l - INDEPENDENCE OF THE TRACE OF MONODROMY

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Abstract. Let X be a proper smooth variety over a local field K of mixed characteristics and let l be a prime number different from the characteristic of the residue field of K. Let I_K be the inertia subgroup of $G_K := \mathrm{Gal}(\overline{K}/K)$. Our main result is the l-independence of the alternating sum of traces of $g \in I_K$ on $H^i(\overline{X}, \mathbb{Q}_l)$ and its comparison with the traces on p -adic cohomology.

0.Introduction

Let K be a complete discrete valuation field with finite residue field $\mathbb{F}_{p^h}$, G_K the absolute Galois group of K, I_K the inertia subgroup of G_K and W_K the Weil group of K. Recall that the Weil group is a subgroup of G_K defined by the following exact sequence:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & I_K & \longrightarrow & W_K & \stackrel{u}{\longrightarrow} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/h\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \bigcap & & \bigcap & & \| & & \\ 0 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) & \longrightarrow & \mathrm{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p) & \longrightarrow & 0, \end{array}$$

where the map u is defined by $g \longmapsto f^{u(g)}$ for the geometric Frobenius $f : x \longmapsto x^{1/p}$ in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ and $\overline{F}$ denotes the separable closure of F for any field F. We define a subset W_K^+ of W_K to be

$$W_K^+ := \{g \in W_K | u(g) \geq 0\}.$$

Let X be a variety over K (Throughout the paper, a variety X over a field K means a reduced irreducible scheme X separated and of finite type over K). Throughout this paper, $\overline{X}$ means the scalar extension $X \underset{K}{\otimes} \overline{K}$. We denote by l a prime number $\neq p$. We consider the traces of the action of elements of W_K^+ or W_K on the compact support etale cohomology $H_c^i(\overline{X}, \mathbb{Q}_l) := \varprojlim_n H_c^i(\overline{X}, \mathbb{Z}/l^n\mathbb{Z}) \underset{\mathbb{Z}_l}{\otimes} \mathbb{Q}_l$. Let us recall the following classical conjecture.

Conjecture([S-T]). For any variety X over K and $g \in W_K^+$, $\mathrm{Tr}(g^*; H_c^i(\overline{X}, \mathbb{Q}_l))$ is a rational integer which is independent of the choice of l.

Remark. If X is a d-dimensional proper smooth variety, the conjecture above holds for $i = 0, 1, 2d-1, 2d$ [SGA7-1]. If X has good reduction, the above conjecture is true for any i due to the Weil conjecture proved by P. Deligne [De].

In this paper, we shall prove the following weak versions of the conjecture:

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Proposition A (Proposition 2.1). The trace $\mathrm{Tr}(g^*; H^i_c(\overline{X}, \mathbb{Q}_l))$ is an algebraic integer for any variety X over K and any $g \in W_K^+$.

Theorem B (Theorem 2.4). The alternating sum $\sum_{i=0}^{2d}(-1)^i \mathrm{Tr}(g^*; H^i_c(\overline{X}, \mathbb{Q}_l))$ is a rational integer which is independent of l for any variety X over K and any $g \in W_K^+$.

Remark. Even if K is a complete discrete valuation field whose residue field k is a perfect field not necessarily of finite order, the results of Proposition A and Theorem B are valid by defining W_K^+ to be the inertia group I_K.

For certain types of varieties, Theorem 2.4 implies the above conjecture (see Corollary 2.5).

We see the l-independence for the Swan conductor as a direct consequence of Theorem B. The definition of Swan conductor is reviewed in §2. Then the following statement is an immediate corollary of Theorem B.

Corollary C (Corollary 2.6). The alternating sum of Swan conductors $\sum_{i=0}^{2d}(-1)^i \mathrm{Sw}(X, K, l, i)$ is an integer which is independent of the choice of l.

When K is of mixed characteristics, we compare the traces for p-adic cohomologies and the traces for l-adic cohomologies (Theorem 3.5) in addition to the above result. Assume that X is a proper and smooth over K. The etale cohomology $H^i(\overline{X}, \mathbb{Q}_p) := \varprojlim_n H^i(\overline{X}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a potentially semi-stable representation by the works of de Jong [dJ1] and T. Tsuji [T] (see [Be] p307 for example). Fontaine [F3] attached a p-adic representation of Weil(-Deligne) group $\widehat{D}_{\mathrm{pst}}(V)$ to a (p-adic) potentially semi-stable representation V (We recall the definition of $\widehat{D}_{\mathrm{pst}}$ and the representation of the Weil group on it in §3). Thus we apply the functor $\widehat{D}_{\mathrm{pst}}$ on $H^i(\overline{X}, \mathbb{Q}_p)$. Then $\widehat{D}_{\mathrm{pst}}(H^i(\overline{X}, \mathbb{Q}_p))$ gives a "good" p-adic representation of the Weil group W_K which completes the family of l-adic cohomologies for $l \neq p$. We can prove:

Theorem D (Theorem 3.1). For $g \in W_K^+$ and a proper smooth variety X over K, we have the following equality between rational integers:

$$\sum_i (-1)^i \mathrm{Tr}(g^*; H^i(\overline{X}, \mathbb{Q}_l)) = \sum_i (-1)^i \mathrm{Tr}\Big(g^*; \widehat{D}_{\mathrm{pst}}\big(H^i(\overline{X}, \mathbb{Q}_p)\big)\Big).$$

For the proof of Theorem B, the result by A. J. de Jong concerning semi-stable reduction plays an essential role. We reduce the problem for general varieties to the problem for strict semi-stable varieties with a group action by using de Jong's result. Then we apply the weight spectral sequence by Rapoport-Zink [R-Z] and the Lefschetz trace formula to see the l-independences of the alternating sum of traces for these strict semi-stable varieties with a group action. Theorem D is proved similarly by using a p-adic analogue of the weight spectral sequence constructed by A. Mokrane [Mo].

The plan of this paper is as follows. In §1, we shall prove the l-independence of alternating sum of traces for strict semi-stable varieties with a group action. In §2, we combine the results stated in the previous section and prove Theorem B. In §3, we shall prove Theorem D.

1. l-INDEPENDENCE FOR STRICT SEMI-STABLE VARIETIES WITH A GROUP ACTION

In this section, we see the l-independence of the alternating sum of the traces of Weil group on l-adic cohomologies in the case of strict semi-stable varieties with a group action. The result of this section is used in the next section to prove the l-independence of traces for general varieties. In fact, by using a result of de Jong, we reduce the general cases to the cases treated in this section. The reader can consult [I] and [R-Z] for details on the weight spectral sequence which is used in this section.

Let L be a complete discrete valuation field with perfect residue field k' and let π_L be a prime element of the integer ring $\mathcal{O}_L$. Let X' be a d-dimensional proper smooth variety over L with strict semi-stable reduction. That is X' has a proper flat regular model $\mathcal{X}'$ over the integer ring $\mathcal{O}_L$ which is isomorphic to $\mathcal{O}_L[t_1, \cdots, t_d]/(t_1 \cdots t_r - \pi_L)$ locally for the etale topology such that the special fiber $Y' = \mathcal{X}' \underset{\mathcal{O}_L}{\otimes} k'$ is a union of smooth varieties. We assume that X' is geometrically reduced and geometrically irreducible over L. We denote a strict semi-stable model of X' over $\mathcal{O}_L$ by $\mathcal{X}' = \mathcal{X}'_{\mathcal{O}_L}$. We consider a finite group G which acts on $\mathcal{X}'$. Here, the action of G is not necessarily trivial on the base $\mathcal{O}_L$. We denote the kernel of the action of G on the base $\mathcal{O}_L$ by H and we put $K := L^G$. We define $\widetilde{G}$ to be the fiber product $G \underset{G/H}{\times} \mathrm{Gal}(\overline{K}/K)$. We see that $\mathcal{X}'_{\mathcal{O}_{\overline{L}}} := \mathcal{X}' \underset{\mathcal{O}_L}{\otimes} \mathcal{O}_{\overline{K}}$ has a natural action of a group $\widetilde{G}$. When the residue field k of the field K is finite, we define a subgroup $\widetilde{W}$ of $\widetilde{G}$ (resp. a subset $\widetilde{W}^+$ of $\widetilde{W}$) to be the inverse image of the Weil group $W_K \subset G_K$ (resp. W_K^+) by the natural projection $\widetilde{G} \longrightarrow G_K$ (see §0 for the definition of W_K and W_K^+). As the definition of $\widetilde{G}$, $\widetilde{W}$ is also written as $G \underset{G/H}{\times} W_K$. When the residue field k is infinite, we define $W_K = W_K^+ = I_K$. The representation of $\widetilde{W}$ on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)(l \neq p)$ is simply the restriction of the action of $\widetilde{G}$ on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)$ which is induced from the action of $\widetilde{G}$ on $X' \underset{L}{\otimes} \overline{K}$ above. The main result in this section is as follows:

Proposition 1.1. Let the assumptions and the notations be as above. Then the alternating sum $\sum_i (-1)^i \operatorname{Tr}(\tilde{g}^*; H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l))$ is a rational integer which is independent of l for any $\tilde{g} \in \widetilde{W}^+$.

Proof of Proposition 1.1. Recall the weight spectral sequence ([R-Z]):

$$\text{(A)} \qquad {}_W E_1^{a,b} = H^{a+b}(\overline{Y}', \mathrm{gr}_{-a}^W R\Psi(\mathbb{Q}_l)) \Rightarrow E^{a+b} = H^{a+b}(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)$$

where Y' is the closed fiber of strict semi-stable model $\mathcal{X}' = \mathcal{X}'_{\mathcal{O}_L}$, $\overline{Y}'$ means the scalar extension $Y' \underset{k'}{\otimes} \overline{k}'$, and $R\Psi(\mathbb{Q}_l)$ is the complex of the nearby cycles.

The action of $\widetilde{G}$ on $\overline{Y}'$ is induced from the action of $\widetilde{G}$ on $\mathcal{X}'_{\mathcal{O}_L}$. $\widetilde{G}$ acts on $\mathrm{gr}_{-a}^W R\Psi(\mathbb{Q}_l)$ by the construction of $\mathrm{gr}_{-a}^W R\Psi(\mathbb{Q}_l)$. Thus $\widetilde{G}$ acts on both the E_1-term and the E^{a+b}-term of the spectral sequence (A). The following lemma can be verified immediately from the construction of the above spectral sequence.

Lemma 1.2. The spectral sequence (A) is $\widetilde{G}$-equivariant, that is, the differentials ${}_W E_r^{p,q} \longrightarrow {}_W E_r^{p+r,q-r+1}$ in the spectral sequence (A) are $\widetilde{G}$-equivariant, the filtration of E^n-terms are stable under $\widetilde{G}$-action and the isomorphism ${}_WE_\infty^{p,q} \cong \mathrm{Gr}_q E^{p+q}$ is $\widetilde{G}$-equivariant.

Now we describe E_1-term of the spectral sequence (A). Since we are assuming that X' has a strict semi-stable model $\mathcal{X}'$ over $\mathcal{O}_L$, the special fiber $Y' = \bigcup_{1\le i\le e} Y'_i$ is a global sum of smooth irreducible divisors Y'_i. Let us define the variety ${Y'}^{(m)}$ by ${Y'}^{(m)} = \coprod_{1\le i_1<\ldots<i_m\le e}\left(Y'_{i_1}\bigcap\ldots\bigcap Y'_{i_m}\right)$ for each integer $1 \le m \le \min(d+1, e)$. The variety ${Y'}^{(m)}$ is smooth variety of dimension $d+1-m$ over k' by definition. It is known that E_1-term of the spectral sequence (A) is presented as

$$ {}_WE_1^{-r,n+r} = \bigoplus_{q,r+q\ge 0} H^{n-r-2q}(\overline{Y}'^{(r+1+2q)}, \mathbb{Q}_l)(-r-q). $$

Using the above presentation and Lemma 1.2, we have

$$ \begin{aligned} \sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; E^i) &= \sum_{a,b}(-1)^{a+b}\mathrm{Tr}(\widetilde{g}^*; E_1^{a,b}) \\ &= \sum_n \Big(\sum_{s\ge 0, t\ge 0} (-1)^n \mathrm{Tr}(\widetilde{g}^*; H^{n-s-t}(\overline{Y}'^{(1+s+t)}, \mathbb{Q}_l)(-t)) \Big) \\ &= \sum_{u\ge 0}\Bigg(\sum_{0\le t\le u} \Big(\sum_n (-1)^n \mathrm{Tr}(\widetilde{g}^*; H^{n-u}(\overline{Y}'^{(1+u)}, \mathbb{Q}_l)(-t)) \Big)\Bigg). \end{aligned} $$

Thus, we have only to prove the following lemma in order to complete the proof of Proposition 1.1.

Lemma 1.3. For any $\widetilde{g} \in \widetilde{W}^+$, $\sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; H^i(\overline{Y}'^{(m)}, \mathbb{Q}_l))$ is independent of the choice of l.

Proof. Let f_{ab} be the absolute Frobenius on $\overline{Y}'^{(m)}$. Let us denote the composite $\widetilde{W} = \widetilde{W}_K \longrightarrow W = W_K \xrightarrow{u} \mathbb{Z}$ by $\widetilde{u}$. Then $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$ is a geometric endomorphism on $\overline{Y}'^{(m)}$ (i.e. the action is defined over $\overline{k}'$). It is known that f_{ab} acts trivially on $H^i(\overline{Y}'^{(m)}, \mathbb{Q}_l)$ (See [M] VI, Remark 13.5, for example). Thus we can use the Lefschetz trace formula (See [M] VI, Theorem 12.3, for example):

$$ (\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})}\circ\widetilde{g}} \cdot \Delta) = \sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; H^i(\overline{Y}'^{(m)}, \mathbb{Q}_l)), $$

where Δ is the diagonal cycle in $\overline{Y}'^{(m)} \times \overline{Y}'^{(m)}$ and $\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})}\circ\widetilde{g}}$ is the graph of the endomorphism $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$. We see the left hand side is independent of l. Thus the proof of Lemma 1.3 (and Proposition 1.1) is completed.

2. l -independence of the alternating sum of traces of $H_c^i(\overline{X}, \mathbb{Q}_l)$

In this section, we shall prove that the traces of W_K^+ on l-adic cohomologies are algebraic integers (Proposition 2.1) and that the alternating sum of the traces on l-adic cohomology is independent of l (Theorem 2.4). We also state several consequences of Theorem 2.4.

Let K be a complete discrete valuation field with perfect residue field k and let X be any variety over K (As defined in the introduction, a variety X over a field K means a reduced irreducible scheme X separated and of finite type over K). When k is infinite, we define W_K and W_K^+ by $W_K = W_K^+ = I_K$. The following proposition is a consequence of de Jong's result:

Proposition 2.1. Let X be a variety over a complete discrete valuation field K with a perfect residue field k of characteristic $p \neq l$ and let g be an element of W_K^+. Then the eigenvalues of the action of g on $H_c^i(\overline{X}, \mathbb{Q}_l)$ are algebraic integers. Especially, $\mathrm{Tr}(g^*; H_c^i(\overline{X}, \mathbb{Q}_l))$ is an algebraic integer for any variety X over K and any $g \in W_K^+$.

Before proving the proposition, we give the following lemma:

Lemma 2.2. Let X, X_1 be d-dimensional varieties over K. Assume that X and X_1 are birational to each other. Let g be an element of W_K^+. Then

(1) Assume that the eigenvalues of the action of g on $H_c^i(\overline{Z}, \mathbb{Q}_l)$ are algebraic integers for any variety Z with dimension $< d$. If the eigenvalues of the action of g on $H_c^i(\overline{X}, \mathbb{Q}_l)$ are algebraic integers, then the same thing is true for the action of g on $H_c^i(\overline{X}_1, \mathbb{Q}_l)$.

(2) Assume that the alternating sum $\sum\limits_i (-1)^i \mathrm{Tr}(g^*; H_c^i(\overline{Z}, \mathbb{Q}_l))$ is a rational number which is independent of l for any variety Z with dimension $< d$. If $\sum\limits_i (-1)^i \mathrm{Tr}(g^*; H_c^i(\overline{X}, \mathbb{Q}_l))$ is a rational number which is independent of l, then so is $\sum\limits_i (-1)^i \mathrm{Tr}(g^*; H_c^i(\overline{X}_1, \mathbb{Q}_l))$.

Proof. Since X and X_1 are birational to each other, we have a common nonempty open subscheme U defined over K. Now we have exact sequences of compact support etale cohomologies:

$$\cdots \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_l) \longrightarrow H_c^i(\overline{X}, \mathbb{Q}_l) \longrightarrow H_c^i(\overline{X} - \overline{U}, \mathbb{Q}_l) \longrightarrow \cdots,$$
$$\cdots \longrightarrow H_c^i(\overline{U}, \mathbb{Q}_l) \longrightarrow H_c^i(\overline{X}_1, \mathbb{Q}_l) \longrightarrow H_c^i(\overline{X}_1 - \overline{U}, \mathbb{Q}_l) \longrightarrow \cdots.$$

Thus the difference between $H_c^i(\overline{X}, \mathbb{Q}_l)$ and $H_c^i(\overline{X}_1, \mathbb{Q}_l)$ are described in terms of $H_c^i(\overline{X} - \overline{U}, \mathbb{Q}_l)$ and $H_c^i(\overline{X}_1 - \overline{U}, \mathbb{Q}_l)$. Since the dimensions of $X - U$ and $X_1 - U$ are both strictly smaller than $\dim X = \dim X_1$, we deduce (1) and (2) from the assumptions of (1) and (2).

Proof of Proposition 2.1. We prove the proposition by an induction argument with respect to the dimension of X. By using Lemma 2.2 (1), we may assume that X is proper over K. Further we easily reduce to the case where X is geometrically reduced and geometrically irreducible over K. This is seen as follows. Let K' be the largest radicial extension of K contained in $K(X)$. Then the reduced scheme $(X \underset{K}{\otimes} K')^{\mathrm{red}}$ associated to $X \underset{K}{\otimes} K'$ is geometrically reduced over K'. The absolute Galois group G_K is isomorphic to $G_{K'}$ and through this isomorphism, $H_c^i(\overline{X}, \mathbb{Q}_l)$ is identified with $H_c^i((X \underset{K}{\otimes} K')$ as $G_{K'}$-module. As $G_{K'}$-modules, we have

$$H_c^i(\overline{X}, \mathbb{Q}_l) \cong H_c^i((X \underset{K}{\otimes} K') \underset{K'}{\otimes} \overline{K'}, \mathbb{Q}_l) \cong H_c^i((X \underset{K}{\otimes} K')^{\mathrm{red}} \underset{K'}{\otimes} \overline{K'}, \mathbb{Q}_l).$$

The last equality comes from the fact that a nilpotent closed immersion induces an isomorphism on etale cohomologies. Hence we reduce the proof to the geometrically reduced variety $(X \underset{K}{\otimes} K')^{\mathrm{red}}$ over K'. Next, let us consider about the assumption of geometrically irreducibility. Let K'' be the intersection $\overline{K} \cap K(X)$. Then X has a structure as a variety over K'' and $X_{/K''}$ is geometrically irreducible over K''. Now $H^i_c(X \underset{K}{\otimes} \overline{K}, \mathbb{Q}_l)$ is the induced representation:

$$\mathrm{Ind}^{G_{K''}}_{G_K} H^i_c(X_{/K''} \underset{K''}{\otimes} \overline{K}, \mathbb{Q}_l) \cong H^i_c(X_{/K''} \underset{K''}{\otimes} \overline{K}, \mathbb{Q}_l) \underset{\mathbb{Q}_l[G_{K''}]}{\otimes} \mathbb{Q}_l[G_K].$$

Hence we have only to prove the assertion of Proposition 2.1 for the geometrically irreducible variety $X_{/K''}$ over K'' instead of the variety X over K.

The following lemma is deduced by combining the results in [dJ1] and [dJ2] (We combine Theorem 5.9 and Proposition 5.11 of [dJ2] and we apply 2.16 and 4.16 of [dJ1]).

Lemma 2.3. Let X be a proper, geometrically reduced and geometrically irreducible variety over K and let $\mathcal{X}$ be a proper flat model of X over the integer ring $\mathcal{O}_K$. Then we have the following commutative diagram:

$$\begin{array}{ccc} & & \mathcal{X}' \\ & \overset{\psi}{\swarrow} & \downarrow \\ \mathcal{X} & \leftarrow & \mathcal{X} \underset{\mathcal{O}_K}{\otimes} \mathcal{O}_L, \end{array}$$

where

(1) L is a finite extension of K.
(2) $X' = \mathcal{X}' \underset{\mathcal{O}_L}{\otimes} L$ is a proper smooth variety over L with strict semi-stable reduction and $\mathcal{X}'$ is a strict semi-stable model of X'.
(3) ψ is a Galois alteration with Galois group G, that is $\mathcal{X}'$ has an action of a finite group G and $\mathcal{X}'$ has the quotient $\mathcal{X}'/G$ and the induced finite extension of the function fields $K(\mathcal{X}'/G)/K(\mathcal{X})$ is purely inseparable.
(4) The morphisms in the diagram are G-equivariant. (Note that the action of G on the base $\mathcal{O}_L$ of $\mathcal{X}'$ is not necessarily trivial.)

Further X' is geometrically reduced and geometrically irreducible over L.

For a finite extension K' of K, we easily see that if the proposition is true for the scalar extension $X \underset{K}{\otimes} K'$ over K' the proposition is true for X/K. Applying Lemma 2.3 above, we can take a finite extension L of K and a morphism $X' \longrightarrow X \underset{K}{\otimes} L$ over L such that X' has strict semi-stable reduction over L. Further a finite group H acts on the morphism $X' \longrightarrow X \underset{K}{\otimes} L$ in such a way that H acts trivially on $X \underset{K}{\otimes} L$ and that the induced finite extension of function fields $K(X'/H)/K(X \underset{K}{\otimes} L)$ is purely inseparable. Then for any $g \in {W_L}^+$, the eigenvalues of the action of g on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)$ are algebraic integers due to the Weil conjecture [De] and the weight spectral sequence of Rapoport-Zink [R-Z]. Hence the eigenvalues of the action of $g \in {W_L}^+$ on the H-invariant subspace

$$H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)^H = H^i((X' \underset{L}{\otimes} \overline{K})/H, \mathbb{Q}_l) = H^i((X'/H) \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l),$$

are algebraic integers. Let us remark about the above equalities. The second equality is easy since the automorphism H is defined over L. Let us discuss the first one. Let us denote the quotient map $X' \underset{L}{\otimes} \overline{K} \longrightarrow (X' \underset{L}{\otimes} \overline{K})/H$ by q. Since q is a finite morphism, we have $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Z}/l^n\mathbb{Z}) \cong H^i((X' \underset{L}{\otimes} \overline{K})/H, q_*(\mathbb{Z}/l^n\mathbb{Z}))$ for any $n \geq 1$. The group H acts on the cohomology $H^i((X' \underset{L}{\otimes} \overline{K})/H, q_*(\mathbb{Z}/l^n\mathbb{Z}))$ through the sheaf $q_*(\mathbb{Z}/l^n\mathbb{Z})$ and does not act on the scheme $(X' \underset{L}{\otimes} \overline{K})/H$. By an easy spectral sequence argument, we see that $H^i((X' \underset{L}{\otimes} \overline{K})/H, q_*(\mathbb{Z}/l^n\mathbb{Z}))^H$ is isomorphic to $H^i((X' \underset{L}{\otimes} \overline{K})/H, (q_*(\mathbb{Z}/l^n\mathbb{Z}))^H)$ modulo a finite abelian group with exponent equal or less than $\sharp H$. On the other hand, we see that $H^i((X' \underset{L}{\otimes} \overline{K})/H, (q_*(\mathbb{Z}/l^n\mathbb{Z}))^H)$ is isomorphic to $H^i((X' \underset{L}{\otimes} \overline{K})/H, \mathbb{Z}/l^n\mathbb{Z})$ since $(q_*(\mathbb{Z}/l^n\mathbb{Z}))^H \cong \mathbb{Z}/l^n\mathbb{Z}$. By taking limit with respect to n and by taking tensor product with $\mathbb{Q}_l$ over $\mathbb{Z}_l$, we obtain:

$$
\begin{aligned}
H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)^H &\cong \varprojlim_n H^i((X' \underset{L}{\otimes} \overline{K})/H, q_*(\mathbb{Z}/l^n\mathbb{Z}))^H \underset{\mathbb{Z}_l}{\otimes} \mathbb{Q}_l \\
&\cong \varprojlim_n H^i((X' \underset{L}{\otimes} \overline{K})/H, (q_*(\mathbb{Z}/l^n\mathbb{Z}))^H) \underset{\mathbb{Z}_l}{\otimes} \mathbb{Q}_l \cong H^i((X' \underset{L}{\otimes} \overline{K})/H, \mathbb{Q}_l).
\end{aligned}
$$

Hence the first equality follows.

Since the extension $K(X'/H)/K(X \underset{K}{\otimes} L)$ is purely inseparable, we have a sufficiently small open subscheme U'_L (resp. U_L) of X'/H (resp. $X \underset{K}{\otimes} L$) and the following commutative diagram:

$$
\begin{array}{ccc}
U'_L & \longrightarrow & X'/H \\
\big\downarrow & & \big\downarrow \\
U_L & \longrightarrow & X \underset{K}{\otimes} L,
\end{array}
$$

such that the left vertical map is radicial surjection. Hence the induced homomorphism $H^i_c(U_L \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l) \longrightarrow H^i_c(U'_L \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)$ is an isomorphism. Then the proof of Proposition 2.1 is completed by Lemma 2.2 (1) and the assumption of the induction.

Theorem 2.4. Let X be a variety over a complete discrete valuation field K with a perfect residue field k of characteristic $p \neq l$.

Then $\sum_i (-1)^i \mathrm{Tr}(g^*; H^i_c(\overline{X}, \mathbb{Q}_l))$ is a rational integer which is independent of the choice of l for any $g \in W_K^+$.

Proof of Theorem 2.4. We prove the theorem by an induction argument with respect to the dimension of X. We may assume that X is proper over K by Lemma 2.2. We easily reduce to the case where X is geometrically reduced and geometrically irreducible over K. So we take X', L and G as in Lemma 2.3.

Now for $g \in \widetilde{W}^+/H \cong W_K^+$, we have

$$\begin{aligned}
\mathrm{Tr}(g^*; H_c^i((X'/G) \underset{K}{\otimes} \overline{K}, \mathbb{Q}_l)) &= \mathrm{Tr}(g^*; H_c^i((X'/H) \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)) \\
&= \mathrm{Tr}(g^*; H_c^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)^H) \\
&= \frac{1}{\sharp H}\Big(\sum_{\substack{\widetilde{g} \mapsto g \\ \widetilde{g} \in \widetilde{W}^+}} \mathrm{Tr}(\widetilde{g}^*; H_c^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l))\Big).
\end{aligned}$$

Here, the first equality follows from the definition of X', G, L and H. The second one is shown by the same reason as the proof of Proposition 2.1. The third equality is just an argument on linear representations of finite groups. Let V be a finite dimensional $\mathbb{Q}_l$-vector space with $\widetilde{W}$-action and let H be a finite normal subgroup of $\widetilde{W}$. The fact we need is that

$$\mathrm{Tr}(g^*; V^H) = \frac{1}{\sharp H}\big(\sum_{h \in H} \mathrm{Tr}\big(\widetilde{g}_0^* h^*; V\big)\big) \quad \text{for } g \in \widetilde{W}/H,$$

where $\widetilde{g}_0 \in \widetilde{W}$ is a fixed lifting of $g \in \widetilde{W}/H$. Then the right hand side of the above sequence is $\frac{1}{\sharp H}\big(\sum_{h \in H} \mathrm{Tr}\big(\widetilde{g}_0^* h^*; V\big)\big)$. If we regard the elements $\widetilde{g}_0^* h^*$ as elements in the endomorphism ring $\mathrm{End}(V)$ and consider the sum $\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*$ in $\mathrm{End}(V)$, we see:

$$\frac{1}{\sharp H}\big(\sum_{h \in H} \mathrm{Tr}\big(\widetilde{g}_0^* h^*; V\big)\big) = \mathrm{Tr}\big(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*; V\big).$$

Since $(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V^H} = g^*$ in $\mathrm{End}(V^H)$, we obtain:

$$\begin{aligned}
&\mathrm{Tr}\big((\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*); V\big) \\
&\quad = \mathrm{Tr}\big((\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V^H}; V^H\big) + \mathrm{Tr}\big((\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V/V^H}; V/V^H\big) \\
&\quad = \mathrm{Tr}\big(g^*; V^H\big) + \mathrm{Tr}\big((\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V/V^H}; V/V^H\big).
\end{aligned}$$

To see that $\mathrm{Tr}\big((\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V/V^H}; V/V^H\big)$ is zero, it suffices to see that

$$(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V/V^H} = 0 \quad \text{in } \mathrm{End}(V/V^H),$$

or equivalently, $(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*)|_{V} \in \mathrm{End}(V)$ maps V to the subspace V^H of V. But for $v \in V$ and any $h' \in H$, we see:

$$\begin{aligned}
(h')^* \cdot \big(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* h^*\big)v &= \big(\frac{1}{\sharp H}\sum_{h \in H} \widetilde{g}_0^* (\widetilde{g}_0^{*\,-1} h'^* \widetilde{g}_0^*) h^*\big)v \\
&= \big(\frac{1}{\sharp H}\sum_{h'' \in H} \widetilde{g}_0^* h''^*\big)v.
\end{aligned}$$

Hence $(\frac{1}{\sharp H}\sum_{h\in H}\widetilde{g}_0^* h^*)v$ belongs to V^H. Thus the third equality follows. Because of Proposition 1.1, we see that the alternating sum $\sum_i(-1)^i\mathrm{Tr}(g^*;H_c^i((X'/G)\underset{K}{\otimes}\overline{K},\mathbb{Q}_l))$ is a rational number which is independent of l.

Then we see that $\sum_i(-1)^i\mathrm{Tr}(g^*;H_c^i(\overline{X},\mathbb{Q}_l))$ is a rational number which is independent of l by the same argument as that in the proof of Proposition 2.1 by using Lemma 2.2 (2). On the other hand, $\sum_i(-1)^i\mathrm{Tr}(g^*;H_c^i(\overline{X},\mathbb{Q}_l))$ is also an algebraic integer because of Proposition 2.1. Thus the proof of Theorem 2.4 is completed.

Corollary 2.5. Let the assumptions and the notations be as in Theorem 2.4 and assume one of the following two cases:

(1) X is a proper smooth surface over K.
(2) X is a smooth complete intersection variety over any K.

Then $\mathrm{Tr}(g^*;H^i(\overline{X},\mathbb{Q}_l))$ is a rational integer which is independent of the choice of l for any i and any $g\in W_K^+$.

Remark to Corollary 2.5. For $g\in W_K$ and X as in Corollary 2.5, we easily see that $\mathrm{Tr}(g^*;H^i(\overline{X},\mathbb{Q}_l))$ is a rational number which is independent of l. Equivalently, the characteristic polynomial $\det(1-g^*T;H^i(\overline{X},\mathbb{Q}_l))$ is a polynomial with coefficients in $\mathbb{Q}$ which is independent of l.

Proof of Corollary 2.5. For X given in Corollary 2.5, traces of W_K^+ on $H^0(\overline{X},\mathbb{Q}_l)$ and $H^4(\overline{X},\mathbb{Q}_l)$ are trivially independent of l. As for $H^1(\overline{X},\mathbb{Q}_l)$ and $H^3(\overline{X},\mathbb{Q}_l)$, we can apply Theorem 4.4 (a) of P.459 in [SGA7-1] to the Albanese variety and the Picard variety of X in order to see l-independence of traces of W_K^+. Thus by Theorem 2.4 stated above, we get the desired result. The case of a smooth complete intersection variety is done in a similar way since the cohomologies of complete intersection varieties are the same as those of projective spaces except in the degree equal to the dimension of the variety.

The l-independence of the alternating sum of Swan conductors is also a consequence of Theorem 2.4. The definition of Swan conductor and the precise statement of our result are as follows:

Let I_K be the inertia subgroup of G_K. By the monodromy theorem of Grothendieck (cf. Appendix in [S-T] and exposée 1 of [SGA7-1]), we have an open subgroup $J\subset I_K$ such that the action of J on $H_c^i(\overline{X},\mathbb{Q}_l)$ is unipotent. Let $\phi_i(g)$ be the character of the action of $g\in I_K/J$ on $H_c^i(\overline{X},\mathbb{Q}_l)$. Then $\mathrm{Sw}(X,K,l,i)$ is defined as

$$\mathrm{Sw}(X,K,l,i):=\left(\frac{1}{\sharp I_K/J}\sum_{g\in I_K/J}\Big(b_{I_K/J}(g^{-1})\cdot\phi_i(g)\Big)\right),$$

where $b_{I_K/J}$ is the Swan character of I_K/J (cf. [Se2], 19.1). It is known that $\mathrm{Sw}(X,K,l,i)$ is a well-defined integer (cf. [Se1] Chap.6 and [O]). Since the alternating sum $\sum_i(-1)^i\phi_i(g)$ is independent of l due to Theorem 2.4, we have the following:

Corollary 2.6. Let the assumptions and the notations be as in Theorem 2.4. Then $\sum_i(-1)^i\,\mathrm{Sw}(X,K,l,i)$ is an integer which is independent of the choice of l.

The following corollary is a direct consequence of Corollary 2.6.

Corollary 2.7. Let the assumptions and the notations be as in Theorem 2.4. Assume one of the following cases.

(1) X is a proper smooth surface over K.
(2) X is a smooth complete intersection variety over K.

Then $\mathrm{Sw}(X, K, l, i)$ is a rational integer which is independent of l for any i.

Remark. By Lemma 1.11 of [M] Chap.5 §1, the mod l version of Corollary 2.6 is verified exactly in the same way.

3. Comparison between p and l

In this section, we prove that the alternating sum of traces on l-adic cohomology and the alternating sum of traces on p-adic cohomology coincide if X is a proper smooth variety over a complete discrete valuation field K of mixed characteristics (Theorem 3.1). Further, we can compare the traces on l-adic side and traces on p-adic side in the case where X is not necessarily proper nor smooth over K if we work on the level of Grothendieck groups (Theorem 3.5).

Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$ and P_0 the fraction field of $W(\overline{k})$. We denote by σ the Frobenius morphism of P_0 which induces $x \longmapsto x^p$ on $\overline{k}$. We denote the category of p-adic representations by $\underline{\mathrm{Rep}}(G_K)$ and the subcategory of $\underline{\mathrm{Rep}}(G_K)$ consisting of potentially semi-stable representations by $\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)$, (See [F2] for the definition of potentially semi-stable representations). By [F3], for a potentially semi-stable representation V,

$$\text{a finite dimensional } P_0\text{-vector space} \qquad \widehat{D}_{\mathrm{pst}}(V) := \varinjlim_{F \in \mathcal{F}} (B_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} V)^F$$

is constructed. Here, $\mathcal{F}$ is the set of open subgroups of the inertia group I_K and B_{st} is the ring of periods of varieties with semi-stable reduction([F1]). We denote the functor $\widehat{D}_{\mathrm{pst}}$ by $\widehat{D}_K$ to specify the base field K. The P_0-vector space $\widehat{D}_K(V)$ has σ-linear G_K-action, that is, $g(x \cdot d) = \sigma^{u(g)}(x) \cdot g(d)$ for $g \in G_K, x \in P_0$ and $d \in \widehat{D}_K(V)$. Here, the map $u : G_K \longrightarrow \widehat{\mathbb{Z}}$ is defined to be the composite map:

$$G_K \to G_k \hookrightarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \mathrm{Gal}(P_0/\mathbb{Q}_p) \cong \widehat{\mathbb{Z}}$$

when the residue field k is finite (the last isomorphism is normalized so that the geometric Frobenius $f : x \to x^{1/p}$ goes to $1 \in \widehat{\mathbb{Z}}$) and is defined to be the zero map when k is infinite. Further, $\widehat{D}_K(V)$ is equipped with a σ -linear map φ: $\widehat{D}_K(V) \longrightarrow \widehat{D}_K(V)$. We denote by $\underline{D}_K$ the category of finite dimensional P_0-vector spaces with a σ-linear action of G_K and φ as above.

P_0 -linear representation of W_K on $\widehat{D}_K(V)$ is defined by the map

$$W_K \longrightarrow \mathrm{Aut}_{P_0}(\widehat{D}_K(V)) \qquad (g \longmapsto \varphi^{u(g)} \cdot g).$$

Let X be a proper smooth variety of over K. Combining the results of de Jong[dJ1] and T. Tsuji[T], we see that $H^i(\overline{X}, \mathbb{Q}_p)$ is a potentially semi-stable representation for a proper smooth variety X over K (cf. [Be] p307). Now we have the following theorem:

Theorem 3.1. Let K be a complete discrete valuation field of mixed characteristics with perfect residue field k of characteristic p and let X be a proper smooth variety over K. Then for any $g \in W_K^+ := \{g \in W_K | u(g) \geq 0\}$ and any prime number $l \neq p$, we have an equality of rational integers

$$\sum_i (-1)^i \mathrm{Tr}(g^*; H^i(\overline{X}, \mathbb{Q}_l)) = \sum_i (-1)^i \mathrm{Tr}(g^*; \widehat{D}_{\mathrm{pst}}(H^i(\overline{X}, \mathbb{Q}_p))).$$

Here, the l-adic representation of W_K on $H^i(\overline{X}, \mathbb{Q}_l)$ is simply the restriction of the representation of G_K and the P_0-linear representation of W_K on $\widehat{D}_{\mathrm{pst}}(H^i(\overline{X}, \mathbb{Q}_p))$ is the one given above.

Proof of Theorem 3.1. We have only to prove that the trace on l-adic side and the trace on p-adic side coincide as a rational number. Then the result follows from Proposition 2.1. Let $\mathcal{X} = \mathcal{X}_{\mathcal{O}_K}$ be a proper flat model of X_K over the integer ring $\mathcal{O}_K$. Again, as in §2, we have a Galois alteration with the Galois group G (see Lemma 2.3 (3) for the definition of Galois alteration):

$$\mathcal{X}' \longrightarrow \mathcal{X}_{\mathcal{O}_K},$$

where $\mathcal{X}'$ is a proper flat regular variety over $\mathcal{O}_L$ with strict semi-stable reduction(Here, L is a finite extension of K). We denote the generic fiber $\mathcal{X}'_{\mathcal{O}_L} \underset{\mathcal{O}_L}{\otimes} L$ by X'. Let Y' be the special fiber of $\mathcal{X}'_{\mathcal{O}_L}$. We denote the residue field of L by k'. $\mathcal{X}'_{\mathcal{O}_L}$ has the action of $\widetilde{G}$ through the action of the quotient $G = \widetilde{G}/\mathrm{Gal}(\overline{K}/L)$. Hence $\widetilde{G}$ acts on Y' and we see that the logarithmic crystalline cohomology $H^i_{\text{log crys}}(Y'/W(k'))$ is a finitely generated $W(k')$-module equipped with the action of $\widetilde{G}$ (Here, $W(k')$ denotes the Witt ring of k' and $H^i_{\text{log crys}}(Y'/W(k'))$ denotes the Hyodo-Kato cohomology D_∞ in [H-K], (3.2)). ¿From the assumption of Galois alteration, the group G acts on $H^i_{\text{log crys}}(Y'/W(k'))$ and $\mathrm{Gal}(\overline{K}/K)$ acts on P_0 through the quotient $\mathrm{Gal}(K^{nr}/K)$. Hence $\widetilde{G} = G \underset{G/H}{\times} \mathrm{Gal}(\overline{K}/K)$ acts on $H^i_{\text{log crys}}(Y'/W(k')) \underset{W(k')}{\otimes} P_0$. On the other hand, since $\widetilde{G}$ again acts on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)$, $\widetilde{G}$ also acts on $\widehat{D}_{\mathrm{pst}}(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))$ by functoriality. Put

$$\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)) := \varinjlim_{F' \in \mathcal{F}'} (B_{\mathrm{st}} \underset{\mathbb{Q}_p}{\otimes} H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))^{F'},$$

where $\mathcal{F}'$ is the set of open subgroups of I_L. $\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))$ is a finite dimensional P_0-vector space with a σ -linear $\widetilde{G}$-action and is equipped with a σ-linear map

$$\varphi : \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)) \longrightarrow \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)).$$

A P_0-linear representation of $\widetilde{W}$ on $\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))$ is defined by the map:

$$\widetilde{W} \longrightarrow \mathrm{Aut}_{P_0}(\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))) \ \ (\widetilde{g} \longmapsto \varphi^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}),$$

where $\widetilde{u}$ is the map defined in the proof of Lemma 1.3. We have the following proposition, which is a p-adic analogue of Proposition 1.1:

Proposition 3.2. Let the notations and the assumptions be as above. Let $\widetilde{g}$ be an element of $\widetilde{W}^{+}$. Then we have the following equality between rational integers:

$$\sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)) = \sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)).$$

Proof of Proposition 3.2. By the C_{st} conjecture proved by T. Tsuji [T], we have the following lemma:

Lemma 3.3. We have the following isomorphism between finite dimensional P_0-vector spaces.

$$H^i_{\log \text{ crys}}(Y'/W(k')) \underset{W(k')}{\otimes} P_0 \cong \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)).$$

This isomorphism is compatible with the action of $\widetilde{G}$. The endomorphism on the left hand side which is induced by the endomorphism φ on the right hand term $\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))$ comes from $f_{\mathrm{ab}} \underset{\sigma|_{W(k')}}{\otimes} \sigma|_{P_0}$ for the absolute Frobenius f_{ab} : $Y' \longrightarrow Y'$.

Mokrane([Mo]) constructed the following spectral sequence, which is a p-adic analogue of the spectral sequence of Rapoport-Zink:

$$\text{(B)}\ E'^{-r,n+r}_1 = \underset{q, r+q \geq 0}{\oplus} H^{n-r-2q}_{\mathrm{crys}}(Y'^{(r+1+2q)}/W(k'))(-r-q)$$
$$\Longrightarrow E'^n = H^n_{\log \text{ crys}}(Y'/W(k')).$$

As explained in §1, $\widetilde{G}$ acts on Y' through its finite quotient $G = \widetilde{G}/\mathrm{Gal}(\overline{K}/L)$. So we see that $\widetilde{G}$ acts on both E_1-term and E^n-term of the spectral sequence (B). The following lemma is a direct consequence from the construction of the spectral sequence (B).

Lemma 3.4. The spectral sequence (B) is $\widetilde{G}$-equivariant (see Lemma 1.2 for the definition of a $\widetilde{G}$-equivariant spectral sequence).

The P_0-linear representation of $\widetilde{W}$ on $\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))$ is defined by $\varphi^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$. Hence the representation of $\widetilde{g} \in \widetilde{W}^+$ on

$$H^i_{\log \text{ crys}}(Y'/W(k')) \underset{W(k')}{\otimes} P_0 = \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)),$$

is given by the action induced by $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$. On the other hand, the representation of $\widetilde{W}$ on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)(l \neq p)$ is simply the restriction of the $\widetilde{G}$-action. Recall that the absolute Frobenius f_{ab} induces an identity map on $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)$. Thus we have only to compare the actions on both cohomologies induced by $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$ in order to prove Proposition 3.2. We see that $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$ gives a geometric endomorphism of Y' defined over k'.
By using the spectral sequence (A), we have:

$$\sum_i (-1)^i \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l)) = \sum_{a,b} (-1)^{a+b} \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; {}_W E_1^{a,b}).$$

By using the spectral sequence (B), we have:

$$\sum_i (-1)^i \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^i_{\mathrm{crys}}(Y'/W(k')) \underset{W(k')}{\otimes} P_0)$$
$$= \sum_{a,b} (-1)^{a+b} \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; E'^{a,b}_1).$$

By using Lefschetz trace formulas for etale cohomology, we see that:

$$\begin{aligned}
&\sum_{a,b} (-1)^{a+b} \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; {}_W E_1^{a,b}) \\
&= \sum_{u \geq 0} \left(\sum_{0 \leq t \leq u} \left(\sum_n (-1)^n \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^{n-u}(\overline{Y}'^{(1+u)}, \mathbb{Q}_l)(-t)) \right) \right) \\
&= \sum_{u \geq 0} \left((u+1) \cdot \left(\sum_n (-1)^n \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^{n-u}(\overline{Y}'^{(1+u)}, \mathbb{Q}_l)) \right) \right) \\
&= \sum_{u \geq 0} \left((u+1) \cdot (-1)^u \cdot \left(\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}} \cdot \Delta_{Y'^{(1+u)}} \right) \right),
\end{aligned}$$

where $\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}}$ is the graph for the endomorphism $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$ of $Y'^{(1+u)}$ and $\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}} \cdot \Delta_{Y'^{(1+u)}}$ denotes the intersection number of the graph $\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}}$ and the diagonal cycle $\Delta_{Y'^{(1+u)}}$ in $Y'^{(1+u)} \times Y'^{(1+u)}$. The second equality holds since the endomorphism $f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}$ on $\overline{Y}'^{1+u}$ is defined over $\overline{k}'$. The third equality comes from Lefschetz trace formula. In the same way, using the Lefschetz trace formula for crystalline cohomology (see [G-M] and [K] Proposition 1.3.6 for the trace formula for crystalline cohomology), we have the following equality in the crystalline cohomology side:

$$\sum_{a,b} (-1)^{a+b} \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; E'^{a,b}_1) = \sum_{u \geq 0} \left((u+1) \cdot (-1)^u \cdot \left(\Gamma_{f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g}} \cdot \Delta_{Y'^{(1+u)}} \right) \right).$$

Hence, we see:

$$\sum_i (-1)^i \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l))$$
$$= \sum_i (-1)^i \mathrm{Tr}((f_{\mathrm{ab}}^{\widetilde{u}(\widetilde{g})} \circ \widetilde{g})^*; H^i_{\mathrm{crys}}(Y'/W(k')) \underset{W(k')}{\otimes} P_0).$$

Finally, the proof of Proposition 3.2 is completed by Lemma 3.3.

Let $\underline{D}_L$ be the category of finite dimensional P_0-vector spaces D equipped with a σ-linear $\widetilde{G}$-action and a σ-linear map $\varphi : D \longrightarrow D$. Then we see:

$$\widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p))^H = \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)^H) = \widehat{D}_L(H^i((X'/G) \underset{K}{\otimes} \overline{K}, \mathbb{Q}_p)).$$

Thus

$$\begin{aligned}\sum_i (-1)^i \mathrm{Tr}(g^*; H^i(\overline{X'/G}, \mathbb{Q}_l)) &= \frac{1}{\sharp H} \sum_{\substack{\widetilde{g}\mapsto g\\ \widetilde{g}\in \widetilde{W}^+}} \sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_l))\\ &= \frac{1}{\sharp H} \sum_{\substack{\widetilde{g}\mapsto g\\ \widetilde{g}\in \widetilde{W}^+}} \sum_i (-1)^i \mathrm{Tr}(\widetilde{g}^*; \widehat{D}_L(H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)))\\ &= \sum_i (-1)^i \mathrm{Tr}(g^*; \widehat{D}_L(H^i(\overline{X'/G}, \mathbb{Q}_p))).\end{aligned}$$

In the above equation, the first equality is shown by the same argument as the argument in the proof of Theorem 2.4 (Note that the compact support etale cohomology and usual etale cohomology are the same since X'/G and X' are proper varieties). The third one is similar if we note that $\widehat{D}_L : \underline{\mathrm{Rep}}_{\mathrm{pst}}(\widetilde{G}) \longrightarrow \underline{D}_L$ is an exact functor. Finally, the second one is obtained by Proposition 3.2.

The varieties X and X'/G are birational to each other due to Lemma 2.3 (3). Hence we can take a common nonempty open subscheme U of X and X'/G. As in the argument in the proof of Lemma 2.2, we want to reduce to the varieties with smaller dimensions. But the differences from the case of Lemma 2.2 are that we do not know whether $H^i_c(\overline{U}, \mathbb{Q}_p)$, $H^i_c(\overline{X} - \overline{U}, \mathbb{Q}_p)$ and $H^i_c(\overline{X'/G} - \overline{U}, \mathbb{Q}_p)$ are potentially semi-stable representations and that we do not have a comparison theorem for compact support cohomology. In order to avoid these difficulties, we consider the Grothendieck groups of the representations.

Let $K_0(\underline{\mathrm{Rep}}(G_K))$ (resp. $K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)), K_0(\underline{D}_K)$, etc.) be the Grothendieck group of $\underline{\mathrm{Rep}}(G_K)$ (resp. $\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K), \underline{D}_K$, etc.). The functor $\widehat{D}_{\mathrm{pst}}$ induces a morphism from the Grothendieck group $K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$ of the category $\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)$ to the Grothendieck group of the category of P_0-linear representations of Weil group W_K. For a p-adic representation V in $\underline{\mathrm{Rep}}(G_K)$, we denote its class in $K_0(\underline{\mathrm{Rep}}(G_K))$ by $[V]$. By the above remark, if $[V]$ lies in $K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$, we are able to apply $\widehat{D}_{\mathrm{pst}}$ to $[V]$. Then $\widehat{D}_{\mathrm{pst}}([V])$ defines an element of the Grothendieck group of the category of P_0-linear representations of Weil group W_K and we are able to consider the trace $\mathrm{Tr}(g^*; \widehat{D}_{\mathrm{pst}}([V]))$ for $g \in W_K$. We are going to work with these Grothendieck groups. One of the advantages of working with $K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$ instead of $\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)$ is that if $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is an exact sequence in $\underline{\mathrm{Rep}}(G_K)$ with $V', V'' \in \underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)$, then $[V] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$ whereas $V \notin \underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)$ in general.

We complete the proof of Theorem 3.1 by the following theorem.

Theorem 3.5. Let K be a complete discrete valuation field of mixed characteristics with perfect residue field k. Then the following statement holds.
(1) Let X be any variety over K. Then the class $[H^i_c(\overline{X}, \mathbb{Q}_p)] \in K_0(\underline{\mathrm{Rep}}(G_K))$ lies in $K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$.
(2) For any $g \in W_K^+ := \{g \in W_K | u(g) \geq 0\}$ and any prime number $l \neq p$, we have an equality of rational integers:

$$\sum_i (-1)^i \mathrm{Tr}(g^*; H^i_c(\overline{X}, \mathbb{Q}_l)) = \sum_i (-1)^i \mathrm{Tr}(g^*; \widehat{D}_{\mathrm{pst}}([H^i_c(\overline{X}, \mathbb{Q}_p)])).$$

We prove the following lemma before proving Theorem 3.5:

Lemma 3.6. Let K be a complete discrete valuation field of mixed characteristics with perfect residue field and let X and X_1 be d-dimensional varieties over K birational to each other. Assume that $[H^i_c(\overline{Z}, \mathbb{Q}_p)] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$ for any variety Z with dimension $< d$ over K. Then, for $[H^i_c(\overline{X}, \mathbb{Q}_p)], [H^i_c(\overline{X}_1, \mathbb{Q}_p)] \in K_0(\underline{\mathrm{Rep}}(G_K))$, we have

$$[H^i_c(\overline{X}, \mathbb{Q}_p)] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)) \Longleftrightarrow [H^i_c(\overline{X}_1, \mathbb{Q}_p)] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K)).$$

Proof. Recall that a subrepresentation or a quotient representation of a potentially semi-stable representation is also potentially semi-stable ([F2]). Consequently, if V' is a quotient or a sub-representation of a representation V such that $[V] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$, then $[V'] \in K_0(\underline{\mathrm{Rep}}_{\mathrm{pst}}(G_K))$. Then, by the arguments of compact support cohomology as in the proof of Lemma 2.2, we obtain the desired result.

Proof of Theorem 3.5. We prove the theorem by induction argument with respect to the dimension of X. By Lemma 3.6, we may assume that X is proper over K. We take X', L and G as given in Lemma 2.3. Then $H^i((X'/G) \underset{K}{\otimes} \overline{K}, \mathbb{Q}_p)$ is potentially semi-stable since it is a subrepresentation of a potentially semi-stable representation $H^i(X' \underset{L}{\otimes} \overline{K}, \mathbb{Q}_p)$. Thus (1) is true due to Lemma 3.6.

In the level of the Grothendieck group, we have

$$\begin{aligned}\sum_i (-1)^i [H^i_c(\overline{X}, \mathbb{Q}_p)] - \sum_i (-1)^i [H^i_c(\overline{X'/G}, \mathbb{Q}_p)] \\ = \sum_i (-1)^i [H^i_c(\overline{X} - \overline{U}, \mathbb{Q}_p)] - \sum_i (-1)^i [H^i_c(\overline{X'/G} - \overline{U}, \mathbb{Q}_p)].\end{aligned}$$

Hence, by (1), we have:

$$\begin{aligned}\sum_i (-1)^i \widehat{D}_K([H^i_c(\overline{X}, \mathbb{Q}_p)]) - \sum_i (-1)^i \widehat{D}_K([H^i_c(\overline{X'/G}, \mathbb{Q}_p)]) \\ = \sum_i (-1)^i \widehat{D}_K([H^i_c(\overline{X} - \overline{U}, \mathbb{Q}_p)]) - \sum_i (-1)^i \widehat{D}_K([H^i_c(\overline{X'/G} - \overline{U}, \mathbb{Q}_p)]).\end{aligned}$$

Since traces are additive with respect to exact sequences, we may consider our problem in the level of Grothendieck groups. Then we finish the proof of Theorem 3.1 by induction as in the proof of Theorem 2.1.

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