CONTROL THEOREM FOR BLOCH-KATO'S SELMER GROUPS OF p-ADIC REPRESENTATIONS

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ABSTRACT. We give a sufficient condition for the Selmer group of the p-adic representation associated to a motive over a number field to be controlled in the cyclotomic $\mathbb{Z}_p$-extension. Then we apply this result to the Selmer group of various Galois representations. For example, we treat the representation of a modular form or the symmetric power of an elliptic curve.

1. INTRODUCTION

In [14], Mazur discussed a generalization of classical Iwasawa theory for the class group of a number field to the Selmer group of an elliptic curve over a number field. Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of an algebraic number field. We denote the n-th layer of F_∞/F by F_n. For the p-primary Selmer group $\mathrm{Sel}(E/F_n)\{p\}$ of an elliptic curve E over F, we have the following exact sequence:

$$0 \longrightarrow E(F_n)\otimes_{\mathbb{Z}}\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}(E/F_n)\{p\} \longrightarrow \mathrm{Sh}(E/F_n)\{p\} \longrightarrow 0,$$

where $E(F_n)$ is the F_n-valued points of E and $\mathrm{Sh}(E/F_n)\{p\}$ is the p-primary subgroup of the Tate-Shafarevich group of E/F_n. Mazur proved the following theorem.

Mazur's Control Theorem ([14]). *Let E be an elliptic curve over a number field F. Assume that E has good ordinary reduction at all places of F dividing p. Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension and let Γ_n be $\mathrm{Gal}(F_\infty/F_n)$. Then the kernel and the cokernel of the restriction map:*

$$\mathrm{Sel}(E/F_n)\{p\} \xrightarrow{f_n} \mathrm{Sel}(E/F_\infty)\{p\}^{\Gamma_n}$$

are finite groups whose orders are bounded independently of n.

We are interested in the behavior of the Selmer group $\mathrm{Sel}(E/F_n)\{p\}$ in the cyclotomic $\mathbb{Z}_p$-extension because $\mathrm{Sel}(E/F_n)\{p\}$ contains important arithmetic information of E/F_n such as the Mordell-Weil rank of $E(F_n)$. By Mazur's control theorem stated above, knowing the behavior of $\mathrm{Sel}(E/F_n)\{p\}$ is equivalent to knowing the behavior of $\mathrm{Sel}(E/F_\infty)\{p\}^{\Gamma_n}$. We denote by Λ the power series ring $\mathbb{Z}_p[[X]]$. By fixing a topological generator γ of the group $\Gamma = \mathrm{Gal}(F_\infty/F)$, we have an isomorphism $\Lambda \xrightarrow{\sim} \mathbb{Z}_p[[\Gamma]]$, $1+X \longmapsto \gamma$. Then $\mathrm{Sel}(E/F_\infty)\{p\}$ is naturally endowed with a Λ-module structure via the above isomorphism $\Lambda \cong \mathbb{Z}_p[[\Gamma]]$. It is conjectured that the Pontrjagin dual $\mathrm{Sel}(\widehat{E/F_\infty})\{p\}$ of $\mathrm{Sel}(E/F_\infty)\{p\}$ is a finitely generated torsion Λ-module and that the Fitting ideal is equal to the ideal generated by the p-adic L-function of E (Main Conjecture proposed by Mazur). Assuming the main conjecture, the behavior of $\mathrm{Sel}(E/F_\infty)\{p\}^{\Gamma_n}$ when n varies is related to some invariants of the p-adic L-function, which are computable compared to those of the Selmer group. Hence we will be able to know the behavior of $\mathrm{Sel}(E/F_n)\{p\}$.

In this paper, we discuss a generalization of Mazur's theorem stated above to more general p-adic representations.

To a motive M over a number field F, we associate a free $\mathbb{Z}_p$-module T with continuous action of $G_F = \mathrm{Gal}(\overline{F}/F)$ and the discrete G_F-module $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. For example, we associate $T = H^i_{et}(Z \underset{F}{\otimes} \overline{F}, \mathbb{Z}_p(r))/(\text{torsion part})$ to the motive $H^i(Z, \mathbb{Z}(r))$, where Z is a proper smooth variety over F. For the discrete Galois module $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$, Bloch and Kato [1] defined the Selmer group $\mathrm{Sel}^{\mathrm{BK}}(F, A)$. Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of a number field. We want to control these Selmer groups since $\mathrm{Sel}^{\mathrm{BK}}(F_n, A)$ is expected to have fruitful arithmetic information related to M. Under the assumption that V is ordinary, crystalline and critical in the sense of [4], it is conjectured that the Selmer group $\mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)$ is Λ-cotorsion and that its Fitting ideal is related to the ideal generated by the p-adic L-function for M (This is the Main Conjecture for motives, which is a generalization of Mazur's conjecture stated above). By this philosophy of the Main Conjecture for motives, it is important to control the difference between $\mathrm{Sel}^{\mathrm{BK}}(F_n, A)$ and $\mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n}$ in order to study various arithmetic properties of $\mathrm{Sel}^{\mathrm{BK}}(F_n, A)$.

We say that a Selmer group is controlled in the cyclotomic $\mathbb{Z}_p$-extension F_∞/F if the kernel and the cokernel of the restriction map

$$\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \xrightarrow{f_n} \mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n}$$

are finite groups whose orders are bounded independently of n. We shall give a sufficient condition for this Selmer group to be controlled. The result is as follows:

Theorem A (Theorem 2.4)**.** *Let V be a p-adic representation of G_F for an odd prime number p and let Σ be a finite set of primes containing all primes of F dividing p. We assume that V is unramified outside Σ. Let T be a G_F-stable lattice of V. We denote the representation V/T by A. Let F_∞/F be a $\mathbb{Z}_p$-extension of F.*
(1) *Assume that $H^0(F_n, V) = 0$ for all n. Then the kernel of the restriction map:*

$$\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \xrightarrow{f_n} \mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n},$$

is finite and bounded independently of n.
(2) *Assume that F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension. Assume further the following conditions at each place v of F_∞ over p.*

- (i) *The p-adic representation V is ordinary at the prime of F lying under v.*
- (ii) *We have $D_{\mathrm{crys},n}(V/\mathrm{Fil}^1_v V)^{\varphi=1} = D_{\mathrm{crys},n}((\mathrm{Fil}^1_v V)^*(1))/(\varphi-1)D_{\mathrm{crys},n}((\mathrm{Fil}^1_v V)^*(1)) = 0$ for each n, where $D_{\mathrm{crys},n}(\ \ )$ means $(\ \ \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_{F_{n,v}}}$.*
- (iii) *The groups $H^0(F_{\infty,v}, (\mathrm{Fil}^1_v T)^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1))$ and $H^0(F_{\infty,v}, (T/\mathrm{Fil}^1_v T) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ are finite, where $F_{\infty,v}$ is the direct limit $\bigcup_{n\geq 0} F_{n,v}$.*

Then the cokernel of the restriction map f_n is a finite group whose order is bounded independently of n.

As an attempt to generalize classical Iwasawa theory to Iwasawa theory for motives, explicit examples primarily being the representations associated to modular forms or the symmetric powers of elliptic curves are actively studied. We apply our Theorem A to these representations. More concretely, we treat the following representations.

Let $f \in S_k(\Gamma_1(N))$ be a newform of even weight $k \geq 2$ and of level N prime to p such that p-th Fourier coefficient $a_p(f)$ is a p-adic unit. We denote by V_f the p-adic representation of $G_{\mathbb{Q}}$ of weight $1-k$ associated to f. Then $V_f(r)$ is a critical twist of V_f if and only if $2-k \leq r \leq 0$. Let E be an elliptic curve over $\mathbb{Q}$ such that E has good ordinary reduction at p. Let $T_p(E)$ be the p-Tate module of E and let $V_p(E)$ be $T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. For an odd positive integer d, we denote by V_d the d-th symmetric power $\mathrm{Sym}^d V_p(E)$. The representation V_d has the only one critical twist $V_d \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\frac{-d+1}{2})$. We take V_f or $V_d \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\frac{-d+1}{2})$ as V. Let T be a $G_{\mathbb{Q}}$-stable lattice of $V$and A the discrete $G_{\mathbb{Q}}$-module $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. We denote by $\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A)$ Bloch-Kato's Selmer group for A (see §2 for the definition of $\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A)$). As a corollary of Theorem A, we obtain the following theorem.

Theorem B (Proposition 3.1, Proposition 3.4). *Let V be $V_d(\frac{-d+1}{2})$ or $V_f(r)$ with $2-k \leq r \leq 0$. Then the restriction map $\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A) \longrightarrow \mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_\infty, A)^{\Gamma_n}$ has finite kernel and cokernel which are bounded independently of n.*

Plan. The plan of this paper is as follows. In §2, we give a sufficient condition for Bloch-Kato's Selmer groups to be controlled in the cyclotomic $\mathbb{Z}_p$-extension. In §3, we control the Selmer group for a modular form or the symmetric power of an elliptic curve as corollaries of the result of §2. In §4, we compare Bloch-Kato's Selmer groups with another important Selmer groups defined by Greenberg.

Notation. For a field K, we denote $\mathrm{Gal}(\overline{K}/K)$ by G_K where $\overline{K}$ is the separable closure of K. When k is a local field, we denote the inertia subgroup of G_k by I_k. For any commutative ring R, we denote by $R^\times$ the group of invertible elements in R. If M is a finitely generated free $\mathbb{Z}_p$-module (resp. $\mathbb{Q}_p$-module), we denote the linear dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ (resp. $\mathrm{Hom}_{\mathbb{Q}_p}(M, \mathbb{Q}_p)$) by M^*. For a $\mathbb{Z}_p$-extension F_∞/F and a place v of F_∞, we denote by $F_{n,v}$ the completion of F_n at the place of F_n which is under v and denote by $I_{n,v}$ the inertia subgroup of $G_{F_{n,v}}$. For a finite set Σ of primes of F, we denote by Σ_n the set of primes of F_n over the primes in Σ. We denote the direct limit $\bigcup_{n\geq 0} F_{n,v}$ by $F_{\infty,v}$. Note that $F_{\infty,v}$ does not mean the completion of F_∞ at the prime v. Throughout the paper, we assume that the fixed prime number p is odd.

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2. Control Theorem for Bloch-Kato's Selmer Groups

In this section, we give a sufficient condition for a p-primary Bloch-Kato's Selmer group to be controlled in a $\mathbb{Z}_p$-extension F_∞/F.

Let us fix the notations. If V(resp. T) is a p-adic representation(resp. $\mathbb{Z}_p$-adic representation) of G_F or G_{F_v}, then we denote $\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ (resp. $\mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$) by V^*(resp. T^*). For a p-adic field k, Fontaine (see [8]) defines important rings called rings of p-adic periods $B_{\mathrm{crys}} = B_{\mathrm{crys},k}, B_{\mathrm{dR}} = B_{\mathrm{dR},k}$, etc. The ring B_{crys} is a $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$-algebra equipped with a continuous G_k-action, a G_k-stable filtration and a Frobenius operator φ where $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. The ring B_{dR} is a $\overline{\mathbb{Q}}_p$-algebra equipped with a continuous G_k-action and a G_k-stable filtration and is a discrete valuation field. Using rings of p-adic periods, Fontaine also defines the filtered module $\mathrm{D}_{\mathrm{crys}}(V)$ (resp. $\mathrm{D}_{\mathrm{dR}}(V)$) by $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_k}$ (resp. $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_k}$). The module $\mathrm{D}_{\mathrm{crys}}(V)$ is a finite dimensional k_0-vector space with $\dim_{k_0} \mathrm{D}_{\mathrm{crys}}(V) \leq \dim_{\mathbb{Q}_p} V$ and is equipped with Frobenius operator φ, where k_0 is the maximal unramified subfield of k. The module $\mathrm{D}_{\mathrm{dR}}(V)$ is a finite dimensional k-vector space with $\dim_k \mathrm{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$ and is equipped with a decreasing filtration:

$$\cdots \supset \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) \supset \mathrm{Fil}^{i+1} \mathrm{D}_{\mathrm{dR}}(V) \supset \cdots,$$

such that

$$\begin{cases} \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = \mathrm{D}_{\mathrm{dR}}(V) & \text{for } i \ll 0, \\ \mathrm{Fil}^i \mathrm{D}_{\mathrm{dR}}(V) = 0 & \text{for } i \gg 0. \end{cases}$$

A p-adic representation V is called crystalline (resp. de Rham) if $\dim_{k_0} \mathrm{D}_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V$ (resp. $\dim_k \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$). We recall the local conditions for the Selmer group introduced by Bloch and Kato [1].

Definition 2.1. Let T be a $\mathbb{Z}_p$-adic representation of G_{F_v}, and let V (resp. A) be a continuous G_{F_v}-module $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$).

(1) Assume $v \nmid p$. Then $H^1_f(F_v, V)$ is defined by:

$$H^1_f(F_v, V) = \mathrm{Ker}\big[H^1(F_v, V) \longrightarrow H^1(I_{F_v}, V)\big].$$

(2) Assume $v|p$. Then $H^1_f(F_v, V)$ is defined by:

$$H^1_f(F_v, V) = \mathrm{Ker}\big[H^1(F_v, V) \longrightarrow H^1(F_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})\big].$$

Similarly we define:

$$H^1_e(F_v, V) = \mathrm{Ker}\big[H^1(F_v, V) \longrightarrow H^1(F_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}^{\varphi=1})\big],$$
$$H^1_g(F_v, V) = \mathrm{Ker}\big[H^1(F_v, V) \longrightarrow H^1(F_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})\big].$$

(We have the relation $H^1_e(F_v, V) \subset H^1_f(F_v, V) \subset H^1_g(F_v, V)$.)

(3) At a prime v of F, we define:

$$H^1_*(F_v, T) = i^{-1} H^1_*(F_v, V),$$
$$H^1_*(F_v, A) = pr H^1_*(F_v, V),$$

where $*$ is e, f, g, and i, pr are the following maps:

$$H^1(F_v, T) \xrightarrow{i} H^1(F_v, V) \xrightarrow{pr} H^1(F_v, A).$$

For a $\mathbb{Z}_p$-adic representation T of G_F, Bloch-Kato's Selmer group $\mathrm{Sel}^{\mathrm{BK}}(F, A)$ is defined as follows:

$$\mathrm{Sel}^{\mathrm{BK}}(F, A) := \mathrm{Ker}\left[H^1(F, A) \longrightarrow \prod_v \frac{H^1(F_v, A)}{H^1_f(F_v, A)}\right].$$

For a $\mathbb{Z}_p$-extension F_∞/F of F, we define the Selmer group over F_∞ to be:

$$\mathrm{Sel}^{\mathrm{BK}}(F_\infty, A) := \varinjlim_n \mathrm{Sel}^{\mathrm{BK}}(F_n, A).$$

Definition 2.2. Let V be a p-adic representation of G_k. Then V satisfies *Panchishkin condition* if:

(i) V is a de Rham representation.

(ii) There exists an exact sequence of p-adic representations of G_k:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

such that $V' \otimes \mathbb{C}_p \cong \underset{i>0}{\oplus} \mathbb{C}_p(i)^{\oplus k_i}$ and $V'' \otimes \mathbb{C}_p \cong \underset{i\leq 0}{\oplus} \mathbb{C}_p(i)^{\oplus k_i}$ as G_k-modules where $k_i \in \mathbb{Z}_{\geq 0}$.

Remark 2.3. (1) Ordinary representations in the sense of [9] satisfy the Panchishkin condition.

(2) Let V be a de Rham representation. Then for $n \gg 0$, $V(n)$ and $V(-n)$ satisfy the Panchishkin condition.

(3) We call the exact sequence which appeared in (ii) of Definition 2.2, *the Panchishkin filtration for V*. We see $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V'') = \mathrm{D}_{\mathrm{dR}}(V'')$ and $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V') = 0$. If V satisfies the Panchishkin condition, then $V^*(1)$ also satisfies the Panchishkin condition. The Panchishkin filtration for $V^*(1)$ is given by:

$$0 \longrightarrow V''^*(1) \longrightarrow V^*(1) \longrightarrow V'^*(1) \longrightarrow 0.$$

We have the following theorem:

Theorem 2.4. *Let V be a p-adic representation of G_F for an odd prime number p and let Σ be a finite set of primes containing the set Σ_p of all primes of F dividing p. We assume that V is unramified outside Σ. Let T be a G_F-stable lattice of V. We denote the representation V/T by A. Let F_∞/F be a $\mathbb{Z}_p$-extension.*

(1) *Assume that $H^0(F_n, V) = 0$ for all n. Then the kernel of the restriction map:*

$$\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \xrightarrow{f_n} \mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n},$$

is finite and bounded independently of n.

(2) *Assume that F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension. We further assume the following conditions at each place v of F_∞ over p.*

(i) *The p-adic representation V satisfies the Panchishkin condition at the prime of F under v.*

(ii) *For each n, we have*

$$D_{\mathrm{crys},n}(V'')^{\varphi=1} = D_{\mathrm{crys},n}(V'^*(1))/(\varphi-1)D_{\mathrm{crys},n}(V'^*(1)) = 0,$$

where $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is the Panchishkin filtration of V as a G_{F_v}-module and $D_{\mathrm{crys},n}(\ \ )$ means $(\ \ \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_{F_{n,v}}}$.

(iii) *Let $0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$ be the exact sequence defined by the Panchishkin filtration. Then $H^0(F_{\infty,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^0(F_{\infty,v}, T'^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1))$ are finite groups, where $F_{\infty,v}$ is the direct limit $\underset{n\geq 0}{\cup} F_{n,v}$.*

Then the cokernel of the restriction map f_n is a finite group whose order is bounded independently of n.

Remark 2.5. Assume that V is the p-adic realization of a pure motive M over F. If the motive M does not have the component with weight 0, then the assumption of Theorem 2.4 (1) is satisfied. If the motive M has no component with weight -2 nor weight 0 and M arises from a variety which has a good reduction at all primes above p, then the condition (ii) of Theorem 2.4 (2) is satisfied due to [13] and the crystalline conjecture proved by Faltings, Tsuji and Niziol (see [5], [20] and [15]).

Remark 2.6. Let $T := T_p(E)$ be the p-Tate module of an elliptic curve E over a number field F which has good ordinary reduction at all places of F over p, and let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension of F. Then Theorem 2.4 recovers Mazur's result mentioned in §1, since $\mathrm{Sel}^{\mathrm{BK}}(F_n, A)$ coincides with the classical p-primary Selmer group for the elliptic curve E over F_n by [1, Example 3.11].

Before proving the theorem, we prepare some lemmas.

Lemma 2.7. *Let A be a discrete G_F-module and let Σ a finite set of primes of F which contains all primes above p. Assume that A is unramified outside Σ. Let F_n be the n-th layer of F_∞/F. Then we have:*

$$H^1(F_\Sigma/F_n, A) = \mathrm{Ker}\big[H^1(F_n, A) \xrightarrow{\alpha_n} \prod_{v \notin \Sigma_n} H^1(I_{n,v}, A)\big]$$

as a submodule of $H^1(F_n, A)$, where F_Σ is the maximal extension of F which is unramified outside Σ, the set Σ_n is the primes of F_n above the primes in Σ of F and $I_{n,v}$ is the inertia subgroup of $G_{F_{n,v}}$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(F_\Sigma/F_n, A) & \longrightarrow & H^1(F_n, A) & \xrightarrow{\mathrm{res}_{F_\Sigma}} & H^1(F_\Sigma, A) \\ & & & & {\scriptstyle\alpha_n}\big\downarrow & & \big\downarrow{\scriptstyle\gamma_n} \\ & & & & \prod_{v \notin \Sigma_n} H^1(I_{n,v}, A) & \xrightarrow{\beta_n} & \prod_{\overline{v} \notin \overline{\Sigma}} H^1(I_{F_{\Sigma,\overline{v}}}, A), \end{array}$$

where $\overline{\Sigma}$ is the primes of F_Σ above the primes in Σ of F. The map β_n is easily seen to be injective since F_Σ/F is unramified outside Σ. Since $\mathrm{Gal}(\overline{F}/F_\Sigma)$ acts trivially on A, we see that $H^1(\overline{F}/F_\Sigma, A) = \mathrm{Hom}(\mathrm{Gal}(\overline{F}/F_\Sigma), A)$. By the same reason, we have $H^1(I_{F_{\Sigma,\overline{v}}}, A) = \mathrm{Hom}(I_{F_{\Sigma,\overline{v}}}, A)$. Thus γ_n must be injective by the maximality of F_Σ. We see that $H^1(F_\Sigma/F_n, A) = \mathrm{Ker}(\alpha_n)$ by easy diagram chasing. □

Lemma 2.8. *Let v be a place of F_∞ which does not divide p. Then the kernel of the restriction map: $\dfrac{H^1(F_{n,v}, A)}{H^1_f(F_{n,v}, A)} \longrightarrow \dfrac{H^1(F_{\infty,v}, A)}{H^1_f(F_{\infty,v}, A)}$ is a finite group whose order is bounded independently of n, where $H^1_f(F_{\infty,v}, A)$ is the direct limit $\varinjlim_m H^1_f(F_{m,v}, A)$.*

Proof. Let us denote by $H^1_{\mathrm{ur}}(F_{n,v},A)$ the module $\mathrm{Ker}[H^1(F_{n,v},A)\longrightarrow H^1(I_{n,v},A)]$. The map $\dfrac{H^1(F_{n,v},A)}{H^1_{\mathrm{ur}}(F_{n,v},A)}\longrightarrow\dfrac{H^1(F_{\infty,v},A)}{H^1_{\mathrm{ur}}(F_{\infty,v},A)}$ is injective since $F_{\infty,v}/F_{n,v}$ is unramified for $v\nmid p$. Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & H^1_f(F_{n,v},V) \arrow[r, "pr'"] \arrow[d] & H^1_{\mathrm{ur}}(F_{n,v},A)_{\mathrm{div}} \arrow[d] & \\
0 \arrow[r] & H^1(F_{n,v},T)/\mathrm{tor} \arrow[r] \arrow[d] & H^1(F_{n,v},V) \arrow[r, "pr"] \arrow[d] & H^1(F_{n,v},A)_{\mathrm{div}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(I_{n,v},T)/\mathrm{tor} \arrow[r] & H^1(I_{n,v},V) \arrow[r] & H^1(I_{n,v},A)_{\mathrm{div}} \arrow[r] & 0.
\end{tikzcd}$$

Since v does not divide p, the group $H^1(I_{n,v},T)/(\mathrm{tor})$ is finitely generated over $\mathbb{Z}_p$. Therefore the group $H^1_f(F_{n,v},A)$ coincides with the maximal divisible part $H^1_{\mathrm{ur}}(F_{n,v},A)_{\mathrm{div}}$ of $H^1_{\mathrm{ur}}(F_{n,v},A)$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \dfrac{H^1_{\mathrm{ur}}(F_{n,v},A)}{H^1_{\mathrm{ur}}(F_{n,v},A)_{\mathrm{div}}} \arrow[r] \arrow[d] & \dfrac{H^1(F_{n,v},A)}{H^1_f(F_{n,v},A)} \arrow[r] \arrow[d] & \dfrac{H^1(F_{n,v},A)}{H^1_{\mathrm{ur}}(F_{n,v},A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \dfrac{H^1_{\mathrm{ur}}(F_{\infty,v},A)}{H^1_{\mathrm{ur}}(F_{\infty,v},A)_{\mathrm{div}}} \arrow[r] & \dfrac{H^1(F_{\infty,v},A)}{H^1_f(F_{\infty,v},A)} \arrow[r] & \dfrac{H^1(F_{\infty,v},A)}{H^1_{\mathrm{ur}}(F_{\infty,v},A)} \arrow[r] & 0.
\end{tikzcd}$$

Thus it suffices to bound the cotorsion part of $H^1_{\mathrm{ur}}(F_{n,v},A)=H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},A^{I_{n,v}})$ independently of n. We note that $A^{I_{n,v}}$ does not depend on n since primes outside p are unramified for any $\mathbb{Z}_p$-extension of a number field. Thus we denote $A^{I_{n,v}}$ by $\overline{A}$. We have the exact sequence:

$$0\longrightarrow\overline{A}_{\mathrm{div}}\longrightarrow\overline{A}\longrightarrow\overline{A}_{\mathrm{fin}}\longrightarrow 0,$$

where $\overline{A}_{\mathrm{div}}$ (resp. $\overline{A}_{\mathrm{fin}}$) is the maximal divisible subgroup (resp. the largest cotorsion quotient) of $\overline{A}$. By taking cohomology of this exact sequence, we have:

$$H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},\overline{A}_{\mathrm{div}})\longrightarrow H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},\overline{A})\longrightarrow H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},\overline{A}_{\mathrm{fin}}).$$

The image of $H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},\overline{A}_{\mathrm{div}})$ is a cofree $\mathbb{Z}_p$-module. Hence the order of the cotorsion part of $H^1_{\mathrm{ur}}(F_{n,v},A)=H^1(F^{\mathrm{ur}}_{n,v}/F_{n,v},\overline{A})$ is bounded by the order of $\overline{A}_{\mathrm{fin}}$ because the cotorsion part is a subgroup of the $\mathrm{Gal}(F^{\mathrm{ur}}_{n,v}/F_{n,v})$-coinvariant of $\overline{A}_{\mathrm{fin}}$. □

Lemma 2.9. *Let V be a p-adic representation of G_F which satisfies Panchishkin condition. Let T be a G_F-stable lattice of V. We denote the Panchishkin filtration of T by*

$$0\longrightarrow T'\longrightarrow T\longrightarrow T''\longrightarrow 0.$$

Assume that $H^0(F_{n,v},V^*(1))=0$ for each $n\geq 0$ and each $v\in\Sigma_{p,n}$. Then the following statements hold for each $v\in\Sigma_{p,n}$:

(a) *We have an exact sequence:*

$$0 \longrightarrow H^1_f(F_{n,v}, T''^*(1)) \longrightarrow H^1_f(F_{n,v}, T^*(1)) \longrightarrow H^1_f(F_{n,v}, T'^*(1)).$$

(b) *Assume that* $D_{\mathrm{crys},n}(V'^*(1))/(\varphi - 1)D_{\mathrm{crys},n}(V'^*(1))$ *is zero. Then, we have:*

$$H^1_f(F_{n,v}, T'^*(1)) = H^0(F_{n,v}, T'^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1)).$$

Proof. First, we note that

$$H^1_f(F_{n,v}, T) = \mathrm{Ker}[H^1(F_{n,v}, T) \longrightarrow H^1(F_{n,v}, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})]$$

for any torsion free $\mathbb{Z}_p$-adic representation T of $G_{F_{n,v}}$. By the assumption of the lemma, we have:

$$H^0\big(F_{n,v}, T'^*(1)\big) = H^0\big(F_{n,v}, V'^*(1)\big) = 0$$

for each n. Then we have the following diagram:

$$\begin{array}{ccccccc} 0 & \to & H^1(F_{n,v}, T''^*(1)) & \to & H^1(F_{n,v}, T^*(1)) & \to & H^1(F_{n,v}, T'^*(1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (V'^*(1) \otimes B_{\mathrm{crys}})^{G_{F_{n,v}}} & \to & H^1(V''^*(1) \otimes B_{\mathrm{crys}}) & \to & H^1(V^*(1) \otimes B_{\mathrm{crys}}) & \to & H^1(V'^*(1) \otimes B_{\mathrm{crys}}), \end{array}$$

where $H^1(\ \ )$ is defined to be $H^1(F_{n,v},\ \ )$. Since $V^*(1)$ is a de Rham representation, the map $(V'^*(1) \otimes B_{\mathrm{crys}})^{G_{F_{n,v}}} \longrightarrow H^1(V''^*(1) \otimes B_{\mathrm{crys}})$ in the bottom row is a zero map. Hence (a) follows from the above diagram.

Next, we shall prove (b). We take the long exact sequence of continuous Galois cohomology of the short exact sequence:

$$0 \longrightarrow T'^*(1) \longrightarrow V'^*(1) \longrightarrow T'^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1) \longrightarrow 0.$$

Then we have the following exact sequence:

$$H^0(F_{n,v}, V'^*(1)) \xrightarrow{\gamma_n} H^0(F_{n,v}, T'^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow H^1_f(F_{n,v}, T'^*(1)) \xrightarrow{\delta_n} H^1_f(F_{n,v}, V'^*(1)).$$

Note that the cokernel of γ_n is isomorphic to $H^1(F_{n,v}, T'^*(1))_{\mathrm{tor}}$ by [19, Proposition 2.3] and that the kernel of δ_n is isomorphic to $H^1(F_{n,v}, T'^*(1))_{\mathrm{tor}}$ since $H^1_f(F_{n,v}, T'^*(1))$ is the pull-back of $H^1_f(F_{n,v}, V'^*(1))$ via the natural map $H^1_f(F_{n,v}, T'^*(1)) \xrightarrow{i} H^1_f(F_{n,v}, V'^*(1))$. Since we have $H^0(F_{n,v}, V'^*(1)) = 0$ by the assumption of the lemma, we have only to show that $H^1_f(F_{n,v}, V'^*(1)) = 0$. Since $V'^*(1)$ is a de Rham representation, we have:

$$H^1_f(F_{n,v}, V'^*(1))/H^1_e(F_{n,v}, V'^*(1)) \cong \mathrm{D}_{\mathrm{crys},n}(V'^*(1))/(\varphi - 1)\mathrm{D}_{\mathrm{crys},n}(V'^*(1))$$

due to [1, Corollary 3.8.4]. As for the right hand side term, we have

$$\mathrm{D}_{\mathrm{crys},n}(V'^*(1))/(\varphi - 1)\mathrm{D}_{\mathrm{crys},n}(V'^*(1)) = 0$$

by the assumption of the lemma. Hence, by [1, Corollary 3.8.4], we have the following equality:

$$H^1_f\big(F_{n,v}, V'^*(1)\big) = H^1_e\big(F_{n,v}, V'^*(1)\big) \cong \mathrm{D}_{\mathrm{dR}}(V'^*(1))/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V'^*(1)).$$

The right hand side of this equality is 0 by Remark 2.3 (3). So we have $H^1_f(F_{n,v}, V'^*(1)) = 0$. Thus (b) follows. □

Lemma 2.10. *Let $v \in \Sigma_{p,n}$ be a prime of F_n dividing p. Let V'' be a de Rham p-adic representation of $G_{F_{n,v}}$ such that $V'' \otimes \mathbb{C}_p \cong \bigoplus_{i \leq 0} \mathbb{C}_p(i)^{\oplus k_i}$ as $G_{F_{n,v}}$-module and let T'' be a $G_{F_{n,v}}$-stable lattice of V''. Assume that $\mathrm{D}_{\mathrm{crys},n}(V'')^{\varphi=1} = 0$ for each $n \geq 0$. Then we have $H^1_f\big(F_{n,v}, T''^*(1)\big) = H^1\big(F_{n,v}, T''^*(1)\big)$ for each $n \geq 0$.*

Proof. Since $V''^*(1)$ is a de Rham representation, we have:

$$\dim_{\mathbb{Q}_p} H^1_g(F_{n,v}, V''^*(1)) - \dim_{\mathbb{Q}_p} H^1_f(F_{n,v}, V''^*(1)) = \dim_{\mathbb{Q}_p} \mathrm{D}_{\mathrm{crys},n}(V'')^{\varphi=1}.$$

We have $\mathrm{D}_{\mathrm{crys},n}(V'')^{\varphi=1} = 0$ by assumption. Thus

$$\begin{aligned} H^1_g(F_{n,v}, V''^*(1)) &= H^1_f(F_{n,v}, V''^*(1)), \\ \dim_{\mathbb{Q}_p} H^1(F_{n,v}, V''^*(1)) - \dim_{\mathbb{Q}_p} H^1_g(F_{n,v}, V''^*(1)) &= \dim_{\mathbb{Q}_p} H^1_e(F_{n,v}, V'') \\ &= \dim_{\mathbb{Q}_p} (\mathrm{D}_{\mathrm{dR}}(V'')/\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V'')) = 0. \end{aligned}$$

□

Let us return to the proof of Theorem 2.4.

Proof of Theorem 2.4. Let us prove (1). We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sel}^{\mathrm{BK}}(F_n, A) & \xrightarrow{f_n} & \mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n} \\ \bigcap & & \bigcap \\ H^1(F_n, A) & \longrightarrow & H^1(F_\infty, A)^{\Gamma_n}. \end{array}$$

In order to bound $\mathrm{Ker}(f_n)$, it suffices to bound the group:

$$(A^{G_{F_\infty}})_{\Gamma_n} = \mathrm{Ker}\big[H^1(F_n, A) \longrightarrow H^1(F_\infty, A)^{\Gamma_n}\big].$$

Since $A^{G_{F_\infty}}$ is of cofinite type over $\mathbb{Z}_p$, we have the following exact sequence:

$$0 \longrightarrow D \longrightarrow A^{G_{F_\infty}} \longrightarrow E \longrightarrow 0,$$

where D is the maximal divisible subgroup of $A^{G_{F_\infty}}$ and E is the largest finite quotient of $A^{G_{F_\infty}}$. Then it suffices to bound D_{Γ_n} and E_{Γ_n}. For each n, E_{Γ_n} is a finite group whose order is bounded by the order of E. Especially, the order of E_{Γ_n} is bounded independently of n. Assume that $D \xrightarrow{1-\gamma_n} D$ is not surjective. Then $D_{\Gamma_n} = \mathrm{Coker}(1-\gamma_n)$ is infinite since D is divisible. Especially, $\mathrm{corank}_{\mathbb{Z}_p} D_{\Gamma_n} = \mathrm{corank}_{\mathbb{Z}_p} D^{\Gamma_n}$ is not zero. Since D^{Γ_n} is contained in $A^{G_{F_n}}$, this implies that $A^{G_{F_n}}$ is infinite and that $H^0(F_n, V) \neq 0$. However, $H^0(F_n, V)$ is zero by the assumption. This is contradiction and hence D_{Γ_n} must be zero. Thus the proof of (1) is completed.

Next, let us prove (2). By Lemma 2.7, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Sel}^{\mathrm{BK}}(F_n, A) & \longrightarrow & H^1(F_\Sigma/F_n, A) & \xrightarrow{\gamma} & \displaystyle\prod_{v \in \Sigma_n} \frac{H^1(F_{n,v}, A)}{H^1_f(F_{n,v}, A)} \\ & & \Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle g_n} & & \Big\downarrow{\scriptstyle h_n} \\ 0 & \longrightarrow & \mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)^{\Gamma_n} & \longrightarrow & H^1(F_\Sigma/F_\infty, A)^{\Gamma_n} & \longrightarrow & \left(\displaystyle\prod_{v \in \Sigma_\infty} \frac{H^1(F_{\infty,v}, A)}{H^1_f(F_{\infty,v}, A)}\right)^{\Gamma_n}, \end{array}$$

where $H^1_f(F_{\infty,v}, A) = \varinjlim_n H^1_f(F_{n,v}, A)$. By the Hochschild-Serre spectral sequence, the cokernel of g_n is contained in $H^2(F_\infty/F_n, A^{G_{F_\infty}})$. Since the p-cohomological dimension of $\mathrm{Gal}(F_\infty/F_n)$ is 1, g_n must be surjective. Consequently, the module $\mathrm{Coker}(f_n)$ is a sub-quotient of $\mathrm{Ker}(h_n)$ by the snake lemma. Thus we have only to bound the kernel of h_n. We bound the kernel of the map

$$h_{n,v}: \frac{H^1(F_{n,v}, A)}{H^1_f(F_{n,v}, A)} \longrightarrow \left(\frac{H^1(F_{\infty,v}, A)}{H^1_f(F_{\infty,v}, A)}\right)^{\Gamma_n}$$

for each $v \in \Sigma_n$.

As for $v \nmid p$, the kernel of the map $h_{n,v}$ is bounded by Lemma 2.8.

Let us consider the case $v \in \Sigma_{p,\infty}$. By Proposition 3.8 of [1], the kernel of h_n is Pontrjagin dual to the cokernel of the corestriction map:

$$\prod_{v\in\Sigma} \big(\varprojlim_m H^1_f(F_{m,v}, T^*(1))\big) \xrightarrow{b_n} \prod_{v\in\Sigma} H^1_f(F_{n,v}, T^*(1)).$$

Hence it suffices to bound the cokernel of the above corestriction map. Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \varprojlim_m H^1_f\big(F_{m,v}, T''^*(1)\big) & \longrightarrow & \varprojlim_m H^1_f\big(F_{m,v}, T^*(1)\big) & \longrightarrow & \varprojlim_m H^1_f\big(F_{m,v}, T'^*(1)\big) \\
& {\scriptstyle a_n}\big\downarrow & & {\scriptstyle b_n}\big\downarrow & & {\scriptstyle c_n}\big\downarrow \\
0 \longrightarrow & H^1_f\big(F_{n,v}, T''^*(1)\big) & \longrightarrow & H^1_f\big(F_{n,v}, T^*(1)\big) & \longrightarrow & H^1_f\big(F_{n,v}, T'^*(1)\big).
\end{array}$$

Since we have $H^1_f\big(F_{n,v}, T'^*(1)\big) = H^0\big(F_{n,v}, T'^* \otimes \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$ by Lemma 2.9 (b), the cokernel of c_n is a finite group whose order is bounded by the assumption (iii). Hence it suffices to bound the cokernel of a_n. By Lemma 2.10 and the assumption (ii) of the Theorem 2.4, we have:

$$\mathrm{Coker}(a_n) \cong \mathrm{Coker}\big[\varprojlim_m H^1(F_{m,v}, T''^*(1)) \longrightarrow H^1(F_{n,v}, T''^*(1))\big].$$

This is Pontrjagin dual to

$$\begin{aligned}
&\mathrm{Ker}\big[H^1(F_{n,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \varinjlim_m H^1(F_{m,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p)\big] \\
&= \mathrm{Ker}\big[H^1(F_{n,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow H^1(F_{\infty,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p)\big] \\
&= H^1\big(\Gamma_n, H^0(F_{\infty,v}, T'' \otimes \mathbb{Q}_p/\mathbb{Z}_p)\big).
\end{aligned}$$

But the last group is finite and bounded independently of n by the assumption (iii) of the theorem. □

3. Applications of Theorem 2.4

In the last section, we obtained a sufficient condition (Theorem 2.4) for Bloch-Kato's Selmer groups to be controlled in the cyclotomic $\mathbb{Z}_p$-extension. In this section, we apply Theorem 2.4 to the Selmer group associated to a modular form or the symmetric power of an elliptic curve. Let $f \in S_k(\Gamma_1(N))$ be a newform of weight $k \geq 2$ and of level N prime to p such that p-th Fourier coefficient $a_p(f)$ is a $\wp$-adic unit for a fixed prime $\wp$

of $\mathbb{Q}_f = \mathbb{Q}(\{a_n(f)\})$ over p. We denote by V_f the $\wp$-adic representation of $G_{\mathbb{Q}}$ of weight $1-k$ associated to f. Note that this is the linear dual of the original representation constructed by Deligne [3]. For a motive M over $\mathbb{Q}$, the gamma factor $\Gamma_M(s)$ is defined as follows:

$$\Gamma_M(s) = \prod_{i<j} \Gamma(s-i)^{h^{i,j}} \cdot \prod_k \Gamma(\tfrac{s-k}{2})^{h^{k,k(-1)^k}} \cdot \prod_k \Gamma(\tfrac{s-k+1}{2})^{h^{k,k(-1)^{k-1}}},$$

where $h^{i,j}$ is defined to be $\mathbb{C}$-dimension of the Hodge realization $H^{i,j}(M)$ and $h^{k,k(-1)^k}$ (resp. $h^{k,k(-1)^{k-1}}$) is the dimension of $(-1)^k$-eigenspace (resp. $(-1)^{k-1}$-eigenspace) for the complex conjugate on $H^{k,k}(M)$. A motive M over $\mathbb{Q}$ is called *critical* if both $\Gamma_M(s)$ and $\Gamma_{M^*(1)}(s)$ do not have a pole at $s=0$, where the motive $M^*(1)$ is the Kummer dual of M. The p-adic realization $V_p(M)$ of M is called *critical* if M is critical. For Iwasawa theory of p-adic representations, it is important to consider the critical Tate twist (see [9] and [10]). It is known that $V_f(r)$ is critical if and only if $2-k \le r \le 0$ for the above V_f. We have the following theorem by applying Theorem 2.4.

Proposition 3.1. *Let T be a lattice of the representation $V_f(r)$ with $2-k \le r \le 0$. Let A be $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Then the restriction map $\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A) \xrightarrow{f_n} \mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_\infty, A)^{\Gamma_n}$ has finite kernel and cokernel whose orders are bounded independently of n.*

In order to prove Proposition 3.1, we need the following proposition, which was originally proved by Deligne in his letter to Serre. Later, Mazur and Wiles gave another proof by using Hida theoretic method. Since the original proof by Deligne is unpublished, we refer to [22, Theorem 2.1.4] and [11].

Proposition 3.2 (Deligne, Mazur-Wiles)**.** *Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalized newform of weight $k \ge 2$ for $\Gamma_1(N)$. Let ψ is the Nebentypus character for f. Assume that a_p is a $\wp$-unit. Then the representation*

$$\rho_f : \quad G_{\mathbb{Q}} \longrightarrow GL(V_f) = GL_2(\mathbb{Q}_{f,\wp})$$

restricted to the decomposition group D_p at p is equivalent to the representation of the form:

$$\begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix},$$

where ϵ_2 is the unramified character such that $\epsilon_2(\mathrm{Frob}_p) = \alpha$ where α is the p-unit root of $x^2 - a_p x + \psi(p)p^{k-1}$ and Frob_p is the p-th power arithmetic Frobenius.

Proof of Proposition3.1. In order to prove the assertion for the kernel of f_n, we have only to show that $H^0(\mathbb{Q}_n, V_f(r)) = 0$ for each n according to Theorem 2.4 (1).

First, consider the case where the weight of the Galois representation $V_f(r)$ is not zero, that is, we consider the case where $1-k-2r$ is not zero. Then we see that $H^0(\mathbb{Q}_n, V_f(r)) = 0$ by considering the action of the Frobenius Frob_l for $l \not| Np$.

Next, consider the case where f is not of CM-type (We do not give the definition for a cusp form f to be of CM-type, but we refer [16] for the definition). If f is not of CM-type, the image of $G_{\mathbb{Q}}$ in $\mathrm{Aut}(V_f(r))$ is an open subgroup of $\mathrm{Aut}(V_f(r))$ (cf. [16], [17]). Since $G_{\mathbb{Q}_n}$ is open in $G_{\mathbb{Q}}$, the image of $G_{\mathbb{Q}_n}$ is also an open subgroup of $\mathrm{Aut}(V_f(r))$.

Hence, even in the case where the Galois representation $V_f(r)$ has weight zero, we see that $H^0(\mathbb{Q}_n, V_f(r)) = 0$.

Now the case left for us is $V_f(r)$ such that f is of CM-type and $1-k-2r=0$. In this case, there exists a quadratic imaginary extension K of $\mathbb{Q}$ with the property that for any open subgroup H of $G_{\mathbb{Q}}$, the action of H on $V_f(r)$ is irreducible if and only if $H \not\subset G_K$ (cf. [16, Proposition 4.4]). Now since we are assuming that p is odd, $G_{\mathbb{Q}_n}$ is not contained in G_K for any n. This completes the proof of the assertion for $\mathrm{Ker}(f_n)$.

In order to prove the boundedness of the cokernel, it suffices to check the conditions (i)-(iii) of Theorem 2.4 (2). The condition (i) is a consequence of Proposition 3.2 for all $k \geq 2$. We have

$$\det\bigl(x-\varphi^{-1}; \mathrm{D}_{\mathrm{crys}}(V_f)\bigr) = x^2 - a_p x + \psi(p)p^{k-1}$$

due to [13] and [18]. By the assumption that a_p is a $\wp$-adic unit, we can not have $x = \zeta_{p^n} \cdot p^{(k-1)/2}$ for a root of the equation $x^2 - a_p x + \psi(p)p^{k-1}$. Hence we have $\mathrm{D}_{\mathrm{crys},n}(V")^{\varphi=1} = \mathrm{D}_{\mathrm{crys},n}(V'^*(1))/(\varphi-1)\mathrm{D}_{\mathrm{crys},n}(V'^*(1)) = 0$. This completes the proof of (ii). Since V is ordinary at p, V has a filtration:

$$\cdots \supset \mathrm{Fil}_p^i V \supset \mathrm{Fil}_p^{i+1} V \supset \cdots$$

as a $G_{\mathbb{Q}_p}$-module. Hence we have the Panchishkin filtration:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

by putting $T' = \mathrm{Fil}_p^1 V \cap T$. $H^0(\mathbb{Q}_\infty, T''\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $H^0(\mathbb{Q}_\infty, T'^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p(1))$) is contained in $H^0(\mathbb{Q}_p(\zeta_{p^\infty}), T''\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $H^0(\mathbb{Q}_p(\zeta_{p^\infty}), T'^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p(1))$). We see that $G_{\mathbb{Q}_p(\zeta_{p^\infty})}/I_{\mathbb{Q}_p(\zeta_{p^\infty})}$ acts non trivially on $T''\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$ (resp. $T'^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p(1)$) due to Proposition 3.2. Hence $H^0(\mathbb{Q}_p(\zeta_{p^\infty}), T''\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$ (resp. $H^0(\mathbb{Q}_p(\zeta_{p^\infty}), T'^*\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p(1))$) is finite. Thus (iii) follows. $\square$

Besides the representation associated to a modular form, the symmetric power of the representation of an ordinary elliptic curve has been also studied actively as an attempt to generalize Iwasawa theory. We can also apply Theorem 2.4 to the symmetric product of the representation of a good ordinary elliptic curve. Let E be an elliptic curve over $\mathbb{Q}$ such that E has good ordinary reduction at p. Let $T_p(E)$ be the p-Tate module of E. We define $V_p(E)$ (resp. $A_p(E)$) by $V_p(E) := T_p(E)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ (resp. $A_p(E) := T_p(E)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$). For $d \geq 1$, we denote the d-th symmetric power of $V_p(E)$ by V_d. In the case of symmetric power of an elliptic curve, the critical twists are as follows:

Lemma 3.3 ([2], Lemma 2.1.1)**.** *Let E be an elliptic curve over $\mathbb{Q}$. Then*

(1) *If d is an odd positive integer, then $V_d(\frac{-d+1}{2})$ is the only critical twist of V_d.*
(2) *If d is an even positive integer such that $\frac{d}{2}$ is odd, then $V_d(\frac{-d}{2})$ and $V_d(\frac{-d}{2}+1)$ are the only critical twists of V_d.*
(3) *If d is an even positive integer such that $\frac{d}{2}$ is even, then V_d has no critical twist.*

In the case 1 of Lemma 3.3, we have the following result by applying Theorem 2.4 (We can not apply Theorem 2.4 to the case 2. see Remark 3.5).

Proposition 3.4. *Let E be an elliptic curve over $\mathbb{Q}$ which has good ordinary reduction at p. We denote by V the d-th symmetric power $\mathrm{Sym}^d V_p(E)(\frac{-d+1}{2})$ where d is an odd positive integer. Then the restriction map*

$$\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A) \xrightarrow{f_n} \mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_\infty, A)^{\Gamma_n}$$

has finite kernel and cokernel which are bounded independently of n.

Proof of Proposition 3.4. We have $H^0(\mathbb{Q}_n, V)$ is zero for all n since V has weight -1. Thus the boundedness of the kernels follows from Theorem 2.4 (1).

Let us prove the boundedness of the cokernel. Let α be the p-unit root of the equation $x^2 - a_p x + p$, where a_p is $p + 1 - \sharp E(\mathbb{F}_p)$. The filtration on $V_d(\frac{-d}{2} + 1)$ as a $G_{\mathbb{Q}_p}$-module is given as follows:

$$(\mathrm{Fil}_p^{\frac{-d+1}{2}} V/\mathrm{Fil}_p^{\frac{-d+1}{2}} V) \otimes \mathbb{Q}_p(\frac{d-1}{2}) \cong \mathbb{Q}_p(\alpha^{-d})$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(\mathrm{Fil}_p^{\frac{-d+1}{2}+i} V/\mathrm{Fil}_p^{\frac{-d+1}{2}+i+1} V) \otimes \mathbb{Q}_p(\frac{d-1}{2} - i) \cong \mathbb{Q}_p(\alpha^{-d+2i})$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(\mathrm{Fil}_p^{\frac{d+1}{2}} V/\mathrm{Fil}_p^{\frac{d+3}{2}} V) \otimes \mathbb{Q}_p(\frac{-d+1}{2}) \cong \mathbb{Q}_p(\alpha^{d}),$$

where $\mathbb{Q}_p(\alpha^j)$ is a one dimensional $\mathbb{Q}_p$-vector space with $G_{\mathbb{Q}_p}$-action on which $G_{\mathbb{Q}_p}$ is unramified and the arithmetic Frobenius $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$ has the eigenvalue α^j. We check the conditions (i)-(iii) of Theorem 2.4 (2). $T_p(E)$ is a crystalline representation since E has good reduction at p. Since V is an ordinary representation, V satisfies the Panchishkin condition. Hence the condition (i) is satisfied. By [13], we see that the roots of $\det(x - \varphi^{-1}; D_{\mathrm{crys},n}(V))$ have the complex absolute value $p^{1/2}$. We see that $V = V^*(1)$ by the Weil pairing for E. Thus the condition (ii) is satisfied. The Panchishkin filtration for T:

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

is given by $T' = \mathrm{Fil}_p^1 T$, $T'' = T/\mathrm{Fil}_p^1 T$. Then we can check the condition (iii) by the same argument as the proof of Proposition 3.1 □

Remark 3.5. Our method can not treat the case where d is even positive number. Recent results of Hida [12] treats these cases by a complete different method from ours.

4. Comparison of Selmer Groups

In this section, we compare Bloch-Kato's Selmer group and Greenberg's Selmer group. The comparison of these two Selmer groups over a finite number field F is already done by Flach ([6] and [7]). Our subject here is the comparison of these two Selmer groups over a n-th layer F_n of a $\mathbb{Z}_p$-extension F_∞/F_n when n varies.

Let V be a p-adic representation of G_F which satisfies the Panchishkin condition at each prime v of F over p. We fix a G_F-stable lattice T of V and we denote V/T by A.

Greenberg's Selmer group is simply defined as follows:

$$\mathrm{Sel}^{\mathrm{Gr}}(F_n, A) = \mathrm{Ker}[H^1(F_\Sigma/F_n, A) \longrightarrow \prod_{v \in \Sigma_{p,n}} H^1(I_{n,v}, A/\mathrm{F}_v^+ A) \oplus \prod_{v \in \Sigma_n \setminus \Sigma_{p,n}} H^1(I_{n,v}, A)],$$

where $\mathrm{F}_v^+ A \subset A$ is a filtration as a $G_{F_{n,v}}$-module which is defined by Panchishkin filtration (see Remark 2.3) and Σ_n (resp. $\Sigma_{p,n}$) is the primes of F_n above Σ (resp. Σ_p). The following result is shown by Flach in his papers [6] and [7].

Proposition 4.1 (Flach)**.** *Let V be a p-adic representation of G_F which is unramified outside Σ. Assume that V satisfies Panchishkin condition at each prime v of F over p. Let T be a G_F-stable lattice of V. Then we have the following:*

(1) *There exists a natural injection*

$$\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \hookrightarrow \mathrm{Sel}^{\mathrm{Gr}}(F_n, A).$$

for each $n \geq 0$. Further, the $\mathbb{Z}_p$-corank of the cokernel of the above map is bounded by

$$\sum_{v \in \Sigma_{p,n}} \big[\dim_{\mathbb{Q}_p} H^1_g(F_{n,v}, V) - \dim_{\mathbb{Q}_p} H^1_f(F_{n,v}, V)\big].$$

Especially the cokernel of the above restriction map is finite when $H^1_f(F_v, V)$ coincides with $H^1_g(F_v, V)$ for each prime $v \in \Sigma_{p,n}$ of F_n.

(2) *Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension. Then the $\mathbb{Z}_p$-corank of the cokernel of*

$$\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \hookrightarrow \mathrm{Sel}^{\mathrm{Gr}}(F_n, A)$$

is bounded.

His proof is written only for ordinary representations and he does not treat general p-adic representations with Panchishkin condition. But, the same proof works for representations satisfying the Panchishkin condition. The statement 1 is proved in [7, Theorem 3]. The statement 2 is proved in [6, Lemma 2.9].

In order to control the Selmer group, we need to bound the finite error terms. Our result is as follows:

Proposition 4.2. *Let V be a p-adic representation of G_F which is unramified outside Σ. Assume that V satisfies Panchishkin condition at each prime v of F over p. Let T be a G_F-stable lattice of V. Let F_∞/F be the cyclotomic $\mathbb{Z}_p$-extension. Assume further the following conditions:*

(a) *For each $n \geq 0$ and each prime $v \in \Sigma_{p,n}$ of F_n, $H^1_f(F_{n,v}, V)$ is equal to $H^1_g(F_{n,v}, V)$.*

(b) *For each $v \in \Sigma_{p,\infty}$, $H^0\big(F_{\infty,v}, (\mathrm{F}_v^+ T)^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$, $H^0\big(F_{\infty,v}, T^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$ and $H^0(F_{\infty,v}, A/\mathrm{F}_v^+ A)$ are finite, where $0 \longrightarrow \mathrm{F}_v^+ T \longrightarrow T \longrightarrow T/\mathrm{F}_v^+ T \longrightarrow 0$ is the Panchishkin filtration of T as a G_{F_v}-module.*

Then the order of the cokernel of the injection $\mathrm{Sel}^{\mathrm{BK}}(F_n, A) \hookrightarrow \mathrm{Sel}^{\mathrm{Gr}}(F_n, A)$ is finite and bounded independently of n. Especially, $\mathrm{Sel}^{\mathrm{BK}}(F_\infty, A)$ is a finite index subgroup of $\mathrm{Sel}^{\mathrm{Gr}}(F_\infty, A)$.

Proof. For the proof, we have only to compare the local condition at each prime $v \in \Sigma_n$ of F_n which appears in the definition of these two Selmer groups.

First, we consider the case $v \nmid p$. For a prime $v \in \Sigma_n \setminus \Sigma_{p,n}$ of F_n, the local condition of Greenberg's Selmer group at v is given by the module $H^1_{\mathrm{ur}}(F_{n,v}, A)$ defined in the proof of Lemma 2.8. The group $H^1_f(F_{n,v}, A)$ is the maximal divisible subgroup of $H^1_{\mathrm{ur}}(F_{n,v}, A)$ and the cotorsion part of $H^1_{\mathrm{ur}}(F_{n,v}, A)$ is a finite group whose order is bounded independently of n as shown in the proof of Lemma 2.8.

Next, we consider the case $v \in \Sigma_{p,n}$. In this case, it is shown by Flach that $H^1_g(F_{n,v}, V)$ coincides with $\mathrm{Ker}[H^1(F_{n,v}, V) \longrightarrow H^1(I_{n,v}, V/\mathrm{F}^+_v V)]$. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^1_g(F_{n,v}, V) & \overset{p_{n,*}}{\to} & H^1_{\mathrm{Gr}}(F_{n,v}, A) & & \\
 & & \downarrow & & d_n\downarrow & & \\
\dfrac{H^1(F_{n,v}, T)}{H^1(F_{n,v}, T)_{\mathrm{tor}}} & \hookrightarrow & H^1(F_{n,v}, V) & \to & H^1(F_{n,v}, A) & \to H^2(F_{n,v}, T)_{\mathrm{tor}} \\
a_n\downarrow & & b_n\downarrow & & c_n\downarrow & & \\
\left(\dfrac{H^1(I_{n,v}, T/\mathrm{F}^+_v T)}{H^1(I_{n,v}, T/\mathrm{F}^+_v T)_{\mathrm{tor}}}\right)^{G_{n,v}} & \hookrightarrow & H^1(I_{n,v}, V/\mathrm{F}^+_v V)^{G_{n,v}} & \to & H^1(I_{n,v}, A/\mathrm{F}^+_v A)^{G_{n,v}}, & &
\end{array}$$

where $G_{n,v} := G_{F_{n,v}}/I_{n,v}$ and the module $H^1_{\mathrm{Gr}}(F_{n,v}, A)$ is defined to be the kernel of the map c_n. The map b_n is decomposed as:

$$H^1(F_{n,v}, V) \xrightarrow{b'_n} H^1(F_{n,v}, V/\mathrm{F}^+_v V) \xrightarrow{b''_n} H^1(I_{n,v}, V/\mathrm{F}^+_v V)^{G_{n,v}}.$$

The cokernel of b'_n is the dual of $H^0(F_{n,v}, (\mathrm{F}^+_v V)^* \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(1))$ which is zero by the assumption (b) of the proposition. Since the group $G_{n,v} = G_{F_{n,v}}/I_{n,v}$ has cohomological dimension one, the map b''_n is surjective by the inflation-restriction sequence. Hence b_n is surjective. Now, we have the following exact sequence by the snake lemma:

$$0 \longrightarrow \mathrm{Coker}(a_n) \longrightarrow \mathrm{Coker}(p_{n,*}) \longrightarrow H^2(F_{n,v}, T)_{\mathrm{tor}}.$$

The last term $H^2(F_{n,v}, T)_{\mathrm{tor}}$ is finite and bounded by the local Tate duality and by the assumption (b). Thus, we have only to bound the group $\mathrm{Coker}(a_n)$ to bound the difference of the local conditions of Greenberg's Selmer group and Bloch-Kato's Selmer group at the primes above p.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^1(F_{n,v}, T)_{\mathrm{tor}} & \hookrightarrow & H^1(F_{n,v}, T) & \to & \dfrac{H^1(F_{n,v}, T)}{H^1(F_{n,v}, T)_{\mathrm{tor}}} & \longrightarrow & 0 \\
\downarrow & & \underline{a}_n\downarrow & & a_n\downarrow & & \\
H^1(I_{n,v}, T/\mathrm{F}^+ T)^{G_{n,v}}_{\mathrm{tor}} & \hookrightarrow & H^1(I_{n,v}, T/\mathrm{F}^+_v T)^{G_{n,v}} & \overset{e_n}{\to} & \left(\dfrac{H^1(I_{n,v}, T/\mathrm{F}^+_v T)}{H^1(F_{n,v}, T/\mathrm{F}^+_v T)_{\mathrm{tor}}}\right)^{G_{n,v}}. & &
\end{array}$$

The group $\mathrm{Coker}(e_n)$ is a submodule of $H^1(G_{n,v}, H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}})$. Hence we have the exact sequence:

$$\mathrm{Coker}(\underline{a}_n) \longrightarrow \mathrm{Coker}(a_n) \longrightarrow H^1(G_{n,v}, H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}}).$$

The group $H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}}$ is the largest cotorsion quotient of $H^0(I_{n,v}, A/\mathrm{F}_v^+A)$. We denote by $D_{n,v}$ the maximal divisible subgroup of $H^0(I_{n,v}, A/\mathrm{F}_v^+A)$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_{n,v} & \longrightarrow & H^0(I_{n,v}, A/\mathrm{F}_v^+A) & \longrightarrow & H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}} & \longrightarrow & 0 \\
 & & {\scriptstyle 1-g_{n,v}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1-g_{n,v}} & & \Big\downarrow{\scriptstyle 1-g_{n,v}} & & \\
0 & \longrightarrow & D_{n,v} & \longrightarrow & H^0(I_{n,v}, A/\mathrm{F}_v^+A) & \longrightarrow & H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}} & \longrightarrow & 0,
\end{array}$$

where $g_{n,v}$ is a topological generator of the cyclic group $G_{n,v}$. The cokernel of the right vertical map is $H^1(G_{n,v}, H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}})$. The kernel of the middle vertical map is $H^0(F_{n,v}, A/\mathrm{F}_v^+A)$, which is finite and bounded by the assumption (b) of the proposition. Hence the cokernel of the middle vertical map is finite and bounded. Consequently, $H^1(G_{n,v}, H^1(I_{n,v}, T/\mathrm{F}_v^+T)_{\mathrm{tor}})$ is a finite group whose order is bounded independently of n. As for the map $\underline{a}_n$, it is decomposed as

$$H^1(F_{n,v}, T) \xrightarrow{\underline{a}'_n} H^1(F_{n,v}, T/\mathrm{F}_v^+T) \xrightarrow{\underline{a}''_n} H^1(I_{n,v}, T/\mathrm{F}_v^+T)^{G_{n,v}}.$$

The map $\underline{a}''_n$ is surjective by the same argument as that for the surjectivity of b''_n. Hence, we have only to bound $\mathrm{Coker}(\underline{a}'_n)$ independently of n. $\mathrm{Coker}(\underline{a}'_n)$ is a submodule of $H^2(F_{n,v}, \mathrm{F}_v^+T)$. Since the group $H^2(F_{n,v}, \mathrm{F}_v^+T)$ is the Pontrjagin dual of $H^0\big(F_{n,v}, (\mathrm{F}_v^+T)^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p(1)\big)$, it is finite and bounded independently of n by the assumption (b). Consequently, $\mathrm{Coker}(\underline{a}_n)$ is finite and bounded inedependently of n. This completes the proof of the proposition. □

We obtain the following corollary by applying Proposition 4.2 to the representations considered in §3.

Corollary 4.3. *Let V be $V_f(r)$ with $2-k \leq r \leq 0$ or $V_d(\frac{-d+1}{2})$ defined in §3. We take a $G_{\mathbb{Q}}$-stable lattice T of V and a discrete Galois module $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. Then the natural inclusion map:*

$$\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_n, A) \xrightarrow{a_n} \mathrm{Sel}^{\mathrm{Gr}}(\mathbb{Q}_n, A)$$

has finite cokernel which is bounded independently of n. Especially, $\mathrm{Sel}^{\mathrm{BK}}(\mathbb{Q}_\infty, A)$ is a finite index subgroup of $\mathrm{Sel}^{\mathrm{Gr}}(\mathbb{Q}_\infty, A)$.

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