ON THE TWO-VARIABLE IWASAWA MAIN CONJECTURE

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Abstract. This paper is a continuation of the papers [O3] and [O4], where we showed one of the inequalities between the characteristic ideal of the Selmer group and the ideal of the p-adic L-function predicted by the two-variable Iwasawa Main Conjecture for a nearly ordinary Hida deformation $\mathcal{T}$ (see [O4] and §1 of this paper for the conjecture). In this paper, we study several properties of the Selmer group and the p-adic L-function solving some of open questions raised in [O3]. As applications, we examine the Main Conjecture for certain given cusp forms.

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1. Introduction

In this section, we shall introduce our results for the Iwasawa theory on Hida deformations obtained in [O3] and [O4]. We will also give a slight modification (see Theorem 2 and Remark 1.4) of our Euler system theory to give an application in §8.

To introduce our results, let us recall briefly Hida's nearly ordinary modular deformations.

We fix a prime number $p \geq 3$ and a norm compatible system $\{\zeta_{p^n}\}_{n\geq 1}$ of primitive p^n-th roots of unity throughout the paper. Let Γ be the Galois group $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ of the rational number field $\mathbb{Q}$. We denote by Γ' the group of diamond operators for the tower of modular curves $\{Y_1(p^t)\}_{t\geq 1}$. We have the canonical isomorphisms:

$$\Gamma \underset{\chi}{\overset{\sim}{\longrightarrow}} 1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times, \quad \Gamma' \underset{\kappa}{\overset{\sim}{\longrightarrow}} 1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times.$$

Fix a topological generator γ (resp. γ') of Γ (resp. Γ'). For later convenience, we choose γ and γ' so that $\chi(\gamma) = \kappa(\gamma')$. From now on, we fix an embedding of an algebraic closure $\overline{\mathbb{Q}}$ into the field $\mathbb{C}$ of complex numbers and an embedding of $\overline{\mathbb{Q}}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of the field $\mathbb{Q}_p$ of p-adic numbers simultaneously. We also fix a natural number N prime to p.

Let $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ be the quotient of the universal ordinary Hecke algebra $\mathbb{H}^{\mathrm{ord}}_{Np^\infty}$ with tame conductor N, which corresponds to a certain Λ-adic eigen cusp form $\mathcal{F}$. The algebra $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ is a local domain finite flat over $\mathbb{Z}_p[[\Gamma']]$. Then (the $\mathcal{F}$-component of) Hida's nearly ordinary Hecke algebra $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is defined to be the formal tensor product of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ and the cyclotomic Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$. By this, $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is isomorphic to $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}[[\Gamma]]$ and is a local domain finite flat over $\mathbb{Z}_p[[\Gamma\times\Gamma']]$. Let Σ be the finite set of places of $\mathbb{Q}$ consists of $\{\infty\}$ and the primes dividing Np. In his celebrated paper [H2], Hida constructs a big continuous Galois representation $\rho :\ G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}}(\mathcal{T}^{(0)}_{\mathcal{F}})$ unramified outside Σ, where $\mathcal{T}^{(0)}_{\mathcal{F}}$ is a finitely generated torsion-free module of generic rank two over $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. The representation $\mathcal{T}^{(0)}_{\mathcal{F}}$ is presented as $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}\widehat{\otimes}\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$, where $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ is a finitely generated torsion-free module of generic rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ with continuous $G_{\mathbb{Q}}$-action, $\widetilde{\chi}$ is the universal cyclotomic character $G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]^\times$ and $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ is a rank one free $\mathbb{Z}_p[[\Gamma]]$-module on which $G_{\mathbb{Q}}$ acts via the character $\widetilde{\chi}$. The trace of the Frobenius element $\mathrm{Fr}_l \in G_{\mathbb{Q}}$ acting on $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ is equal to the Fourier coefficient $A_l(\mathcal{F})$ of $\mathcal{F}$ for every primes $l \notin \Sigma$. Let $\mathfrak{M}$ be the maximal ideal of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ and let $\mathbb{F}$ be a finite residue field $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/\mathfrak{M}$. The residual representation of $\mathcal{T}^{(0)}_{\mathcal{F}}$ is defined to be a rank two $\mathbb{F}$-module with semi-simple $G_{\mathbb{Q}}$-action where the trace of Fr_l is congruent to $A_l(\mathcal{F})$ modulo $\mathfrak{M}$ for every primes $l \notin \Sigma$. Such residual representation of $\mathcal{T}^{(0)}_{\mathcal{F}}$ is always known to exist by Hida (cf. [MW, §9]) and is unique up to isomorphism by Chebotarev density theorem. Throughout the paper, we always assume the following condition unless otherwise stated:

(**Ir**) The residual representation of $\mathcal{T}^{(0)}_{\mathcal{F}}$ is an irreducible $G_{\mathbb{Q}}$-module.

The condition (**Ir**) implies that $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ (resp. $\mathcal{T}^{(0)}_{\mathcal{F}}$) is free of rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ (resp. $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$). Let us recall the following definition:

Definition 1.1. Let w be an integer. A point $\mathfrak{I} \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}, \overline{\mathbb{Q}}_p)$ is called *an arithmetic point of weight* w if there exists an open subgroup U of Γ' such that the restriction $\mathfrak{I}|_U :\ U \hookrightarrow \mathbb{Z}_p[[\Gamma']]^\times \hookrightarrow (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^\times \xrightarrow{\mathfrak{I}} \overline{\mathbb{Q}}_p^\times$ sends u to $\kappa^w(u)$ for any $u \in U$. We denote by $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ the set of arithmetic points of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. For an arithmetic point $\mathfrak{I}$ of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$, we will denote by $w(\mathfrak{I})$ the weight of $\mathfrak{I}$. We define a subset $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0} \subset \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ to be $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0} = \{\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}) | w(\mathfrak{I}) \geq 0\}$.

We briefly recall the properties of $\mathcal{T}^{(0)}_{\mathcal{F}}$ (cf. [H2], [Wi2]):

Basic property of nearly ordinary Hida deformations $\mathcal{T}^{(0)}_{\mathcal{F}}$. *Assume the condition* (**Ir**). *The deformation $\mathcal{T}^{(0)}_{\mathcal{F}}$ (resp. $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$) has the following properties*:

1. *For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, there exists a normalized eigen cusp form $f_{\mathfrak{I}}$ of weight $w(\mathfrak{I}) + 2$ and the quotient $\mathbb{T}^{\mathrm{ord}}/\mathrm{Ker}(\mathfrak{I})\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \cong \mathcal{O}^{\oplus 2}_{\mathfrak{I}}$ with $\mathcal{O}_{\mathfrak{I}} := \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/\mathrm{Ker}(\mathfrak{I})$ is isomorphic to $T_{f_{\mathfrak{I}}}$, where $T_{f_{\mathfrak{I}}}$ is the unique lattice of Deligne's Galois representation associated to $f_{\mathfrak{I}}$ (cf.* [De1]). *Thus, $\mathcal{T}^{(0)}_{\mathcal{F}}/(\mathrm{Ker}(\mathfrak{I}), \gamma - \chi^j(\gamma))\mathcal{T}^{(0)}_{\mathcal{F}}$ is isomorphic to $T_{f_{\mathfrak{I}}} \otimes \chi^j$ for each $j \in \mathbb{Z}$ and each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$.*
2. *As a representation of the decomposition group $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ at p, $\mathcal{T}^{(0)}_{\mathcal{F}}$ has a filtration $0 \longrightarrow \mathrm{F}^+\mathcal{T}^{(0)}_{\mathcal{F}} \longrightarrow \mathcal{T}^{(0)}_{\mathcal{F}} \longrightarrow \mathrm{F}^-\mathcal{T}^{(0)}_{\mathcal{F}} \longrightarrow 0$ such that the graded pieces $\mathrm{F}^+\mathcal{T}^{(0)}_{\mathcal{F}}$ and $\mathrm{F}^-\mathcal{T}^{(0)}_{\mathcal{F}}$ are free of rank one over $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.*
3. *Further, $\mathrm{F}^+\mathcal{T}^{(0)}_{\mathcal{F}}$ is isomorphic to $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\alpha})$ as a $G_{\mathbb{Q}_p}$-module, where $\widetilde{\alpha}$ is an unramified character $G_{\mathbb{Q}_p} \longrightarrow (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\times}$ such that $A_p(\mathcal{F}) = \widetilde{\alpha}(\mathrm{Frob}_p) \in \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ satisfies an interpolation property $\mathfrak{I}(A_p(\mathcal{F})) = a_p(f_{\mathfrak{I}})$ for each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ and $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\alpha})$ is a rank one free $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module on which $G_{\mathbb{Q}_p}$ acts via the character $\widetilde{\alpha}$.*

Let ω be the Teichmuller character. We will study the twist $\mathcal{T}^{(i)}_{\mathcal{F}} = \mathcal{T}^{(0)}_{\mathcal{F}} \otimes \omega^i$ for a fixed integer $0 \leq i \leq p-2$, which we call a nearly ordinary deformation. From now on, we will denote $\mathcal{T}^{(i)}_{\mathcal{F}}$ by $\mathcal{T}$ if there causes no possibility of confusion. We would like to study "the Iwasawa theory for $\mathcal{T}$". The space of p-adic characters of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is naturally viewed as a rigid analytic space finite flat over a two dimensional open unit ball in $\mathbb{C}^2_p$. Hence $\mathcal{T}$ corresponds to a family of Galois representations over a two dimensional rigid space. Each "hypersurface" of the space of characters of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is a rigid space of dimension one, which also interests us from a view point of "the Iwasawa theory for deformation spaces". Among infinitely many hypersurfaces, we especially study the following four types of hypersurfaces $\mathcal{T}_J = \mathcal{T}/J\mathcal{T}$ for hight one primes J of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ (see **(a)**, **(b)**, **(c)** and **(d)** below).

(a) Cyclotomic deformations of ordinary cuspforms.
$\mathcal{T}_I = T_{f_{\mathfrak{I}} \otimes \omega^i} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ for a cuspform $f_{\mathfrak{I}} \otimes \omega^i$ of weight $k = w(\mathfrak{I}) + 2$, which is free of rank two over $\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$. Here, $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ and I is a height-one ideal $\mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. This is the case called "the cyclotomic deformation" and was developed by many people since Mazur [Mz] started the Iwasawa theory for the cyclotomic deformation of an ordinary elliptic curve. (see [Gr1], [Gr4] and [MTT], for example)

(b) Ordinary deformation twisted by χ.
$\mathcal{T}_{(\gamma - \chi(\gamma))} = \mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes \chi\omega^i$, which is free of rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, $\mathcal{T}_{(\gamma - \chi(\gamma))}/\mathrm{Ker}(\mathfrak{I})\mathcal{T}_{(\gamma - \chi(\gamma))}$ is isomorphic to $T_{f_{\mathfrak{I}}} \otimes \chi\omega^i$. Hence $\mathcal{T}_{(\gamma - \chi(\gamma))}$ is the interpolation of the $\mathbb{Z}_p(1)$-twists of the Galois representations for $f_{\mathfrak{I}} \otimes \omega^{i-1}$ when $\mathfrak{I}$ varies in $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$.

(c) Ordinary deformation twisted by $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \chi$.

$\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')} = \mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \chi\omega^i$, which is free of rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. Note that $\mathcal{T} = \mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \omega^i$ and that the tensor product is taken through canonical isomorphism $\Gamma \xrightarrow{\sim} \Gamma'$. For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, $\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')}/\mathrm{Ker}(\mathfrak{I})\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')}$ is isomorphic to $T_{f_{\mathfrak{I}}} \otimes \chi^{w(\mathfrak{I})+1}\omega^i$. Hence $\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')}$ is the interpolation of $\mathbb{Z}(1)^{\otimes w(\mathfrak{I})+1}$-twists of the Galois representations of $f_{\mathfrak{I}} \otimes \omega^{i-1-w(\mathfrak{I})}$ when $\mathfrak{I}$ varies in $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$.
(d) One-variable deformation at the diagonal line.
$\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')} = \mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{\frac{1}{2}}) \otimes \chi\omega^i$, which is free of rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. Similarly as above, $\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}$ is the interpolation of $\mathbb{Z}(1)^{\otimes k(\mathfrak{I})/2}$-twists of the Galois representations of $f_{\mathfrak{I}} \otimes \omega^{i-k(\mathfrak{I})/2}$ when $\mathfrak{I}$ runs arithmetic points of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ with $k(\mathfrak{I}) \in 2\mathbb{Z}_{\geq 0}$, where $k(\mathfrak{I}) = w(\mathfrak{I}) + 2$ is the weight of the cuspform $f_{\mathfrak{I}}$. Note that the representations with the above twist correspond to the special value of $L(f_{\mathfrak{I}} \otimes \omega^{i-k(\mathfrak{I})/2}, s)$ at the center of the functional equations when $\mathfrak{I}$ varies.

Some of the Iwasawa theoretic properties of $\mathcal{T}_J$'s are deduced by the method of "specialization" from those of $\mathcal{T}$ (see §4 for such technique and also §6 for results and conjectures in these cases).

In the rest of this section, we will recall the results for the Iwasawa main conjecture principally in the case of $\mathcal{T}$. By Combining [O3, Theorem 3.14] and an optimal normalization of Beilinson-Kato elements done in this paper (Theorem 5.10), we have the following theorem:

Theorem 1 (Corollary 5.16)**.** *Let i be an integer such that $0 \leq i \leq p-2$. We assume the condition* (**Ir**) *for a nearly ordinary deformation $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$. Assume also the following condition:*
(**Nor**) $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *is integrally closed in its fraction field* $\mathrm{Frac}(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}})$.

Then we have an Euler system $\{\mathcal{Z}(r) \in H^1(\mathbb{Q}(\mu_r), \mathcal{T}^*(1))\}$ whose first layer $\mathcal{Z} = \mathcal{Z}(1)$ satisfies the equality

$$\mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}} \left(H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1))/\mathrm{loc}_{/f}(\mathcal{Z})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} \right)_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L^{\mathrm{Ki}}_p(\mathcal{T}))$$

for each height one prime $\mathfrak{l} \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, where $L^{\mathrm{Ki}}_p(\mathcal{T})$ is the Kitagawa's two-variable p-adic L-function [Ki] (*see also Theorem* 5.7 *for the interpolation property of Kitagawa's p-adic L-function*).

Remark 1.2. In the above theorem, the condition that $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is integrally closed is necessary only to assure that the image of Ξ_d is contained in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. Without this condition, the image of Ξ_d is in the fraction field $\mathrm{Frac}(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}})$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ and that the localization $\Xi_d(\mathcal{C})_{\mathfrak{l}}$ of $\Xi_d(\mathcal{C})$ is contained in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}$ for each height one prime $\mathfrak{l} \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. All interpolation properties as above hold without this condition (see the arguments in [O3, §5]).

On the other hand, we associate the Selmer group $\mathrm{Sel}_{\mathcal{T}}$ to $\mathcal{T}$. Let $\mathcal{A} = \mathcal{T} \otimes_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}, \mathbb{Q}_p/\mathbb{Z}_p)$. $\mathrm{Sel}_{\mathcal{T}}$ is defined as a subgroup of $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})$ (see §3.3 for the precise definition). The Pontryagin dual $(\mathrm{Sel}_{\mathcal{T}})^\vee$ of $\mathrm{Sel}_{\mathcal{T}}$ is a finitely generated torsion $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module (cf. Proposition 3.9). We propose the following conjecture:

Conjecture 1.3 (Two-variable Main Conjecture)**.** We assume the condition (**Ir**). We have the equality $\mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(\mathrm{Sel}_{\mathcal{T}})^\vee_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}))$ for each height one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.

In [O4], we proved that the ideal associated to the (localization of) Beilinson-Kato element for $\mathcal{T}$ is contained in the characteristic ideal of $(\mathrm{Sel}_{\mathcal{T}})^\vee$. We restate the result, but with slight modification of the assumptions (see Remark 1.4 below):

Theorem 2 . *We assume that $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is isomorphic to a two-variable power series algebra $\mathcal{O}[[X_1, X_2]]$ over the ring of the integers $\mathcal{O}$ of a certain finite extension of $\mathbb{Q}_p$. Let us assume the condition (**Ir**) for $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$ and the existence of the elements $\tau \in G_{\mathbb{Q}(\mu_{p^\infty})}$ and $\tau' \in G_{\mathbb{Q}}$ which satisfy the following properties:*

(i) *The image of τ under the representation $G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\mathcal{T}) \cong GL_2(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}})$ has a presentation $\begin{pmatrix} 1 & P_\tau \\ 0 & 1 \end{pmatrix}$ under certain choice of basis $\mathcal{T} \cong (\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}})^{\oplus 2}$, where P_τ is a non-zero element of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.*

(ii) *The element $\tau' \in G_{\mathbb{Q}}$ acts on $\mathcal{T}/\mathfrak{M}\mathcal{T}$ via the multiplication by -1.*

Then there exists an integer $m \geq 0$ such that we have the following inequality for each height one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$:

$$\mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(\mathrm{Sel}_{\mathcal{T}})^\vee_{\mathfrak{l}} \leq \mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}\left(H_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1))/\mathrm{loc}_{/f}(\mathcal{Z})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}\right)_{\mathfrak{l}} + \mathrm{ord}_{\mathfrak{l}}(P^m_\tau).$$

Remark 1.4. In the paper [O4], we assumed the following condition (ii$'$) in place of the above condition (ii) :

(ii$'$). The element $\tau' \in G_{\mathbb{Q}}$ acts on $\mathcal{T}$ via the multiplication by -1.

However, the condition (ii) and (ii$'$) are equivalent to each other by the following lemma:

Lemma 1.5. *Let R be a complete Noetherian local ring whose residue field $R/\mathfrak{M}$ is a finite field of characteristic $p > 2$ and let G be a subgroup of $GL_2(R)$. We denote by $\overline{G} \subset GL_2(R/\mathfrak{M})$ the image of G under the reduction map $GL_2(R) \longrightarrow GL_2(R/\mathfrak{M})$. Then G contains a scalar matrix of multiplication by -1 if and only if $\overline{G}$ contains the multiplication by -1.*

We omit the proof of this rather elementary lemma, but we remark that the condition (ii) is easier to check than (ii$'$) (cf. §8 and Claim 8.11).

Finally, our results combining Theorem 1 and Theorem 2 are summarized as follows.

Theorem 3 . *Let us assume the condition (**Ir**). Assume further that $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is isomorphic to a two-variable power series algebra $\mathcal{O}[[X_1, X_2]]$. Then*

(1) *The Pontryagin dual $(\mathrm{Sel}_{\mathcal{T}})^\vee$ of $\mathrm{Sel}_{\mathcal{T}}$ is a finitely generated torsion $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module.*

(2) *Suppose that we have elements $\tau \in G_{\mathbb{Q}(\mu_{p^\infty})}$ and $\tau' \in G_{\mathbb{Q}}$ satisfying the conditions* (i) *and* (ii) *in Theorem 2. Then, there exists an integer m such that we have the following inequality for each height one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$:*

$$\mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(\mathrm{Sel}_{\mathcal{T}})^\vee_{\mathfrak{l}} \leq \mathrm{ord}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(L^{\mathrm{Ki}}_p(\mathcal{T})) + \mathrm{ord}_{\mathfrak{l}}(P^m_\tau).$$

So far, we gave results on the two-variable Iwasawa main conjecture for nearly ordinary deformations $\mathcal{T}$. The above results are applied to the Iwasawa theory for one-variable specializations $\mathcal{T}/J\mathcal{T}$ for height one ideals J of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. In §3.2, Selmer group for $\mathcal{T}/J\mathcal{T}$ are studied using Bloch-Kato's method or Greenberg's one and we have certain comparison between two-definitions. The specialization technique from two-variable to one-variable is discussed in §4.1. Under these preparation, we discuss the one-variable Iwasawa theory for the deformations (**a**), (**b**), (**c**) and (**d**) above. For example, by applying Lemma 6.2 to the case (**a**), we have the following corollary to Theorem 3 (see Corollary 6.4):

Corollary . *Under the same assumption as that of Theorem 2 with $P_{\mathcal{T}}$ a unit in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, the cyclotomic Iwasawa Main conjecture (I.M.C.) by Mazur-Tate-Teitelbaum for one of arithmetic specializations $f_{\mathfrak{I}_0}$ of $\mathcal{F}$ with $w(\mathfrak{I}_0) \geq 0$ is true if and only if the cyclotomic I.M.C. is true for every specializations $f_{\mathfrak{I}}$ of $\mathcal{F}$ with $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$.*

Thus we have an infinite p-adic family of modular forms where the cyclotomic I.M.C. is true. Note that under the assumption with $\mu = 0$, a recent preprint [EPW] also proves such a p-adic family of modular forms where the cyclotomic I.M.C. is true. Though our corollary does not recover all the results of [EPW], we have an advantage that we do not need any assumption on μ-invariants.

As far as we know, the one-variable Main conjecture in the cases (**b**), (**c**) and (**d**) are not known nor formulated previously. According to the discussion in the preceding sections, we formulate these conjecture in §6. We refer the reader to §6 for the detail on these new conjectures and our results.

Since only few things are known for the two-variable Iwasawa theory, we would like explicit examples which help us to develop our future perspective. As an attempt, we will start from the case of Ramanujan's cuspform $\Delta = q\prod_{n\geq 1}(1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z}))$. For each prime number p such that $p \nmid a_p(\Delta)$, we have a unique Λ-adic newform $\mathcal{F}(\Delta)$ which contains Δ at weight 12. For each integer i with $0 \leq i \leq p-2$, we have a nearly ordinary deformation $\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}$.

Proposition 1.6. *Let $p \geq 11$ be a prime number with $p \nmid a_p(\Delta)$. Assume that $1 \leq i \leq 11$ and $p \leq 10,000$. Let $\mathcal{T}$ be $\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}$.*

1. *Except for $(p,i) = (11,1)$, $(23,1)$ and $(691,1)$, we have $\mathrm{Sel}_{\mathcal{T}} = 0$ and $L_p(\mathcal{T})$ is a unit.*
2. *When $p = 11$ and $i = 1$, $\mathrm{Sel}^{\vee}_{\mathcal{T}}$ is isomorphic to $\mathbb{Z}_p[[\Gamma \times \Gamma']]/(\gamma^2 - \kappa^2(\gamma')\gamma')$ and we have the equality of ideal $(\gamma^2 - \kappa^2(\gamma')\gamma') = (L_p(\mathcal{T}))$.*

Remark 1.7. Thus, especially the two-variable Main conjecture of Δ holds for all $p \leq 10,000$ and $0 \leq i \leq 10$ except for $(p,i) = (23,1)$ and $(p,i) = (691,1)$. For $(p,i) = (23,1)$, it is easy to see that $L^{\mathrm{Ki}}_p(\mathcal{T})$ is not a unit by the interpolation property in Theorem 5.7 since $a_p(\Delta) - 1 \equiv 0$ modulo 23. The image of mod 23 representation for Δ is dihedral and thus the condition (ii) in Theorem 2 is not satisfied. It is our future project to generalize the results in [O4] so that Theorem 2 is true in the case $p = 23$. For $p = 691$, the residual representation is not irreducible any more (the condition (**Ir**) is not satisfied) and the choice of lattice $\mathcal{T}$ is not unique for a given $\mathcal{F}$. We treat the Main conjecture for residually reducible deformations in a forthcoming paper [O5].

Notations. For an integer r, we denote by μ_r the group of r-th roots of unity and denote by $\mathbb{Q}(\mu_r)$ the field obtained by adjoining μ_r to the rational number field $\mathbb{Q}$. We often denote by $\mathbb{Q}(\mu_{p^\infty})$ the field obtained by adjoining all p-power roots of unity to the rational number field $\mathbb{Q}$. For any Galois extension $L/\mathbb{Q}$ and a prime number q which is unramified in $L/\mathbb{Q}$, we denote by $\mathrm{Frob}_q \in \mathrm{Gal}(L/\mathbb{Q})$ (resp. $\varphi_q \in \mathrm{Gal}(L/\mathbb{Q})$) (a conjugate class of) a geometric (resp. arithmetic) Frobenius element at q.

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2. Local monodromy on $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$

For later use in §3 and §4, we study the action $\rho_{\mathcal{F}}$ of the inertia group I_v at $v \in \Sigma \setminus \{p, \infty\}$ on the Hida deformation $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ associated to a Λ-adic newform $\mathcal{F}$ introduced in §1. We will keep the notations in the previous section. The result of this section is summarized in Theorem 2.3. The reader who is mainly interested in the Selmer group or in the p-adic L-function can skip this section by admitting Theorem 2.3.

We prepare the following lemma:

Lemma 2.1. *Let $G \subset \mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ be a finite subgroup. For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, G is mapped into $\mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}})$ under the specialization $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}) \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}})$.*

Proof. By fixing a basis of $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$, we have isomorphisms $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \cong (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\oplus 2}$ and $T_{f_{\mathfrak{I}}} \cong (\mathcal{O}_{\mathfrak{I}})^{\oplus 2}$. Suppose that there exists an element $g \in G$ which is mapped to a trivial element on $\mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}}) \cong GL_2(\mathcal{O}_{\mathfrak{I}})$. Since the order of g is finite, by extending the coefficients of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ if necessary, we may assume that g is conjugate to a diagonal matrix $\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \in GL_2(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with u and u' roots of unity. This completes the proof since the roots of unity in $(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\times}$ is disjoint from $\mathrm{Ker}[(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\times} \twoheadrightarrow (\mathcal{O}_{\mathfrak{I}})^{\times}]$. □

Since $v \in \Sigma \setminus \{p, \infty\}$, the action of I_v on $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ is non-trivial. By the above lemma, we consider the following case:

(A) The image $\rho_{\mathcal{F}}(I_v)$ in $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ is a finite subgroup.

In this case, the action of I_v on $(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^{*} = \mathrm{Hom}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}, \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ also factors through a finite quotient of I_v. Hence there exist a finite flat extension $\mathcal{O}$ of $\mathbb{Z}_p$ contained in $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ and a finite type $\mathcal{O}$-module M with $\mathrm{rank}_{\mathrm{Frac}(\mathcal{O})}(M \otimes_{\mathcal{O}} \mathrm{Frac}(\mathcal{O})) \leq 1$ such that the coinvariant quotient $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^{*})_{I_v}$ is isomorphic to $M \otimes_{\mathcal{O}} \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$.

Next, we discuss the following case:

(B) The image $\rho_{\mathcal{F}}(I_v)$ in $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ is an infinite subgroup.

In this case, it is not difficult to see that there exists an arithmetic point $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ such that the action of I_v on $T_{f_{\mathfrak{I}}} \cong \mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}/\mathrm{Ker}(\mathfrak{I})\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ does not factor through a finite quotient of I_v. Let us fix one such $\mathfrak{I}_0 \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ for a while. We note that the action of I_v on $T_{f_{\mathfrak{I}_0}}$ can be infinite only when the local automorphic representation $\pi_v(\mathfrak{I}_0)$ of $GL_2(\mathbb{Q}_v)$ associated to $f_{\mathfrak{I}_0}$ is a special representation. Hence the local Galois representation $G_{\mathbb{Q}_v} \longrightarrow GL_2(\mathcal{O}_{\mathfrak{I}})$ for $f_{\mathfrak{I}_0}$ is represented by a matrix $\begin{pmatrix} \chi & * \\ 0 & \chi' \end{pmatrix}$ such that $\chi|_{I_v} = \chi'|_{I_v}$ and $\chi'\chi^{-1} = |\ |^{\pm}$, where $|\ |$ is the absolute value character $G_{\mathbb{Q}_v} \longrightarrow G^{\mathrm{ab}}_{\mathbb{Q}_v} \xrightarrow{\sim} \mathbb{Q}_v^{\times} \longrightarrow |\ |_v$. Since a finite-order character of $G_{\mathbb{Q}_v}$ is always the localization of a finite-order character of $G_{\mathbb{Q}}$, we have a Dirichlet character η_0 with v-primary conductor such that the action of I_v on $T_{f_{\mathfrak{I}_0}} \otimes \eta_0 = T_{f_{\mathfrak{I}_0} \otimes \eta_0}$ is unipotent. Let us now recall the structure on the inertia group I_v. The group I_v has the filtration $P \subset Q \subset I_v$ such that P is the maximal pro-v subgroup of I_v and I_v/Q is isomorphic to $\mathbb{Z}_p$. Since Q/P is isomorphic to $\prod_{l \neq v,p} \mathbb{Z}_l$, Q has no non-trivial p-primary subquotient. This immediately implies the following lemma:

Lemma 2.2. *Let $v \in \Sigma \setminus \{p, \infty\}$.*

1. *The image $\rho_{\mathcal{F}}(Q)$ is a finite subgroup of $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$.*
2. *For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, the group $\rho_{\mathcal{F}}(Q)$ is mapped into $\mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}})$ under the specialization $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}) \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}})$.*

Proof. For the proof, we note that the prime-to-p part of $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ is finite and that the kernel of $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}) \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_{\mathfrak{I}}}(T_{f_{\mathfrak{I}}})$ is a pro-p group. □

Since the action of I_v on $T_{f_{\mathfrak{I}} \otimes \eta_0}$ is unipotent, the subgroup Q acts trivially on $T_{f_{\mathfrak{I}} \otimes \eta_0}$. By Lemma 2.2, Q acts trivially on $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes \eta_0 = \mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0}$, where $\mathcal{F} \otimes \eta_0$ is the Λ-adic newform obtained as the twist of $\mathcal{F}$ by η_0. Let γ be a topological generator of $I_v/Q \cong \mathbb{Z}_p$. By assumption, the action of γ on $T_{f_{\mathfrak{I}_0} \otimes \eta_0}$ is represented by a non-trivial unipotent matrix. Let $(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes \eta_0)^{\mathrm{ss}}$ be the semi-simplification of as an I_v-module. Then, the action of γ on $(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0})^{\mathrm{ss}} = (\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \otimes \eta_0)^{\mathrm{ss}}$ is represented by a matrix $\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}$ with $a, a' \in (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\times}$. If a or a' is not a root of unity, there exists $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ such that the action of I_v on $(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0})^{\mathrm{ss}}/\mathrm{Ker}(\mathfrak{I})(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0})^{\mathrm{ss}} = (T_{f_{\mathfrak{I}}})^{\mathrm{ss}}$ is of infinite order. It is impossible for a representation of I_v associated to a cuspform. Hence a and a' are roots of unity. Since a and a' are congruent to 1 modulo $\mathrm{Ker}(\mathfrak{I}_0)$, we show that $a = a' = 1$ by similar discussion as Lemma 2.1 for $(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0})^{\mathrm{ss}}$. Thus, the action of γ on $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F} \otimes \eta_0}$ is represented by a unipotent matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$. Recall that the v-order of the tame conductor of $f_{\mathfrak{I} \otimes \eta_0}$ is constant when $\mathfrak{I}$ varies in $\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ by applying [H1, Corollary 3.7] to $\mathcal{F} \otimes \eta_0$. Thus $\mathfrak{I}(b)$ are not zero for every $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. We conclude that $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}$ is isomorphic to $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(1-u) \oplus \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(b, 1-u)$, where u is a root of unity which generates the group of the values of η_0.

Summarizing the above argument, we have the following theorem.

Theorem 2.3. *Let $v \in \Sigma \setminus \{p, \infty\}$.*

1. *If the image of I_v on $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ is finite, there exist a finite flat extension $\mathcal{O}$ of $\mathbb{Z}_p$ contained in $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ and a finite type $\mathcal{O}$-module M with $\mathrm{rank}_{\mathrm{Frac}(\mathcal{O})}(M \otimes_{\mathcal{O}} \mathrm{Frac}(\mathcal{O})) \leq 1$ such that the coinvariant quotient $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}$ is isomorphic to $M \otimes_{\mathcal{O}} \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$.*
2. *If the image of I_v on $\mathrm{Aut}_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}}(\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})$ is infinite, $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}$ is isomorphic to $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(1-u) \oplus \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(b, 1-u)$ where b is an element in $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ such that $\mathfrak{I}(b) \neq 0$ for every $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ and u is a certain root of unity in $(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\times}$ ($u = 1$ is possible).*

The following remark explains Theorem 2.3 from the theory of admissible representations and the local Langlands correspondence for GL_2.

Remark 2.4. In the case **(A)** of this section, the admissible representation $\pi_v(\mathfrak{I})$ of $GL_2(\mathbb{Q}_p)$ corresponding to $f_{\mathfrak{I}}$ is a supercuspidal representation or a principal series at each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. Further, if $\pi_v(\mathfrak{I})$ is a supercuspidal representation (resp. a principal series) at one of $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, $\pi_v(\mathfrak{I})$ are supercuspidal representations (resp. a principal series) at every $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. In the case **(B)**, $\pi_v(\mathfrak{I})$ is a special representation at each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$.

3. Selmer groups for Galois deformations

In this section, we review the definition of Selmer groups for a two-variable nearly ordinary deformation $\mathcal{T}$ and for its various specializations $\mathcal{T}/\mathfrak{A}\mathcal{T}$ by ideals $\mathfrak{A} \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. We also give some fundamental properties on these Selmer groups.

Let $\mathcal{A}$ be the discrete Galois representation $\mathcal{T} \otimes_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}, \mathbb{Q}_p/\mathbb{Z}_p)$. We denote by $\mathbb{Q}_\Sigma$ the maximal Galois extension of $\mathbb{Q}$ which is unramified outside Σ.

3.1. **Selmer groups over discrete valuation rings.** Let (j,k) be a pair of integers satisfying $1 \leq j \leq k-1$ and let $\Delta^{(j,k)}_{s,t} = (\gamma^{p^s} - \chi^j(\gamma^{p^s}), \gamma'^{p^t} - \kappa^{k-2}(\gamma'^{p^t}))$ be a height-two ideal of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. We denote by $A^{(j,k)}_{s,t}$ the $\Delta^{(j,k)}_{s,t}$-torsion part $\mathcal{A}[\Delta^{(j,k)}_{s,t}]$ of $\mathcal{A}$, which is identified with $(\mathcal{T}/\Delta^{(j,k)}_{s,t}\mathcal{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Note that

1. $\mathcal{T}/\Delta^{(j,k)}_{s,t}\mathcal{T}$ is free of finite rank over $\mathbb{Z}_p$.
2. The p-adic representation $(\mathcal{T}/\Delta^{(j,k)}_{s,t}\mathcal{T}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to $\bigoplus_f (V_f \otimes \chi^j\omega^i) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma/\Gamma^{p^s}](\widetilde{\chi})$, where f runs ordinary eigen cusp forms of weight k for $\Gamma_1(Np^t)$ such that the residual representation for f are isomorphic to that of $\mathcal{T} \otimes \omega^{-i}$. Here, $\mathbb{Z}_p[\Gamma/\Gamma^{p^s}](\widetilde{\chi})$ is a free $\mathbb{Z}_p[\Gamma/\Gamma^{p^s}]$-module of rank one on which $G_{\mathbb{Q}}$ acts via the tautological character $\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma/\Gamma^{p^s} \hookrightarrow \mathbb{Z}_p[\Gamma/\Gamma^{p^s}]^\times$.

For any $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-module T which is free of finite rank over $\mathbb{Z}_p$, Selmer groups are defined as a subgroup of $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, A)$, where $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Once we fix a local condition $H^1_?(\mathbb{Q}_v, A) \subset H^1(\mathbb{Q}_v, A)$ at each $v \in \Sigma \setminus \{\infty\}$, we define a Selmer group $\mathrm{Sel}^?_T$ as follows:

$$\mathrm{Sel}^?_T = \mathrm{Ker}\left[H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, A) \longrightarrow \prod_{v \in \Sigma} \frac{H^1(\mathbb{Q}_v, A)}{H^1_?(\mathbb{Q}_v, A)} \right] \tag{1}$$

For $v \in \Sigma \setminus \{p, \infty\}$, one of the local conditions is given by the unramified part

$$H^1_{\mathrm{ur}}(\mathbb{Q}_v, A) = \mathrm{Ker}[H^1(\mathbb{Q}_v, A) \longrightarrow H^1(I_v, A)],$$

where I_v is the inertia subgroup at v. Let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We define "the finite part":

$$H^1_f(\mathbb{Q}_v, A) = \mathrm{pr}(H^1_{\mathrm{ur}}(\mathbb{Q}_v, V)),$$

where $\mathrm{pr} : H^1(\mathbb{Q}_v, V) \longrightarrow H^1(\mathbb{Q}_v, A)$ is the map induced by the projection map $V \twoheadrightarrow A = V/T$ of $G_{\mathbb{Q}_v}$-modules and $H^1_{\mathrm{ur}}(\mathbb{Q}_v, V) = \mathrm{Ker}[H^1(\mathbb{Q}_v, V) \longrightarrow H^1(I_v, V)]$.

We also give local conditions at p.

1. Greenberg's local condition $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A) \subset H^1(\mathbb{Q}_p, A)$ is defined to be

$$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, A) \longrightarrow H^1(I_p, \mathrm{F}^- A)\right].$$

2. Bloch-Kato defined $H^1_f(\mathbb{Q}_p, A)$ called "the finite part" to be $H^1_f(\mathbb{Q}_p, A) = \mathrm{pr}(H^1_f(\mathbb{Q}_p, V))$, where

$$H^1_f(\mathbb{Q}_p, V) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})\right],$$

by using the ring of p-adic periods B_{crys} defined by Fontaine (cf. [Bu]).

Selmer groups $\mathrm{Sel}^{\mathrm{BK}}_T$ and $\mathrm{Sel}^{\mathrm{Gr}}_T$ according to [BK] and [Gr1] are defined by the following condition (cf. equation (1) in this subsection).

	$\mathrm{Sel}^{\mathrm{BK}}_T$	$\mathrm{Sel}^{\mathrm{Gr}}_T$
$H^1_?(\mathbb{Q}_v, A)$ for $v \in \Sigma \setminus \{p, \infty\}$	H^1_f	H^1_{ur}
$H^1_?(\mathbb{Q}_p, A)$	$H^1_f(\mathbb{Q}_p, A)$	$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A)$

Recall that we have the following proposition (cf. [O3, §4]):

Proposition 3.1. *Let us assume that $1 \leq j \leq k-1$. Then $H^1_f(\mathbb{Q}_p, A^{(j,k)}_{s,t})$ is the maximal divisible subgroup of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A^{(j,k)}_{s,t})$ for each pair of integers $(s,t) \geq (0,0)$.*

We have the following corollary of Proposition 3.1:

Corollary 3.2. *Let us assume that $1 \leq j \leq k-1$. We denote by $T^{(j,k)}_{s,t}$ the representation $\mathcal{T}/\Phi^{(j,k)}_{s,t}\mathcal{T}$, which is free of finite rank over $\mathbb{Z}_p$ for each $(s,t) \geq (0,0)$. Then $\mathrm{Sel}^{\mathrm{BK}}_{T^{(j,k)}_{s,t}}$ is a subgroup of $\mathrm{Sel}^{\mathrm{Gr}}_{T^{(j,k)}_{s,t}}$ with finite index.*

Remark 3.3. Let T be a $G_{\mathbb{Q}}$-module which is a quotient $\mathcal{T}/J\mathcal{T}$ by a height-two ideal (not necessarily a prime ideal) $J \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. Assume that there is a pair (j,k) with $1 \leq j \leq k-1$ such that T is dominated by $T^{(j,k)}_{s,t}$ for sufficiently large s,t. Since T is free of finite rank over $\mathbb{Z}_p$, we define $\mathrm{Sel}^{\mathrm{BK}}_T$ as in the previous subsection. We define also $\mathrm{Sel}^{\mathrm{Gr}}_T$ by means of the $G_{\mathbb{Q}_p}$-stable filtration F^+T induced from $\mathrm{F}^+\mathcal{T}$. Then, the same results as Proposition 3.1 and Corollary 3.2 hold.

3.2. **Selmer groups over one-variable Iwasawa algebras.** In this subsection, we give Selmer groups for specializations $\mathcal{T}_J = \mathcal{T}/J\mathcal{T}$ at height-one primes J of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ in the cases **(a)**, **(b)**, **(c)** and **(d)** in §1. Recall that J is given as follows in each case:
(**a**) J is $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ for $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. (**b**) J is $(\gamma - \chi(\gamma)) \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. (**c**) J is $(\gamma - \kappa(\gamma')\gamma') \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. (**d**) J is $(\gamma^2 - \kappa^2(\gamma')\gamma') \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.
In the cases (**a**), $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is isomorphic to $\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$. In the cases (**b**), (**c**) and (**d**), $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is isomorphic to $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. The Greenberg-type Selmer group $\mathrm{Sel}^{\mathrm{Gr}}_J$ for $\mathcal{T}_J$ is defined by:

$$\mathrm{Sel}^{\mathrm{Gr}}_J = \mathrm{Ker}\left[H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J]) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J])} \oplus \bigoplus_{v \in \Sigma\setminus\{p,\infty\}} \frac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])}\right],$$

In each of the above four cases, let us take a system of height-one ideals (not necessarily prime ideals) $\{H_u \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}\}_{u \geq 1}$of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with the following properties:

1. We have $H_u \supset H_{u+1}$ for each $u \geq 1$ and $\bigcap_{u \geq 1} H_u = 0$.

2. $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/(J,H_u)$ is finite flat over $\mathbb{Z}_p$ for each $u \geq 1$.

For each $u \geq 1$, $\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}/(J,H_u)\mathcal{T}}$ is defined as in 3.1 by using the filtration $\mathrm{F}^{\pm}(\mathcal{T}/(J,H_u)\mathcal{T}) := \mathrm{F}^{\pm}\mathcal{T}/(J,H_u)\mathrm{F}^{\pm}\mathcal{T}$. Further, $\mathrm{Sel}^{\mathrm{Gr}}_J$ is isomorphic to $\varinjlim_{u\geq 1}\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}/(J,H_u)\mathcal{T}}$ by definition. On the other hand, the Bloch-Kato-type Selmer group for $\mathcal{T}_J$ is defined via a certain system of height-one ideals $\{H_u \subset \mathbb{H}^{\text{n.o}}_{\mathcal{F}}\}_{u\geq 1}$ in $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$ and might depend on the choice apriori. For a fixed natural number, we will make the following choice of a system $\{H_u\}_{u\geq 1}$ of height-one ideals:

$$\begin{cases} \{H_s\}_{s\geq 1} = \{\Phi^{(j)}_s = (\gamma^{p^s} - \chi^j(\gamma^{p^s}))\}_{s\geq 1} & \text{in the case } (\mathbf{a}), \\ \{H_t\}_{t\geq 1} = \{\Psi^{(k)}_t = (\gamma'^{p^t} - \kappa^{k-2}(\gamma'^{p^t}))\}_{t\geq 1} & \text{in the case } (\mathbf{b}), (\mathbf{c}) \text{ or } (\mathbf{d}). \end{cases}$$

We define the Bloch-Kato type Selmer group as follows:

$$\begin{cases} \mathrm{Sel}^{\mathrm{BK},(j)}_J = \varinjlim_s \mathrm{Sel}^{\mathrm{BK}}_{\mathcal{T}/(J,\Phi^{(j)}_s)\mathcal{T}} & \text{in the case } (\mathbf{a}), \\ \mathrm{Sel}^{\mathrm{BK},(k)}_J = \varinjlim_t \mathrm{Sel}^{\mathrm{BK}}_{\mathcal{T}/(J,\Psi^{(k)}_t)\mathcal{T}} & \text{in the case } (\mathbf{b}), (\mathbf{c}) \text{ or } (\mathbf{d}), \end{cases}$$

(In the case (**a**), we assume that $1 \leq j \leq w(\mathfrak{I}) + 1$)

Let $\mathrm{Div}(M)$ be the maximal divisible subgroup for an abelian group M. We have the following proposition:

Proposition 3.4. *We assume the condition* (**Ir**) *for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$ *with* $0 \leq i \leq p-2$. *Let* J *be a height-one ideal of* $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$ *determined at the beginning of* 3.2 *according to which of the cases* (**a**), (**b**), (**c**) *and* (**d**) *we consider. Then,*

1. $\mathrm{Sel}^{\mathrm{BK},(?)}_J$ *is a* $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$*-submodule of* $\mathrm{Sel}^{\mathrm{Gr}}_J$. *The Pontryagin dual* $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ *of* $\mathrm{Sel}^{\mathrm{Gr}}_J$ *is a finitely generated* $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/J$*-module.* $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ *is torsion over* $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/J$ *except in the case* (**d**) (*cf. Remark* 3.5).
2. *In the case* (**a**) *with* $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$, *we have* :
$$\mathrm{Sel}^{\mathrm{Gr}}_J/\mathrm{Sel}^{\mathrm{BK},(j)}_J \cong \begin{cases} (\mathcal{O}_{\mathfrak{I}})^\vee & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0 \text{ and } a_p(f_{\mathfrak{I}}) = 1, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0 \text{ or } a_p(f_{\mathfrak{I}}) \neq 1. \end{cases}$$
3. *We have* :
$$\begin{cases} \mathrm{Sel}^{\mathrm{Gr}}_J/\mathrm{Sel}^{\mathrm{BK},(k)}_J \cong W_J & \text{in the cases } (\mathbf{b}) \text{ and } (\mathbf{c}), \\ \mathrm{Sel}^{\mathrm{Gr}}_J/\mathrm{Sel}^{\mathrm{BK},(k)}_J \hookrightarrow W_J & \text{in the cases } (\mathbf{d}), \end{cases}$$
where $W_J = \varinjlim_t \dfrac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,\Psi^{(k)}_t])}{\mathrm{Div}\big(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,\Psi^{(k)}_t])\big)} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \left(\Big(((\mathcal{T}^*_J)_{I_v})^{G_{\mathbb{Q}_v}}\Big)_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}\text{-tor}}\right)^\vee.$
Further, the component of W_J *at each prime is given as follows:*
(i) *For each* $v \in \Sigma \setminus \{p,\infty\}$, *we have*:
$$\big((\mathcal{T}^*_J)_{I_v}\big)^{G_{\mathbb{Q}_v}} \cong \begin{cases} \big(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \otimes \chi^{-1}\omega^{-i}\big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{b}), \\ \big(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \chi^{-1}\omega^{-i}\big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{c}), \\ \Big(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-\frac{1}{2}}) \otimes \chi^{-1}\omega^{-i}\Big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{d}). \end{cases}$$

(ii) *We have* $\varinjlim_t \dfrac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, \Psi^{(k)}_t])}{\mathrm{Div}\big(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, \Psi^{(k)}_t])\big)} \twoheadrightarrow H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathcal{A}[J])$. *When* $i \neq 1$ *is satisfied or when* $a_p(f_{\mathfrak{I}}) \neq 1$ *are satisfied for every* $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, *we have* $\varinjlim_t \dfrac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, \Psi^{(k)}_t])}{\mathrm{Div}\big(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, \Psi^{(k)}_t])\big)} \cong H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- \mathcal{A}[J])$. *We have :*

$$H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- \mathcal{A}[J]) \cong \begin{cases} \big(\big((\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(\gamma'-1)\big)[A_p(\mathcal{F})-1]\big)^\vee & \textit{in the cases } (\mathbf{b}) \textit{ and } (\mathbf{d}) \textit{ with } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 3.5. 1. In the case ($\mathbf{d}$), $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ is not necessarily a torsion $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module. We refer the reader to the section 6 for more information.

2. Let us note that $a_p(f_{\mathfrak{I}}) = 1$ happens only when $w(\mathfrak{I}) = 0$. In the case ($\mathbf{a}$), the difference in the second statement is rather known to the experts as "trivial zero" phenomena at least when $f_{\mathfrak{I}}$ is associated to an elliptic curve.
3. The group $\Big(\big((\mathcal{T}^*_J)_{I_v}\big)^{G_{\mathbb{Q}_v}}\Big)_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}\text{-tor}}$ is shown to be zero if certain conditions are satisfied in the case **(B)** of the section 2. In fact, we have an extension as follows in the case **(B)**:

$$0 \longrightarrow \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{\frac{1}{2}}\chi\omega\psi) \longrightarrow ((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \longrightarrow \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{\frac{1}{2}}\psi)/(b)\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{\frac{1}{2}}\psi) \longrightarrow 0,$$

where ψ is a Dirichlet character and $b \in \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ is a non-zero element such that the ideal (b) is prime to every height-one ideals $\Phi^{(k)}_t$ when $k \geq 2$ and $t \geq 0$ varies. Hence, we have:

$$\Big(\big((\mathcal{T}^*_J)_{I_v}\big)^{G_{\mathbb{Q}_v}}\Big)_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}\text{-tor}} \cong \begin{cases} \big(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{\frac{1}{2}}\chi^{-1}\psi\omega^{-i})/(b)\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{\frac{1}{2}}\chi^{-1}\psi\omega^{-i})\big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{b}), \\ \big(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{-\frac{1}{2}}\chi^{-1}\psi\omega^{-i})/(b)\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\widetilde{\chi}^{-\frac{1}{2}}\chi^{-1}\psi\omega^{-i})\big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{c}), \\ \big(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\chi^{-1}\psi\omega^{-i})/(b)\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}(\chi^{-1}\psi\omega^{-i})\big)^{G_{\mathbb{Q}_v}} & \text{in the case } (\mathbf{d}). \end{cases}$$

We see that $\Big(\big((\mathcal{T}^*_J)_{I_v}\big)^{G_{\mathbb{Q}_v}}\Big)_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}\text{-tor}}$ is trivial when $\psi = 1$ and ($\mathbf{c}$) are satisfied or when ($\mathbf{d}$) is satisfied. We expect that $\Big(\big((\mathcal{T}^*_J)_{I_v}\big)^{G_{\mathbb{Q}_v}}\Big)_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}\text{-tor}}$ is trivial in other cases. However we have not studied it.

Proof. Recall that $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ is known to be a finitely generated torsion module over $\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$ in the case ($\mathbf{a}$) by results of Kato-Rubin (cf. [R1], [Ka2]). In the cases ($\mathbf{b}$) and ($\mathbf{c}$), it is not difficult to see that $\mathrm{Sel}_{\mathcal{T}/(J,\Psi^{(k)}_t)\mathcal{T}} \longrightarrow \mathrm{Sel}_J[\Psi^{(k)}_t]$ has finite kernel and cokernel. If $k \geq 3$, $\mathrm{Sel}_{\mathcal{T}/(J,\Psi^{(k)}_t)\mathcal{T}}$ is finite by Kato-Rubin. Since $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee/\Psi^{(k)}_t(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ is finite, $(\mathrm{Sel}^{\mathrm{Gr}}_J)^\vee$ is torsion over $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J \cong \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. This gives the first assertion.

We will show the other assertions in the rest of the proof. Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Sel}_J^{\mathrm{BK},(?)} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J]) & \xrightarrow{\mathrm{loc}_J^{\mathrm{BK}}} & \displaystyle\bigoplus_{v\in\Sigma\setminus\{\infty\}} \varinjlim_u \frac{H^1(\mathbb{Q}_v, \mathcal{A}[J,H_u])}{H^1_f(\mathbb{Q}_v, \mathcal{A}[J,H_u])} \\
 & & \downarrow & & \| & & \downarrow \gamma_J \\
0 & \longrightarrow & \mathrm{Sel}_J^{\mathrm{Gr}} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J]) & \longrightarrow & \displaystyle\frac{H^1(\mathbb{Q}_p, \mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J])} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \frac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])}.
\end{array}$$

As we will see in Theorem 3.10, the map $\mathrm{loc}_J^{\mathrm{BK}}$ is surjective in the cases (**a**), (**b**) and (**c**). By the snake lemma, we have:

$$\begin{cases} \mathrm{Sel}_J^{\mathrm{Gr}}/\mathrm{Sel}_J^{\mathrm{BK},(k)} \cong \mathrm{Ker}(\gamma_J) & \text{in the cases } (\mathbf{a}), (\mathbf{b}) \text{ and } (\mathbf{c}) , \\ \mathrm{Sel}_J^{\mathrm{Gr}}/\mathrm{Sel}_J^{\mathrm{BK},(k)} \hookrightarrow \mathrm{Ker}(\gamma_J) & \text{in the case } (\mathbf{d}). \end{cases}$$

Let us denote $\mathrm{Ker}(\gamma_J)$ by W_J. By Proposition 3.1,

$$\begin{aligned} W_J &\cong \varinjlim_u \frac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,H_u])}{\mathrm{Div}(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,H_u]))} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \varinjlim_u \frac{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J,H_u])}{\mathrm{Div}(H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J,H_u]))} \\ &\cong \varinjlim_u \frac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,H_u])}{\mathrm{Div}(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J,H_u]))} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \varinjlim_u \frac{(\mathcal{A}[J,H_u]^{I_v})_{G_{\mathbb{Q}_v}}}{\mathrm{Div}\big((\mathcal{A}[J,H_u]^{I_v})_{G_{\mathbb{Q}_v}}\big)}. \end{aligned}$$

The Pontryagin dual of $\varinjlim_u \dfrac{(\mathcal{A}[J,H_u]^{I_v})_{G_{\mathbb{Q}_v}}}{\mathrm{Div}\big((\mathcal{A}[J,H_u]^{I_v})_{G_{\mathbb{Q}_v}}\big)}$ is $\varprojlim_u \Big(\big((\mathcal{T}_J^*)_{I_v}/H_u(\mathcal{T}_J^*)_{I_v}\big)^{G_{\mathbb{Q}_v}} \Big)_{\mathbb{Z}_p\text{-tor}}$, where $(\ )_{\mathbb{Z}_p\text{-tor}}$ means the torsion-part as a $\mathbb{Z}_p$-module and $\mathcal{T}_J^* = \mathrm{Hom}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J}(\mathcal{T}_J, \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J)$. Recall that

$$\mathcal{T}_J^* \cong \begin{cases} (T_{f_{\mathfrak{I}}})^* \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) & \text{in the case } (\mathbf{a}), \\ (\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^* \otimes \chi^{-1}\omega^{-i} & \text{in the case } (\mathbf{b}), \\ (\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^* \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \chi^{-1}\omega^{-i} & \text{in the case } (\mathbf{c}), \\ (\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^* \otimes_{\mathbb{Z}_p[[\Gamma']]} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-\frac{1}{2}}) \otimes \chi^{-1}\omega^{-i} & \text{in the case } (\mathbf{d}), \end{cases}$$

where $(\ )^*$ means the $\mathbb{Z}_p$-linear dual in the first line and $(\ )^*$ means the $\mathbb{Z}_p[[\Gamma']]$-linear dual in each of other three cases.

In the case (**a**) with $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, we have $\big((\mathcal{T}_J^*)_{I_v}\big)_{(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J)\text{-tor}} \cong \big((T^*_{f_{\mathfrak{I}}})_{I_v}\big)_{\mathbb{Z}_p\text{-tor}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1})$. Since $\big((T^*_{f_{\mathfrak{I}}})_{I_v}\big)_{\mathbb{Z}_p\text{-tor}}$ is a finite abelian group and the image of $G_{\mathbb{Q}_v}$ on $\mathrm{Aut}(\mathbb{Z}_p/(p^n)[[\Gamma]](\widetilde{\chi}^{-1}))$ is infinite for any n, $\Big(\big((\mathcal{T}_J^*)_{I_v}\big)_{(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J)\text{-tor}}\Big)^{G_{\mathbb{Q}_v}}$ must be zero. By the structure of $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}$ studied in §2 the associated primes in $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}} \cong \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J$ of $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}$ are different from every prime factors of the images of $\{\Psi^{(k)}_t\}_{t\geq 1}$ in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J$

for (**b**), (**c**) and (**d**). Hence, we have:

$$\varprojlim_t \left(\left((\mathcal{T}_J^*)_{I_v} / \Psi_t^{(k)} (\mathcal{T}_J^*)_{I_v} \right)^{G_{\mathbb{Q}_v}} \right)_{\mathbb{Z}_p\text{-tor}} \cong \varprojlim_t \left(\left((\mathcal{T}_J^*)_{I_v} / \Psi_t^{(k)} (\mathcal{T}_J^*)_{I_v} \right)_{\mathbb{Z}_p\text{-tor}} \right)^{G_{\mathbb{Q}_v}}$$
$$\cong \varprojlim_t \left(\left((\mathcal{T}_J^*)_{I_v} \right)_{(\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/J)\text{-tor}} / \Psi_t^{(k)} \left((\mathcal{T}_J^*)_{I_v} \right)_{(\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/J)\text{-tor}} \right)^{G_{\mathbb{Q}_v}}$$
$$\cong \left(\left((\mathcal{T}_J^*)_{I_v} \right)_{(\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/J)\text{-tor}} \right)^{G_{\mathbb{Q}_v}} \cong \left(\left((\mathcal{T}_J^*)_{I_v} \right)^{G_{\mathbb{Q}_v}} \right)_{(\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/J)\text{-tor}}.$$

The proof for the contribution of local terms outside p in the second and the third assertions is completed.

Next, we discuss the group $\varinjlim_u \dfrac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, H_u])}{\mathrm{Div}\big(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, H_u])\big)}$. By definition, we have the following exact sequence:

$$0 \longrightarrow H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, H_u]) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{A}[J, H_u]) \xrightarrow{a_u} H^1(I_p, \mathrm{F}^- \mathcal{A}[J, H_u])^{G_{\mathbb{Q}_p}},$$

where the latter map a_u is decomposed as follows for each u:

$$H^1(\mathbb{Q}_p, \mathcal{A}[J, H_u]) \xrightarrow{a'_u} H^1(\mathbb{Q}_p, \mathrm{F}^- \mathcal{A}[J, H_u]) \xrightarrow{a''_u} H^1(I_p, \mathrm{F}^- \mathcal{A}[J, H_u])^{G_{\mathbb{Q}_p}}.$$

Hence we have the following extension:

$$(2) \quad \varinjlim_u \frac{\mathrm{Ker}(a'_u)}{\mathrm{Div}(\mathrm{Ker}(a'_u))} \longrightarrow \varinjlim_u \frac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, H_u])}{\mathrm{Div}\big(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J, H_u])\big)}$$
$$\longrightarrow \varinjlim_u \frac{\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)}{\mathrm{Div}\big(\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)\big)} \longrightarrow 0.$$

The first group $\dfrac{\mathrm{Ker}(a'_u)}{\mathrm{Div}(\mathrm{Ker}(a'_u))}$ is a quotient of $\dfrac{H^1(\mathbb{Q}_p, \mathrm{F}^+ \mathcal{A}[J, H_u])}{\mathrm{Div}\big(H^1(\mathbb{Q}_p, \mathrm{F}^+ \mathcal{A}[J, H_u])\big)}$, which is isomorphic to:

$$H^2(\mathbb{Q}_p, \mathrm{F}^+ \mathcal{T}_J / H_u \mathrm{F}^+ \mathcal{T}_J)_{\mathbb{Z}_p\text{-tor}} \cong \left((\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} / H_u (\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} \right)_{\mathbb{Z}_p\text{-tor}}.$$

In the case (**a**) with $J = \mathrm{Ker}(\mathfrak{I}) \mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$, we have:

$$\mathrm{F}^+ \mathcal{T}_J(-1) \cong \mathcal{O}_{\mathfrak{I}}[[\Gamma]](\omega^{i-1} \chi^{-1} \widetilde{\chi}) \otimes_{\mathcal{O}_{\mathfrak{I}}} \mathcal{O}_{\mathfrak{I}}(\alpha_{\mathfrak{I}}),$$

where $\mathcal{O}_{\mathfrak{I}}[[\Gamma]](\omega^{i-1} \chi^{-1} \widetilde{\chi})$ is a free $\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$-module of rank one on which $G_{\mathbb{Q}_p}$ acts via $\omega^{i-1} \chi^{-1} \widetilde{\chi}$ and $\mathcal{O}_{\mathfrak{I}}(\alpha_{\mathfrak{I}})$ is a free $\mathcal{O}_{\mathfrak{I}}$-module of rank one on which $G_{\mathbb{Q}_p}$ acts via the unramified character $\alpha_{\mathfrak{I}} : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\mathfrak{I}}^{\times}$ given by $\alpha_{\mathfrak{I}}(\mathrm{Frob}_p) = a_p(f_{\mathfrak{I}})$. Since I_p acts on $\mathrm{F}^+ \mathcal{T}_J(-1) / \mathfrak{M} \mathrm{F}^+ \mathcal{T}_J(-1)$ via ω^{i-1}, we have:

$$\begin{cases} (\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} / \Phi_s^{(j)} (\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} = 0 & i \neq 1, \\ (\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} / \Phi_s^{(j)} (\mathrm{F}^+ \mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}} \cong \mathcal{O}_{\mathfrak{I}} / (a_p(f_{\mathfrak{I}}) - 1) & i = 1. \end{cases}$$

We recall the following lemma:

Lemma 3.6. *Let M be a finite $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/J$-module in the case* (**a**), (**b**), (**c**) *or* (**d**). *Then we have $\varinjlim_u (M/H_u M) = 0$.*

By this lemma, we have $\varinjlim_u \Big((\mathrm{F}^+\mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}}/H_u(\mathrm{F}^+\mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}}\Big)_{\mathbb{Z}_p\text{-tor}} = 0$ in the case (**a**). Hence, $\varinjlim_u \dfrac{\mathrm{Ker}(a'_u)}{\mathrm{Div}(\mathrm{Ker}(a'_u))}$ in the equation (2) is trivial in the case (**a**).

In the cases (**b**), (**c**) and (**d**), $(\mathrm{F}^+\mathcal{T}_J(-1))_{G_{\mathbb{Q}_p}}$ is finite when $i \neq 1$ is satisfied or when $a_p(f_{\mathfrak{I}}) \neq 1$ are satisfied for every $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. This again implies that $\varinjlim_u \dfrac{\mathrm{Ker}(a'_u)}{\mathrm{Div}(\mathrm{Ker}(a'_u))}$ in (2) is trivial under these assumptions by Lemma 3.6.

For the proof of Proposition 3.4, we need to show that

$$(3) \qquad \varinjlim_u \frac{\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)}{\mathrm{Div}\big(\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)\big)} \cong H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J]).$$

We have the following claim:

Claim 3.7. In the cases (**a**), (**b**), (**c**) and (**d**), $\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)$ is finite for every u.

On the other hand, we have also the following claim:

Claim 3.8. For any height-one ideal $I \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, $H^1(\mathbb{Q}_p, \mathcal{A}[I]) \stackrel{a}{\longrightarrow} H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[I])$ is surjective.

We will finish the proof of Proposition 3.4 by using these claims. By Claim 3.7, we have:

$$\varinjlim_u \frac{\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)}{\mathrm{Div}\big(\mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u)\big)} \cong \varinjlim_u \mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u).$$

By Claim 3.8, we have:

$$\varinjlim_u \mathrm{Ker}(a''_u) \cap \mathrm{Im}(a'_u) \cong \varinjlim_u \mathrm{Ker}(a''_u) \cong H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J]).$$

This completes the proof of (3). Let us finally calculate the group $H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J])$. In the case (**a**) with $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, we have

$$\mathrm{F}^-\mathcal{A}[J]^{I_p} \cong \begin{cases} (\mathcal{O}_{\mathfrak{I}})^{\vee} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0, \end{cases}$$

on which Frob_p acts via multiplication by $a_p(f_{\mathfrak{I}})^{-1}$. In the cases (**b**) and (**d**), we have

$$\mathrm{F}^-\mathcal{A}[J]^{I_p} \cong \begin{cases} (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}/(\gamma' - 1))^{\vee} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0, \end{cases}$$

on which Frob_p acts via multiplication by $A_p(\mathcal{F})^{-1}$. In the case (**c**), we have

$$\mathrm{F}^-\mathcal{A}[J]^{I_p} \cong \begin{cases} (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^{\vee} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0, \end{cases}$$

on which Frob_p acts via multiplication by $A_p(\mathcal{F})^{-1}$. This completes the proof of Proposition 3.4 since $A_p(\mathcal{F}) \in \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ is not a root of unity.

In the rest of the proof, we finish the proof of two claims above.

Proof of Claim 3.7:

It suffices to show that:

$$\mathrm{Im}\left[H^1(\mathbb{Q}_p, V_{J,H_u}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^- V_{J,H_u})\right] \cap H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- V_{J,H_u}) = 0$$

for every u, where $V_{J,H_u} = (\mathcal{T} \otimes_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}} \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/(J, H_u)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In the case (**a**) with $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$, we have:

$$V_{J,\Phi^{(j)}_s} = \left(V_{f_{\mathfrak{I}}} \otimes \chi^j \omega^i\right) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma/\Gamma^{p^s}](\widetilde{\chi}).$$

The inertia group I_p acts on $\mathrm{F}^- V_{f_{\mathfrak{I}}} \otimes \chi^j \omega^i$ via the character $\chi^{j-1-w(\mathfrak{I})}$ modulo a finite character. We have:

$$\left(\left(\mathrm{F}^- V_{f_{\mathfrak{I}}} \otimes \chi^j \omega^i\right) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma/\Gamma^{p^s}](\widetilde{\chi})\right)^{I_p} = \begin{cases} 0 & \text{if } j \neq w(\mathfrak{I})+1 \text{ or } (\mathrm{F}^-\mathcal{A}[\mathfrak{M}] \otimes \omega^i)^{I_p} = 0, \\ K_{\mathfrak{I}}(\alpha_{\mathfrak{I}}^{-1}) & \text{otherwise,} \end{cases}$$

where $K_{\mathfrak{I}}(\alpha_{\mathfrak{I}}^{-1})$ is a vector space of rank one over $K_{\mathfrak{I}} = \mathcal{O}_{\mathfrak{I}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ on which $G_{\mathbb{Q}_p}$ acts via $\alpha_{\mathfrak{I}}^{-1}$. This implies that $H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- V_{H,\Phi^{(j)}_s}) = \left((\mathrm{F}^- V_{H,\Phi^{(j)}_s})^{I_p}\right)_{G_{\mathbb{Q}_p}}$ is trivial if $a_p(f_{\mathfrak{I}}) \neq 1$. Suppose that $a_p(f_{\mathfrak{I}}) = 1$. This happens only when $w(\mathfrak{I}) = 0$. Note that $w(\mathfrak{I}) = 0$ implies $j = 1$ and $V_{f_{\mathfrak{I}}} \subset H^1_{et}(B_{\mathfrak{I}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$ for certain abelian variety $B_{\mathfrak{I}}$ over $\mathbb{Q}$. Let $\mathbb{Q}_{f_{\mathfrak{I}}}$ be a finite extension of $\mathbb{Q}$ obtained by adjoining all Fourier coefficients $a_n(f_{\mathfrak{I}})$ of $f_{\mathfrak{I}}$. The field $K_{\mathfrak{I}}$ is naturally identified with a direct-summand of $\mathbb{Q}_{f_{\mathfrak{I}}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Since $a_p(f_{\mathfrak{I}}) = 1$, there exists an abelian variety $B'_{\mathfrak{I}}$ over $\mathbb{Q}_p$ with the following properties:

1. $B'_{\mathfrak{I}}$ is isogenious to a sub abelian variety of $B_{\mathfrak{I}} \otimes \mathbb{Q}_p$ of dimension $d = [K_{\mathfrak{I}} : \mathbb{Q}_p]$ over $\mathbb{Q}_p$.
2. $B'_{\mathfrak{I}}$ has totally multiplicative reduction over $\mathbb{Q}_p$.
3. $H^1_{et}(B'_{\mathfrak{I}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_p)$ is isomorphic to $V_{f_{\mathfrak{I}}}$ as a $G_{\mathbb{Q}_p}$-module.

Recall that $V_{f_{\mathfrak{I}}} \otimes \chi\omega \cong V_p({B'_{\mathfrak{I}}}^{\mathrm{t}}) := T_p({B'_{\mathfrak{I}}}^{\mathrm{t}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where ${B'_{\mathfrak{I}}}^{\mathrm{t}}$ is the dual abelian variety of $B'_{\mathfrak{I}}$ and $T_p({B'_{\mathfrak{I}}}^{\mathrm{t}})$ is the p-Tate module $\varprojlim_n {B'_{\mathfrak{I}}}^{\mathrm{t}}(\overline{\mathbb{Q}}_p)[p^n]$. Since ${B'_{\mathfrak{I}}}^{\mathrm{t}}$ has totally multiplicative reduction over $\mathbb{Q}_p$, ${B'_{\mathfrak{I}}}^{\mathrm{t}}(\overline{\mathbb{Q}}_p)$ is isomorphic to $(\overline{\mathbb{Q}}^\times_p)^d/P$ as $G_{\mathbb{Q}_p}$-module using the Tate's uniformization of $B'_{\mathfrak{I}}$ when $d = 1$ or its generalization by Mumford (cf. [FC, appendix]) when $d > 1$, where P is subgroup of $(\overline{\mathbb{Q}}^\times_p)^d$ which is mapped into a free $\mathbb{Z}$-module of rank d in $\mathbb{Q}^{\oplus d}$ via the composite $P \hookrightarrow (\overline{\mathbb{Q}}^\times_p)^d \xrightarrow{\mathrm{ord}_p} \mathbb{Q}^{\oplus d}$. Since $a_p(f_{\mathfrak{I}}) = 1$, $\alpha_{\mathfrak{I}}$ is a trivial character. Hence P is contained in $(\mathbb{Q}^\times_p)^d$ and $\mathrm{F}^+ V_p({B'_{\mathfrak{I}}}^{\mathrm{t}}) := (\mathrm{F}^+ V_{f_{\mathfrak{I}}} \otimes \chi\omega) \cap V_p({B'_{\mathfrak{I}}}^{\mathrm{t}})$ (resp. $\mathrm{F}^- V_p({B'_{\mathfrak{I}}}^{\mathrm{t}}) := V_p({B'_{\mathfrak{I}}}^{\mathrm{t}})/\mathrm{F}^+ V_p({B'_{\mathfrak{I}}}^{\mathrm{t}})$) is isomorphic to $\mathbb{Q}_p(\chi\omega)^{\oplus d}$ (resp. $\mathbb{Q}_p^{\oplus d}$). By Shapiro's lemma on induced Galois representations, we have:

$$\begin{aligned} &\mathrm{Im}\left[H^1(\mathbb{Q}_p, V_{J,\Phi^{(1)}_s}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^- V_{J,\Phi^{(1)}_s})\right] \cap H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- V_{J,\Phi^{(1)}_s}) \\ &= \mathrm{Im}\left[H^1(\mathbb{Q}_{p,s}, V_p({B'_{\mathfrak{I}}}^{\mathrm{t}})) \longrightarrow H^1(\mathbb{Q}_{p,s}, \mathrm{F}^- V_p({B'_{\mathfrak{I}}}^{\mathrm{t}}))\right] \cap H^1_{\mathrm{ur}}(\mathbb{Q}_{p,s}, \mathrm{F}^- V_p({B'_{\mathfrak{I}}}^{\mathrm{t}})) \end{aligned}$$

where $\mathbb{Q}_{p,s}$ is the unique Galois extension of $\mathbb{Q}_p$ contained in $\mathbb{Q}_p(\mu_{p^{s+1}})$ with $\mathrm{Gal}(\mathbb{Q}_{p,s}/\mathbb{Q}_p) \cong \mathbb{Z}/(p^s)$. By the properties of P mentioned above, we have $P \cong \prod_{1 \leq h \leq d} q_h^{\mathbb{Z}} \subset (\mathbb{Q}^\times_p)^d$ with

$\mathrm{ord}_p(q_h) > 0$ for each $1 \leq i \leq d$. Via the following identification:

$$H^1(\mathbb{Q}_{p,s}, \mathrm{F}^- V_p({B'_{\mathfrak{J}}}^{\mathrm{t}})) \cong H^1(\mathbb{Q}_{p,s}, \mathbb{Q}_p^{\oplus d}) \cong \bigoplus_{1 \leq h \leq d} \mathrm{Hom}(G_{\mathbb{Q}_{p,s}}, \mathbb{Q}_p),$$

the image of $H^1(\mathbb{Q}_{p,s}, V_p({B'_{\mathfrak{J}}}^{\mathrm{t}})) \longrightarrow H^1(\mathbb{Q}_{p,s}, \mathrm{F}^- V_p({B'_{\mathfrak{J}}}^{\mathrm{t}}))$ is equal to:

$$\bigoplus_{1 \leq h \leq d} \mathrm{Hom}(\mathrm{Gal}(F^{(h)}_{\infty,s}/\mathbb{Q}_{p,s}), \mathbb{Q}_p) \subset \bigoplus_{1 \leq h \leq d} \mathrm{Hom}(G_{\mathbb{Q}_{p,s}}, \mathbb{Q}_p),$$

where $F^{(h)}_{\infty,s}/\mathbb{Q}_{p,s}$ is the Galois extension of $\mathbb{Q}_{p,s}$ characterized as follows:

1. $\mathrm{Gal}(F^{(h)}_{\infty,s}/\mathbb{Q}_{p,s})$ is isomorphic to $\mathbb{Z}_p$.
2. When F runs finite extensions of $\mathbb{Q}_{p,s}$ contained in $F^{(h)}_{\infty,s}$, $\bigcap_F \mathrm{Norm}_{F/\mathbb{Q}_{p,s}}(F^\times)$ coincides with $\mu_{p-1} \cdot q_h^{\mathbb{Z}} \subset \mathbb{Q}_{p,s}^\times$.

Since we have

$$\begin{aligned} &\left(\bigoplus_{1 \leq h \leq d} \mathrm{Hom}(\mathrm{Gal}(F^{(h)}_{\infty,s}/\mathbb{Q}_{p,s}), \mathbb{Q}_p) \right) \cap H^1_{\mathrm{ur}}(\mathbb{Q}_{p,s}, \mathbb{Q}_p^{\oplus d}) \\ = &\left(\bigoplus_{1 \leq h \leq d} \mathrm{Hom}(\mathrm{Gal}(F^{(h)}_{\infty,s}/\mathbb{Q}_{p,s}), \mathbb{Q}_p) \right) \cap \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_{p,s}^{\mathrm{ur}}/\mathbb{Q}_{p,s}), \mathbb{Q}_p^{\oplus d}) = 0, \end{aligned}$$

we complete the proof in the case (**a**).

Let us denote by $\chi_{\mathfrak{J}} : G_{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p^\times$ the character $G_{\mathbb{Q}} \twoheadrightarrow \Gamma' \hookrightarrow (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^\times \xrightarrow{\mathfrak{J}} \overline{\mathbb{Q}}_p^\times$ for each $\mathfrak{J} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$. We have:

$$V_{J,\Psi^{(k)}_t} \cong \begin{cases} \bigoplus_{f_{\mathfrak{J}}} (V_{f_{\mathfrak{J}}} \otimes \chi\chi_{\mathfrak{J}}\omega^i) & \text{in the case } (\mathbf{b}), \\ \bigoplus_{f_{\mathfrak{J}}} (V_{f_{\mathfrak{J}}} \otimes \chi\chi_{\mathfrak{J}}^{\frac{1}{2}}\omega^i) & \text{in the cases } (\mathbf{c}) \text{ and } (\mathbf{d}), \end{cases}$$

where $\mathfrak{J}$ runs arithmetic points such that $\mathrm{Ker}(\mathfrak{J})$ contains the ideal $(\gamma'^{p^t} - \kappa^{k-2}(\gamma'^{p^t})) \subset \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. The group $\mathrm{Im}\left[H^1(\mathbb{Q}_p, V_{J,\Psi^{(k)}_t}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^- V^{(k)}_{J,t}) \right] \cap H^1_{\mathrm{ur}}(\mathbb{Q}_p, \mathrm{F}^- V_{J,\Psi^{(k)}_t})$ is trivial by the same argument with that of the case (**a**). This completes the proof of Claim 3.7 in the cases (**b**), (**c**) and (**d**).

<u>Proof of Claim 3.8</u>:

The cokernel of a is a submodule of $H^2(\mathbb{Q}_p, \mathrm{F}^+\mathcal{A}[J])$, which is the Pontryagin dual of $(((\mathrm{F}^+\mathcal{T})^*(-1))_J)^{G_{\mathbb{Q}_p}}$. Since $((\mathrm{F}^+\mathcal{T})^*(-1))_{G_{\mathbb{Q}_p}}$ has support whose codimension is greater than or equal to two, $(((\mathrm{F}^+\mathcal{T})^*(-1))_J)^{G_{\mathbb{Q}_p}}$ must be zero for any height-one prime $J \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. Consequently, the map a is surjective. This completes the proof of Claim 3.8. □

3.3. **Selmer groups over the two-variable Iwasawa algebra.** For each j, k with $1 \leq j \leq k-1$, we define $\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}} \subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})$ in the same way as previous ones by using the filtration $\mathrm{F}^+\mathcal{A}$. We define $\mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}}$ to be $\mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}} = \varinjlim_{s,t} \mathrm{Sel}^{\mathrm{BK}}_{T^{(j,k)}_{s,t}}$ where

$T^{(j,k)}_{s,t}$ and $\mathrm{Sel}^{\mathrm{BK}}_{T^{(j,k)}_{s,t}}$ are as given in §3.1. A priori, $\mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}}$ might depend on the choice of (j,k). However, we have the following proposition :

Proposition 3.9. *Assume the condition* (**Ir**) *above. We have the following statements*:

(1) *Selmer groups* $\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}}$ *and* $\mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}}$ *are equal as subgroups of* $H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathcal{A})$. *Especially, the definition of* $\mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}}$ *does not depend on the choice of* (j,k) *as above.*

(2) *The Pontryagin dual of* $(\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}})^\vee$ *is a torsion module over* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.

Proof. The first statement is implicitly proved in the reference [O3]. We recall briefly how to use the result in [O3]. Recall the following diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Sel}^{\mathrm{BK},(j,k)}_{\mathcal{T}} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathcal{A}) & \longrightarrow & \displaystyle\bigoplus_{v\in\Sigma\setminus\{\infty\}} \varinjlim_{s,t} \frac{H^1(\mathbb{Q}_v,A^{(j,k)}_{s,t})}{H^1_f(\mathbb{Q}_v,A^{(j,k)}_{s,t})} \\
& \downarrow & & \| & & \downarrow \\
0 \longrightarrow & \mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathcal{A}) & \longrightarrow & \displaystyle\frac{H^1(\mathbb{Q}_p,\mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p,\mathcal{A})} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \frac{H^1(\mathbb{Q}_v,\mathcal{A})}{H^1_{\mathrm{ur}}(\mathbb{Q}_v,\mathcal{A})}.
\end{array}$$

For $v\in\Sigma\setminus\{p,\infty\}$, $\varinjlim_{s,t} H^1_f(\mathbb{Q}_v,A^{(j,k)}_{s,t})$ is a subgroup of $H^1_{\mathrm{ur}}(\mathbb{Q}_v,\mathcal{A})$ by definition. We have $H^1_{\mathrm{ur}}(\mathbb{Q}_v,\mathcal{A})\cong H^1(\mathbb{Q}^{\mathrm{ur}}_v/\mathbb{Q}_v,\mathcal{A}^{I_v})$ by the Inflation-Restriction sequence. By Shapiro's lemma, $H^1(\mathbb{Q}^{\mathrm{ur}}_v/\mathbb{Q}_v,\mathcal{A}^{I_v})$ is isomorphic to $H^1(\mathbb{Q}^{\mathrm{ur}}_v/\mathbb{Q}_{v,\infty},(\mathcal{A}[\gamma-1])^{I_v})$. Here $\mathbb{Q}_{v,\infty}$ is the unique sub-extension of $\mathbb{Q}_v(\mu_{p^\infty})/\mathbb{Q}_v$ such that $\mathrm{Gal}(\mathbb{Q}_{v,\infty}/\mathbb{Q}_v)\cong\mathbb{Z}_p$. Note that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ur}}_v/\mathbb{Q}_{v,\infty})$ is isomorphic to $\prod_{l\neq p}\mathbb{Z}_l$ and that $(\mathcal{A}[\gamma-1])^{I_v}$ is a p-torsion group. Hence we have $\varinjlim_{s,t} H^1_f(\mathbb{Q}_v,A^{(j,k)}_{s,t}) = H^1_{\mathrm{ur}}(\mathbb{Q}_v,\mathcal{A}) = 0$ for any $v\in\Sigma\setminus\{p,\infty\}$. On the other hand, $\varinjlim_{s,t} H^1_f(\mathbb{Q}_p,A^{(j,k)}_{s,t}) = H^1_{\mathrm{Gr}}(\mathbb{Q}_p,\mathcal{A})$ by [O3, Corollary 4.13]. This completes the proof of (1). □

Remark on the Notation . *By Proposition* 3.9 (1), $\mathrm{Sel}^{\mathrm{Gr}}_{\mathcal{T}}$ *and* $\mathrm{Sel}^{\mathrm{BK}}_{\mathcal{T}}$ *coincide to each other for a two-variable nearly ordinary deformation* $\mathcal{T}$. *Hence we denote the Selmer group for* $\mathcal{T}$ *by* $\mathrm{Sel}_{\mathcal{T}}$ *from now on. For various specializations* $\mathcal{T}_J$ *of* $\mathcal{T}$, *we mainly study* $\mathrm{Sel}^{\mathrm{Gr}}_J$ *rather than* $\mathrm{Sel}^{\mathrm{BK}}_J$ *because of the simplicity of the definition of* $\mathrm{Sel}^{\mathrm{Gr}}_J$. *We denote* $\mathrm{Sel}^{\mathrm{Gr}}_J$ *by* Sel_J *for short if there causes no confusion* (*note that* $\mathrm{Sel}^{\mathrm{Gr}}_J$ *and* $\mathrm{Sel}^{\mathrm{BK}}_J$ *are different in general*).

3.4. **Surjectivity of localization maps.** In this subsection, we give surjectivity of localization maps from semi-global Galois cohomologies to certain Galois cohomologies at decomposition groups (Theorem 3.10 and Corollary 3.12). The result on this section was used in previous subsections in §3 and will be used in §§4, 6 and 7.

Let $\mathcal{R}$ be a "deformation ring" and let M be a rank two Galois representation over $\mathcal{R}$. In this subsection, we will treat one of the following situations:

1. $\mathcal{R}=\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ and $M=\mathcal{T}$.
2. $\mathcal{R}=\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$ and $M=\mathcal{T}_I$ where $I=\mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ for $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$ (the one-variable deformation (**a**) introduced in §1).

3. $\mathcal{R} = \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ and $M = \mathcal{T}_{(\gamma - \chi(\gamma))}$ or $M = \mathcal{T}_{(\gamma - \kappa(\gamma')\gamma')}$ ((**b**) or (**c**) introduced in §1).

The following theorem is obtained from a variant of the Global duality theorem in our situation.

Theorem 3.10. *Let $\mathcal{R}$ and M be one of the above. Then the localization map:*

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, M \otimes_{\mathcal{R}} \mathcal{R}^\vee) \longrightarrow \bigoplus_{v \in \Sigma \setminus \{\infty\}} \varinjlim_H \frac{H^1(\mathbb{Q}_v, M \otimes_{\mathcal{R}} \mathcal{R}^\vee)}{H^1_f(\mathbb{Q}_v, M \otimes_{\mathcal{R}} \mathcal{R}^\vee[H])}$$

is surjective, where H runs height-two ideals $\Delta^{(j,k)}_{s,t}$ for $s, t \geq 0$ with fixed j, k in the case 1 above, H runs height-one ideals $\Phi^{(j)}_s \subset \mathcal{O}_{\mathfrak{I}}[[\Gamma]]$ for $s \geq 0$ in the case 2, and H runs $\Psi^{(k)}_t \subset \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ for $t \geq 0$ in the case 3.

Proof. By the global duality theorem, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Sel}^{\mathrm{BK}}_{M/HM} \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, M \otimes_{\mathcal{R}} \mathcal{R}^\vee[H]) \longrightarrow \bigoplus_{v \in \Sigma \setminus \{\infty\}} \frac{H^1(\mathbb{Q}_v, M \otimes_{\mathcal{R}} \mathcal{R}^\vee)}{H^1_f(\mathbb{Q}_v, M \otimes_{\mathcal{R}} \mathcal{R}^\vee[H])} \longrightarrow (\varprojlim_{H,n} \mathrm{Sel}^{\mathrm{BK}}_{M^\vee(1)[H,p^n]})^\vee,$$

where $\mathrm{Sel}^{\mathrm{BK}}_{M^\vee(1)[H,p^n]}$ is defined to be:

$$\mathrm{Sel}^{\mathrm{BK}}_{M^\vee(1)[H,p^n]} = \mathrm{Ker}\left[H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, M^\vee(1)[H,p^n]) \longrightarrow \bigoplus_{v \in \Sigma \setminus \{\infty\}} \frac{H^1(\mathbb{Q}_v, M^\vee(1)[H,p^n])}{H^1_f(\mathbb{Q}_v, M^\vee(1)[H,p^n])} \right].$$

Note that the local condition $H^1_f(\mathbb{Q}_v, M^\vee(1)[H,p^n]) \subset H^1(\mathbb{Q}_v, M^\vee(1)[H,p^n])$ for a finite Galois module $M^\vee(1)[H,p^n]$ is defined to be the pull-back of $H^1_f(\mathbb{Q}_v, M^\vee(1)[H]) \subset H^1(\mathbb{Q}_v, M^\vee(1)[H])$ via the natural map $H^1(\mathbb{Q}_v, M^\vee(1)[H,p^n]) \longrightarrow H^1(\mathbb{Q}_v, M^\vee(1)[H])$.

Since we assume the condition (**Ir**) given in §1, $\mathcal{R}$ is an Gorenstein algebra in each case 1, 2 or 3. Hence we have an involution $\iota : \mathcal{R} \longrightarrow \mathcal{R}$ which coincides with the canonical involution $g \mapsto g^{-1}$ of $\mathbb{Z}_p[[\Gamma \times \Gamma']] \subset \mathcal{R}$ (resp. $\mathbb{Z}_p[[\Gamma]] \subset \mathcal{R}$, $\mathbb{Z}_p[[\Gamma']] \subset \mathcal{R}$) for $g \in \Gamma \times \Gamma'$ (resp. $g \in \Gamma$, $g \in \Gamma'$) in the case 1 (resp. 2, 3). Let us denote by M^ι a free $\mathcal{R}$-module of rank two $\varprojlim_{H,n} M^\vee(1)[H,p^n]$. By (**Ir**), the natural restriction map $\mathrm{Sel}^{\mathrm{BK}}_{M^\vee(1)[H,p^n]} \longrightarrow \mathrm{Sel}^{\mathrm{BK}}_{M^\iota}[H,p^n]$ is injective, where $\mathrm{Sel}^{\mathrm{BK}}_{M^\iota} = \varinjlim_H \mathrm{Sel}^{\mathrm{BK}}_{M^\iota/HM^\iota}$. Thus, it suffices to show that $\varprojlim_{H,n} \mathrm{Sel}^{\mathrm{BK}}_{M^\iota}[H,p^n]$ is zero in order to have the desired surjectivity. We refer to [O3, §5] for the above facts and the following lemma:

Lemma 3.11. *As the following $\mathcal{R}$-linear isomorphism:*

$$\mathrm{Sel}^{\mathrm{BK}}_{M^\iota}[H,p^n] \cong \mathrm{Hom}_{\mathcal{R}/(H,p^n)}\big((\mathrm{Sel}^{\mathrm{BK}}_{M^\iota})^\vee/(H,p^n)(\mathrm{Sel}^{\mathrm{BK}}_{M^\iota})^\vee, \mathcal{R}/(H,p^n)\big)^\iota$$

where $(\ \ )^\iota$ means the twist of an $\mathcal{R}$-module structure via the involution ι.

Since $\mathrm{Sel}^{\mathrm{BK}}_{M^\iota}$ is a torsion $\mathcal{R}$-module, the proof is completed. □

Corollary 3.12. *Let $\mathcal{R}$ and M be one of the pair given in this subsection. Then the localization map:*

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, M\otimes_{\mathcal{R}}\mathcal{R}^\vee) \longrightarrow \frac{H^1(\mathbb{Q}_p, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \frac{H^1(\mathbb{Q}_v, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)}$$

is surjective.

Proof. The corollary is a consequence of Theorem 3.10 because $\varinjlim_{H} H^1_f(\mathbb{Q}_v, M\otimes_{\mathcal{R}}\mathcal{R}^\vee[H])$ is contained in $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)$ (resp. $H^1_{\mathrm{ur}}(\mathbb{Q}_v, M\otimes_{\mathcal{R}}\mathcal{R}^\vee)$) when $v=p$ (resp. $v\in\Sigma\setminus\{p,\infty\}$). $\square$

4. Control theorem for Greenberg's Selmer groups

For a Galois representation $M\cong\mathcal{R}^d$ of $G_{\mathbb{Q}}$ and a prime ideal J of $\mathcal{R}$, we have the natural restriction map between Selmer groups $\mathrm{Sel}_{M/JM}\xrightarrow{\mathrm{res}_J}\mathrm{Sel}_M[J]$ (if they are defined). What we call the control theorem is the type of problems (or theorems) where we study the kernel and the cokernel of res_J (or equivalently its Pontryagin dual $(\mathrm{Sel}_M)^\vee/J(\mathrm{Sel}_M)^\vee\longrightarrow(\mathrm{Sel}_{M/JM})^\vee$). For a family M over a one-variable algebra $\mathcal{R}$ and its specialization to a zero-variable algebra (i.e. a discrete valuation ring) $\mathcal{R}/J$, the control theorem was already studied in [O2].

In this section, we study the control theorem for a nearly ordinary deformation $\mathcal{T}$ or quotient representations of $\mathcal{T}$. Throughout the section, we denote by N_J the quotient N/JN for an $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module N and an ideal J of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ for short. We will always assume (**Ir**) throughout the section. The assertions on $\mathrm{Coker}(\mathrm{res}_J)$ holds without (**Ir**). However, the assertion on $\mathrm{Ker}(\mathrm{res}_J)$ might be modified if we replace (**Ir**) by a weaker condition. Though it is not difficult, we decide not to do it in order to avoid unnecessarily complicate description. We refer [O2] for the idea of such argument in the case without (**Ir**).

4.1. From two-variable to one-variable.

First, we discuss the specialization of the two-variable $\mathrm{Sel}_{\mathcal{T}}$ to some of important one-variable deformations.

Proposition 4.1. *Assume the condition* (**Ir**) *for $\mathcal{T}=\mathcal{T}^{(i)}_{\mathcal{F}}$. Let J be a height-one prime ideal of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ and let res_J be the restriction map $\mathrm{Sel}_J\longrightarrow\mathrm{Sel}_{\mathcal{T}}[J]$, where $\mathrm{Sel}_J=\mathrm{Sel}_{\mathcal{T}/J\mathcal{T}}$. Then the map res_J is injective.* $\mathrm{Coker}(\mathrm{res}_J)$ *is a sub-quotient of the following group $\mathcal{L}_J$:*

$$\mathcal{L}_J=\begin{cases}(\mathrm{F}^-\mathcal{A}[\gamma-\kappa(\gamma')\gamma']_J)^{G_{\mathbb{Q}_p}}\oplus\bigoplus_{v\in\Sigma\setminus\{p,\infty\}}((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p}\neq 0,\\ \bigoplus_{v\in\Sigma\setminus\{p,\infty\}}((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p}=0,\end{cases}$$

where M_J means M/JM for an $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module M. If we have further the surjectivity of the localization map:

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathcal{A}[J])\xrightarrow{\mathrm{loc}_J}\frac{H^1(\mathbb{Q}_p,\mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p,\mathcal{A}[J])}\oplus\bigoplus_{v\in\Sigma\setminus\{p,\infty\}}\frac{H^1(\mathbb{Q}_v,\mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v,\mathcal{A}[J])},$$

then, the cokernel of res_J *is isomorphic to $\mathcal{L}_J$.*

Proof. Let us recall a diagram as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}_J & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J]) & \xrightarrow[\mathrm{loc}_J]{} & Y(J) \\ & & \downarrow{\scriptstyle \mathrm{res}_J} & & \downarrow{\scriptstyle \alpha_J} & & \downarrow{\scriptstyle \beta_J} \\ 0 & \longrightarrow & \mathrm{Sel}_{\mathcal{T}}[J] & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})[J] & \longrightarrow & Y[J], \end{array}$$

where

$$Y(J) = \frac{H^1(\mathbb{Q}_p, \mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J])} \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \frac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])}$$

$$Y = \frac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})} \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \frac{H^1(\mathbb{Q}_v, \mathcal{A})}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A})}.$$

By the condition (**Ir**), the map α_J is injective. Consequently, res_J is injective. By the snake lemma and by the injectivity of α_J, $\mathrm{Coker}(\mathrm{res}_J)$ is isomorphic to a submodule of $\mathrm{Ker}(\beta_J)$. Further, we have $\mathrm{Coker}(\mathrm{res}_J) \cong \mathrm{Ker}(\beta_J)$ if loc_J is surjective. Hence we have only to show that $\mathrm{Ker}(\beta_J)$ is isomorphic to $\mathcal{L}_J$. By the Inflation-Restriction sequence, it is easy to see that the kernel of $\dfrac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])} \longrightarrow \dfrac{H^1(\mathbb{Q}_v, \mathcal{A})}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A})}$ is $((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}}$ at each $v \in \Sigma \setminus \{p, \infty\}$. In the rest of the proof, we will concentrate on the map β_J restricted to p-part. Let us consider the exact sequence:

$$0 \longrightarrow H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J]) \longrightarrow H^1(\mathbb{Q}_p, \mathcal{A}[J]) \longrightarrow H^1(I_p, \mathrm{F}^-\mathcal{A}[J])^{G_{\mathbb{Q}_p}}.$$

Note that the second map $H^1(\mathbb{Q}_p, \mathcal{A}[J]) \longrightarrow H^1(I_p, \mathrm{F}^-\mathcal{A}[J])^{G_{\mathbb{Q}_p}}$ decomposes as:

$$H^1(\mathbb{Q}_p, \mathcal{A}[J]) \xrightarrow{a} H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[J]) \xrightarrow{b} H^1(I_p, \mathrm{F}^-\mathcal{A}[J])^{G_{\mathbb{Q}_p}}.$$

The map a is surjective as is shown in Claim 3.8 and the map b is surjective since the cohomological dimension of $G_{\mathbb{Q}_p}/I_p$ is one. Thus, we have $\dfrac{H^1(\mathbb{Q}_p, \mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J])} \cong H^1(I_p, \mathrm{F}^-\mathcal{A}[J])^{G_{\mathbb{Q}_p}}$.

By a similar argument, we have $\dfrac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})} \cong H^1(I_p, \mathrm{F}^-\mathcal{A})^{G_{\mathbb{Q}_p}}$. This gives

$$\mathrm{Ker}\left[\frac{H^1(\mathbb{Q}_p, \mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J])} \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})}\right] \cong ((\mathrm{F}^-\mathcal{A})^{I_p}/J(\mathrm{F}^-\mathcal{A})^{I_p})^{G_{\mathbb{Q}_p}}.$$

We complete the proof since $(\mathrm{F}^-\mathcal{A})^{I_p} \cong \begin{cases} \mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma'] & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0. \end{cases}$ □

We will apply Proposition 4.1:

Proposition 4.2. *Assume the condition* (**Ir**) *for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$. *Let us consider height-one primes* $J \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *as follows in the following four cases*:
(**a**) J *is* $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *for* $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. (**b**) J *is* $(\gamma - \chi(\gamma)) \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. (**c**) J *is* $(\gamma - \kappa(\gamma')\gamma') \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. (**d**) J *is* $(\gamma^2 - \kappa^2(\gamma')\gamma') \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.

Then, the restriction map $\mathrm{res}_J : \mathrm{Sel}_J \longrightarrow \mathrm{Sel}_{\mathcal{T}}[J]$ *are injective in every four cases and we have:*

$$\begin{cases} \mathrm{Coker}(\mathrm{res}_J) \cong (U_J)^\vee & \textit{in the case } (\mathbf{a}) \textit{ with } J = \mathrm{Ker}(\mathfrak{I}), \\ \mathrm{Coker}(\mathrm{res}_J) \cong (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^\vee[A_p(\mathcal{F})-1] & \textit{in the case } (\mathbf{c}) \textit{ with } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ \mathrm{Coker}(\mathrm{res}_J) = 0 & \textit{otherwise,} \end{cases}$$

where $U_J = \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \left(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \omega^{-i}\right)^{G_{\mathbb{Q}_p}}$.

Remark 4.3. For each $v \in \Sigma \setminus \{p,\infty\}$, $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})]$ is always finite and is trivial except certain special cases (cf. Theorem 2.3).

Proof. By Proposition 4.1, res_J is injective and we have:

$$\mathrm{Coker}(\mathrm{res}_J) = \begin{cases} (\mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma']_J)^{G_{\mathbb{Q}_p}} \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} ((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} ((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0. \end{cases}$$

Except in the case $(\mathbf{c})$, $\mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma']_J$ is zero. In the case $(\mathbf{c})$, $\mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma']_J = \mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma']$ is a cofree $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module of rank one with unramified $G_{\mathbb{Q}_p}$-action on which Frob_p acts via the multiplication of $A_p(\mathcal{F})$ (see §1 for $A_p(\mathcal{F})$). Hence we have $(\mathrm{F}^-\mathcal{A}[\gamma - \kappa(\gamma')\gamma']_J)^{G_{\mathbb{Q}_p}} \cong (\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})^\vee[A_p(\mathcal{F})-1]$ in this case.

Next, we discuss local terms at $v \in \Sigma \setminus \{p,\infty\}$. Recall that

$$\mathcal{A} \cong (\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \omega^i) \otimes_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

(see the beginning of §1 for $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$). Since I_v acts trivially on $\mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \omega^i$, we have:

$$\left(((\mathcal{A}^{I_v})_J)^{G_{\mathbb{Q}_v}}\right)^\vee \cong \left(\left(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \omega^{-i}\right)[J]\right)_{G_{\mathbb{Q}_v}}.$$

In the cases $(\mathbf{b})$, $(\mathbf{c})$ and $(\mathbf{d})$, $\left(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \omega^{-i}\right)[J]$ is clearly zero. In the case $(\mathbf{a})$ for $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with certain $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, we have:

$$\left(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v} \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \omega^{-i}\right)[J] = ((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})] \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}^{-1}) \otimes \omega^{-i}.$$

This completes the proof. ☐

4.2. **From one-variable to discrete valuation case.** In [O2], we studied control theorems of the Selmer groups for one-variable Galois deformations when they are specialized into various representations over discrete valuation rings. In this subsection, we restrict ourselves to one-variable deformations inside Hida deformations in order to have more precise and complete result. By applying the fundamental diagram and the snake lemma as in 4.1, we prove also the Control theorem in this case.

Proposition 4.4. *Assume the condition* $(\mathbf{Ir})$ *for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$. *Let* J *and* J' *be two different height-one prime ideals of* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *and let* $\mathrm{res}_{J'}$ *be the restriction map* $\mathrm{Sel}_{\mathcal{T}/(J,J')\mathcal{T}'} \longrightarrow$

$\mathrm{Sel}_J[J']$, *where* $\mathrm{Sel}_J = \mathrm{Sel}_{\mathcal{T}/J\mathcal{T}}$. *Then the map* $\mathrm{res}_{J'}$ *is injective. The cokernel of* $\mathrm{res}_{J'}$ *is a sub-quotient of the following group* $\mathcal{L}_J$:

$$\mathcal{L}_J = \begin{cases} \left((\mathrm{F}^-\mathcal{A}[J]^{I_p})_{J'}\right)^{G_{\mathbb{Q}_p}} \oplus \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} ((\mathcal{A}[J]^{I_v})_{J'})^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} ((\mathcal{A}[J]^{I_v})_{J'})^{G_{\mathbb{Q}_v}} & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0, \end{cases}$$

By calculating the term $\mathcal{L}_J$ in each case, we have the following proposition:

Proposition 4.5. *Assume the condition* (**Ir**) *for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$. *We consider the four cases* (**a**), (**b**), (**c**) *and* (**d**) *with the same* J*'s as Proposition* 4.2. *We consider another height-one ideal* J' :

$$J' = \begin{cases} (\gamma - \chi^j(\gamma)) \text{ for a certain } j \text{ with } 1 \leq j \leq w(\mathfrak{I}) + 1 & \text{in the case } (\mathbf{a}), \\ \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} \text{ for certain } \mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0} & \text{in the cases } (\mathbf{b}), (\mathbf{c}) \text{ and } (\mathbf{d}). \end{cases}$$

The kernels and the cokernels of $\mathrm{res}_{J'} : \mathrm{Sel}_{\mathcal{T}/(J,J')\mathcal{T}} \longrightarrow \mathrm{Sel}_J[J']$ *are given as follows*:

1. *The restriction map* $\mathrm{res}_{J'}$ *is injective in each of* (**a**), (**b**), (**c**) *and* (**d**).
2. *In the cases* (**a**), (**b**), (**c**) *and* (**d**) *with* $\mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0$, $\mathrm{Coker}(\mathrm{res}_{J'})$ *is a sub-quotient of a finite group* Z *which is given as follows*:

$$Z = \begin{cases} \bigoplus_{v\in\Sigma\setminus\{p,\infty\}} \left(((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})]\right)^{\vee} & \text{in the cases } (\mathbf{b}), (\mathbf{c}) \text{ and } (\mathbf{d}), \\ 0 & \text{in the case } (\mathbf{a}). \end{cases}$$

In the cases (**a**), (**b**), (**c**) *and* (**d**) *with* $\mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0$, $\mathrm{Coker}(\mathrm{res}_{J'})$ *is a sub-quotient of the following group:*

$$\begin{cases} (\mathcal{O}_{\mathfrak{I}})^{\vee}[1 - a_p(f_{\mathfrak{I}})] \oplus Z & \text{in the cases } (\mathbf{a}), (\mathbf{b}) \text{ and } (\mathbf{d}), \\ Z & \text{in the case } (\mathbf{c}) \end{cases}$$

Remark 4.6. 1. Note that $\mathcal{T}/(J,J')\mathcal{T}$ is equal to $T_{f_{\mathfrak{I}}} \otimes \chi^j\omega^i$ (resp. $T_{f_{\mathfrak{I}}} \otimes \chi\omega^i$, $T_{f_{\mathfrak{I}}} \otimes \chi^{w(\mathfrak{I})+1}\omega^i$, $T_{f_{\mathfrak{I}}} \otimes \chi^{\frac{w(\mathfrak{I})}{2}+1}\omega^i$) in the case (**a**) (resp. (**b**), (**c**), (**d**)).

2. The control theorem in the case (**a**) was studied in various references (e.g [Gr4]) when $f_{\mathfrak{I}}$ is associated to an elliptic curve E. Note that there has been contribution of the local Tamagawa number of E at every $v \in \Sigma \setminus \{p,\infty\}$ to $\mathrm{Coker}(\mathrm{res}_{J'})$ in the above mentioned references (e.g [Gr4]). Whereas, there are no such contribution in our result. This is because $\mathrm{Sel}_{T_{f_{\mathfrak{I}}}\otimes\chi\omega}$ is isomorphic to the classical Selmer group for E only after divided by a finite abelian group whose order is related to the local Tamagawa number of E at v.

5. Two-variable p-adic L-function

In this section, we discuss the two-variable p-adic L-function for a nearly ordinary deformation $\mathcal{T}$ through Beilinson-Kato elements. The construction will be done by using the two-variable Coleman map (Theorem 5.3) which translates a norm compatible elements to a measure. The key of the section is an optimization of two-variable Beilinson-Kato element given in Theorem 5.10. The results of this section make clear the relation between Kitagawa's two-variable p-adic L-function [Ki] and our Euler system construction, modifying previous constructions [O3] and [FC] which was not well-optimized in general.

5.1. **Review on the work on** [O3]**.** In order to introduce Beilinson-Kato elements, we need to prepare notations. For a normalized eigen cusp form $f = \sum_{n>0} a_n(f)q^n$ of weight $k \geq 2$, we denote by $\mathbb{Q}_f$ a finite extension of $\mathbb{Q}$ obtained by adjoining all Fourier coefficients of f to $\mathbb{Q}$. We denote by $\overline{f} = \sum_{n>0} a_n(f)^\sigma q^n$ the dual modular form of f where σ is a complex conjugation. The dual modular form $\overline{f}$ is known to be a Hecke eigen cusp form of weight k with Neben character dual of that of f. The field $\mathbb{Q}_f$ is equal to $\mathbb{Q}_{\overline{f}}$. We associate the de Rham realization $V_{\mathrm{dR}}(f)$ to f. The de Rham realization $V_{\mathrm{dR}}(f)$ has the following properties:

1. $V_{\mathrm{dR}}(f)$ is a two dimensional vector space over $\mathbb{Q}_f$ and is equipped with a de Rham filtration $\mathrm{Fil}^i V_{\mathrm{dR}}(f) \subset V_{\mathrm{dR}}(f)$, which is a decreasing filtration of $\mathbb{Q}_f$-vector spaces.
2. We have $\mathrm{Fil}^0 V_{\mathrm{dR}}(f) = V_{\mathrm{dR}}(f)$ and $\mathrm{Fil}^k V_{\mathrm{dR}}(f) = \{0\}$. For each j such that $1 \leq j \leq k-1$, $\mathrm{Fil}^j V_{\mathrm{dR}}(f)$ is naturally identified with one-dimensional $\mathbb{Q}_f$-vector space $\mathbb{Q}_f \cdot f$.
3. Let $\mathfrak{I}$ be an arithmetic point of weight $w(\mathfrak{I}) \geq 0$. For each j such that $1 \leq j \leq k-1$, $\mathrm{Fil}^{k-j} V_{\mathrm{dR}}(\overline{f}_\mathfrak{I}) \otimes_{\mathbb{Q}_{f_\mathfrak{I}}} K_\mathfrak{I}$ is naturally identified with $\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*_{f_\mathfrak{I}} \otimes \chi^{1-j}\omega^{1-i})$, where $K_\mathfrak{I}$ is the p-adic completion of $\mathbb{Q}_{f_\mathfrak{I}}$ with respect to the fixed embedding $\mathbb{Q}_{f_\mathfrak{I}} \subset \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $V_{f_\mathfrak{I}}$ is $T_{f_\mathfrak{I}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $(\ \ )^*$ means the $\mathbb{Q}_p$-linear dual here.

For each $1 \leq j \leq w(\mathfrak{I})+1$, we denote by $\overline{\delta}^{\mathrm{dR}}_\mathfrak{I}$ the $\mathbb{Q}_{f_\mathfrak{I}}$-basis of $\mathrm{Fil}^{w(\mathfrak{I})+2-j} V_{\mathrm{dR}}(\overline{f}_\mathfrak{I})$ sent to $\overline{f}_\mathfrak{I}$ under the natural identification $\mathrm{Fil}^{w(\mathfrak{I})+2-j} V_{\mathrm{dR}}(\overline{f}_\mathfrak{I}) = \mathbb{Q}_{f_\mathfrak{I}} \cdot \overline{f}_\mathfrak{I}$. Let $\mathcal{D}$ be an $\mathbb{H}^{\mathrm{ord}}_\mathcal{F}$-module $(\mathbb{H}^{\mathrm{ord}}_\mathcal{F}(\widetilde{\alpha}) \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p)^{G_{\mathbb{Q}_p}}$ and let $\delta_{\mathbb{Q}_p(1)}$ be the inverse image of $1 \in \mathbb{Q}_p$ via the isomorphism $D_{\mathrm{dR}}(\mathbb{Q}_p(1)) \xrightarrow{\sim} \mathbb{Q}_p$ determined by a fixed norm compatible system $\{\zeta_{p^n}\}_{n \geq 1}$ of p^n-th roots of unity. We recall the following properties (see [O3, §3] for the proof):

Lemma 5.1. (1) *$\mathcal{D}$ is a free $\mathbb{H}^{\mathrm{ord}}_\mathcal{F}$-module of rank one.*

(2) *$\mathcal{D}/\mathrm{Ker}(\mathfrak{I})\mathcal{D}$ is the canonical lattice of $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+ V_{f_\mathfrak{I}}) = \mathrm{D}_{\mathrm{crys}}(\mathrm{F}^+ V_{f_\mathfrak{I}})$ for each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_\mathcal{F})_{\geq 0}$.*

(3) *For each $(j, \mathfrak{I})$ such that $1 \leq j \leq w(\mathfrak{I}) + 1$, we have the canonical isomorphism $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+ V_{f_\mathfrak{I}} \otimes \chi^j \omega^i) \cong \mathrm{D}_{\mathrm{dR}}(V_{f_\mathfrak{I}} \otimes \chi^j \omega^i)/\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{f_\mathfrak{I}} \otimes \chi^j \omega^i)$*

(4) *The fixed norm compatible system $\{\zeta_{p^n}\}_{n \geq 1}$ induces the following isomorphism:*

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+ V_{f_\mathfrak{I}}) \xrightarrow[\otimes \delta^{\otimes j}_{\mathbb{Q}_p(1)}]{\sim} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+ V_{f_\mathfrak{I}} \otimes \chi^j \omega^j) \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+ V_{f_\mathfrak{I}} \otimes \chi^j \omega^i),$$

where $0 \leq i \leq p-2$

Definition 5.2. Fix an $\mathbb{H}^{\mathrm{ord}}_\mathcal{F}$-basis d of $\mathcal{D}$. For each $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_\mathcal{F})_{\geq 0}$, we define a de Rham p-adic period $C_{p,\mathfrak{I},d} \in \overline{\mathbb{Q}}_p$ to be

$$C_{p,\mathfrak{I},d} = \langle \overline{\delta}^{\mathrm{dR}}_\mathfrak{I}, d_\mathfrak{I} \otimes \delta^{\otimes j}_{\mathbb{Q}_p(1)} \rangle \tag{4}$$

where j is an integer satisfying $1 \leq j \leq w(\mathfrak{I})+1$, $\langle\ ,\ \rangle$ is the pairing :

$$(5)\ \ \langle\ ,\ \rangle:\ \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V^*_{f_\mathfrak{I}}\otimes\chi^{1-j}\omega^{1-i})\times\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{f_\mathfrak{I}}\otimes\chi^j\omega^i)\longrightarrow\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1))\otimes K_\mathfrak{I}\cong K_\mathfrak{I}$$

induced by the identification of Lemma 5.1 (3) and the de Rham paring:

$$\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V^*_{f_\mathfrak{I}}\otimes\chi^{1-j}\omega^{1-i})\times\mathrm{D}_{\mathrm{dR}}(V_{f_\mathfrak{I}}\otimes\chi^j\omega^i)/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{f_\mathfrak{I}}\otimes\chi^j\omega^i)\longrightarrow\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1))\otimes K_\mathfrak{I}.$$

and $d_\mathfrak{I}\in\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{f_\mathfrak{I}})$ is the specialization modulo $\mathrm{Ker}(\mathfrak{I})$ of $d\in\mathcal{D}$ (cf. Lemma 5.1 (2)).

The p-adic period $C_{p,\mathfrak{I},d}$ does not depend on j and depends only on d and a fixed norm compatible system $\{\zeta_{p^n}\}_{n\geq 1}$ of p^n-th roots of unity.

Theorem 5.3. [O3, Theorem 3.14] *Let i be an integer such that $0\leq i\leq p-2$. We assume the condition* (**Ir**) *for a nearly ordinary deformation $\mathcal{T}=\mathcal{T}^{(i)}_\mathcal{F}$. Assume further that $\mathbb{H}^{\mathrm{n.o}}_\mathcal{F}$ is integrally closed in its fraction field* $\mathrm{Frac}(\mathbb{H}^{\mathrm{n.o}}_\mathcal{F})$*. Fix an $\mathbb{H}^{\mathrm{ord}}_\mathcal{F}$-basis d of $\mathcal{D}=(\mathbb{H}^{\mathrm{ord}}_\mathcal{F}(\widetilde{\alpha})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}^{\mathrm{ur}}_p)^{G_{\mathbb{Q}_p}}$ (Lemma 5.1 (1)). Then we have a map $\Xi_d:\ H^1_{/f}(\mathbb{Q}_p,\mathcal{T}^*(1))\longrightarrow\mathbb{H}^{\mathrm{n.o}}_\mathcal{F}$ which has the following properties:*

1. *The map Ξ_d is an $\mathbb{H}^{\mathrm{n.o}}_\mathcal{F}$-linear pseudo-isomorphism.*
2. *Let $\mathcal{C}\in H^1_{/f}(\mathbb{Q}_p,\mathcal{T}^*(1))$. For each $(j,\mathfrak{I})$ such that $1\leq j\leq w(\mathfrak{I})+1$ and for each finite order character η of Γ, $(\chi^j\eta\circ\mathfrak{I})(\Xi_d(\mathcal{C}))$ is equal to*:

$$\left(1-\frac{(\omega^{i-j}\eta)(p)p^{j-1}}{a_p(f_\mathfrak{I})}\right)\left(1-\frac{(\omega^{i-j}\eta^{-1})(p)a_p(f_\mathfrak{I})}{p^j}\right)^{-1}\times\left(\frac{p^{j-1}}{a_p(f_\mathfrak{I})}\right)^{q(i,j,\eta)}G(\omega^{j-i}\eta)\left\langle\exp^*((\chi^j\eta\circ\mathfrak{I})(\mathcal{C})),d\right\rangle,$$

where $(\chi^j\circ\mathfrak{I})(\mathcal{C})\in H^1_{/f}(\mathbb{Q}_p,T^*_{f_\mathfrak{I}}(1)\otimes\omega^{-i}\chi^{-j})$ is the specialization of $\mathcal{C}$ via $\chi^j\circ\mathfrak{I}$, $q(i,j,\eta)$ is the p-order of the conductor of $\omega^{j-i}\eta$ and $G(\omega^{j-i}\eta)$ is the Gauss sum for $\omega^{j-i}\eta$.

5.2. **p-adic periods at weight two.** In this subsection, we study the p-adic periods $C_{p,\mathfrak{I},d}$ in the special cases where $w(\mathfrak{I})=0$. We fix an $\mathbb{H}^{\mathrm{ord}}_\mathcal{F}$-basis d of $\mathcal{D}$ throughout §5.2. The main result of §5.2 is as follows:

Proposition 5.4. *Let $\mathcal{T}=\mathcal{T}^{(i)}_\mathcal{F}$ be a nearly ordinary deformation. Then $C_{p,\mathfrak{I},d}$ is a p-adic unit for any $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_\mathcal{F})_{\geq 0}$.*

Proof. For an arithmetic point $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_\mathcal{F})$ with $w(\mathfrak{I})=0$, let $B_\mathfrak{I}$ be the abelian variety associated to the normalized weight two eigen cuspform $f_\mathfrak{I}$. $B_\mathfrak{I}$ is an abelian variety of dimension $g=[\mathbb{Q}_{f_\mathfrak{I}}:\mathbb{Q}]$ over $\mathbb{Q}$ and we have an injection $\mathbb{Q}_{f_\mathfrak{I}}\hookrightarrow\mathrm{End}_\mathbb{Q}(B_\mathfrak{I})\otimes\mathbb{Q}$. Since $f_\mathfrak{I}$ is ordinary at p, there exists an abelian variety $B'_\mathfrak{I}$ over $\mathbb{Q}_p$ with the following properties (see [Wi1, §2.2]):

1. $B'_\mathfrak{I}$ is isogenious to a subabelian variety of $B_\mathfrak{I}\otimes\mathbb{Q}_p$ with $d=\dim(B'_\mathfrak{I})=[K_\mathfrak{I}:\mathbb{Q}_p]$ over $\mathbb{Q}_p$.
2. $B'_\mathfrak{I}$ has totally multiplicative reduction or good ordinary reduction over $\mathbb{Q}_p$.
3. $H^1_{et}(B'_\mathfrak{I}\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p,\mathbb{Q}_p)$ is isomorphic to $V_{f_\mathfrak{I}}$ as a $G_{\mathbb{Q}_p}$-module.

Let ${B'_{\mathfrak{J}}}^{\mathrm{t}}$ be the dual abelian variety of $B'_{\mathfrak{J}}$. We denote by $\mathcal{B}$ the p-divisible group over $\mathbb{Q}_p$ associated to ${B'_{\mathfrak{J}}}^{\mathrm{t}}$ with its connected part $\mathcal{B}^0$. We see that

$$\begin{aligned}\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}\big(V^*_{f_{\mathfrak{J}}} \otimes \chi^{1-j}\omega^{1-i}\big) &\cong \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}\big(H^1_{et}({B'_{\mathfrak{J}}}^{\mathrm{t}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_p)\big)\\ &\cong \mathrm{Fil}^1 H^1_{\mathrm{dR}}({B'_{\mathfrak{J}}}^{\mathrm{t}}) \cong \mathbb{D}(\mathcal{B}^0),\end{aligned}$$

where $H^1_{\mathrm{dR}}({B'_{\mathfrak{J}}}^{\mathrm{t}})$ means the de Rham cohomology of ${B'_{\mathfrak{J}}}^{\mathrm{t}}$ and $\mathbb{D}(\mathcal{B}^0)$ is the Dieudonee module for $\mathcal{B}^0$. By the second statement of "Basic property of nearly ordinary Hida deformations $\mathcal{T}^{(0)}_{\mathcal{F}}$" stated in §1, we see that

$$\begin{aligned}\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}\big(V^*_{f_{\mathfrak{J}}} \otimes \chi^{1-j}\omega^{1-i}\big) &\cong \mathrm{D}_{\mathrm{dR}}\big(\mathrm{F}^-V^*_{f_{\mathfrak{J}}} \otimes \chi^{1-j}\omega^{1-i}\big)\\ &\cong \mathrm{D}_{\mathrm{dR}}(K_{\mathfrak{J}}(\alpha^{-1})).\end{aligned}$$

By Definition 5.2, $C_{p,\mathfrak{J},d}$ is a p-adic unit if and only if the $K_{\mathfrak{J}}$-basis of $\mathbb{D}(\mathcal{B}^0)$ induced by $\overline{\delta}^{\mathrm{dR}}_{\mathfrak{J}}$ gives an integral basis of $\mathrm{D}_{\mathrm{dR}}(K_{\mathfrak{J}}(\alpha^{-1})) = (K_{\mathfrak{J}}(\alpha^{-1}) \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}}^{\mathrm{ur}}_p)^{G_{\mathbb{Q}_p}}$ with respect to the lattice $(\mathcal{O}_{\mathfrak{J}}(\alpha^{-1}) \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}^{\mathrm{ur}}_p)^{G_{\mathbb{Q}_p}}$. This is clear since $\mathcal{B}^0$ is of multiplicative type. □

5.3. **Beilinson-Kato element.** Let $H^1_{\mathrm{B}}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^{k-2}(R^1p_*A))$ be a Betti cohomology and let $H^1_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^{k-2}(R^1p_*A))$ be a compact support Betti cohomology, where $p : \mathcal{E} \to Y_1(M)$ is the universal elliptic curve over the affine modular curve $Y_1(M)$ and A is a submodule of $\mathbb{C}$. To each normalized eigen newform $f \in S_k(\Gamma_1(M))$ of weight $k \geq 2$, we associate the Betti realization $V_{\mathrm{B}}(f)$. $V_{\mathrm{B}}(f)$ is defined to be $H^1_{\mathrm{B}}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^{k-2}(R^1p_*\mathbb{Q}_f))[\pi_f]$ (resp. $H^1_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^{k-2}(R^1p_*\mathbb{Q}_f))[\pi_f]$), where $[\pi_f]$ means a direct summand cut out by the kernels of $T_l - a_l(f)$ with Hecke operators $T_l \in \mathrm{End}_{\mathbb{Q}_f}(S_k(\Gamma_1(M); \mathbb{Q}_f))$ for all prime l. The Betti realization $V_{\mathrm{B}}(f)$ has the following properties:

1. $V_{\mathrm{B}}(f)$ is a two-dimensional vector space over $\mathbb{Q}_f$ and is equipped with a natural action of complex conjugate σ, whose $\pm$-eigen space $V_{\mathrm{B}}(f)^{\pm}$ is one-dimensional over $\mathbb{Q}_f$.
2. We have a period map $\mathrm{Per}^{\pm} : \mathrm{Fil}^j V_{\mathrm{dR}}(f) \otimes_{\mathbb{Q}_f} \mathbb{C} \xrightarrow{\sim} V_{\mathrm{B}}(f)^{\pm} \otimes_{\mathbb{Q}_f} \mathbb{C}$ for each $1 \leq j \leq k-1$.

Let us denote by $\mathcal{H}$ the local system on $Y_1(M)_{\mathbb{C}}$ whose fiber $\mathcal{H}_s$ is $H_1(\mathcal{E}_s, \mathbb{Z})$ at $s \in Y_1(M)_{\mathbb{C}}$. Let $\varphi : \mathfrak{H} \longrightarrow Y_1(M)_{\mathbb{C}}$ be the uniformization map. The stalk of $\mathcal{H}$ at $\varphi(yi)$ is identified with $\mathbb{Z} + \mathbb{Z}yi$ for any $y \in \mathbb{R}_{>0}$. We denote by β be the element of $\Gamma((0, i\infty), \varphi^{-1}(\mathcal{H}))$ which corresponds to $1 \in \mathbb{Z}$.

Definition 5.5. 1. Let $\delta^{\mathrm{B},w}_0$ be the element of $H^{\mathrm{B}}_1(Y_1(M)_{\mathbb{C}}; \{\mathrm{cusps}\}, \mathrm{Sym}^w(\mathcal{H}))$ which represents a path $(0, \infty)$ and β^w. By abuse of notation, we also denote by $\delta^{\mathrm{B},w}_0$ the image via the map :

$$H^{\mathrm{B}}_1(Y_1(M)_{\mathbb{C}}; \{\mathrm{cusps}\}, \mathrm{Sym}^w(\mathcal{H}_{\mathbb{Q}_f})) \xrightarrow{\sim} H^1_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^w(R^1p_*\mathbb{Q}_f)) \twoheadrightarrow V_{\mathrm{B}}(f).$$

2. Let $V_{\mathrm{B}}(f) \times V_{\mathrm{B}}(f) \xrightarrow{\langle\ ,\ \rangle_{\mathrm{B}}} \mathbb{Q}_f$ be the pairing induced from the Poincare duality :

$$H^1_{\mathrm{B}}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^w(R^1p_*\mathbb{Q}_f)) \times H^1_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}}, \mathrm{Sym}^w(R^1p_*\mathbb{Q}_f)) \longrightarrow H^2_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}}, \mathbb{Q}_f) \cong \mathbb{Q}_f.$$

Let $\langle\ ,\ \rangle_\infty$ be the extension of $\langle\ ,\ \rangle_{\mathrm{B}}$ as follows :

$$V_{\mathrm{B}}(f)\otimes_{\mathbb{Q}_f}\mathbb{C}\times V_{\mathrm{B}}(f)\otimes_{\mathbb{Q}_f}\mathbb{C}\xrightarrow{\langle\ ,\ \rangle_\infty}\mathbb{C}.$$

Recall that $\langle \mathrm{Per}^+(f_{\mathfrak{I}}),\delta_0^{\mathrm{B},w}\rangle_\infty$ is equal to $\displaystyle\int_0^\infty f(\sqrt{-1}y)dy=L(f,1)$.

In [Ki], Kitagawa constructed modules of Λ-adic modular symbols $\mathcal{B}^\pm$, which has the following properties:

1. $\mathcal{B}^\pm$ is a finitely generated $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-modules whose generical ranks are one.
2. For each $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$, $\mathcal{B}^\pm/\mathrm{Ker}(\mathfrak{I})\mathcal{B}^\pm$ is a lattice of $V_{\mathrm{B}}(f_{\mathfrak{I}})\otimes_{\mathbb{Q}_{f_{\mathfrak{I}}}}\widehat{\mathbb{Q}}_{f_{\mathfrak{I}}}$.

Definition 5.6. Let $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. Then $f_{\mathfrak{I}}$ is a newform in $S_{w(\mathfrak{I})+2}(\Gamma_1(M))$ for a certain multiple M of N. Let $\mathcal{O}_{f_{\mathfrak{I}}}$ be the ring of integers of $\mathbb{Q}_{f_{\mathfrak{I}}}$. Choose an $\mathcal{O}_{f_{\mathfrak{I}}}$-basis $\delta_{\mathfrak{I}}^{\mathrm{B},\pm}$ of the natural $\mathcal{O}_{f_{\mathfrak{I}}}$-lattice $H^1_{\mathrm{B},c}(Y_1(M)_{\mathbb{C}},\mathrm{Sym}^w(R^1p_*\mathcal{O}_{f_{\mathfrak{I}}}))\cap V_{\mathrm{B}}(f_{\mathfrak{I}})^\pm$ of $V_{\mathrm{B}}(f_{\mathfrak{I}})^\pm$.

1. We define a complex period $C^\pm_{\infty,\mathfrak{I}}\in\mathbb{C}$ to be $C^\pm_{\infty,\mathfrak{I}}=\langle\mathrm{Per}^\pm(\delta^{\mathrm{dR}}_{\mathfrak{I}}),\delta^{\mathrm{B},\pm}_{\mathfrak{I}}\rangle_\infty$.
2. Let $\langle\ ,\ \rangle_p$ be the extension of $\langle\ ,\ \rangle_{\mathrm{B}}$ as follows :

$$V_{\mathrm{B}}(f_{\mathfrak{I}})\otimes_{\mathbb{Q}_{f_{\mathfrak{I}}}}\overline{\mathbb{Q}}_p\times V_{\mathrm{B}}(f_{\mathfrak{I}})\otimes_{\mathbb{Q}_{f_{\mathfrak{I}}}}\overline{\mathbb{Q}}_p\xrightarrow{\langle\ ,\ \rangle_p}\overline{\mathbb{Q}}_p.$$

   We define a p-adic period $C^\pm_{p,\mathfrak{I},b}\in\overline{\mathbb{Q}}_p$ to be $C^\pm_{p,\mathfrak{I},b}=\langle b^\pm_{\mathfrak{I}},\delta^{\mathrm{B},\pm}_{\mathfrak{I}}\rangle_p$.

Theorem 5.7. [Ki, Theorem 1.1] *Let us fix an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis b of $\mathcal{B}^{(-1)^i}$. Then we have a two-variable p-adic L-function $L^{\mathrm{Ki}}_p(\mathcal{T})\in\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with the following interpolation properties*:

$$\begin{aligned}&(\chi^j\eta\circ\mathfrak{I})(L^{\mathrm{Ki}}_{p,b}(\mathcal{T}))/C_{p,\mathfrak{I},b}\\&=(-1)^{j-1}(j-1)!\left(1-\frac{(\omega^{i-j}\eta)(p)p^{j-1}}{a_p(f_{\mathfrak{I}})}\right)\left(\frac{p^{j-1}}{a_p(f_{\mathfrak{I}})}\right)^{q(i,j,\eta)}G(\omega^{j-i}\eta^{-1})\frac{L(f_{\mathfrak{I}},\omega^{i-j}\eta,j)}{(2\pi\sqrt{-1})^{j-1}C^{(-1)^i}_{\infty,\mathfrak{I}}},\end{aligned}$$

where $q(i,j,\eta)$ is the p-order of the conductor of $\omega^{j-i}\eta^{-1}$ and $G(\omega^{j-i}\eta^{-1})$ is the Gauss sum for $\omega^{j-i}\eta^{-1}$.

Proposition 5.8. [Ka2] *Assume the condition* (**Ir**). *Let us fix an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis b of $\mathcal{B}^{(-1)^i}$ and an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis d of $\mathcal{D}$. Fix an arithmetic point $\mathfrak{I}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $w(\mathfrak{I})=0$. Then we have an Euler system $\{\mathcal{Z}_{\mathfrak{I}}(r)\in H^1(\mathbb{Q}(\mu_r)_\Sigma/\mathbb{Q}(\mu_r),\mathcal{T}^*(1)_I)\}$ whose first layer $\mathcal{Z}_{\mathfrak{I}}=\mathcal{Z}_{\mathfrak{I}}(1)$ satisfies the following properties*:

1. *For each finite order character η of Γ, $(\exp^*\circ\mathrm{loc}_{/f})(\eta(\mathcal{Z}_{\mathfrak{I}}))$ is contained in* $\mathrm{Fil}^1V_{\mathrm{dR}}(\overline{f}_{\mathfrak{I}}\otimes\omega^{1-i}\eta^{-1})\subset\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V^*_{f_{\mathfrak{I}}}\otimes\omega^{1-i}\eta^{-1})$.
2. *Further,* $(\exp^*\circ\mathrm{loc}_{/f})(\eta(\mathcal{Z}_{\mathfrak{I}}))$ *is equal to* $\dfrac{C_{p,\mathfrak{I},b}}{C_{p,\mathfrak{I},d}}\dfrac{L_{(p)}(f_{\mathfrak{I}},\omega^{i-1}\eta,1)}{C^{(-1)^i}_{\infty,\mathfrak{I}}}\cdot\overline{\delta}^{\mathrm{dR}}_{\mathfrak{I}}$.

(*We denote by $\mathbb{Q}(\mu_r)_\Sigma$ the maximal Galois extension of $\mathbb{Q}(\mu_r)$ unramified outside primes over Σ*)

Remark 5.9. By taking the projective limit of the elements in Galois cohomology groups obtained via Chern character from Beilinson-Kato elements in K_2-group of $Y_1(Np)\otimes\mathbb{Q}(\mu_{rp^s})$, we have an Euler system $\{\mathcal{Z}_{\mathfrak{I},0}(r)\in H^1(\mathbb{Q}(\mu_r)_\Sigma/\mathbb{Q}(\mu_r),\mathcal{T}^*(1)_{\mathfrak{I}})\}$ where r runs square-free natural numbers prime to p. The above Euler system $\mathcal{Z}_{\mathfrak{I}}(r)$ is optimally normalized at $\mathfrak{I}$ and is obtained as a summation $\sum_{\xi} c_{\xi} \mathcal{Z}_{\mathfrak{I},0}(r)^{\xi}$ multiplied by $C_{p,\mathfrak{I},b}/C_{p,\mathfrak{I},d}$, where $\mathcal{Z}_{\mathfrak{I},0}(r)^{\xi}$ is the twist of $\mathcal{Z}_{\mathfrak{I},0}(r)$ by $\xi \in SL_2(\mathbb{Z})$ and c_{ξ} are rational integers. For such optimal normalization for a fixed $f_{\mathfrak{I}}$, we refer to [Ka2, § 12].

We will give the following optimization of the two-variable Beilinson-Kato element.

Theorem 5.10. *Let us fix an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis b of $\mathcal{B}^{(-1)^i}$ and an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis d of $\mathcal{D}$. Then we have an Euler system $\{\mathcal{Z}(r) \in H^1(\mathbb{Q}(\mu_r)_{\Sigma}/\mathbb{Q}(\mu_r), \mathcal{T}^*(1))\}$ and the specialization of the first layer $\mathcal{Z} = \mathcal{Z}(1)$ at each arithmetic point $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $w(\mathfrak{I}) = 0$ and at each finite order character η of Γ satisfies the following properties*:

1. $(\exp^* \circ \mathrm{loc}_{/f})((\eta \circ \mathfrak{I})(\mathcal{Z}))$ *is contained in* $\mathrm{Fil}^1 V_{\mathrm{dR}}(\overline{f}_{\mathfrak{I}} \otimes \omega^{1-i}\eta^{-1}) \subset \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*_{f_{\mathfrak{I}}} \otimes \omega^{1-i}\eta^{-1})$.
2. *Further,* $(\exp^* \circ \mathrm{loc}_{/f})((\eta \circ \mathfrak{I})(\mathcal{Z}))$ *is equal to* $\dfrac{C_{p,\mathfrak{I},b}}{C_{p,\mathfrak{I},d}} \dfrac{L_{(p)}(f_{\mathfrak{I}}, \omega^{i-1}\eta, 1)}{C^{(-1)^i}_{\infty,\mathfrak{I}}} \cdot \overline{\delta}^{\mathrm{dR}}_{\mathfrak{I}}$, *where* $L_{(p)}(f_{\mathfrak{I}}, \omega^{i-1}\eta, s)$ *is the $\omega^{i-1}\eta$-twist of the Hecke L-function for $f_{\mathfrak{I}}$ whose p-factor is removed.*

Remark 5.11.
1. The construction of $\mathcal{Z}(r)$ will be done by "gluing" of the elements $\mathcal{Z}_{\mathfrak{I}}(r)$ given in Proposition 5.8 for various $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $w(\mathfrak{I}) = 0$ by using Lemma 5.12 below.
2. Though the interpolation property is given only for $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $w(\mathfrak{I}) = 0$, $(\exp^* \circ \mathrm{loc}_{/f})((\eta \circ \mathfrak{I})(\mathcal{Z}))$ is related to an optimal L-value even when $w(\mathfrak{I}) > 0$.

Proof of Theorem 5.10. Let $\mathfrak{S} = \{I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} | \mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}), w(\mathfrak{I}) = 0\}$. We denote by $\mathfrak{A}$ a subset of the set of height one ideals of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ as follows:

$$\mathfrak{A} = \left\{ J = \bigcap\nolimits_{I \in S} I \;\middle|\; S \subset \mathfrak{S}, \sharp S < \infty \right\}.$$

Note that $J \cap J' \in \mathfrak{A}$ for any $J, J' \in \mathfrak{A}$ and that the intersection $\bigcap J$ for infinitely many $J \in \mathfrak{A}$ is zero.

Lemma 5.12. *For each natural number r and for each $J, J' \in \mathfrak{A}$, we have the exact sequence*:

$$0 \longrightarrow H^1(\mathcal{T}^*(1)_{J \cap J'}) \longrightarrow H^1(\mathcal{T}^*(1)_J) \oplus H^1(\mathcal{T}^*(1)_{J'}) \longrightarrow H^1(\mathcal{T}^*(1)_{J+J'}),$$

where $H^1(M)$ is $H^1(\mathbb{Q}(\mu_r)_{\Sigma}/\mathbb{Q}(\mu_r), M)$ in the above sequence.

In the following, we only construct $\mathcal{Z} = \mathcal{Z}(1) \in H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}, \mathcal{T}^*(1))$ with two desired properties stated in Theorem 5.10. The construction for general r is done basically in the same way using Lemma 5.12. We need the following Claim for the proof.

Claim 5.13. Let $J \in \mathfrak{A}$. Then there exists an element $\mathcal{Z}_J$ such that $(\exp^* \circ \mathrm{loc}_{/f})((\eta \circ \mathfrak{I})(\mathcal{Z}_J))$ satisfy two properties stated in Theorem 5.10 for all arithmetic points $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $\mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} \supset J$ and for all finite order characters η of Γ.

In fact, $\mathcal{Z}$ is obtained as $\varprojlim_{J \in A} \mathcal{Z}_J \in H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}, \mathcal{T}^*(1))$ when J runs a directed subset $A \subset \mathfrak{A}$ such that $\bigcap_{J \in A} J = 0$. Hence we will prove the above claim in the rest of the proof. The proof proceeds by induction with respect to the numbers of arithmetic points $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ with $\mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} \supset J$. By Proposition 5.8 the claim holds when $J = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ for an arithmetic point $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$. Now we take arbitrary ideal $J \in \mathfrak{A}$ at which Claim 5.13 is true. We will prove Claim 5.13 for $J \cap I$ where $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ for an arithmetic point $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$ such that $w(\mathfrak{I}) = 0$ and $J \not\subset I$. Let us denote $\mathcal{T}^*/(J, \mathrm{Ker}(\eta))\mathcal{T}^*$ by $T^*_{J,\eta}$ and let us denote the continuous Galois cohomology $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, M)$ by $H^1(M)$ for short. Then, we have the following diagram for each finite order character η of Γ:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(\mathcal{T}^*(1)_{J\cap I}) & \xrightarrow[a_{I,J}]{} & H^1(\mathcal{T}^*(1)_J) \oplus H^1(\mathcal{T}^*(1)_I) & \xrightarrow[b_{I,J}]{} & H^1(\mathcal{T}^*(1)_{J+I}) \\
& \eta\downarrow & & \eta\downarrow & & \downarrow\eta \\
& H^1(T^*_{J\cap I,\eta}(1)) & \longrightarrow & H^1(T^*_{J,\eta}(1)) \oplus H^1(T^*_{I,\eta}(1)) & \longrightarrow & H^1(T^*_{f_{\mathfrak{I}}\otimes\eta}(1)_{\overline{\mathcal{J}}}),
\end{array}$$

where $a_{I,J}$ sends $x \in H^1(\mathcal{T}^*(1)_{J\cap I})$ to $x_J \oplus x_I \in H^1(\mathcal{T}^*(1)_J) \oplus H^1(\mathcal{T}^*(1)_I)$ and $b_{I,J}$ sends $x \oplus y \in H^1(\mathcal{T}^*(1)_J) \oplus H^1(\mathcal{T}^*(1)_I)$ to $x_{J+I} - y_{J+I} \in H^1(\mathcal{T}^*(1)_{J+I})$. Let us consider the following morphism:

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*_{f_{\mathfrak{I}}\otimes\eta}(1)) \xrightarrow{\mathrm{loc}_{/f}} \frac{H^1(\mathbb{Q}_p, T^*_{f_{\mathfrak{I}}\otimes\eta}(1))}{H^1_f(\mathbb{Q}_p, T^*_{f_{\mathfrak{I}}\otimes\eta}(1))} \xrightarrow{\langle d,\ \rangle_{\mathrm{dR}}\circ\exp^*} (\eta\circ\mathfrak{I})(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}) \tag{6}$$

The element $(\eta\circ\mathfrak{I}')(\mathcal{Z}_J) \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*_{f_{\mathfrak{I}'}\otimes\eta}(1))$ (resp. $\eta(\mathcal{Z}_I) \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*_{f_{\mathfrak{I}}\otimes\eta}(1))$) is mapped to $v_{\mathfrak{I}',\eta} := \dfrac{C_{p,\mathfrak{I}',b}}{C_{p,\mathfrak{I}',d}} \dfrac{L_{(p)}(f_{\mathfrak{I}'}, \eta, 1)}{C^{(-1)^i}_{\infty,\mathfrak{I}'}}$ when $J \subset \mathrm{Ker}(\mathfrak{I}')\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ (resp. $\mathfrak{I} = \mathfrak{I}'$). The following lemma is obtained by Euler system argument using the Beilinson-Kato element and by a Rohlich's result (cf. [Ka2]):

Lemma 5.14. *Under the condition* (**Ir**)*, the map in* (6) *is injective when the conductor of η is sufficiently large.*

In fact, since $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T^*_{f_{\mathfrak{I}}\otimes\eta}(1))$ has no non-zero torsion by (**Ir**), the kernel of (6) is non-zero if and only if $\mathrm{Sel}_{T^*_{f_{\mathfrak{I}}\otimes\eta}(1)}$ is an infinite abelian group. This happens only for finitely many η by Kato-Rubin and Rohlich.

Since values $v_{\mathfrak{I}',\eta}$ and $v_{\mathfrak{I}'',\eta}$ are congruent to each other modulo $(\mathrm{Ker}(\mathfrak{I}')+\mathrm{Ker}(\mathfrak{I}''), \mathrm{Ker}(\eta))$, $\eta \circ b_{I,J}(\mathcal{Z}_J \oplus \mathcal{Z}_I)$ is zero for each finite order character η of Γ with sufficiently large conductor.

Lemma 5.15. *Let I be a height one ideal of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ generated by a height one ideal of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. Then the intersection $\bigcap_\eta \mathrm{Ker}(\eta) \subset H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{T}^*(1)_I)$ is trivial when η runs infinitely many finite order characters of Γ.*

Since $\mathcal{Z}_J \oplus \mathcal{Z}_I$ is mapped to zero via $b_{I,J}$ by Lemma 5.14 and Lemma 5.15, we have an element $\mathcal{Z}_{J\cap I} \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{T}^*(1)_{J\cap I})$ such that $a_{I,J}(\mathcal{Z}_{J\cap I}) = \mathcal{Z}_J \oplus \mathcal{Z}_I$. By construction, $\mathcal{Z}_{J\cap I}$ satisfies desired properties for Claim 5.13. This completes the proof. □

Corollary 5.16. *Let us fix an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis b of $\mathcal{B}^{(-1)^i}$ and an $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-basis d of $\mathcal{D}$. Then we have an Euler system $\{\mathcal{Z}_{b,d}(r) \in H^1(\mathbb{Q}(\mu_r)_\Sigma/\mathbb{Q}(\mu_r), \mathcal{T}^*(1))\}$ such that $L_p(\mathcal{T}) := \Xi_d(\mathcal{Z}_{b,d}(1)) \in \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ is equal to the two-variable p-adic L-function $L^{\mathrm{Ki}}_{p,b}(\mathcal{T})$ by Kitagawa.*

6. Iwasawa Main conjectures for various specializations of $\mathcal{T}$

In this section, we formulate and discuss the Iwasawa Main Conjecture for various one-variable specializations $\mathcal{T}_J$ of $\mathcal{T}$. Especially, we will discuss how to obtain a result on the one-variable Iwasawa theory on $\mathcal{T}_J$ from the two-variable Iwasawa theory on $\mathcal{T}$ and vice versa. Recall the following definition:

Definition 6.1. Let R be a Noetherian local domain such that R is integrally closed in the fraction field $\mathrm{Frac}(R)$ of R. A finitely generated torsion R-module M is called *pseudo-null* if $\mathrm{length}_{R_{\mathfrak{l}}}(M_{\mathfrak{l}}) = 0$ for every height-one prime $\mathfrak{l}$ in R or equivalently $\mathrm{Supp}_R(M)$ has codimension greater than one in $\mathrm{Spec}(R)$.

To study a relation between the two-variable Iwasawa main conjecture for $\mathcal{T}$ and the one-variable Iwasawa main conjecture for each $\mathcal{T}_J$, the following lemma plays an important role:

Lemma 6.2. *Assume the conditions* (**Ir**) *and* (**Nor**) *for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$. *Let us consider a height-one prime* $J \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *as follows in the following three cases*:
(**a**) J *is* $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *for* $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. (**b**) J *is* $(\gamma - \chi(\gamma)) \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. (**c**) J *is* $(\gamma - \kappa(\gamma')\gamma') \subset \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$.
Then, $(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathrm{null}}/J(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathrm{null}}$ *is pseudo-null* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J$*-module.*

Proof. Let us consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Sel}_{\mathcal{T}} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}) & \xrightarrow{\mathrm{loc}} & Y & \longrightarrow & 0 \\ & & \downarrow{\times J} & & \downarrow{\times J} & & \downarrow{\times J} & & \\ 0 & \longrightarrow & \mathrm{Sel}_{\mathcal{T}} & \longrightarrow & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}) & \xrightarrow{\mathrm{loc}} & Y & \longrightarrow & 0, \end{array}$$

where $Y = \dfrac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})} \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \dfrac{H^1(\mathbb{Q}_v, \mathcal{A})}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A})}$. The cokernel of the middle vertical map is a subgroup of $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J])$, which is zero since $\mathrm{Sel}_{\mathcal{T}/J\mathcal{T}}$ is a cotorsion $\mathcal{O}_{\mathfrak{I}}[[\Gamma]]$-module. By the snake lemma, $(\mathrm{Sel}_{\mathcal{T}})/J(\mathrm{Sel}_{\mathcal{T}})$ is isomorphic to the cokernel of the map $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})[J] \xrightarrow{\mathrm{loc}[J]} Y[J]$. We compare $\mathrm{Coker}(\mathrm{loc}[J])$ with the cokernel of

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[J]) \xrightarrow{\mathrm{loc}_J} Y(J)$$

where $Y(J) = \dfrac{H^1(\mathbb{Q}_p, \mathcal{A}[J])}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J])} \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \dfrac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])}$. We consider another diagram:

$$(7) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(\mathrm{loc}_J) & \longrightarrow & Y(J) & \longrightarrow & \mathrm{Coker}(\mathrm{loc}_J) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{w} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(\mathrm{loc}[J]) & \longrightarrow & Y[J] & \longrightarrow & \mathrm{Coker}(\mathrm{loc}[J]) & \longrightarrow & 0. \end{array}$$

By Corollary 3.12, $\mathrm{Coker}(\mathrm{loc}_J)$ is zero in the above commutative diagram. Let us consider another diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}[J]) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{A}[J]) \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \dfrac{H^1(\mathbb{Q}_v, \mathcal{A}[J])}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A}[J])} & \xrightarrow[q]{} & Y(J) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow t & & \downarrow w & & \\
0 & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})[J] & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{A})[J] \oplus \bigoplus_{v \in \Sigma \setminus \{p,\infty\}} \dfrac{H^1(\mathbb{Q}_v, \mathcal{A})}{H^1_{\mathrm{ur}}(\mathbb{Q}_v, \mathcal{A})}[J] & \xrightarrow[q']{} & Y[J] & &
\end{array}$$

Note that the upper horizontal map q is surjective by definition. The cokernel of the lower horizontal map q' is a subgroup of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})/JH^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$ by the snake lemma. Since the Pontryagin dual of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$ is a torsion-free $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module by [O2, Corollary 4.13], the map q' has to be surjective. Hence, the surjectivity of the middle vertical map w of the diagram (7) follows since t is easily seen to be surjective. We have shown that $(\mathrm{Sel}_{\mathcal{T}})/J(\mathrm{Sel}_{\mathcal{T}})$ is zero, or equivalently, we have shown that $(\mathrm{Sel}_{\mathcal{T}})^{\vee}[J]$ is zero by taking the Pontryagin dual. By a simple argument(cf. [O4, Lemma 3.1]), $(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathrm{null}}/J(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathrm{null}}$ is pseudo-null $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J$-module since $(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathrm{null}}[J]$ is a pseudo-null $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/J$-module. This completes the proof. □

(a) Iwasawa Main conjecture for $\mathcal{T}_I$.
Let $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. The specialization of $\mathcal{T}$ at I is the cyclotomic deformation of $f_{\mathfrak{I}}$ as we saw in §1. By Mazur-Tate-Teitelbaum, we have $L_p^{\mathrm{MTT}}(\mathcal{T}_I) \in \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$ which has the following interpolation property for each finite order character η of Γ and for each integer $1 \leq j \leq w(\mathfrak{I}) + 1$:

$$\chi^j\eta(L_p^{\mathrm{MTT}}(\mathcal{T}_I)) = \left(1 - \frac{(\omega^{i-j}\eta)(p)p^{j-1}}{a_p(f_{\mathfrak{I}})}\right)\left(\frac{p^{j-1}}{a_p(f_{\mathfrak{I}})}\right)^{q(i,j,\eta)} G(\omega^{j-i}\eta)\frac{L(f_{\mathfrak{I}}, \omega^{i-j}, j)}{(2\pi\sqrt{-1})^{j-1}C^{\pm}_{\infty,\mathfrak{I}}},$$

where $C^{\pm}_{\infty,\mathfrak{I}}$ is a complex period given by Definition 5.6. Note that the ideal $(L_p^{\mathrm{MTT}}(\mathcal{T}_I))$ is well-defined since $C^{\pm}_{\infty,\mathfrak{I}}$ is unique up to multiplication by a unit in $\mathcal{O}_{f_{\mathfrak{I}}}$. Recall that $\mathrm{Sel}_I^{\vee}$ is a cotorsion $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$-module (cf. §3.2).

Conjecture 6.3. Let $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. We have the following equality:

$$\mathrm{length}_{(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I)_{\mathfrak{l}}}(\mathrm{Sel}_I^{\vee})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p^{\mathrm{MTT}}(\mathcal{T}_I)),$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$.

As a corollary of Theorem 3 in §1, we have the following result:

Corollary 6.4. *Let $I = \mathrm{Ker}(\mathfrak{I})\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ with $\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}$. For every prime $v \in \Sigma \setminus \{p, \infty\}$, we assume that $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})]$ is trivial. Then,*

1. *The two-variable main conjecture* (*Conjecture* 1.3) *implies Conjecture* 6.3 *for $f_{\mathfrak{I}}$.*
2. *Assume further the conditions listed in Theorem* 2 *of* §1 *with P_{τ} a unit in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. Then Conjecture* 6.3 *for $f_{\mathfrak{I}}$ implies the two-variable main conjecture* (*Conjecture* 1.3).

Remark 6.5. 1. Concerning our condition on the triviality of $((\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}})^*)_{I_v}[\mathrm{Ker}(\mathfrak{I})]$, we refer to Theorem 2.3 for detailed information on when it is trivial. We expect that the conclusion of Corollary 6.4 is true without any assumption. However we do not prove it at the moment (cf. Remark 3.5).

2. Let $\mathcal{F}$ be a Λ-adic eigen form satisfying the conditions listed in Theorem 2 of §1. If the conjecture 6.3 is true for a classical specialization $f_{\mathfrak{I}_0}$ in $\mathcal{F}$ with $w(\mathfrak{I}_0) \geq 0$, the conjecture 6.3 is true for every specializations $f_{\mathfrak{I}}$ of $\mathcal{F}$ with $w(\mathfrak{I}) \geq 0$. Thus we have an infinite family of modular forms $\{f_{\mathfrak{I}}\}_{\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}}$ where the conjecture 6.3 is true. A recent paper [EPW] also proves a similar result on the conjecture 6.3 for infinite family $\{f_{\mathfrak{I}}\}_{\mathfrak{I} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\geq 0}}$ with further deep interpretation of λ-invariants under the assumption $\mu = 0$ for the μ-invariant of $f_{\mathfrak{I}_0}$. The advantage of our result above is that we do not have to assume $\mu = 0$.

Proof. The restriction map $\mathrm{Sel}_I \longrightarrow \mathrm{Sel}_{\mathcal{T}}[I]$ is an isomorphism by Proposition 4.2 and by the assumption of Corollary 6.4. By Lemma 6.2, the image of $\mathrm{char}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}}(\mathrm{Sel}_{\mathcal{T}})^{\vee}$ in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$ is equal to $\mathrm{char}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I}(\mathrm{Sel}_{\mathcal{T}}[I])^{\vee}$. The image of $L^{\mathrm{Ki}}_p(\mathcal{T}) \in \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$ is equal to $L^{\mathrm{MTT}}_p(\mathcal{T}_I)$ up to multiplication of a unit in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$ since the p-adic period $C_{p,\mathfrak{I},b}$ is a p-adic unit. Thus we obtain the first assertion. For the second assertion, note that we have the following inequality:

$$\mathrm{length}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(\mathrm{Sel}_{\mathcal{T}})^{\vee}_{\mathfrak{l}} \leq \mathrm{ord}_{\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F},\mathfrak{l}}}(L^{\mathrm{Ki}}_p(\mathcal{T}))$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ by Theorem 3 in §1. If Conjecture 6.3 is true for $f_{\mathfrak{I}}$, we have the following equality:

$$\mathrm{length}_{(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I)_{\mathfrak{l}}}\left((\mathrm{Sel}_{\mathcal{T}})^{\vee}/I(\mathrm{Sel}_{\mathcal{T}})^{\vee}\right)_{\mathfrak{l}} = \mathrm{ord}_{(\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I)_{\mathfrak{l}}}(L^{\mathrm{Ki}}_p(\mathcal{T}))$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/I$ by the same argument as in the proof of the first assertion. Thus we complete the proof of the second assertion. □

(b) Iwasawa Main conjecture for $\mathcal{T}_{(\gamma-\chi(\gamma))}$.

$\mathrm{Sel}_{(\gamma-\chi(\gamma))}$ is a cotorsion $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module (cf. §3.2). On the other hand, we define $L_p(\mathcal{T}_{(\gamma-\chi(\gamma))})$ to be the image of $L^{\mathrm{Ki}}_p(\mathcal{T})$ in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/(\gamma-\chi(\gamma)) = \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$. The one-variable Iwasawa main conjecture is formulated as follows:

Conjecture 6.6. We have the following equality:

$$\mathrm{length}_{(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\mathfrak{l}}}(\mathrm{Sel}^{\vee}_{(\gamma-\chi(\gamma))})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}_{(\gamma-\chi(\gamma))})),$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$.

We have the following corollary of Theorem 3 in §1:

Corollary 6.7. 1. *The two-variable main conjecture* (*Conjecture* 1.3) *implies Conjecture* 6.6*.*

2. *Assume further the conditions listed in Theorem* 2 *of* §1 *with P_{τ} a unit in $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. Then Conjecture* 6.6 *implies the two-variable main conjecture* (*Conjecture* 1.3)*.*

The proof is done in the same manner as the case (**a**) above by using Lemma 6.2.

(c) Iwasawa Main conjecture for $\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')}$.

$\mathrm{Sel}_{(\gamma-\kappa(\gamma')\gamma')}$ is a cotorsion $\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$-module (cf. §3.2). On the other hand, we define $L_p(\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')})$ to be the image of $L_p^{\mathrm{Ki}}(\mathcal{T})$ in $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/((\gamma-\kappa(\gamma')\gamma')) = \mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$.

Conjecture 6.8. Let $\mathcal{T} = \mathcal{T}_{\mathcal{F}}^{(i)}$. We have the following equality:

$$\mathrm{length}_{(\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}})_{\mathfrak{l}}}(\mathrm{Sel}^{\vee}_{(\gamma-\kappa(\gamma')\gamma')})_{\mathfrak{l}} + e_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}_{(\gamma-\kappa(\gamma')\gamma')})),$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$, where

$$e_{\mathfrak{l}} = \begin{cases} \mathrm{ord}_{\mathfrak{l}}(1-A_p(\mathcal{F})) & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} \neq 0, \\ 0 & \text{if } \mathrm{F}^-\mathcal{A}[\mathfrak{M}]^{I_p} = 0. \end{cases}$$

A corollary of Theorem 3 in §1 is given as follows:

Corollary 6.9.
1. *The two-variable main conjecture* (*Conjecture* 1.3) *implies Conjecture* 6.8.
2. *Assume further the conditions listed in Theorem* 2 *of* §1 *with* $P_{\mathcal{T}}$ *a unit in* $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$. *Then Conjecture* 6.8 *implies the two-variable main conjecture* (*Conjecture* 1.3).

The proof is done in the same manner as the case (**a**) and (**b**) by using Lemma 6.2.

(d) Iwasawa Main conjecture for $\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}$. The Selmer group for the diagonal specialization $\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}$ is not a cotorsion $\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$-module in general.

Conjecture 6.10.
1. Let $\mathcal{T} = \mathcal{T}_{\mathcal{F}}^{(i)}$. The group $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^{\vee}$ has rank one or zero as an $\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$-module.
2. Assume that $T_0 \otimes \omega^{i'}$ is isomorphic to its Kummer dual $(T_0 \otimes \omega^{i'})^*(1)$ with certain $0 \leq i' \leq p-2$, where $T_0 := \mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}/(\gamma'-1)\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}$. In this case, we have

$$\mathrm{rank}_{\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}}(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^{\vee} = \begin{cases} 1 & \text{if } \epsilon(l) = -1 \text{ for every } l > 0, \\ 0 & \text{if } \epsilon(l) = 1 \text{ for every } l > 0, \end{cases}$$

where $\epsilon(l)$ is the sign of the functional equation of L-function for a specialization of $\mathcal{F} \otimes \omega^{i'-i}$ explained in Remark 6.11 below.

Remark 6.11.
1. Suppose that $T_0 \otimes \omega^{i'}$ is isomorphic to the Kummer dual of itself. For each l, we put $P_l = \gamma' - \kappa^{a(l)}(\gamma')$ for each integer $l > 0$ with $a(l) := 2(i'-i) + 2l(p-1)$. For each $l > 0$, $\mathcal{T}/(P_l)\mathcal{T}$ is isomorphic to the Tate-twist $T_{f_l}(\frac{a(l)}{2}+1)$ of the Deligne's Galois representation T_{f_l} for an eigen cuspform f_l of weight $2+a(l)$. The sign $\epsilon(l) = \pm 1$ is the sign of the functional equation $\Lambda(f_l, s) = \epsilon(l)\Lambda(f_l, 2+a(l)-s)$ where $\Lambda(f_l, s)$ is the Hecke L-function for f_l with its Γ-factor.
2. The phenomena for the generic rank on the line $(\gamma^2 - \kappa^2(\gamma')\gamma')$ was first studied and conjectured at least under the condition as in 1 (see [NP, §0] for example). We believe that such phenomena is always true even in the case without the functional equation.

Suppose that $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^{\vee}$ is a torsion $\mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$-module. We define $L_p(\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')}) \in \mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$ to be the specialization of the two-variable p-adic L-function $L_p^{\mathrm{Ki}}(\mathcal{T}) \in \mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$ via $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}} \longrightarrow \mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(\gamma^2-\kappa^2(\gamma')\gamma') \cong \mathbb{H}_{\mathcal{F}}^{\mathrm{ord}}$, in this case.

Conjecture 6.12. Suppose that $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee$ is a torsion $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module. Then, we have the following equality:

$$\mathrm{length}_{(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{(\gamma^2-\kappa^2(\gamma')\gamma')})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}_{(\gamma-\chi(\gamma))})),$$

for each height-one prime $\mathfrak{l}$ of $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$.

Corollary 6.13. *Suppose that* $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee$ *is a torsion* $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$*-module.*

1. *The two-variable main conjecture* (*Conjecture* 1.3) *implies Conjecture* 6.12.
2. *Assume further the conditions listed in Theorem* 2 *of* §1 *with* P_τ *a unit in* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$. *Then Conjecture* 6.12 *implies the two-variable main conjecture* (*Conjecture* 1.3).

Finally, in a general case where $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee$ is not necessarily a torsion $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$-module, we propose the following Iwasawa Main Conjecture:

Conjecture . *Suppose that* $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee$ *is an* $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$*-module with generic rank* $r = \dim_{\mathrm{Frac}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})}(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee \otimes_{\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}} \mathrm{Frac}(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})$. *Let* $\mathfrak{X}$ *be the* $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$*-torsion part of the* $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$*-module* $(\mathrm{Sel}_{(\gamma^2-\kappa^2(\gamma')\gamma')})^\vee$. *Then the following statements hold:*

1. *The order* $\mathrm{ord}_{(\gamma^2-\kappa^2(\gamma')\gamma')}(L_p(\mathcal{T}))$ *is equal to* r.
2. *For every height-one prime* $\mathfrak{l}$ *of* $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$, *we have:*

$$\mathrm{length}_{(\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}})_{\mathfrak{l}}}(\mathfrak{X})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')})),$$

where $L_p(\mathcal{T}_{(\gamma^2-\kappa^2(\gamma')\gamma')})$ *is defined to be the image of* $L_p(\mathcal{T})/(\gamma^2-\kappa^2(\gamma')\gamma')^r$ *via* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}} \longrightarrow \mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}/(\gamma^2-\kappa^2(\gamma')\gamma') \cong \mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$.

7. Pseudo-null submodule

In this section, we give a sufficient condition (Proposition 7.1) for $(\mathrm{Sel}_{\mathcal{T}})^\vee$ to have no non-trivial pseudo-null submodule. Our proof relies on the method in Greenberg's paper [Gr1] (see also Remark 7.2). The result in this section is used in §8 in order to study examples where we can determine the structure of the Selmer group. In this section, we do not assume necessarily the condition (**Ir**). In stead of (**Ir**), we will assume the following condition:
(**Fr**) $\mathbb{T}^{\mathrm{ord}}_{\mathcal{F}}$ (resp. $\mathcal{T}^{(i)}_{\mathcal{F}}$) is free of rank two over $\mathbb{H}^{\mathrm{ord}}_{\mathcal{F}}$ (resp. $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$).
As remarked in §1, (**Ir**) implies (**Fr**). Since we find no reference for the pseudo-null submodule of the Selmer group for a Galois deformation, we decide to assume only a weaker condition (**Fr**) in this section for our later use. Our main proposition here is as follows:

Proposition 7.1. *Let* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}}$ *be a nearly ordinary deformation satisfying the condition* (**Fr**) *and let* Σ *be the set of ramified places for* $\mathcal{T}$ (*see* §1 *for the notation*). *Assume the following conditions:*

1. $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ *is a regular local ring.*
2. Σ *consists only of* $\{p, \infty\}$.

Then $(\mathrm{Sel}_{\mathcal{T}})^\vee$ *has no non-trivial pseudo-null* $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$*-submodule.*

Remark 7.2. Under similar assumptions, Greenberg [Gr1, Proposition 5] has proved that the Pontryagin dual of the Selmer group for the cyclotomic deformation of an ordinary p-adic representation $T \cong \mathbb{Z}_p^d$ has no pseudo-null $\mathbb{Z}_p[[\Gamma]]$-submodule when T is unramified outside $\{p, \infty\}$. Our proof follows the idea of [Gr1, Proposition 5]. Since we treat two-variable case, there causes technical difficulties in order to imitate his argument over the cyclotomic (one-variable) Iwasawa algebra.

Before giving the proof, we prepare several lemmas. Though our main proposition stated above treats only the case where Σ consists only of $\{p, \infty\}$, we allow Σ to contain primes other than p in the most part of this section unless we state it.

First, we prove the following lemma known as the weak Leopoldt conjecture for $\mathcal{T}$.

Lemma 7.3. $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}) = 0$.

Proof. Note that $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})$ is equal to the inductive limit $\varinjlim_{s,t} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)})$ for any pair (j,k) with $1 \leq j \leq k-1$, where $A_{s,t}^{(j,k)}$ is the module defined in §3.1. From now on we assume further that $2j \neq k$. It suffices to show that $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}) = 0$ for every s, t under this condition. Since the Galois group $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$ has cohomological dimension *two*, $H^3(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}[p])$ is zero. By the natural exact sequence:

$$H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}[p]) \longrightarrow H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}) \xrightarrow{\times p} H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}) \longrightarrow H^3(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)}[p]) \longrightarrow \cdots,$$

$H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)})$ must be a p-divisible abelian group. On the other hand, by a Kato's result [Ka2, §14], $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)})$ is finite for each s, t under the above assumption on (j,k). Hence $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, A_{s,t}^{(j,k)})$ must be zero. □

Lemma 7.4. *Assume that* $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$ *is regular and that* $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$ *satisfies the condition* (**Fr**). $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})^\vee$ *has no nontrivial pseudo-null* $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$*-submodule.*

Proof. Let N be the largest pseudo-null submodule of $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})^\vee$. Let h be an arbitrary irreducible element of $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$. By taking the short exact sequence:

$$0 \longrightarrow \mathcal{A}[h] \longrightarrow \mathcal{A} \xrightarrow{\times h} \mathcal{A} \longrightarrow 0,$$

and by using Lemma 7.3, we have

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})/(h)H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}) \cong H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[h]). \tag{8}$$

By a similar argument as that used in the proof of Lemma 7.3 depending on the Galois cohomological dimension of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$, $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A}[h])^\vee$ is shown to be torsion-free over the local domain $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(h)$. Consequently, $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})^\vee[h]$ must be a torsion-free $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(h)$-module by taking the Pontryagin dual of the equation (8). $N[h]$ is also a torsion-free $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(h)$-module since $N[h]$ is a sub $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(h)$-module of $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathcal{A})^\vee[h]$. On the other hand, the torsion part $N[h]$ of N for the height one prime (h) is a torsion $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}/(h)$-module because N is a pseudo-null $\mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$-module. Thus $N[h]$ is zero for any irreducible element $h \in \mathbb{H}_{\mathcal{F}}^{\mathrm{n.o}}$. This completes the proof for $N = 0$. □

A finitely generated $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module V is called *reflexive* if the canonical homomorphism $V \longrightarrow V^{**} := \mathrm{Hom}_{\mathbb{H}^{\text{n.o}}_{\mathcal{F}}}(\mathrm{Hom}_{\mathbb{H}^{\text{n.o}}_{\mathcal{F}}}(V, \mathbb{H}^{\text{n.o}}_{\mathcal{F}}), \mathbb{H}^{\text{n.o}}_{\mathcal{F}})$ is an isomorphism. We have the following lemma:

Lemma 7.5. *Assume that $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$ is a regular local ring. Then, $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee$ is a reflexive $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module.*

Proof. First, we show that $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee$ is torsion-free over $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$. By taking the Pontryagin dual, it is equivalent to the statement that $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})$ is a divisible $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module. Consider the long exact sequence of the $G_{\mathbb{Q}_p}$-cohomology of

$$0 \longrightarrow \mathrm{F}^-\mathcal{A}[h] \longrightarrow \mathrm{F}^-\mathcal{A} \xrightarrow{\times h} \mathrm{F}^-\mathcal{A} \longrightarrow 0,$$

for an irreducible element $h \in \mathbb{H}^{\text{n.o}}_{\mathcal{F}}$, $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})/(h)H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})$ is a $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-submodule of $H^2(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[h])$. By the local Tate duality theorem, $H^2(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[h])$ is the Pontryagin dual of $H^0(\mathbb{Q}_p, (\mathrm{F}^-\mathcal{A}(-1))^\vee/(h)(\mathrm{F}^-\mathcal{A}(-1))^\vee)$. This group must be zero, because $\big((\mathrm{F}^-\mathcal{A}(-1))^\vee\big)_{G_{\mathbb{Q}_p}}$ has support whose codimension is equal or greater than two. Hence $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})$ is a divisible $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module. Since the Pontryagin dual $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee$ has no non-trivial $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-torsion submodule, the structure theorem of finitely generated $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-modules (cf. Proposition (5.17) and Proposition (5.1.8) in [NSW]) gives us an exact sequence:

$$0 \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee \longrightarrow V \longrightarrow Z \longrightarrow 0,$$

where V is a reflexive $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module and Z is a pseudo-null $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}$-module. Let $h' \in \mathbb{H}^{\text{n.o}}_{\mathcal{F}}$ be an arbitrary non zero irreducible element and let us consider the snake lemma to the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee & \longrightarrow & V & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \times h'} & & \Big\downarrow{\scriptstyle \times h'} & & \Big\downarrow{\scriptstyle \times h'} & & \\
0 & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee & \longrightarrow & V & \longrightarrow & Z & \longrightarrow & 0.
\end{array}$$

By applying the snake lemma to the above diagram, we have an injection

$$Z[h'] \hookrightarrow (H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})[h']^\vee)_{\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/(h')\text{-tor}}.$$

Since $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})[h']^\vee$ is naturally an $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/(h')$-submodule of $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[h'])^\vee$, this also gives us an injection

$$Z[h'] \hookrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[h'])^\vee_{\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/(h')\text{-tor}}.$$

By a similar argument as above, we show that $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}[h'])^\vee$ is a torsion free $\mathbb{H}^{\text{n.o}}_{\mathcal{F}}/(h')$-module by the condition 2 of the lemma. Hence Z must be zero. This completes the proof. ◻

Lemma 7.6. *Assume three conditions stated in Proposition* 7.1. *Then, we have the following exact sequence:*

$$0 \longrightarrow \mathrm{Sel}_{\mathcal{T}} \longrightarrow H^1(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}, \mathcal{A}) \xrightarrow{\text{loc}} H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \longrightarrow 0.$$

Proof. By Corollary 3.12, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Sel}_{\mathcal{T}} \longrightarrow H^1(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}, \mathcal{A}) \xrightarrow{\mathrm{loc}} \frac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})} \longrightarrow 0.$$

Since the restriction map $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \longrightarrow H^1(I_p, \mathrm{F}^-\mathcal{A})^{G_{\mathbb{Q}_p}}$ is an isomorphism, we have an exact sequence:

$$0 \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathcal{A})}{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})} \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \longrightarrow H^2(\mathbb{Q}_p, \mathrm{F}^+\mathcal{A}).$$

Note that $H^2(\mathbb{Q}_p, \mathrm{F}^+\mathcal{A})$ is zero by the same argument as the proof of Claim 3.8. This completes the proof of the lemma. □

Lemma 7.7. *Let R be a Noetherian complete regular local ring and let M be an R-module which has the following presentation*

$$0 \longrightarrow W \longrightarrow U \longrightarrow M \longrightarrow 0,$$

where U is a finitely generated R-module which has no non-trivial pseudo-null R-submodule and W is a reflexive R-module. Then M has no nontrivial pseudo-null R-submodule.

Proof. Suppose that the largest pseudo-null R-submodule M_{null} of M is non-trivial. We denote by $U_0 \subset U$ the inverse image of M_{null} via the natural projection $U \twoheadrightarrow M$. Since U has no non-trivial pseudo-null R-submodule, U_0 also has no non-trivial pseudo-null R-submodule. By the structure theorem of finitely generated R-modules (Proposition **??**), we have the following exact sequence:

$$0 \longrightarrow U_0 \longrightarrow E \oplus W' \longrightarrow Z \longrightarrow 0,$$

where E is an elementary torsion R-module, W' is a reflexive R-module and Z is a pseudo-null R-module. Thus, we have also the following exact sequence:

$$0 \longrightarrow W \longrightarrow E \oplus W' \longrightarrow Z' \longrightarrow 0,$$

where Z' is an extension of Z by M_{null}. Especially, Z' is a non trivial pseudo-null R-module. Since W is reflexive and Z' is pseudo-null, E must be trivial. Thus we have an injection $W \hookrightarrow W'$ whose cokernel is a non trivial pseudo-null R-module. Note that $\mathrm{Ext}^1_R(W'/W, R)$ is zero since W'/W is pseudo-null (see [OV, Proposition 3.4] for example). The injection $W \hookrightarrow W'$ induces an isomorphism $\mathrm{Hom}_R(W', R) \xrightarrow{\sim} \mathrm{Hom}_R(W, R)$. Hence $W \hookrightarrow W'$ must be an isomorphism since W and W' are reflexive R-modules. This contradicts to the assumption that M_{null} is non-trivial. The proof is done. □

Finally we give the proof of Proposition 7.1.

Proof of Proposition 7.1. By Lemma 7.6, we have the following exact sequence:

$$0 \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee \longrightarrow H^1(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}, \mathrm{F}^-\mathcal{A})^\vee \longrightarrow (\mathrm{Sel}_{\mathcal{T}})^\vee \longrightarrow 0.$$

The module $H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})^\vee$ is reflexive over $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$ by Lemma 7.5. The $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-module $H^1(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}, \mathrm{F}^-\mathcal{A})^\vee$ has no non-trivial pseudo-null $\mathbb{H}^{\mathrm{n.o}}_{\mathcal{F}}$-submodule by Lemma 7.4. Thus we complete the proof by applying Lemma 7.7. □

8. Examples

In this section, we study examples of two-variable nearly ordinary deformations where we can determine the structure of the Selmer group or we prove the equality in addition to the inequality result proved by using Beilinson-Kato elements.

First, we prepare some preliminary results.

Proposition 8.1. [Gr1, Theorem 2] *Let T be a $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-module which is free of finite rank over $\mathbb{Z}_p$. Suppose that T is ordinary and critical at p. Then we have:*

$$\mathrm{length}_{\mathbb{Z}_p[[\Gamma]]_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{T\otimes\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})})_{\mathfrak{l}} = \mathrm{length}_{\mathbb{Z}_p[[\Gamma]]_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{T^*(1)\otimes\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})})^{\iota}_{\mathfrak{l}},$$

for every height-one primes $\mathfrak{l}$ in $\mathbb{Z}_p[[\Gamma]]_{\mathfrak{l}}$ where ι is the canonical involution of $\mathbb{Z}_p[[\Gamma]]$ induced by $g \mapsto g^{-1}$ for $g \in \Gamma$.

We recall the following lemma:

Lemma 8.2. *Let R be a Noetherian complete regular local ring of Krull dimension $n \geq 2$ and let N be a pseudo-null R-module. Let I be a height one prime of R such that R/I is a regular local ring of Krull dimension $n-1$. Then, we have the following equality for every height-one prime ideals in R/I:*

$$\mathrm{length}_{(R/I)_{\mathfrak{l}}}(N[I]_{\mathfrak{l}}) = \mathrm{length}_{(R/I)_{\mathfrak{l}}}(N/IN)_{\mathfrak{l}}.$$

Especially, $N[I]$ is a pseudo-null R/I-module if and only if N/IN is a pseudo-null R/I-module.

Though this lemma might be known to the experts, we refer the reader to [O4, Lemma 3.1] for the proof if necessary.

Lemma 8.3. *Let R be a Noetherian complete regular local ring of Krull dimension ≥ 2. Let M (resp. N) be a torsion R-module $R/(f)$ (resp. $R/(g)$) with $f \in R$ (resp. $g \in R$). Suppose that we have a family $\{J_l\}_{1\leq l<\infty}$ of non-zero elements of R satisfying the properties*:

1. *We have an injection $M \hookrightarrow \prod_{1\leq l<\infty} M/J_lM$.*
2. *For each i, R/J_l is a regular local ring.*
3. *The modules M/J_lM and N/J_lN are torsion over R/J_l.*
4. *We have $\mathrm{char}_{R/J_l}(M/J_lM) \supset \mathrm{char}_{R/J_l}(N/J_lN)$ for each $l \geq 1$.*

Then, we have $\mathrm{char}_R(M) \supset \mathrm{char}_R(N)$.

Proof. It suffices to show that the image of g via $R \twoheadrightarrow M$ is zero. By the conditions 3 an 4, the image of g via $R \twoheadrightarrow M/J_lM$ is zero for any l. This completes the proof by the condition 1. □

8.1. **Iwasawa Main conjecture for Ramanujan's cusp form.** Let $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ be the unique eigen cusp form of level 1 and weight 12, whose q-expansion is equal to $q\prod_{1\leq n<\infty}(1-q^n)^{24}$. Only known non-ordinary primes for Δ is $p = 2, 3, 5, 7, 2411$ at the moment. For all other primes p, we have the ordinary Λ-adic newform $\mathcal{F}(\Delta) \in \mathbb{Z}_p[[\Gamma']][[q]]$ such that the specialization of $\mathcal{F}$ under $\mathbb{Z}_p[[\Gamma']][[q]] \longrightarrow \mathbb{Z}_p[[q]],\ \gamma' \mapsto \kappa^{10}(\gamma')$ coincides with the q-expansion of the p-stabilization $\Delta^{(p)}$ of Δ (we omit the prime p in the notation

$\mathcal{F}(\Delta)$ unless there is a possibility of confusion). See [H3, §7.6] for the explanation on the Hida family for Δ. The condition (**Nor**) is always satisfied. The two-variable Iwasawa theory for $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}$ at p is of our interest.

Question 8.4. For which ordinary prime p of Δ and for which integer i with $0 \leq i \leq p-2$, the characteristic ideal $\mathrm{char}_{\mathbb{H}^{\mathfrak{m},\circ}_{\mathcal{F}}}(\mathrm{Sel}_{\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}})^\vee$ or the ideal $(L_p(\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}))$ is non-trivial?

Recall that the value $\dfrac{L(\Delta,j)}{(2\pi\sqrt{-1})^{j-1}C^{(-1)^{j-1}}_{\infty,\Delta}}$ is equal to $\dfrac{2^3\cdot 3^4\cdot 5\cdot 7}{691}$, $2^4{\cdot}3$, $2{\cdot}7$, 5^2, 3^2, $2^2{\cdot}5$ when $j=1,\cdots,6$. By the functional equation, we have $\dfrac{L(\Delta,j)}{(2\pi\sqrt{-1})^{j-1}C^{(-1)^{j-1}}_{\infty,\Delta}} = -\dfrac{L(\Delta,12-j)}{(2\pi\sqrt{-1})^{j-1}C^{(-1)^{11-j}}_{\infty,\Delta}}$. Especially, the value $\dfrac{L(\Delta,j)}{(2\pi\sqrt{-1})^{j-1}C^{(-1)^{j-1}}_{\infty,\Delta}}$ is a p-adic unit for every j with $1 \leq j \leq 11$ and for $p \geq 11$ with $p \neq 691$.

Let p be an ordinary prime of Δ where the condition (**Ir**) is satisfied (Especially, $p \neq 691$). For $1 \leq i \leq 11$, we have:

$$
\begin{aligned}
(\chi^i \circ \kappa^{10}) L_p(\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}) &= \left(1 - \frac{p^{i-1}}{a_p(\Delta^{(p)})}\right) \frac{L(\Delta^{(p)}, i)}{(2\pi\sqrt{-1})^{i-1}C^{(-1)^{i-1}}_{\infty,\Delta}} \\
&= \left(1 - \frac{p^{i-1}}{a_p(\Delta^{(p)})}\right)\left(1 - \frac{a_p(\Delta)}{p^i}\right) \frac{L(\Delta, i)}{(2\pi\sqrt{-1})^{i-1}C^{(-1)^{i-1}}_{\infty,\Delta}}.
\end{aligned}
$$

For $i \neq 1$, this is a p-adic unit, hence we have $L_p(\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}) \in \mathbb{Z}_p[[\Gamma \times \Gamma']]^\times$. For $i=1$, $L_p(\mathcal{T}^{(1)}_{\mathcal{F}(\Delta)})$ is a unit if and only if $a_p(\Delta) \nmid 1$ modulo p. As for the structure of the Selmer group, we have the following result.

Lemma 8.5. $(\mathrm{Sel}_{\mathcal{T}})^\vee$ *has no non-trivial pseudo-null* $\Lambda^{(2)}$*-submodule for* $\mathcal{T} = \mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}$.

Proof. It suffices to see that our nearly ordinary deformation $\mathcal{T}$ associated to Δ satisfies two conditions in Proposition 7.1. The condition 1 is deduced by observing the dimension of the space of weight 12 cusp forms (cf. [H3, §7.6]). The condition 2 is clear since Δ has level one. This completes the proof. □

We summarize our argument above in the following proposition:

Proposition 8.6. *Let p be an ordinary prime of Δ where* (**Ir**) *is satisfied.*

1. *When $2 \leq i \leq 11$, $L_p(\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)})$ is trivial and* $\mathrm{Sel}_{\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)}} = 0$.
2. *When $i=1$, $L_p(\mathcal{T}^{(1)}_{\mathcal{F}(\Delta)})$ is non-trivial if and only if $a_p(\Delta) \equiv 1$ modulo p.*

Remark 8.7. 1. For $i=0$ or for $12 \leq i \leq p-2$, we do not have a precise conjecture about when or how often $L_p(\mathcal{T}^{(i)}_{\mathcal{F}(\Delta)})$ is non-trivial.

2. The primes where $a_p(\Delta) \equiv 1$ modulo p is called anomalous primes for Δ. Among smaller primes, $p = 11$ and $p = 23$ are known to be anomalous. We do not know how much other anomalous primes for Δ exist.

According to the above remark, we will investigate the case $p = 11$ in the next subsection.

8.2. **Ramanujan's cusp form at $p = 11$.** In this subsection, we discuss the two-variable Iwasawa theory for $\mathcal{F}(\Delta)$ at $p = 11$, where we have a Hida family $\mathcal{T} = \mathcal{T}^{(1)}_{\mathcal{F}(\Delta)} \cong \mathbb{Z}_p[[\Gamma \times \Gamma']]^{\oplus 2}$ such that

1. The specialization $\mathcal{T}/\Phi^{(1,2)}\mathcal{T}$ is isomorphic to the p-Tate module of $X_0(11)$.
2. $\mathcal{T}/\Phi^{(j,12)}\mathcal{T}$ is a lattice of the representation $T_\Delta(j) \otimes \omega^{1-j}$, where $T_\Delta \cong \mathbb{Z}_p^{\oplus 2}$ is the p-adic Galois representation associated to Δ by Deligne.

From now on, we shall denote $\mathbb{Z}_p[[\Gamma \times \Gamma']]$ by $\Lambda^{(2)}$ for short. Our results on the Iwasawa theory for $\mathcal{T}$ in this section is as follows:

Results on the Iwasawa theory for $\mathcal{T}$. *Let $\mathcal{T} = \mathcal{T}^{(1)}_{\mathcal{F}(\Delta)}$ with $p = 11$.*

(1) *We have* $\mathrm{length}_{\Lambda^{(2)}_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{\mathcal{T}})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}))$ *for every height-one primes $\mathfrak{l}$ in $\Lambda^{(2)}$.*

(2) *We have* $(\mathrm{Sel}_{\mathcal{T}})^\vee \cong \Lambda^{(2)}/(\gamma^2 - \kappa^2(\gamma')\gamma')$.

We will show the statement (2) at first and the equality in (1) will be proved later. Let us take an infinite family of elements $\{P_l \in \Lambda^{(2)}\}_{1 \leq l < \infty}$ given by $P_l = \gamma' - \kappa^{2l(p-1)}(\gamma')$. Then we have the following claim:

Claim 8.8. Let us denote by T_l the representation associated to the ordinary eigen cusp form $f_{2+2l(p-1)}$ of weight $2 + 2l(p-1)$ in the Hida family for Δ.

(1) $\mathcal{T}/(P_l)\mathcal{T}$ is the cyclotomic deformation of $T_l \otimes \omega$.

(2) The natural restriction map $(\mathrm{Sel}_{\mathcal{T}})^\vee/(P_l)(\mathrm{Sel}_{\mathcal{T}})^\vee \longrightarrow \mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}}$ is an isomorphism.

(3) We have the isomorphism $\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}} \cong \mathbb{Z}_p[[\Gamma]]/(\gamma - \chi^{1+l(p-1)}(\gamma))$

Proof of Claim 8.8. (1) is nothing but the definition of $\mathcal{T}$. We have $H^0(\mathbb{Q}, \mathcal{A}) = 0$ by [Se2, 5.5.2] and by argument using Nakayama's lemma (cf. the proof of Claim 8.11), where $\mathcal{A} = \mathcal{T} \otimes_{\Lambda^{(2)}} \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda^{(2)}, \mathbb{Q}_p/\mathbb{Z}_p)$. By definition, the set of ramified primes Σ for $\mathcal{T}$ is $\{p, \infty\}$. Hence, Claim 8.8 (2) is a corollary of Proposition 4.2. Let us show the statement (3) in the rest. $\mathrm{Sel}_{\mathcal{T}/(P_l,p)\mathcal{T}}$ is isomorphic to $\mathrm{Sel}_{\mathcal{T}/(P_0,p)\mathcal{T}}$ for any $l \geq 0$ by congruence property. On the other hand, $\mathcal{T}/(P_0,p)\mathcal{T}$ is the cyclotomic deformation $X_0(11)[11] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi})$ of the group of 11-torsion elements $X_0(11)[11]$ of the modular elliptic curve $X_0(11)$. Hence, by [Gr2], $\mathrm{Sel}_{\mathcal{T}/(P_0,p)\mathcal{T}}$ is isomorphic to $\mathbb{Z}_p[[\Gamma]]/(\gamma - 1, p) \cong \mathbb{Z}/p\mathbb{Z}$. By the control theorem for modulo-(p) reduction which can be proved in the same manner as those in §4, we have an isomorphism $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee/(p)(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee \cong (\mathrm{Sel}_{\mathcal{T}/(P_l,p)\mathcal{T}})^\vee \cong \mathbb{Z}/p\mathbb{Z}$. Since $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee$ has no finite $\mathbb{Z}_p[[\Gamma]]$-submodule (cf. [Gr3, Proposition 10]), $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee$ must be a free $\mathbb{Z}_p$-module of rank one for any $l \geq 0$. Finally, let us denote the action of Γ on $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee$. Recall that $(T_l \otimes \omega) \otimes \chi^{1+l(p-1)}$ is Kummer-dual to itself via Weil pairing. By Proposition 8.1 and Claim 8.8 (1), $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee \otimes \chi^{-1-l(p-1)} \cong \left(\mathrm{Sel}_{(\mathcal{T}/(P_l)\mathcal{T}) \otimes \chi^{1+l(p-1)}}\right)^\vee$ is a free rank-one $\mathbb{Z}_p$-module with trivial Γ-action. Hence we have $(\mathrm{Sel}_{\mathcal{T}/(P_l)\mathcal{T}})^\vee \cong \mathbb{Z}_p(\chi^{1+l(p-1)})$ for every $l \geq 1$, where $\mathbb{Z}_p(\chi^{1+l(p-1)})$ is a free rank one $\mathbb{Z}_p$-module on which Γ acts via $\chi^{1+l(p-1)}$. This completes the proof of Claim 8.8 (3). $\square$

Let us prove the following claim:

Claim 8.9. We have $\mathrm{length}_{\Lambda^{(2)}_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{\mathcal{T}})_{\mathfrak{l}} \leq \mathrm{ord}_{\mathfrak{l}}(\gamma^2 - \kappa^2(\gamma')\gamma')$ for every height-one primes $\mathfrak{l}$ in $\Lambda^{(2)}$.

Proof. Let g be an element of $\Lambda^{(2)}$ such that $(g) = \mathrm{char}_{\Lambda^{(2)}}(\mathrm{Sel}_{\mathcal{T}})^\vee$. Since $(\mathrm{Sel}_{\mathcal{T}})^\vee$ has no non-trivial pseudo-null $\Lambda^{(2)}$-submodule, we have an injection $(\mathrm{Sel}_{\mathcal{T}})^\vee \hookrightarrow \Lambda^{(2)}/(g)$ with a pseudo-null cokernel. We may replace $(\mathrm{Sel}_{\mathcal{T}})^\vee$ by $\Lambda^{(2)}/(g)$ to prove the claim. Let us apply Lemma 8.3 for $R = \Lambda^{(2)}$, $M = \Lambda^{(2)}/(\gamma^2 - \kappa^2(\gamma')\gamma')$ and $N = \Lambda^{(2)}/(g)$. Let $P_l = \gamma' - \kappa^{2l(p-1)}(\gamma') \in \Lambda^{(2)}$ as above. Note that each P_l is contained in $\mathbb{Z}_p[[\Gamma']]$. Since P_l's are relatively prime to each other, we have an injection:

$$\mathbb{Z}_p[[\Gamma']] \hookrightarrow \prod_{1 \leq l < \infty} \mathbb{Z}_p[[\Gamma']]/(P_l). \tag{9}$$

On the other hand, M is finite flat of degree two over $\mathbb{Z}_p[[\Gamma']]$. Hence by applying the base extension $\otimes_{\mathbb{Z}_p[[\Gamma']]}M$ to (9), we have an injection $M \hookrightarrow \prod_{1 \leq l < \infty} M/(P_l)M$. Thus we have shown the condition 1 of Lemma 8.3. The condition 2 is satisfied since P_l's are polynomials of degree one. $\square$

Claim 8.10. We have $\mathrm{length}_{\Lambda^{(2)}_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{\mathcal{T}})_{\mathfrak{l}} = \mathrm{ord}_{\mathfrak{l}}(\gamma^2 - \kappa^2(\gamma')\gamma')$ for every height-one primes $\mathfrak{l}$ in $\Lambda^{(2)}$.

Proof. Let us consider the specialization at $k = 2$. The image of the two ideals $(\gamma^2 - \kappa^2(\gamma')\gamma') \subset \Lambda^{(2)}$ and $(g) \subset \Lambda^{(2)}$ in $\Lambda^{(2)}/(\gamma' - 1) = \mathbb{Z}_p[[\Gamma]]$ are both equal to $(\gamma - 1)$. $\square$

Hence $(\mathrm{Sel}_{\mathcal{T}})^\vee$ is a torsion $\Lambda^{(2)}$-module whose characteristic ideal is $(\gamma^2 - \kappa^2(\gamma')\gamma')$. By Greenberg, $(\mathrm{Sel}_{\mathcal{T}})^\vee/(P_0)(\mathrm{Sel}_{\mathcal{T}})^\vee \cong (\mathrm{Sel}_{T_{f_2}\otimes\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})})^\vee$ is isomorphic to $\mathbb{Z}_p$. Especially, $(\mathrm{Sel}_{\mathcal{T}})^\vee/(P_0)(\mathrm{Sel}_{\mathcal{T}})^\vee$ is a cyclic module over $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[\Gamma \times \Gamma']]/(P_0)$. By Nakayama's lemma, $(\mathrm{Sel}_{\mathcal{T}})^\vee$ has to be a cyclic module over $\mathbb{Z}_p[[\Gamma \times \Gamma']]$. Consequently, we have $(\mathrm{Sel}_{\mathcal{T}})^\vee \cong \Lambda^{(2)}/(\gamma^2 - \kappa^2(\gamma')\gamma')$.

Next, we shall study Iwasawa Main Conjecture for this $\mathcal{T}$. Theorem 1 and Theorem 2 given in §1 implies the following claim:

Claim 8.11. We have $\mathrm{length}_{\Lambda^{(2)}_{\mathfrak{l}}}(\mathrm{Sel}^\vee_{\mathcal{T}})_{\mathfrak{l}} \leq \mathrm{ord}_{\mathfrak{l}}(L_p(\mathcal{T}))$ for every height-one primes $\mathfrak{l}$ in $\Lambda^{(2)}$.

Proof of Claim 8.11. We shall check the conditions (i) and (ii) in Theorem 2 for $\mathcal{T}$. By the nearly ordinary condition of $\mathcal{T}$, the image of $G_{\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty})}$ is contained in the group $\left\{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix} \in GL_2(\Lambda^{(2)})\right\}$. Let us consider also the action of $G_{\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty})}$ on $\mathcal{T}/\mathfrak{M}\mathcal{T} \cong \mathbb{F}_p^{\oplus 2}$ which is contained in the group $\left\{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix} \in GL_2(\mathbb{F}_p)\right\}$. Recall that the residual representation $\mathcal{T}/\mathfrak{M}\mathcal{T}$ is isomorphic to the group of 11-torsion points of $X_0(11)$ by properties of $\mathcal{T}$ introduced at the beginning of §8.2. Since $X_0(11)$ has split multiplicative reduction at 11, we have a $G_{\mathbb{Q}_p}$-equivariant isomorphism $X_0(11)(\overline{\mathbb{Q}}_p) \cong \overline{\mathbb{Q}}_p^\times/q^{\mathbb{Z}}$ with $q \in p\mathbb{Z}_p$ by the uniformization theory by Tate. Hence we have:

$$0 \longrightarrow \mu_p \longrightarrow \mathcal{T}/\mathfrak{M}\mathcal{T} \longrightarrow q^{\mathbb{Z}}/q^{\frac{1}{p}\mathbb{Z}} \longrightarrow 0.$$

We find $\tau \in G_{\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty})}$ such that the image $\begin{pmatrix} 1 & p_\tau \\ 0 & 1 \end{pmatrix}$ of τ in $\mathrm{Aut}(\mathcal{T}/\mathfrak{M}\mathcal{T}) \cong GL_2(\mathbb{F}_p)$ satisfies $p_\tau \neq 0$ because $q = 11^5 u$ with $u \in \mathbb{Z}_{11}^\times$. Thus τ is presented as $\begin{pmatrix} 1 & P_\tau \\ 0 & 1 \end{pmatrix}$ under certain choice of basis $\mathrm{Aut}(\mathcal{T}) \cong GL_2(\Lambda^{(2)})$, where P_τ is a unit of $\Lambda^{(2)}$.

For the condition (ii), $G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\mathcal{T}/\mathfrak{M}\mathcal{T})$ contains an element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ by the surjectivity of the representation of $G_{\mathbb{Q}}$ on the group of 11-torsion points of $X_0(11)$ shown in [Se2, 5.5.2]. This completes the proof by Theorem 2. □

Since we already have an inequality as in Claim 8.11, it suffices to see that $\mathrm{char}_{\Lambda^{(2)}}(\mathrm{Sel}_{\mathcal{T}})^\vee$ modulo $(\gamma'-1)$ is equal to the ideal of $\mathbb{Z}_p[[\Gamma]]$ generated by $L_p(\mathcal{T})$ modulo $\gamma'-1$. By Claim 8.10, the ideal $\mathrm{char}_{\Lambda^{(2)}}(\mathrm{Sel}_{\mathcal{T}})^\vee$ modulo $(\gamma'-1)$ of $\mathbb{Z}_p[[\Gamma]]$ is equal to $(\gamma^2-1) = (\gamma-1)$ (note that $\gamma+1$ is a unit in $\mathbb{Z}_p[[\Gamma]]$). On the other hand, by the interpolation property given in Theorem 1, $L_p(\mathcal{T})$ modulo $\gamma'-1$ is equal to $C_{p,2} \times L_p^{\mathrm{MTT}}(f_2) \in \mathbb{Z}_p[[\Gamma]]$, where $L_p^{\mathrm{MTT}}(f_2) \in \mathbb{Z}_p[[\Gamma]]$ is the p-adic L-function by Mazur-Tate-Teitelbaum [MTT]. Since $C_{p,2}$ is a p-adic unit by Proposition 5.4, it suffices to prove the following claim to have the equality of the Iwasawa Main conjecture for $\mathcal{T}$:

Claim 8.12. We have the equality $(L_p^{\mathrm{MTT}}(f_2)) = (\gamma - \chi(\gamma))$ in $\mathbb{Z}_p[[\Gamma]]$.

Proof. We denote by $g \in \mathbb{Z}_p[[\Gamma]]$ to be the quotient $L_p^{\mathrm{MTT}}(f_2)/(\gamma-\chi(\gamma))$. We would like to show that g is a unit in $\mathbb{Z}_p[[\Gamma]]$. For any element $h \in \mathbb{Z}_p[[\Gamma]]$, we regard h to be the function on $\mathbb{Z}_p$ by setting $h(s) = \chi^s(h)$ for $s \in \mathbb{Z}_p$. The trivial zero conjecture [MTT] which was already proved by Greenberg-Stevens [GS] gives us an equality as follows:

$$L_p^{\mathrm{MTT}}(f_2)(s)'|_{s=1} = \chi(\gamma)\log_p(\chi(\gamma)) \times g(s)|_{s=1} = \mathcal{L}_p \times \frac{L(f_2,1)}{C_{\infty,2}^+}, \tag{10}$$

where $\mathcal{L}_p \in \mathbb{Q}_p$ is the L-invariant defined to be $\log_p(q)/\mathrm{ord}_p(q)$ by using the Tate period q for the Tate curve $X_0(11)_{/\mathbb{Q}_p}$. By numerical calculation, we have $\mathrm{ord}_p(\mathcal{L}_p) = 1 = \mathrm{ord}_p(\chi(\gamma)\log_p(\chi(\gamma)))$ for $X_0(11)$ (cf. [MTT, §13]). Consequently, $g(s)|_{s=1} \in \mathbb{Z}_p$ is a unit. By Weierstrass preparation theorem, $g \in \mathbb{Z}_p[[\Gamma]]$ must be a unit. This completes the proof of Claim 8.12. □

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