable in the sense of Fontaine. In particular V_λ is a de Rham representation of G_K. Since a potentially de Rham representation is a de Rham representation (see [Bu]) the Galois representation V_λ is a de Rham representation of $G_{\mathbb{Q}_p}$. Let us prove that $\mathrm{Fil}^0 D_{\mathrm{dR}}(F^+V_\lambda) = (F^+V_\lambda \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{G_{\mathbb{Q}_p}} = 0$. We have an exact sequence of G_K-modules:

$$0 \longrightarrow K_\lambda(\alpha^{(1)}\chi^{a_3}) \longrightarrow F^+V_\lambda \longrightarrow K_\lambda(\alpha^{(2)}\chi^{a_3-a_2-1}) \longrightarrow 0$$

where $\alpha^{(i)}$ is the specialization of $\widetilde{\alpha}^{(i)}$ by λ and $K_\lambda = \mathrm{Frac}(\lambda(R^{\mathrm{n.ord}}))$ is the field of definition of V_λ. Since the action of the inertia subgroup of G_K on both $K_\lambda(\alpha^{(1)}\chi^{a_3})$ and $K_\lambda(\alpha^{(2)}\chi^{a_3-a_2-1})$ is given by a strictly positive power of the cyclotomic character, we have

$$(K_\lambda(\alpha^{(1)}\chi^{a_3}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{G_{\mathbb{Q}_p}} = (K_\lambda(\alpha^{(2)}\chi^{a_3-a_2-1}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{G_{\mathbb{Q}_p}} = 0$$

thanks to the result of Tate [Ta] (3.3) Thm. 2. Hence $(F^+V_\lambda \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{G_{\mathbb{Q}_p}} = 0$, so we have a K_λ-linear injection

$$D_{\mathrm{dR}}(F^+V_\lambda) \longrightarrow D_{\mathrm{dR}}(V_\lambda)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V_\lambda).$$

Furthermore, we have $\dim_{K_\lambda} D_{\mathrm{dR}}(V_\lambda) = 4$ since V_λ is a de Rham representation and we have $\dim_{K_\lambda} D_{\mathrm{dR}}(F^+V_\lambda) = 2$ since a sub-representation of a de Rham representation is also a de Rham representation. By observing the Hodge-Tate weights under the condition $a_1 \geq a_3 - 2 \geq a_2 > 0$ and $a_1 > a_2$, we see that $\dim_{K_\lambda} \mathrm{Fil}^0 D_{\mathrm{dR}}(V_\lambda) = 2$. Hence the above injection is an isomorphism. The assertion about $D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda)$ is proved in the same way. □

Let us show that Theorem 3.2 is deduced from the theorem above:

Proof of Theorem 6.1 ⇒ *Theorem* 3.2. By Proposition 6.2, $\mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda)$ is canonically isomorphic to $D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda)$. Further, by the condition **III**, the kernel $H^1_f(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$ of the dual exponential map $\exp^* : H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda)$ coincides with the image of $H^1(\mathbb{Q}_p, F^+\overline{V}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$. Again, by the condition **III**, $H^2(\mathbb{Q}_p, F^+\overline{V}^{(a)}_\lambda)$ is trivial, which implies that the cokernel of $H^1(\mathbb{Q}_p, F^+\overline{V}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda)$ is naturally isomorphic to $H^1(\mathbb{Q}_p, F^-\overline{V}^{(a)}_\lambda)$. Thus, we have the natural commutative diagram:

$$\begin{array}{ccc} H^1_{/f}(\mathbb{Q}_p, \overline{V}^{(a)}_\lambda) & \xrightarrow{\exp^*} & \mathrm{Fil}^0 D_{\mathrm{dR}}(\overline{V}^{(a)}_\lambda) \\ \| & & \| \\ H^1(\mathbb{Q}_p, F^-\overline{V}^{(a)}_\lambda) & \xrightarrow[\exp^*]{} & D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda). \end{array}$$

By the same reason, we identify $H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$ and $H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)})$. We define $\overline{\Xi}^{(a)}$ to be the following composite map:

$$\bigwedge^2 H^1_{/f}(\mathbb{Q}_p, \overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \xrightarrow{\sim} \bigwedge^2 H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \xrightarrow{\overline{\Xi}^{(a)}_-} \overline{\mathcal{D}}^{(a)}.$$

Then the map $\Xi^{(a)}$ satisfies the desired interpolation property and this completes the proof. $\square$

Let $\overline{\mathbb{U}}^{(a)} = R^{\text{n.ord}}((\widetilde{\alpha}^{(1)})^{-1}\chi\widetilde{\chi}_3^{-1}\omega^{1-a})$ be an $R^{\text{n.ord}}[G_{\mathbb{Q}_p}]$-module defined at (4). Let $\mathbb{U}^{(a)} = R^{\text{n.ord}}(\widetilde{\alpha}^{(1)}\widetilde{\chi}_3\omega^a)$ be the $R^{\text{n.ord}}$-linear Kummer dual of $\overline{\mathbb{U}}^{(a)}$. Let us define

$$\mathcal{D}_{\mathbb{U}^{(a)}} = \Big((R^{\text{ord}}(\widetilde{\alpha}^{(1)})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\text{ur}})^{G_{\mathbb{Q}_p}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_3]]\Big) \otimes_{\mathbb{Z}_p} D_{\text{crys}}(\mathbb{Z}_p(1))^{\omega^a}, \tag{9}$$

$$\mathcal{D}_{\overline{\mathbb{U}}^{(a)}} = \Big((R^{\text{ord}}((\widetilde{\alpha}^{(1)})^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\text{ur}})^{G_{\mathbb{Q}_p}}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_3]]^{\iota}\Big) \otimes_{\mathbb{Z}_p} D_{\text{crys}}(\mathbb{Z}_p)^{\omega^{1-a}}, \tag{10}$$

where $D_{\text{crys}}(\mathbb{Z}_p(1))^{\omega^a}$ (resp. $D_{\text{crys}}(\mathbb{Z}_p)^{\omega^{1-a}}$) is the canonical lattice of $D_{\text{crys},\mathbb{Q}_p(\zeta_p)}(\mathbb{Q}_p(1))^{\omega^a}$ (resp. $D_{\text{crys},\mathbb{Q}_p(\zeta_p)}(\mathbb{Q}_p)^{\omega^{1-a}}$).

By [O1, Lemma 3.3] and by the fact that the action on $\mathbb{U}^{(a)}$ is given by universal cyclotomic character on the group G_3 modulo some unramified character, we have the lemma as follows:

Lemma 6.3. *The $R^{\text{n.ord}}$-module $\mathcal{D}_{\mathbb{U}^{(a)}}$ (resp. $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}}$) is free of rank one for each integer a with $1 \le a \le p-1$. Further, for any arithmetic specialization λ of $R^{\text{n.ord}}$, $\mathcal{D}_{\mathbb{U}^{(a)}} \otimes_{R^{\text{n.ord}}} \lambda(R^{\text{n.ord}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ (resp. $\mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \otimes_{R^{\text{n.ord}}} \lambda(R^{\text{n.ord}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$) is naturally identified with $D_{\text{dR}}(\mathbb{U}^{(a)}_\lambda)$ (resp. $D_{\text{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda)$).*

The following theorem is one of the two theorems used to prove Theorem 6.1, the main theorem of this section.

Theorem 6.4. *Assume the conditions* **III** *and* **IV**. *Then, for each integer with $1 \le a \le p-1$, there exists an $R^{\text{n.ord}}$-linear isomorphism*

$$\Xi_{\overline{\mathbb{U}}^{(a)}} : H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) \longrightarrow \mathcal{D}_{\overline{\mathbb{U}}^{(a)}}$$

such that, for every arithmetic specialization λ of weight (a_1, a_2, a_3) with $a_3 \ge 0$, we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}) & \xrightarrow{\Xi_{\overline{\mathbb{U}}^{(a)}}} & \mathcal{D}_{\overline{\mathbb{U}}^{(a)}} \\ \lambda\big\downarrow & & \big\downarrow\lambda \\ H^1(\mathbb{Q}_p, \overline{\mathbb{U}}^{(a)}_\lambda) & \xrightarrow[\overline{m}_\lambda]{} & D_{\text{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda), \end{array}$$

where the map $\overline{m}_\lambda$ is equal to:

$$(-1)^{a_3}a_3!\left(\frac{\widetilde{\alpha}^{(1)}_\lambda(\text{Frob}_p)}{p^{a_3}}\right)^{-s}\left(1 - \frac{p^{a_3}\phi(p)}{\widetilde{\alpha}^{(1)}_\lambda(\text{Frob}_p)}\right) \times \exp^*$$

and ϕ is the finite order character $\lambda|_{G_3}\chi_3^{-a_3}$ of G_3 and s is the p-order of the conductor of ϕ.

Theorem 6.4 is reduced to Proposition 6.5 as follows.

Proposition 6.5. *Assume the conditions* **III** *and* **IV**. *Then, for each integer with* $1 \leq a \leq p-1$, *we have an* $R^{\mathrm{n.ord}}$*-linear isomorphism*

$$\Xi_{\mathbb{U}^{(a)}} : \mathcal{D}_{\mathbb{U}^{(a)}} \longrightarrow H^1(\mathbb{Q}_p, \mathbb{U}^{(a)})$$

such that, for every arithmetic specialization λ *of weight* (a_1, a_2, a_3) *with* $a_3 \geq 0$, *we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{U}^{(a)}} & \xrightarrow{\Xi_{\mathbb{U}^{(a)}}} & H^1(\mathbb{Q}_p, \mathbb{U}^{(a)}) \\ \downarrow & & \downarrow \\ D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda) & \xrightarrow[m_\lambda]{} & H^1(\mathbb{Q}_p, \mathbb{U}^{(a)}_\lambda) \end{array}$$

where the map m_λ *is equal to:*

$$(-1)^{a_3} a_3! \left(\frac{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a_3}} \right)^{-s} \left(1 - \frac{p^{a_3}\phi(p)}{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)} \right) \times \exp$$

and ϕ *is the finite order character* $\lambda|_{G_3}\chi_3^{-a_3}$ *of* G_3 *and* s *is the* p*-order of the conductor of* ϕ.

We will prove Proposition 6.5 later in this section. Before giving the proof of the implication Prop. 6.5 ⇒ Thm. 6.4, we prepare the following lemma.

Lemma 6.6. *Assume the condition* **II**. *Let* $I^{(a_1,a_2,a_3)}_{l,m,n,u}$ *be the height four ideal* $(I^{(a_1,a_2,a_3)}_{l,m,n}, p^u)$ *of* $R^{\mathrm{n.ord}}$. *Let* S *be a cofinitely generated* $R^{\mathrm{n.ord}}$*-module and let* $S^\vee$ *be the Pontryagin dual of* S. *Then there exists an isomorphism*

$$\varprojlim_{l,m,n,u} S[I^{(a_1,a_2,a_3)}_{l,m,n,u}] \simeq \mathrm{Hom}_{R^{\mathrm{n.ord}}}(S^\vee, R^{\mathrm{n.ord}}).$$

Proof. For the proof of this lemma, it suffices to give the following isomorphisms for each $l, m, n, u \geq 0$:

$$\begin{aligned} S[I^{(a_1,a_2,a_3)}_{l,m,n,u}] &= \mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(S^\vee / I^{(a_1,a_2,a_3)}_{l,m,n,u}, \mathbb{Z}/p^u\mathbb{Z}) \\ &\simeq \mathrm{Hom}_{R^{\mathrm{n.ord}}_{l,m,n,u}}(S^\vee / I^{(a_1,a_2,a_3)}_{l,m,n,u}, R^{\mathrm{n.ord}}_{l,m,n,u}). \end{aligned}$$

Since $\mathrm{Hom}_{R^{\mathrm{n.ord}}_{l,m,n,u}}(S^\vee / I^{(a_1,a_2,a_3)}_{l,m,n,u}, R^{\mathrm{n.ord}}_{l,m,n,u})$ is isomorphic to $\mathrm{Hom}_{R^{\mathrm{n.ord}}}(S^\vee, R^{\mathrm{n.ord}}) \otimes_{R^{\mathrm{n.ord}}} R^{\mathrm{n.ord}}_{l,m,n,u}$ the lemma is proved by taking the projective limit with respect to l, m, n, u once we have the above identities. The first equality is nothing but the definition of the Pontryagin dual. In fact, since S is equal to $(S^\vee)^\vee$, we have

$$S[I^{(a_1,a_2,a_3)}_{l,m,n,u}] = \mathrm{Hom}_{\mathbb{Z}_p}(S^\vee, \mathbb{Q}_p/\mathbb{Z}_p)[I^{(a_1,a_2,a_3)}_{l,m,n,u}] = \mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(S^\vee / I^{(a_1,a_2,a_3)}_{l,m,n,u}, \mathbb{Z}/p^u\mathbb{Z}).$$

The last isomorphism is due to the fact that $R^{\mathrm{n.ord}}_{l,m,n,u}$ is a zero dimensional Gorenstein ring thanks to the assumption **II**. For the fundamental properties of zero dimensional Gorenstein rings and the modules over such, we refer the reader to [E]. □

Let us prove Theorem 6.4 assuming Proposition 6.5.

Proof of Proposition 6.5 $\Rightarrow$ *Theorem* 6.4. Let $\mathcal{A}_{\mathbb{U}^{(a)}}$ be $\mathbb{U}^{(a)}\otimes_{R^{\text{n.ord}}}(R^{\text{n.ord}})^\vee$ where $(R^{\text{n.ord}})^\vee$ is the Pontryagin dual of $R^{\text{n.ord}}$.

We have the following map:

$$\begin{aligned} H^1(\mathbb{Q}_p,\mathbb{U}^{(a)}) &\xrightarrow{\sim} \varprojlim_{l,m,n,u} H^1(\mathbb{Q}_p,\mathbb{U}^{(a)}/I^{(a_1,a_2,a_3)}_{l,m,n,u}) \\ &\xrightarrow{\sim} \varprojlim_{l,m,n,u} H^1(\mathbb{Q}_p,\mathcal{A}_{\mathbb{U}^{(a)}}[I^{(a_1,a_2,a_3)}_{l,m,n,u}]) \\ &\xrightarrow{\sim} \varprojlim_{l,m,n,u} H^1(\mathbb{Q}_p,\mathcal{A}_{\mathbb{U}^{(a)}})[I^{(a_1,a_2,a_3)}_{l,m,n,u}] \\ &\xrightarrow{\sim} \mathrm{Hom}_{R^{\text{n.ord}}}(H^1(\mathbb{Q}_p,\overline{\mathbb{U}}^{(a)}),R^{\text{n.ord}}) \end{aligned}$$

Note that, the first map is nothing but the definition and the second map is obtained by the assumption **II**. In fact, $R^{\text{n.ord}}$ is regular, hence Gorenstein, and we have an isomorphism $R^{\text{n.ord}}_{l,m,n,u}\simeq \mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(R^{\text{n.ord}}_{l,m,n,u},\mathbb{Z}/p^u\mathbb{Z})$ as an $R^{\text{n.ord}}$-module. The third map is defined naturally but the fact that this map is an isomorphism is due to the assumption **III**. The last isomorphism is due to Lemma 6.6 and the fact that $H^1(\mathbb{Q}_p,\overline{\mathbb{U}}^{(a)})$ is the Pontryagin dual of $H^1(\mathbb{Q}_p,\mathcal{A}_{\mathbb{U}^{(a)}})$.

We define an $R^{\text{n.ord}}$-linear map as follows:

$$\begin{aligned} H^1(\mathbb{Q}_p,\mathbb{U}^{(a)}) &\xrightarrow{\sim} \mathrm{Hom}_{R^{\text{n.ord}}}(H^1(\mathbb{Q}_p,\overline{\mathbb{U}}^{(a)}),R^{\text{n.ord}}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{R^{\text{n.ord}}}(\mathcal{D}_{\overline{\mathbb{U}}^{(a)}},R^{\text{n.ord}}) \end{aligned}$$

where the first map is the one obtained above and the second map is the $R^{\text{n.ord}}$-linear dual of $\overline{\Xi}_{\overline{\mathbb{U}}^{(a)}}$ given by Proposition 6.5. We define the $R^{\text{n.ord}}$-linear map $\Xi_{\mathbb{U}^{(a)}}$ to be the composite map:

$$H^1(\mathbb{Q}_p,\mathbb{U}^{(a)}) \longrightarrow \mathrm{Hom}_{R^{\text{n.ord}}}(\mathcal{D}_{\overline{\mathbb{U}}^{(a)}},R^{\text{n.ord}}) \cong \mathcal{D}_{\mathbb{U}^{(a)}}.$$

□

From now on, for a moment, we prepare for the proof of Proposition 6.5.

By taking the formal tensor product $\widehat{\otimes}_{\mathbb{Z}_p}R^{\text{ord}}(\widetilde{\alpha}^{(1)})$ of the map obtained in Proposition 4.3, we obtain the following proposition:

Proposition 6.7. *We have an $R^{\text{n.ord}}$-linear homomorphism:*

$$\Xi^{\text{ur}}_{\mathbb{U}^{(a)}}:\mathcal{D}_{\mathbb{U}^{(a)}}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}^{\text{ur}}_p \longrightarrow \frac{H^1(\mathbb{Q}^{\text{ur}}_p,\mathbb{U}^{(a)})}{H^0(\mathbb{Q}^{\text{ur}}_p,R^{\text{ord}}(\widetilde{\alpha}^{(1)}\omega^a))}$$

such that we have the following commutative diagram for every arithmetic character λ of weight (a_1,a_2,a_3) satisfying $a_3\geq 0$:

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{U}^{(a)}}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}^{\text{ur}}_p & \xrightarrow{\Xi^{\text{ur}}_{\mathbb{U}^{(a)}}} & \dfrac{H^1(\mathbb{Q}^{\text{ur}}_p,\mathbb{U}^{(a)})}{H^0(\mathbb{Q}^{\text{ur}}_p,R^{\text{ord}}(\widetilde{\alpha}^{(1)}\omega^a))} \\ \big\downarrow & & \big\downarrow \\ D^{\text{ur}}_{\text{dR}}(\mathbb{U}^{(a)}_\lambda) & \xrightarrow{m_\lambda} & H^1(\mathbb{Q}^{\text{ur}}_p,\mathbb{U}^{(a)}_\lambda), \end{array}$$

where m_λ is the map

$$(-1)^{a_3} a_3! \exp \circ \left(\frac{\sigma}{p^{a_3}}\right)^{-s} \left(1 - \frac{p^{a_3}\phi(p)}{\sigma}\right) \left(1 - \frac{\phi(p)\sigma}{p^{a_3+1}}\right)^{-1},$$

and ϕ is the finite order character $\lambda|_{G_3}\chi_3^{-a_3}$ of G_3 and s is the p-order of the conductor of ϕ.

Proof. In fact, $\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{U}^{(a)})}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)}\omega^a))}$ is isomorphic to $\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_3]](\widetilde{\chi}_3\omega^a))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p(\omega^a))} \widehat{\otimes}_{\mathbb{Z}_p} R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)})$ since $R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)})$ is an unramified representation of $G_{\mathbb{Q}_p}$. We define the desired map $\Xi^{\mathrm{ur}}_{\mathbb{U}^{(a)}}$ to be $\Xi^{\mathrm{ur},(0,0,1)} \otimes 1$. The map $\Xi^{\mathrm{ur}}_{\mathbb{U}^{(a)}}$ is an $R^{\mathrm{n.ord}}$-linear homomorphism because $\Xi^{\mathrm{ur},(0,0,1)}$ is a $\mathbb{Z}_p[[G_3]]$-linear homomorphism and R^{ord} is isomorphic to $R^{\mathrm{ord}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[G_3]]$. The commutativity of the diagram follows from that of Proposition 4.3 since the arithmetic specialization λ of $R^{\mathrm{n.ord}}$ decomposes as $\lambda = \kappa \widehat{\otimes} \eta$ with an arithmetic specialization κ of R^{ord} and an arithmetic character η of $\mathbb{Z}_p[[G_3]]$. Similarly, the $R^{\mathrm{n.ord}}$-module $\mathbb{U}^{(a)}$ is decomposed as $\mathbb{U}^{(a)} = \mathbb{U}^{\mathrm{ord}} \widehat{\otimes} \mathbb{Z}_p[[G_3]](\widetilde{\chi}_3\omega^a)$ where $\mathbb{U}^{\mathrm{ord}}$ is a free R^{ord}-module of rank one on which $G_{\mathbb{Q}_p}$ acts via the unramified character $\widetilde{\alpha}^{(1)}$. As a consequence, we have $\mathbb{U}^{(a)}_\lambda = \mathbb{U}^{\mathrm{ord}}_\kappa \otimes K_\eta(\eta\omega^a)$ where K_η is the finite extension of $\mathbb{Q}_p$ generated by the image of η. Since the exponential map for $\mathbb{U}^{(a)}_\lambda$ is the connecting homomorphism of the cohomology of $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-modules associated to the short exact sequence:

$$0 \longrightarrow \mathbb{U}^{(a)}_\lambda \longrightarrow (B^{f=1}_{\mathrm{crys}} \oplus B^+_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} \mathbb{U}^{(a)}_\lambda \longrightarrow B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{U}^{(a)}_\lambda \longrightarrow 0.$$

We have the following commutative diagram

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p^{\mathrm{ur}}, K_\eta(\eta\omega^a)) \otimes \mathbb{U}^{\mathrm{ord}}_\kappa & \xleftarrow{\exp\otimes 1} & D^{\mathrm{ur}}_{\mathrm{dR}}(K_\eta(\eta\omega^a)) \otimes \mathbb{U}^{\mathrm{ord}}_\kappa \\
\| & & \| \\
H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{U}^{(a)}_\lambda) & \xleftarrow{\exp} & D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)
\end{array}$$

where the map exp on the upper line is the exponential map for the $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module $K_\eta(\eta\omega^a)$ and the map exp on the lower line is the exponential map for the $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module $\mathbb{U}^{(a)}_\lambda$. Hence we obtain the desired commutative diagram. □

Lemma 6.8. *Let a be an integer with $1 \le a \le p-1$ and let λ be an arithmetic specialization of $R^{\mathrm{n.ord}}$ of weight (a_1, a_2, a_3) satisfying $a_1 \ge a_3 - 2 \ge a_2 \ge 0$. The following statements hold:*

(1) *The* $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$*-invariant part of $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$ is equal to $D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$.*

(2) *The operator σ on $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$ induces the multiplication by the Satake parameter $\alpha^{(1)}_p$ on the* $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$*-invariant part $D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$.*

(3) *The restriction map $H^1(\mathbb{Q}_p, \mathbb{U}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{U}^{(a)}_\lambda)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is an isomorphism.*

Proof. The assertion (1) is nothing but the definition of $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$ and $D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$. Let us prove the statement (2). We recall that $\mathbb{U}^{(a)}$ is decomposed as $\overline{\mathbb{U}}^{(a)} = \overline{\mathbb{U}}^{\mathrm{ord}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[G_3]](\widetilde{\chi}_3\omega^a)$ where $\mathbb{U}^{\mathrm{ord}}$ is a representation of rank one over R^{ord} on which $G_{\mathbb{Q}_p}$ acts via $\widetilde{\alpha}^{(1)}$. We have the decomposition $\mathbb{U}^{(a)}_\lambda = \mathbb{U}^{\mathrm{ord}}_\kappa \otimes_{\mathbb{Q}_p} K_\eta(\eta\omega^a)$ corresponding to the decomposition $\lambda = \kappa \widehat{\otimes} \eta$ where κ is an arithmetic specialization of R^{ord} and η is a character of $\mathbb{Z}_p[[G_3]]$, where K_η is the extension of $\mathbb{Q}_p$ generated by the image of η. Since we have $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda) \simeq D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa) \otimes D_{\mathrm{dR}}(K(\eta\omega^a))$ and $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p)} = D_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)$, it suffices to show that the arithmetic Frobenius σ on $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)$ induces the multiplication by $\alpha^{(1)}_p$ on $D_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)$. Since $\mathbb{U}^{\mathrm{ord}}_\kappa$ is an unramified representation of $G_{\mathbb{Q}_p}$, $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)$ is isomorphic to $\mathbb{U}^{\mathrm{ord}}_\kappa \otimes \widehat{\mathbb{Q}}^{\mathrm{ur}}_p$ and the operator σ on $D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa)$ is identified with $1\otimes\sigma$ on $\mathbb{U}^{\mathrm{ord}}_\kappa \otimes \widehat{\mathbb{Q}}^{\mathrm{ur}}_p$, which is equal to $\sigma^{-1}\otimes 1 = \mathrm{Frob}_p \otimes 1$ on the invariant part $D_{\mathrm{dR}}(\mathbb{U}^{\mathrm{ord}}_\kappa) = (\mathbb{U}^{\mathrm{ord}}_\kappa \otimes \widehat{\mathbb{Q}}^{\mathrm{ur}}_p)^{\sigma\otimes\sigma}$. This completes the proof of (2). For the proof of (3), the restriction map $H^1(\mathbb{Q}_p, \mathbb{U}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}^{\mathrm{ur}}_p, \mathbb{U}^{(a)}_\lambda)^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p)}$ is surjective since $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p)$ has cohomological dimension one. The kernel of the restriction map is equal to $H^1(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p, (\mathbb{U}^{(a)}_\lambda)^{G_{\mathbb{Q}^{\mathrm{ur}}_p}})$, which is zero since $(\mathbb{U}^{(a)}_\lambda)^{G_{\mathbb{Q}^{\mathrm{ur}}_p}} = 0$. □

Proof of Proposition 6.5. Recall that, by definition, the $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p)$-invariant part of the module $\widehat{\mathbb{Z}}^{\mathrm{ur}}_p[[G]] \widehat{\otimes}_{\mathbb{Z}_p} R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)}\omega^a)$ is $\mathcal{D}_{\mathbb{U}^{(a)}} \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p[[G]]$. By Proposition 4.3 and Corollary 4.4, we have an exact sequence:

$$0 \longrightarrow \mathcal{D}_{\mathbb{U}^{(a)}} \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p[[G]] \longrightarrow \left(\frac{H^1(\mathbb{Q}^{\mathrm{ur}}_p, \mathbb{U}^{(a)})}{H^0(\mathbb{Q}^{\mathrm{ur}}_p, R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)}\omega^a))} \right) \longrightarrow R^{\mathrm{ord}}(\widetilde{\alpha}^{(1)}\omega^{a-1}\chi^{-1}) \longrightarrow 0.$$

Taking $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_p/\mathbb{Q}_p)$-invariant part of the above sequence, we obtain an isomorphism $\Xi_{\mathbb{U}^{(a)}} : \mathcal{D}_{\mathbb{U}^{(a)}} \stackrel{\sim}{\longrightarrow} H^1(\mathbb{Q}_p, \mathbb{U}^{(a)})$. For each λ we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{Q}^{\mathrm{ur}}_p, \mathbb{U}^{(a)}_\lambda)^{G_{\mathbb{Q}_p}} & \stackrel{\exp}{\longleftarrow} & D^{\mathrm{ur}}_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)^{G_{\mathbb{Q}_p}} \\ \Vert & & \Vert \\ H^1(\mathbb{Q}_p, \mathbb{U}^{(a)}_\lambda) & \stackrel{\exp}{\longleftarrow} & D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda) \end{array}$$

where the map exp in the upper (resp. lower) line is the Bloch-Kato exponential map of $\mathbb{U}^{(a)}_\lambda$ as a $G_{\mathbb{Q}^{\mathrm{ur}}_p}$-module (resp. $G_{\mathbb{Q}_p}$-module). The equalities in the diagram are obtained by Lemma 6.8. The commutativity of the diagram is due to the fact that the exponential map of $\mathbb{U}_\lambda$ as a $G_{\mathbb{Q}^{\mathrm{ur}}_p}$-module (resp. $G_{\mathbb{Q}_p}$-module) is the connecting homomorphism for the Galois cohomology of the short exact sequence:

$$0 \longrightarrow \mathbb{U}^{(a)}_\lambda \longrightarrow (B^{f=1}_{\mathrm{crys}} \oplus B^+_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} \mathbb{U}^{(a)}_\lambda \longrightarrow B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \mathbb{U}^{(a)}_\lambda \longrightarrow 0,$$

of $G_{\mathbb{Q}^{\mathrm{ur}}_p}$-modules (resp. $G_{\mathbb{Q}_p}$-modules). Hence we have the required commutative diagram. □

Let $\overline{\mathbb{S}}^{(a)} = R^{\mathrm{n.ord}}((\widetilde{\alpha}^{(2)})^{-1}\chi^2\widetilde{\chi}_2\widetilde{\chi}_3^{-1}\omega^{1-a})$ be an an $R^{\mathrm{n.ord}}[G_{\mathbb{Q}_p}]$-module defined at (4). Let $\mathbb{S}^{(a)} = R^{\mathrm{n.ord}}(\widetilde{\alpha}^{(2)}\chi^{-1}\widetilde{\chi}_2^{-1}\widetilde{\chi}_3\omega^a)$ be the $R^{\mathrm{n.ord}}$-linear Kummer dual of $\overline{\mathbb{S}}^{(a)}$. We will define similarly the module $\mathcal{D}_{\mathbb{S}^{(a)}}$ which will interpolate lattices of $D_{\mathrm{dR}}(\mathbb{S}^{(a)}_\lambda)$ when λ varies.

However, since the action of $G_{\mathbb{Q}_p}$ on $\mathbb{S}^{(a)}$ is more complicated than that of $\mathbb{U}$, we will first need a careful change of coordinate to find a similar statement. Let us define G_i' for $i = 1, 2, 3$ as follows:

$$\begin{aligned} G_1' &= G_1 \\ G_2' &= \text{the image of } G_2 \longrightarrow G_2 \times G_3, g \mapsto g \times g \\ G_3' &= \text{the image of } G_2 \longrightarrow G_2 \times G_3, g \mapsto g^{-1} \times g \end{aligned}$$

where, we canonically identify G_1, G_2 and G_3 to $1 + p\mathbb{Z}_p$ via characters $\widetilde{\chi}_1$, $\widetilde{\chi}_2$ and $\widetilde{\chi}_3$. We also identify G_2' and G_3' via the canonical character $\widetilde{\chi}_2' : G_2' \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and $\widetilde{\chi}_3' : G_3' \xrightarrow{\sim} 1 + p\mathbb{Z}_p$, by which we define arithmetic characters on G_2' and G_3'. Since $p \neq 2$, this map gives an isomorphism between $G_1 \times G_2 \times G_3$ and $G_1' \times G_2' \times G_3'$. In order to make clear the situation where we consider the new coordinate, we will denote the same ring $R^{\mathrm{n.ord}}$ by $(R^{\mathrm{n.ord}})'$.

By using the new coordinate, we have $\overline{\mathbb{S}}^{(a)} = (R^{\mathrm{n.ord}})'((\widetilde{\alpha}^{(2)})^{-1}\chi^2(\widetilde{\chi}_3')^{-1}\omega^{1-a})$ and $\mathbb{S}^{(a)} = (R^{\mathrm{n.ord}})'(\widetilde{\alpha}^{(2)}\chi^{-1}\widetilde{\chi}_3'\omega^a)$. Then, we define

$$\mathcal{D}_{\mathbb{S}^{(a)}} := \left((R^{\mathrm{n.ord}})'(\widetilde{\alpha}^{(2)})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}\right)^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Z}_p} D_{\mathrm{crys}}(\mathbb{Z}_p(-1))^{\omega^a} \tag{11}$$

$$\mathcal{D}_{\overline{\mathbb{S}}^{(a)}} := \mathrm{Hom}_{(R^{\mathrm{n.ord}})'}(\mathcal{D}_{\mathbb{S}^{(a)}}, (R^{\mathrm{n.ord}})') \tag{12}$$

where $D_{\mathrm{crys}}(\mathbb{Z}_p(2))^{\omega^a}$ is the canonical lattice of ω^a-part $D_{\mathrm{crys}}(\mathbb{Q}_p(-1))^{\omega^a}$ of $D_{\mathrm{crys},\mathbb{Q}(\zeta_p)}(\mathbb{Q}_p(-1))$. As before, by [O1, Lemma 3.3], the $(R^{\mathrm{n.ord}})'$-modules $\mathcal{D}_{\mathbb{S}^{(a)}}$ and $\mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$ are free of rank one.

Lemma 6.9. *For any arithmetic specialization λ of $(R^{\mathrm{n.ord}})'$, $\mathcal{D}_{\mathbb{S}^{(a)}} \otimes_{(R^{\mathrm{n.ord}})'} \lambda((R^{\mathrm{n.ord}})') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $\mathcal{D}_{\overline{\mathbb{S}}^{(a)}} \otimes_{(R^{\mathrm{n.ord}})'} \lambda((R^{\mathrm{n.ord}})') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$) is naturally identified with $D_{\mathrm{dR}}(\mathbb{S}_\lambda^{(a)})$ (resp. $D_{\mathrm{dR}}(\overline{\mathbb{S}}_\lambda^{(a)})$).*

After the change of coordinate above, the proof of the following theorem is done exactly in the same way than the one of Theorem 6.4 and relies principally on Proposition 4.3 so we give us the right to omit it.

Theorem 6.10. *Assume the conditions* **III** *and* **IV**. *Then, for each integer with $1 \leq a \leq p-1$, we have an $(R^{\mathrm{n.ord}})'$-linear isomorphism*

$$\overline{\Xi}^{(a)}_{\overline{\mathbb{S}}^{(a)}} : H^1(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)}) \longrightarrow \mathcal{D}_{\overline{\mathbb{S}}^{(a)}}$$

such that, for every arithmetic specialization λ of weight (a_1', a_2', a_3') with $a_3' \geq -1$, we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, \overline{\mathbb{S}}^{(a)}) & \xrightarrow{\overline{\Xi}_{\overline{\mathbb{S}}^{(a)}}} & \mathcal{D}_{\overline{\mathbb{S}}^{(a)}} \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}_p, \overline{\mathbb{S}}_\lambda^{(a)}) & \xrightarrow[\overline{m}_\lambda]{} & D_{\mathrm{dR}}(\overline{\mathbb{S}}_\lambda^{(a)}) \end{array}$$

where the map $\overline{m}_\lambda$ is equal to:

$$(-1)^{a'_3-1}(a'_3+1)!\left(\frac{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}{p^{a'_3+1}}\right)^{-s}\left(1-\frac{p^{a'_3+1}\phi'(p)}{\widetilde{\alpha}^{(1)}_\lambda(\mathrm{Frob}_p)}\right)\times\exp^*$$

and ϕ' is the finite order character $\lambda|_{G'_3}(\chi'_3)^{-a'_3}$ of G'_3 and s is the p-order of the conductor of ϕ'.

Finally, we finish the proof of Theorem 6.1 combining the results obtained in this section.

Proof of Theorem 6.1. By the definition of the $\overline{\mathbb{S}}^{(a)}$ and $\overline{\mathbb{U}}^{(a)}$ given in (4) and by the exact sequence (8) of the previous section, we will reduce the problem to similar results for interpolation on $\overline{\mathbb{S}}^{(a)}$ and $\overline{\mathbb{U}}^{(a)}$. Recall that we have the exact sequence (8) which induces a canonical isomorphism

$$\bigwedge^2 H^1(\mathbb{Q}_p, F^-\overline{\mathbb{T}}^{\mathrm{n.ord},(a)}) \simeq H^1(\mathbb{Q}_p,\overline{\mathbb{S}}^{(a)})\otimes_{R^{\mathrm{n.ord}}} H^1(\mathbb{Q}_p,\overline{\mathbb{U}}^{(a)}).$$

By taking the Galois cohomology of the sequence (6), we have the following exact sequence for every arithmetic specialization λ:

$$0 \longrightarrow D_{\mathrm{dR}}(\overline{\mathbb{S}}^{(a)}_\lambda) \longrightarrow D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda) \longrightarrow D_{\mathrm{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda) \longrightarrow H^1(\mathbb{Q}_p,\overline{\mathbb{S}}^{(a)}_\lambda\otimes B_{\mathrm{dR}}).$$

However, since $\overline{\mathbb{S}}^{(a)}_\lambda$, $F^-\overline{V}^{(a)}_\lambda$ and $\overline{\mathbb{U}}^{(a)}_\lambda$ are de Rham representations of $G_{\mathbb{Q}_p}$, we have $\dim D_{\mathrm{dR}}(\overline{\mathbb{S}}^{(a)}_\lambda)=\dim D_{\mathrm{dR}}(\overline{\mathbb{U}}^{(a)}_\lambda)=1$ and $\dim D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda)=2$ so that the third map in the exact sequence is surjective, which induces a canonical isomorphism $\bigwedge^2 D_{\mathrm{dR}}(F^-\overline{V}^{(a)}_\lambda)=D_{\mathrm{dR}}(\mathbb{S}^{(a)}_\lambda)\otimes D_{\mathrm{dR}}(\mathbb{U}^{(a)}_\lambda)$. Thus, Theorem 6.1 is reduced to construction of the Coleman map for $\overline{\mathbb{S}}^{(a)}$ and $\overline{\mathbb{U}}^{(a)}$. We complete the proof if we note that a specialization at weight a'_3 with respect to the second coordinate corresponds to a specialization at weight a_3-a_2 with respect to the original coordinate. □

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