Iwasawa Theory for Hida deformations with complex multiplication

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My subject of research is closely related to Greenberg's conjectural program on generalizing Iwasawa theory to families of Galois representations. In his article [G], he proposes the study of Iwasawa theory for families of nearly ordinary Galois representations $\widetilde{T}$ finite over deformation algebras $\widetilde{R}$. Though his plan is a tentative conjecture where the (conjectural) definition of the analytic p-adic L-functions for such $\widetilde{T}$ is still vague, he nevertheless indicates a fascinating direction of research.

In previous work on this subject, I focused on the first non-trivial example with which Greenberg's plan is concerned, namely two-variable Hida deformations. The articles [O1], [O2] and [O3] established Iwasawa theory for Hida deformations $\mathcal{T}$ *without* complex multiplication. A Hida deformation $\mathcal{T}$ is roughly associated to p-adic family of elliptic cuspforms f_k where the weight k of f_k varies in a p-adic parameter space. In the process one discovers new phenomena which do not arise in the usual cyclotomic Iwasawa theory for ordinary motives. For instance, one finds that a detailed study of complex and p-adic periods in families is essential to even cogently formulate the definition of analytic p-adic L-functions in general situations, a fact that does not seem to have been observed before. A consequence of this work is a formulation of Iwasawa theory for general nearly ordinary families of Galois representation that is more precise than before (cf. [O4]) especially with regards the analytic p-adic L-function.

In this work, based on this motivation, we study Iwasawa theory also for Hida deformations $\mathcal{T}$ *with* complex multiplication by an imaginary quadratic field K. For the associated family of f_k, $\mathcal{T}$ with complex multiplication the one where f_k is a lift of a grossencharacter ρ_k of weight $k-1$ on K for each k. Note that grossencharacters are modular forms on the group $GL(1)_{/K}$. From this point of view, Iwasawa theory has been studied previously by Katz, Coates-Wiles, Colmez, Yager, de Shalit, Rubin, Tilouine and other authors using tools from the theory of complex multiplication (eg. the Euler system of elliptic units and the evaluation of Eisenstein series at CM points). On the other hand, forgetting about the complex multiplication, f_k itself is a modular form on the group $GL(2)_{/\mathbb{Q}}$. From such another point of view, we have other tools such as modular symbols and the Beilinson-Kato Euler system, as is also taken in [O1], [O2] and [O3] (Note however that Beilinson-Kato elements cannot be used to bound the size of Selmer group in the CM case because the image of rank two Galois representation for a CM modular form is two small). Thus, there are two completely different approaches to Hida families with complex multiplication and it seems to us that the relation between the approach via $GL(1)_{/K}$ and that via $GL(2)_{/\mathbb{Q}}$ is not so obvious. The purpose of this article is to clarify some aspects of this relation.

More precisely, we will compare the algebraic and analytic p-adic L-functions as well as Iwasawa Main Conjecture for them from these two points of view. The situation is summarized in the following diagram which shows the relations of ideals in a two-variable Iwasawa algebra $A[[\widetilde{\Gamma}]]$ with $\widetilde{\Gamma} \cong \mathbb{Z}_p$ the Galois group of $\mathbb{Z}_p^2$-extension of K and A a finite extension of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$:

(1)
$$\begin{tikzcd}
(L_p^{\mathrm{alg}}(GL(1)_{/K})) \arrow[r, equal, "\text{(a)}"] \arrow[d, equal, "\text{(c)}"'] & (L_p^{\mathrm{anal}}(GL(1)_{/K})) \arrow[d, dotted, dash, "\text{(d)}"] \\
(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) \arrow[r, dotted, dash, "\text{(b)}"'] & (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})).
\end{tikzcd}$$

Two objects on the left are algebraic p-adic L-functions, which are defined to be the characteristic ideals of Selmer groups in each context. The ideal $(L_p^{\mathrm{alg}}(GL(1)_{/K}))$ is the characteristic ideal of the Galois group of certain infinite Galois extension over $\mathbb{Z}_p^2$-extension of K. The ideal $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ is the characteristic ideal of the Pontrjagin dual of the Selmer group $\mathrm{Sel}_{\mathcal{T}}$. Two objects on the right are analytic p-adic L-functions which interpolate critical values of L-functions. The analytic p-adic L-function $L_p^{\mathrm{anal}}(GL(1)_{/K})$ is constructed Katz, Yager, de Shalit, Tilouine. The analytic p-adic L-function $L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})$ is constructed by Kitagawa, Greenberg, Panchishkin, Fukaya and the first author. However, for certain reasons, we have to take the one by Kitagawa which is the best candidate for the Iwasawa Main conjecture.

The relations in the diagram are explained below:

(a) The upper line is two-variable Iwasawa Main Conjecture from the point of view of $GL(1)_{/K}$. The equality is shown by Rubin (cf. [Ru1], [Ru2]) under fairly general conditions.

(b) The lower line is two-variable Iwasawa Main Conjecture from the point of view of $GL(2)_{/\mathbb{Q}}$ which was first formulated in [G, Chapter 4] and later refined by the first author in [O3, Conj. 2.4]. Note that the lower line makes sense for any Hida family, not just the CM ones. In fact, in [O3], we obtained the inequality $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) \supset (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ but only in the non-CM case. However, no equality was known between $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ and $(L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ for the CM case.

(c) In the algebraic side, it is not difficult to show $(L_p^{\mathrm{alg}}(GL(1)_{/K})) = (L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}}))$ by calculation of Galois cohomology.

Now there rests the part (d) the main theme in this paper. $(L_p^{\mathrm{anal}}(GL(1)_{/K}))$ is constructed via the theory of complex multiplication; $(L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ via the theory of modular symbols. These analytic p-adic L-functions are *a priori* different but we conjecture as follows:

Conjecture . *We have the equality of ideal* $(L_p^{\mathrm{anal}}(GL(1)_{/K})) = (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ *at* (d) *in the diagram.*

Note that, according to (a), (b) and (c), in most cases, Conjecture is equivalent to the Iwasawa main conjecture (b) for CM Hida deformations. The main result is the following theorem.

Main Theorem . *The above conjecture is true under certain assumptions, namely the irreducibility of the associated mod p representation and the vanishing of a certain μ-invariant.*

As seen from the diagram (1), we have an immediate corollary as follows.

Corollary . *Iwasawa Main Conjecture* $(L_p^{\mathrm{alg}}(GL(2)_{/\mathbb{Q}})) = (L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}}))$ *at* (b) *in the diagram is true for CM Hida deformations satisfying assumptions in the theorem and the assumption required in Rubin's results* ([Ru1]*,* [Ru2])*.*

Here we give some idea of the principal difficulties involved. Indeed, it may seem at first sight that Conjecture should be more or less immediate, since after all, the two p-adic L-functions interpolate the same set of L-values. However, there are two main obstructions to making such a conclusion. The first is that the periods that occur in the two interpolation formulae are not the same and need to be related to each other. Such "p-integral period relation" is uaually requires a hard work as is done in [Pr1] and [Pr2] recently. First, we established the requisite period relation up to p-adic units. But having done that, one is faced with a second difficulty which may be more formally described as follows. We explain briefly about the second difficulty as well as the argument of the proof. We are given two different elements $F(X_1, X_2)$ and $G(X_1, X_2)$ in $A[[X_1, X_2]]$ ($A[[X_1, X_2]]$ should be $A[[\widetilde{\Gamma}]]$ and $F(X_1, X_2)$, $G(X_1, X_2)$ should be $L_p^{\mathrm{anal}}(GL(1)_{/K})$ and $L_p^{\mathrm{anal}}(GL(2)_{/\mathbb{Q}})$ in the diagram (1)). A priori, we know no divisibility between $F(X_1, X_2)$ and $G(X_1, X_2)$. If we have established the period relation up to p-units above, Weierstrauss preparation shows that the elements $F(X_1, X_2)|_{X_2=(1+p)^n-1}, G(X_1, X_2)|_{X_2=(1+p)^n-1} \in A[[X_1]]$ is equal modulo multiplication by a unit u_n in A for each $n \geq 0$ (Note, however, that there seems to be no systematic choice of the constant u_n for varying n since there will be no canonical choice of modular symbol period). This is not sufficient to deduce the divisibility between $F(X_1, X_2)$ and $G(X_1, X_2)$ and we will can construct a counter example by an example similar to the one we explain below for the specialization of power series algebras in one-variable. Hence, we has developed an argument which allows us to deduce the desired divisibility under the assumption on the Iwasawa μ-invariant for $F(X_1, X_2)|_{X_2=(1+p)^n-1}$ or $G(X_1, X_2)|_{X_2=(1+p)^n-1} \in A[[X_1]]$. This proves our main theorem. The work is written in our paper [OP], which is available quite soon.

References

[G] R. Greenberg, *Iwasawa theory and p-adic deformations of motives*, Proceedings of Symposia in Pure Math. 55 II, 193-223, 1994.

[O1] T. Ochiai, *A generalization of the Coleman map for Hida deformations*, the American Jour. of Mathematics, **125**, 849-892, 2003.

[O2] T. Ochiai, *Euler system for Galois deformation*, Annales de l'Institut Fourier, vol **55**, fascicule 1, pp 113-146, 2005.

[O3] T. Ochiai, *On the two-variable Iwasawa Main conjecture for Hida deformations*, Compositio Mathematica, vol **142**, 1157–1200, 2006.

[O4] T. Ochiai, *p-adic L-functions for Galois deformations and related problems on periods*, in the conference proceeding for the 8th Autumn workshop on Number theory (held in 2005 at Hakuba).

[OP] T. Ochiai, K. Prasanna, *Two-variable Iwasawa theory for Hida family with complex multiplication*, in preparation.

[Pr1] K. Prasanna, *Integrality of a ratio of Petersson norms and level-lowering congruences*, Ann. of Math., Vol. **163**, No. 3, 901-967, 2006.

[Pr2] K. Prasanna, *Arithmetic properties of the Shimura-Shintani-Waldspurger correspondence*, Preprint.

[Ru1] K. Rubin, The "main conjectures" of Iwasawa theory for imaginary quadratic fields, Invent. Math. **103**, no. 1, 25–68. 1991.

[Ru2] K. Rubin, *More "main conjectures" for imaginary quadratic fields*, Elliptic curves and related topics, 23–28, CRM Proc. Lecture Notes, 4, Amer. Math. Soc., Providence, RI, 1994.