A GENERALIZATION OF THE COLEMAN MAP FOR HIDA DEFORMATIONS

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Abstract. In this paper, we give a Coleman/Perrin-Riou type map for an ordinary type deformation and construct a two variable p-adic L-function for a Hida family from the Beilinson-Kato elements.

Contents

1. Introduction

Fix a prime $p \geq 3$. We denote by $\mathbb{Q}(\mu_{p^\infty})$ the extension of the rational number field $\mathbb{Q}$ obtained by adjoining all p-power roots of unity. We fix a complex embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and a p-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ throughout the paper, where $\mathbb{C}$ is the field of complex numbers and $\overline{\mathbb{Q}}_p$ is an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. Let G_∞ be the Galois group of $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}$ and let D_∞ be the group of diamond operators for the tower of modular curves $\{Y_1(p^t)\}_{t\geq 1}$ (see §2). We have the canonical character $\chi : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$ (resp. $\kappa : D_\infty \xrightarrow{\sim} 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$). A character $\eta : G_\infty \to \overline{\mathbb{Q}}_p^\times$ (resp. $\eta' : D_\infty \to \overline{\mathbb{Q}}_p^\times$) is called *an arithmetic character of weight* $w(\eta) \in \mathbb{Z}$ (*resp.* $w(\eta') \in \mathbb{Z}$) if η (resp. η') coincides with the character $\chi^{w(\eta)}$ (resp. $\kappa^{w(\eta')}$) on a sufficiently small open subgroup U (resp. U') of G_∞ (resp. D_∞). For any arithmetic character η (resp. η') of G_∞ (resp. D_∞), let $\mathcal{O}_{\eta,\eta'} = (\eta \times \eta')(\mathbb{Z}_p[[G_\infty \times D_\infty]])$ the finite flat extension of $\mathbb{Z}_p$ obtained by adjoining the values of the character $\eta \times \eta'$. We fix a positive integer N prime to p.

In his celebrated paper [H2], Hida associates a continuous representation $\mathcal{T}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is free of rank two over the complete group algebra $\mathbb{Z}_p[[G_\infty \times D_\infty]]$ to a Λ-adic cusp form $\mathcal{F}$ of level Np^∞. The representation $\mathcal{T}$ has the following properties (see also §2 for more detailed explanation on Hida's theory):

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1. Let $\eta : G_\infty \to \overline{\mathbb{Q}}_p^\times$ (resp. $\eta' : D_\infty \to \overline{\mathbb{Q}}_p^\times$) be a character. Assume that η' is an arithmetic character of weight $w' \geq 0$. We denote by $T_{\eta,\eta'}$ the specialization $\mathcal{T} \otimes_{\mathbb{Z}_p[[G_\infty \times D_\infty]]} \mathcal{O}_{\eta,\eta'}$ of $\mathcal{T}$. Then there exists a cusp form $f_{\eta'} = \sum_{0<n<\infty} a_n(f_{\eta'})q^n$ of weight $w'+2$ and level Np^* such that $T_{\eta,\eta'}$ is isomorphic to $T_{f_{\eta'}} \otimes \eta$ where $T_{f_{\eta'}}$ is the Galois representation associated to $f_{\eta'}$ in the sense of Deligne [De1] and $\otimes\eta$ is the twist by the one dimensional Galois representation associated to η. In this sense, $\mathcal{T}$ is a family of modular representations when the cusp form and the character twist vary.
2. As a representation of the decomposition group $G_{\mathbb{Q}_p}$ at p, the representation $\mathcal{T}$ has a filtration:

$$0 \longrightarrow \mathrm{F}^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathrm{F}^-\mathcal{T} \longrightarrow 0$$

such that $\mathrm{F}^+\mathcal{T}$ and $\mathrm{F}^-\mathcal{T}$ are free $\mathbb{Z}_p[[G_\infty \times D_\infty]]$-modules of rank one. Let $\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow G_\infty \hookrightarrow \mathbb{Z}_p[[G_\infty]]^\times$ be the universal cyclotomic character and let $\widetilde{\alpha} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p[[D_\infty]]^\times$ be the unramified character such that $\eta'(\widetilde{\alpha}(\mathrm{Frob}_p)) = a_p(f_{\eta'})$ for each arithmetic character $\eta' : D_\infty \to \overline{\mathbb{Q}}_p^\times$ of non-negative weight, where Frob_p is the geometric Frobenius element at p. Then $\mathrm{F}^+\mathcal{T}$ is isomorphic to $\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})\widehat{\otimes}\mathbb{Z}_p[[D_\infty]](\widetilde{\alpha})$, where $\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$ (resp. $\mathbb{Z}_p[[D_\infty]](\widetilde{\alpha})$) is the free rank one $\mathbb{Z}_p[[G_\infty]]$-module (resp. $\mathbb{Z}_p[[D_\infty]]$-module) on which $G_{\mathbb{Q}_p}$ acts via $\widetilde{\chi}$ (resp. $\widetilde{\alpha}$). In this sense, $\mathrm{F}^+\mathcal{T}$ interpolates the p-th Fourier coefficient $a_p(f_{\eta'})$ of the cusp form $f_{\eta'}$ when η' varies.

We will construct the Coleman map for this deformation $\mathcal{T}$. Before stating the main results, we prepare some notations.

We define the $\mathbb{Z}_p[[D_\infty]]$-module $\mathcal{D}$ to be $(\mathbb{Z}_p[[D_\infty]](\widetilde{\alpha})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$, where $\widehat{\otimes}_{\mathbb{Z}_p}$ is the formal tensor product over $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension of $\mathbb{Z}_p$. Let $\mathbb{Z}_p(1) = \varprojlim_s \mu_{p^s}$, where μ_{p^s} is the group of p^s-th roots of unity. The absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ acts naturally on $\mathbb{Z}_p(1)$. We denote by $\overline{\mathcal{T}}$ the Kummer dual $\mathrm{Hom}_{\mathbb{Z}_p[[G_\infty \times D_\infty]]}(\mathcal{T}, \mathbb{Z}_p[[G_\infty \times D_\infty]]) \otimes \mathbb{Z}_p(1)$ of $\mathcal{T}$. Let $\overline{T}_{\eta,\eta'}$ (resp. $T_{\eta,\eta'}$) be the specialization of $\overline{\mathcal{T}}$ (resp. $\mathcal{T}$) at a character $\eta \times \eta'$ of $G_\infty \times D_\infty$ and let $\overline{V}_{\eta,\eta'}$ (resp. $V_{\eta,\eta'}$) be the extension $\overline{T}_{\eta,\eta'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (resp. $T_{\eta,\eta'} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$). The module $\mathcal{D}$ defined above has the following properties (see §3 for the proof):

1. The module $\mathcal{D}$ is free of rank one over $\mathbb{Z}_p[[D_\infty]]$.
2. Let η (resp. η') be an arithmetic character of G_∞ (resp. D_∞) satisfying $0 \leq w(\eta) - 1 \leq w(\eta')$. We denote by K_η (resp. $K_{\eta'}$) be the finite extension of $\mathbb{Q}_p$ obtained by adjoining the values of the character η (resp. η') to $\mathbb{Q}_p$. Then $\mathcal{D}_{\eta'} \otimes_{\mathcal{O}_{K_\eta \cap K_{\eta'}}} \mathrm{D}_{\mathrm{dR}}(K_\eta(\eta))$ is naturally identified with Fontaine's filtered module $\mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'})$, where $\mathcal{D}_{\eta'}$ is the specialization of $\mathcal{D}$ at η', $K_\eta(\eta)$ is the one dimensional K_η-vector space on which $G_{\mathbb{Q}}$ acts via η and $\mathcal{O}_{K_\eta \cap K_{\eta'}}$ is the ring of integers of $K_\eta \cap K_{\eta'}$.

For a free $\mathbb{Z}_p$-module T with continuous $G_{\mathbb{Q}_p}$-action, Bloch-Kato (cf. [BK, §3]) defines a subgroup $H^1_f(\mathbb{Q}_p, T)$ of $H^1(\mathbb{Q}_p, T)$ called the finite part. We denote the quotient

$\dfrac{H^1(\mathbb{Q}_p,T)}{H^1_f(\mathbb{Q}_p,T)}$ by $H^1_{/f}(\mathbb{Q}_p,T)$ for short. Kato defines a map:

$$H^1_{/f}(\mathbb{Q}_p,T) \xrightarrow{\exp^*} \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$$

called the dual exponential map (see Definition 3.9).

Let (w,w') be a pair of integers such that $0\leq w-1\leq w'$. We denote by $\overline{T}^{(w,w')}_{s,t}$ the quotient $\overline{T}/\Phi^{(w,w')}_{s,t}\overline{T}$, where $\Phi^{(w,w')}_{s,t}$ is the height two ideal of $\mathbb{Z}_p[[G_\infty\times D_\infty]]$ defined to be the kernel of the homomorphism $\chi^{wp^s}\times\kappa^{w'p^t}:\mathbb{Z}_p[[G_\infty\times D_\infty]]\longrightarrow\overline{\mathbb{Q}}_p$. The projective limit $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(w,w')}_{s,t})$ does not depend on the choice of (w,w') by Corollary 4.13. We denote $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(w,w')}_{s,t})$ by $H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})$. Let Δ be the largest finite subgroup of G_∞ and let $\omega:\Delta\xrightarrow{\sim}(\mathbb{Z}/p\mathbb{Z})^\times\subset\mathbb{Z}_p^\times$ be the Teichmuller character. We define an idempotent $e_i\in\mathbb{Z}_p[\Delta]$ to be $\dfrac{1}{p-1}\sum_{g\in\Delta}\omega^{-i}(g)g$ for each $0\leq i\leq p-2$. For a $\mathbb{Z}_p[[G_\infty\times D_\infty]]$-module M, we have the decomposition $M\cong\bigoplus_{0\leq i\leq p-2}e_i(M)$, where each $e_i(M)$ is naturally regarded as a $\mathbb{Z}_p[[(G_\infty/\Delta)\times D_\infty]]$-module. The module $H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})$ defined above has the following properties (see §4 for the proof):

1. For each $0\leq i\leq p-2$, $e_i(H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}}))$ is a torsion free $\mathbb{Z}_p[[(G_\infty/\Delta)\times D_\infty]]$-module (note that $\mathbb{Z}_p[[(G_\infty/\Delta)\times D_\infty]]$ is an integral domain).
2. Let $\mathrm{Frac}(\mathbb{Z}_p[[G_\infty\times D_\infty]])$ be the total quotient ring of $\mathbb{Z}_p[[G_\infty\times D_\infty]]$. Then $H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})\otimes_{\mathbb{Z}_p[[G_\infty\times D_\infty]]}\mathrm{Frac}(\mathbb{Z}_p[[G_\infty\times D_\infty]])$ is a free $\mathrm{Frac}(\mathbb{Z}_p[[G_\infty\times D_\infty]])$-module of rank one.
3. Let η (resp. η') be an arithmetic character of G_∞ (resp. D_∞) such that $0\leq w(\eta)-1\leq w(\eta')$. Then we have the specialization map $H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})\longrightarrow H^1_{/f}(\mathbb{Q}_p,\overline{T}_{\eta,\eta'})$ at $\eta\times\eta'$ whose cokernel is finite.

From now on throughout the paper, we fix a norm compatible system $\{\zeta_{p^s}\}_{s\geq1}$ of primitive p^s-th roots of unity. Let $\delta_{\mathbb{Q}_p(1)}$ be the inverse image of $1\in\mathbb{Q}_p$ via the isomorphism $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1))\xrightarrow{\sim}\mathbb{Q}_p$ determined by $\{\zeta_{p^s}\}_{s\geq1}$. Let us fix a basis d of the $\mathbb{Z}_p[[D_\infty]]$-module $\mathcal{D}$. Denote by $K_{\eta,\eta'}$ the fraction field of $\mathcal{O}_{\eta,\eta'}$. By the properties of $\mathcal{D}$ stated above, we have a basis $d_{\eta,\eta'}$ of the one dimensional $K_{\eta,\eta'}$-vector space $\mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'})$ induced by the fixed data $\{\zeta_{p^s}\}_{s\geq1}$ and d (see Definition 3.12 for the precise definition of $d_{\eta,\eta'}$) for each η (resp. η') satisfying $0\leq w(\eta)-1\leq w(\eta')$. Our main result is to construct an interpolation of the dual exponential maps when the character $\eta\times\eta'$ of $G_\infty\times D_\infty$ varies. The result is as follows:

Theorem (Theorem 3.13)**.** *Let us fix a basis d of $\mathcal{D}$. Then, we have a $\mathbb{Z}_p[[G_\infty\times D_\infty]]$-linear homomorphism*

$$\overline{\Xi}_d:H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})\longrightarrow\mathbb{Z}_p[[G_\infty\times D_\infty]]$$

satisfying the following properties :

1. *The map* $\overline{\overline{\Xi}}_d$ *is injective and the cokernel of* $\overline{\overline{\Xi}}_d$ *is a pseudo-null* $\mathbb{Z}_p[[D_\infty \times G_\infty]]$*-module.*
2. *Let* $\mathcal{C}$ *be an element of* $H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ *and let* $c_{\eta,\eta'} \in H^1_{/f}(\mathbb{Q}_p, \overline{T}_{\eta,\eta'})$ *be the specialization of* $\mathcal{C}$ *for each arithmetic character* η *(resp.* η'*) of* G_∞ *(resp.* D_∞*). Assume the inequality* $0 \leq w-1 \leq w'$ *for* $w = w(\eta)$ *and* $w' = w(\eta')$*. Then the specialization* $\overline{\overline{\Xi}}_d(\mathcal{C})_{\eta,\eta'} \in \overline{\mathbb{Q}}_p$ *of* $\overline{\overline{\Xi}}_d(\mathcal{C})$ *at the character* $\eta \times \eta'$ *is given by*

$$(-1)^{w-1}(w-1)!\left(\frac{a_p(f_{\eta'})}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_p(f_{\eta'})}\right)\left(1-\frac{a_p(f_{\eta'})\phi(p)}{p^w}\right)^{-1}\langle \exp^*(c_{\eta,\eta'}), d_{\eta,\eta'}\rangle,$$

where $\langle\ ,\ \rangle$ *is the de Rham pairing* :

$$\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\eta'}) \times \mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\eta'}) \longrightarrow \mathrm{D}_{\mathrm{dR}}(K_{\eta,\eta'}(1)) \cong K_{\eta,\eta'},$$

ϕ is the finite order character $\eta \cdot \chi^{-w}$ of G_∞ and s is the p-order of the conductor of ϕ.

Assume that the residual representation $G_{\mathbb{Q}} \longrightarrow GL(\mathcal{T}/(I,p)\mathcal{T}) = GL_2(\mathbb{F}_p)$ is irreducible, where I is the augmentation ideal of $\mathbb{Z}_p[[G_\infty \times D_\infty]]$. By a result of Kato [Ka3], we have an element $\mathcal{Z} \in H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ called the Beilinson-Kato element (see §3). The specialization $z_{\eta,\eta'}$ of $\mathcal{Z}$ is related to a L-value of the modular form $f_{\eta'}$. If we fix a basis d of $\mathcal{D}$, $z_{\eta,\eta'}$ has the property that

$$\langle \exp^*(z_{\eta,\eta'}), d_{\eta,\eta'}\rangle / C_{p,\eta',d} = \frac{G(\phi^{-1},\zeta_{p^s})(2\pi\sqrt{-1})^{w(\eta')-w(\eta)+1}}{\overline{C}^{(-1)^{w(\eta')-w(\eta)+1}\phi(-1)}_{\infty,\eta'}} \times L_{(p)}(f_{\eta'},\phi,w(\eta))$$

where $L_{(p)}(f_{\eta'},\phi,s)$ is the Hecke L-function for the ϕ-twist of $f_{\eta'}$ with its p-factor removed, $C_{p,\eta',d}$ (resp. $\overline{C}^{\pm}_{\infty,\eta'}$) is a p-adic (resp. complex) period (see §3 for the definition of these periods) and $G(\phi^{-1},\zeta_{p^s})$ is the Gauss sum for ϕ^{-1}. The following corollary shows that the image $\overline{\overline{\Xi}}_d(\mathcal{Z}) \in \mathbb{Z}_p[[G_\infty \times D_\infty]]$ of the Beilinson-Kato element $\mathcal{Z}$ gives a two variable p-adic L-function for the Hida deformation corresponding to $\mathcal{Z}$.

Corollary (Theorem 3.17)**.** *Let us fix a basis d of $\mathcal{D}$. Assume that the residual representation $G_{\mathbb{Q}} \longrightarrow GL(\mathcal{T}/(I,p)\mathcal{T}) = GL_2(\mathbb{F}_p)$ is irreducible. Then $\overline{\overline{\Xi}}_d(\mathcal{Z}) \in \mathbb{Z}_p[[G_\infty \times D_\infty]]$ has the following interpolation properties for each arithmetic character η (resp. η') of G_∞ (resp. D_∞) satisfying the inequality $0 \leq w-1 \leq w'$ for $w = w(\eta)$ and $w' = w(\eta')$:*

$$\overline{\overline{\Xi}}_d(\mathcal{Z})_{\eta,\eta'} / C_{p,\eta',d}$$
$$= (-1)^{w-1}(w-1)!\frac{G(\phi^{-1},\zeta_{p^s})(2\pi\sqrt{-1})^{w'-w+1}}{\overline{C}^{(-1)^{w'-w+1}\phi(-1)}_{\infty,\eta'}}\left(\frac{a_p(f_{\eta'})}{p^{w-1}}\right)^{-s}\left(1-\frac{\phi(p)p^{w-1}}{a_p(f_{\eta'})}\right)L(f_{\eta'},\phi,w).$$

Greenberg-Stevens, Kitagawa and Ohta also construct a two-variable p-adic L-function for an ordinary Λ-adic cusp form independently. Their method is to construct a Λ-adic interpolation of modular symbols and their definition of a p-adic period is an error term on the "Betti side". In our case, the definition of a p-adic period is an error term on the "de Rham side". The relation between our p-adic L-function and those of Greenberg-Stevens, Kitagawa and Ohta is not clear at present. One of the advantage of our construction of the p-adic L-function from Euler system is that it is useful to investigate the relation

between a p-adic L-function and a Selmer group (Iwasawa Main conjecture). We give an application of the result in this paper to a two-variable Iwasawa Main conjecture for a Λ-adic cusp form in the paper [O], where we show one of the inequality predicted by the Main conjecture.

Plan. The plan of this paper is as follows. In §2, we recall necessary facts from Hida theory. In §3, we state our main result for nearly ordinary Galois deformations not necessarily limited to Hida deformation. We deduce the result stated above in the case of two variable deformation coming from Hida theory from our main result. In §4, we give the calculation of the limit of local cohomology groups, which is used to show the injectivity of the interpolation map of the main theorem. In §5, we give the proof of the main result.

Notation. For a field K, we denote $\mathrm{Gal}(\overline{K}/K)$ by G_K where $\overline{K}$ is the separable closure of K. Given a finite prime v of a number field F, we denote by Frob_v the geometric Frobenius at v. For a commutative ring S, we denote by $S^\times$ the group of invertible elements in S. We denote by $S(\rho)$ the free S-module of rank one on which G_F acts via a character $\rho : G_F \longrightarrow S^\times$. Throughout the paper, we assume that the fixed integer p is an odd prime number.

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2. Λ-adic Forms and Galois Representations

In this section, we review some fundamental results on Λ-adic cusp forms and their Galois representations.

We keep the notation of the previous section. Let $\mathcal{O} \subset \overline{\mathbb{Q}}_p$ be a commutative ring which is finite flat over $\mathbb{Z}_p$ and let ψ be a Dirichlet character modulo Np^r. We denote by $M_k(\Gamma_1(Np^r), \psi; \mathcal{O})$ (resp. $S_k(\Gamma_1(Np^r), \psi; \mathcal{O})$) the space of modular (resp. cusp) forms of weight k, Neben character ψ and Fourier coefficients in $\mathcal{O}$ for the group $\Gamma_1(Np^r)$.

For each integer $t \geq 1$, the affine modular curve $Y_1(p^t)_{/\mathbb{Q}}$ is the fine moduli of pairs $(E_{/S}, e_{/S})$ of an elliptic curve E over a $\mathbb{Q}$-scheme S and an S-section $e_{/S}$ of order p^t. Recall that the diamond operator $\langle a \rangle$ on $Y_1(p^t)$ is the automorphism on $Y_1(p^t)$ which sends a pair $(E_{/S}, e_{/S})$ to the pair $(E_{/S}, ae_{/S})$ for each $a \in (\mathbb{Z}/p^t\mathbb{Z})^\times$. We denote the p-Sylow subgroup of the group of diamond operators on $Y_1(p^{t+1})$ by D_t. D_t is canonically isomorphic to the group $1 + p(\mathbb{Z}/p^{t+1}\mathbb{Z})(\subset (\mathbb{Z}/p^{t+1}\mathbb{Z})^\times)$. We define the pro-$p$ group D_∞ to be the projective limit $\varprojlim_t D_t$.

Definition 2.1. Let G_∞ and χ be as in the previous section and let D_∞ be as above. Let $\kappa : D_\infty \xrightarrow{\sim} 1 + p\mathbb{Z}_p \hookrightarrow \overline{\mathbb{Q}}_p^\times$ be the canonical character.

(1) A character η (resp. η') of G_∞ (resp. D_∞) is called *an arithmetic character of weight* $w(\eta)$ (resp. $w(\eta')$) if there exists an integer $w(\eta)$ (resp. $w(\eta')$) such that η (resp. η') coincides with $\chi^{w(\eta)}$ (resp. $\kappa^{w(\eta')}$) on a sufficiently small open subgroup U

(resp. U') of G_∞ (resp. D_∞). Let $s(\eta)$ (resp. $s(\eta')$) be the p-order of the conductor of the finite order character $\eta \cdot \chi^{-w(\eta)}$ (resp. $\eta' \cdot \kappa^{-w(\eta')}$). We denote the set of arithmetic characters of G_∞ (resp. D_∞) by $\mathfrak{X}_{\mathrm{arith}}(G_\infty)$ (resp. $\mathfrak{X}_{\mathrm{arith}}(D_\infty)$).

(2) Let R be a local domain finite flat over $\mathbb{Z}_p[[D_\infty]]$. We denote by $\mathfrak{X}(R)$ the set of non-trivial continuous algebra homomorphisms $R \longrightarrow \overline{\mathbb{Q}}_p$. We define a subset $\mathfrak{X}_{\mathrm{arith}}(R)$ of $\mathfrak{X}(R)$ by:

$$\mathfrak{X}_{\mathrm{arith}}(R) = \{\mathfrak{p} \in \mathfrak{X}(R) \mid \text{the character } \mathfrak{p}|_{D_\infty} : D_\infty \to \overline{\mathbb{Q}}_p^\times \text{ is an arithmetic character}\}.$$

We call an element $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(R)$ an arithmetic point of R. The weight of $\mathfrak{p}|_{D_\infty} \in \mathfrak{X}_{\mathrm{arith}}(D_\infty)$ is called the weight of the arithmetic point $\mathfrak{p}$ and is denoted by $w(\mathfrak{p})$. We also denote by $\psi^{(\mathfrak{p})}$ (resp. $s(\mathfrak{p})$) the finite order character $\mathfrak{p}|_{D_\infty} \cdot \kappa^{-w(\mathfrak{p})}$ of D_∞ (resp. the p-order of the conductor of $\psi^{(\mathfrak{p})}$).

Consider a formal q-expansion $\mathcal{F} = \sum_{0 \le n < \infty} A_n(\mathcal{F}) q^n$, where each $A_n(\mathcal{F})$ is an element of a fixed algebraic closure of $\mathrm{Frac}(\mathbb{Z}_p[[D_\infty]])$. For $\mathcal{F}$ as above, we define the subring $\mathbb{H}$ of the algebraic closure of $\mathrm{Frac}(\mathbb{Z}_p[[D_\infty]])$ to be the algebra generated by all $A_n(\mathcal{F})$'s over $\mathbb{Z}_p[[D_\infty]]$. We assume that $\mathbb{H}$ is finite flat over $\mathbb{Z}_p[[D_\infty]]$. For each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$, we denote by $f_\mathfrak{p}$ the formal q-expansion $\sum_{0 \le n < \infty} a_n(f_\mathfrak{p}) q^n$, where $a_n(f_\mathfrak{p}) \in \overline{\mathbb{Q}}_p$ is the image of $A_n(\mathcal{F})$ under the map $\mathfrak{p} : \mathbb{H} \longrightarrow \overline{\mathbb{Q}}_p$.

Definition 2.2. Let ψ_0 be a character defined modulo Np. We call $\mathcal{F}$ *a Λ-adic form of level Np^∞ with Dirichlet character ψ_0* if $f_\mathfrak{p}$ is the q-expansion of a modular form in $M_{w(\mathfrak{p})+2}(\Gamma_1(Np^{s(\mathfrak{p})}), \psi_0 \psi^{(\mathfrak{p})} \omega^{-w(\mathfrak{p})}; \overline{\mathbb{Q}}_p)$ for each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \ge 0$, where ω is the Teichmuller character.

Definition 2.3. Let $r \ge 1$. Let ψ be a character defined modulo Np^r. Then a p-adic cusp form $f \in S_k(\Gamma_1(Np^r), \psi; \mathcal{O})$ is called *a p-stabilized newform of tame conductor N with Dirichlet character ψ* if

1. f is an eigenform of $S_k(\Gamma_1(Np^r), \psi; \mathcal{O})$ for all Hecke operators $T_l (l \nmid Np)$ and $U_{l'} (l' | Np)$ which belong to $\mathrm{End}(S_k(\Gamma_1(Np^r), \psi; \mathcal{O}))$.
2. The newform associated to f has level Np^{r_0} for some r_0, $0 \le r_0 \le r$.
3. The eigenvalue a_p of f for $U_p \in \mathrm{End}(S_k(\Gamma_1(Np^r), \psi; \mathcal{O}))$ is a p-adic unit.

Let f be a newform in $S_k(\Gamma_1(N), \psi; \mathcal{O})$. Assume that the eigenvalue a_p of f for the Hecke operator T_p is a p-adic unit. We define $f^* \in S_k(\Gamma_1(Np), \psi; \mathcal{O})$ by $f^* = f(q) - \beta f(q^p)$, where β is the unique root of $x^2 - a_p x + \psi(p) p^{k-1}$ with p-adic absolute value $|\beta| < 1$. Then f^* is a p-stabilized newform of tame conductor N whose n-th Fourier coefficients equal to that of f for each natural number n prime to p. We call this f^* the p-stabilized newform associated to f.

Definition 2.4. A normalized Λ-adic cusp form $\mathcal{F}$ is *a Λ-adic newform of tame conductor N with Dirichlet character ψ_0* if $f_\mathfrak{p}$ is a weight $w(\mathfrak{p}) + 2$ p-stabilized newform of tame conductor N, level $Np^{s(\mathfrak{p})}$ with character $\psi_0 \psi^{(\mathfrak{p})} \omega^{-w(\mathfrak{p})}$ for each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \ge 0$.

Remark 2.5. Let $\mathcal{F}$ be a Λ-adic newform of the tame conductor N with character ψ_0 modulo Np. Then it is known that the specialized p-stabilized newform $f_{\mathfrak{p}}$ is a newform (that is, $f_{\mathfrak{p}}$ is also new at p) if and only if $s(\mathfrak{p}) > 1$ or $\psi_0\psi^{(\mathfrak{p})}\omega^{-w(\mathfrak{p})}$ restricted to the subgroup $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$ is non-trivial.

Theorem 2.6 (Hida [H1] Corollary 3.2, Corollary 3.7)**.** *Let N be an integer prime to p and let ψ be a Dirichlet character defined modulo Np. Let $f \in S_k(\Gamma_1(Np), \psi; \mathcal{O})$ be a p-stabilized newform of tame conductor N. Then there exist a Λ-adic newform $\mathcal{F}$ of tame conductor N with Dirichlet character $\psi\omega^{k-2}$ and $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) = k-2$ such that $f_{\mathfrak{p}}$ is equal to f.*

Recall the definition of continuity for Galois representations over the field of fractions $\mathbb{K}$ of $\mathbb{H}$ (see [H3, §7.5], for example).

Definition 2.7. The representation $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{K})$ is *continuous* if there exists a finitely generated $\mathbb{H}$-module $\mathbb{T} \subset \mathbb{H}^{\oplus 2}$ which is stable under $G_{\mathbb{Q}}$-action such that $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(\mathbb{T})$ is continuous with respect to the topology of $\mathbb{T}$ defined by the maximal ideal of $\mathbb{H}$ and that $\mathbb{T}\otimes_{\mathbb{H}}\mathbb{K} \cong \mathbb{K}^{\oplus 2}$.

Hida associates a continuous Galois representation over $\mathbb{H}$ to a Λ-adic newform $\mathcal{F}$ as follows:

Theorem 2.8 (Hida [H2])**.** *Let $\mathcal{F}$ be a Λ-adic newform with Dirichlet character ψ_0 modulo Np. Then, there exists a continuous irreducible representation $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{K})$ satisfying the following properties:*

1. *$\rho_{\mathcal{F}}$ is unramified outside Np.*
2. *For the geometric Frobenius element Frob_l at $l \nmid Np$, we have:*
$$\mathrm{Trace}\big(\rho_{\mathcal{F}}(\mathrm{Frob}_l)\big) = A_l(\mathcal{F}),$$
$$\det\big(\rho_{\mathcal{F}}(\mathrm{Frob}_l)\big) = \psi_0(l)\widetilde{\kappa}^{-1}(\bar{l})l^{-1},$$
where $l \mapsto \bar{l}$ is the projection $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1+p\mathbb{Z}_p) \twoheadrightarrow (1+p\mathbb{Z}_p)$ and $\widetilde{\kappa}$ is the tautological character $1+p\mathbb{Z}_p \xrightarrow{\sim} D_\infty \hookrightarrow \mathbb{Z}_p[[D_\infty]]^\times \subset \mathbb{K}^\times$.

In order to study the Iwasawa theory for Hida deformations, it is convenient to assume the following condition:

(**Int**) The representation $\rho_{\mathcal{F}} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{K})$ has a $G_{\mathbb{Q}}$-stable lattice $\mathbb{T}$ which is isomorphic to $\mathbb{H}^{\oplus 2}$.

The condition (**Int**) is satisfied in fairly general situations. Before stating some sufficient conditions for (**Int**) to be satisfied (Proposition 2.10), we introduce some necessary notations.

Definition 2.9. Let $\mathbb{F}$ be the residue field of $\mathbb{H}$ modulo the maximal ideal $\mathcal{M}$ of $\mathbb{H}$. A semi-simple representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ is called *the residual representation* associated to $\mathcal{F}$ if $\overline{\rho}$ is unramified outside Np and the characteristic polynomial of the geometric Frobenius Frob_l for each prime $l \not| Np$ is congruent to
$$X^2 - A_l(\mathcal{F})X - \psi_0(l)\widetilde{\kappa}^{-1}(\bar{l})l^{-1} \ \text{ modulo } \mathcal{M}.$$

Such a residual representation is always known to exist without assuming the condition (**Int**) and it is unique up to isomorphism. The following is a list of cases where the condition (**Int**) is known to be true.

Proposition 2.10 (Mazur-Wiles, Tilouine, Mazur-Tilouine)**.** *The condition* (**Int**) *holds if one of the following conditions is satisfied:*

(1) *The ring* $\mathbb{H}$ *is regular.*
(2) *The tame conductor* N *of* $\mathcal{F}$ *is equal to* 1 *and the ring* $\mathbb{H}$ *is Gorenstein* (*Mazur-Wiles* [MW2] §9).
(3) *Let* a *be a number such that* $\psi_0|_{(\mathbb{Z}/p\mathbb{Z})^\times} = \omega^a$*. Then* $a \not\equiv 0, -1$ *modulo* $p-1$ *and the ring* $\mathbb{H}$ *is Gorenstein* (*Tilouine* [Ti] *Theorem* 4.4).
(4) *The residual representation is irreducible* (*see Mazur-Tilouine* [MT] §2*, Corollary* 6). *We denote this condition by* (**Ir**).

From now on throughout the paper, we will assume the condition (**Int**). We have the following local property of $\rho_{\mathcal{F}}$ due to Mazur and Wiles:

Proposition 2.11 ([Wi] Theorem 2.2.2)**.** *The restriction* $\rho_{\mathcal{F}}|_{G_{\mathbb{Q}_p}}$ *to the decomposition group* $G_{\mathbb{Q}_p}$ *of* $\rho_{\mathcal{F}}$ *has the filtration*:

$$0 \longrightarrow \mathrm{F}^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathrm{F}^-\mathbb{T} \longrightarrow 0$$

such that $\mathrm{F}^+\mathbb{T}$ *and* $\mathrm{F}^-\mathbb{T}$ *are free* $\mathbb{H}$*-modules of rank one. Further,* $G_{\mathbb{Q}_p}$ *acts on* $\mathrm{F}^+\mathbb{T}$ *via the unramified character* $\widetilde{\alpha}$ *such that* $\widetilde{\alpha}(\mathrm{Frob}_p) = A_p(\mathcal{F})$ *for the geometric Frobenius element* Frob_p*.*

Remark 2.12. Note that the normalization of the above proposition is dual to that of the paper [Wi]. In [Wi], the Frobenius element is normalized to be the arithmetic one. We normalize the Frobenius element to be the geometric one throughout this paper.

For each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \geq 0$, the specialization $T_{\mathfrak{p}} = \mathbb{T} \otimes_{\mathbb{H}} \mathfrak{p}(\mathbb{H})$ of $\mathbb{T}$ is isomorphic to the p-adic Galois representation $T_{f_{\mathfrak{p}}}$ associated to $f_{\mathfrak{p}}$ by Deligne [De1]. In this sense, $\rho_{\mathcal{F}}$ is a family of modular Galois representations when the weight of the modular form varies.

Let $\widetilde{\chi}: G_{\mathbb{Q}} \twoheadrightarrow G_\infty \hookrightarrow \mathbb{Z}_p[[G_\infty]]$ be the universal cyclotomic character. We denote by $\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$ the free $\mathbb{Z}_p[[G_\infty]]$-module of rank one on which $G_{\mathbb{Q}}$ acts via the character $\widetilde{\chi}$. The nearly ordinary deformation $\mathcal{T}$ associated to $\mathbb{T}$ is defined to be the formal tensor product $\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$, where the action of $G_{\mathbb{Q}}$ on $\mathcal{T}$ is given by the diagonal one. The representation $\mathcal{T}$ has the following properties:

1. $\mathcal{T}$ is free of rank two over $\mathcal{R} = \mathbb{H}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] = \mathbb{H}[[G_\infty]]$.
2. As a $G_{\mathbb{Q}_p}$-module, we have the filtration:
$$0 \longrightarrow \mathrm{F}^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathrm{F}^-\mathcal{T} \longrightarrow 0,$$
where $\mathrm{F}^+\mathcal{T}$ (resp. $\mathrm{F}^-\mathcal{T}$) is $\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$ (resp. $\mathrm{F}^-\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$).
3. Let $T_{\eta,\mathfrak{p}}$ be the specialization of $\mathcal{T}$ at $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$. Assume that $w(\mathfrak{p})$ is non-negative. Then there exists a cusp form $f_{\mathfrak{p}}$ of weight $w(\mathfrak{p}) + 2$ and $T_{\eta,\mathfrak{p}}$ is isomorphic to $T_{f_{\mathfrak{p}}} \otimes \eta$, where $\otimes\eta$ is the twist by the one dimensional Galois representation corresponding to η.

3. The main result and its application to Hida's Galois deformation

In this section, we state our main results for general nearly ordinary deformations not necessarily limited to the Hida deformation introduced in §2. We give an application to the Hida family in the latter half of this section.

Throughout the first half of the section, let D_∞ be a pro-p group which has the canonical isomorphism $\kappa : D_\infty \xrightarrow{\sim} 1+p\mathbb{Z}_p$ (we do not necessarily assume that D_∞ is the group of diamond operators as in §2). We fix a commutative ring $\mathbb{H}$ which is finite flat over $\mathbb{Z}_p[[D_\infty]]$.

Definition 3.1. Let $\mathbb{T}$ be a free $\mathbb{H}$-module of rank two with continuous $G_\mathbb{Q}$-action. The representation $\mathbb{T}$ is called *an ordinary deformation* if the following conditions are satisfied:

1. The representation $\mathbb{T}$ has a filtration as a $G_{\mathbb{Q}_p}$-module:
$$0 \longrightarrow \mathrm{F}^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathrm{F}^-\mathbb{T} \longrightarrow 0$$
such that $\mathrm{F}^+\mathbb{T}$ and $\mathrm{F}^-\mathbb{T}$ are free rank one $\mathbb{H}$-modules and that the action of $G_{\mathbb{Q}_p}$ on $\mathrm{F}^+\mathbb{T}$ is given by an unramified character $\widetilde{\alpha}$ of $G_{\mathbb{Q}_p}$.
2. There exists a Dirichlet character $\psi_0 : G_\mathbb{Q} \to \mathbb{Z}_p[\psi_0] \hookrightarrow \mathbb{H}^\times$ such that $G_\mathbb{Q}$ acts on the the determinant representation $\det(\mathbb{T}) = \overset{2}{\wedge}\mathbb{T}$ via the character $\psi_0\chi^{-1}\widetilde{\kappa}^{-1}$, where we regard the tautological character $\widetilde{\kappa} : 1+p\mathbb{Z}_p \xrightarrow{\sim} D_\infty \hookrightarrow \mathbb{Z}_p[[D_\infty]]^\times \subset \mathbb{H}^\times$ as a character of $G_\mathbb{Q}$ through the canonical character $\chi : G_\mathbb{Q} \twoheadrightarrow G_\infty \xrightarrow{\sim} 1+p\mathbb{Z}_p$.

Let $\mathcal{R} = \mathbb{H}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] = \mathbb{H}[[G_\infty]]$, which is finite flat over $\mathbb{Z}_p[[G_\infty \times D_\infty]]$. We denote by $\mathcal{T}$ the Galois representation $\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$. We call $\mathcal{T}$ *the nearly ordinary deformation associated to* $\mathbb{T}$.

A nearly ordinary deformation $\mathcal{T}$ defined above is free of rank two over $\mathcal{R}$. The Kummer dual $\overline{\mathcal{T}}$ of $\mathcal{T}$ is defined to be $\mathrm{Hom}_\mathcal{R}(\mathcal{T},\mathcal{R})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$. $\overline{\mathcal{T}}$ has a rank one filtration $\mathrm{F}^+\overline{\mathcal{T}} \subset \overline{\mathcal{T}}$ defined by $\mathrm{F}^+\overline{\mathcal{T}} = \mathrm{Hom}_\mathcal{R}(\mathrm{F}^-\mathcal{T},\mathcal{R})\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$. Let $(w,w') \in \mathbb{Z}\times\mathbb{Z}$ and let $(s,t) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$. We denote by $T_{s,t}^{(w,w')}$ (resp. $\mathrm{F}^+T_{s,t}^{(w,w')}$, $\mathrm{F}^-T_{s,t}^{(w,w')}$, $\overline{T}_{s,t}^{(w,w')}$, $\mathrm{F}^+\overline{T}_{s,t}^{(w,w')}$, $\mathrm{F}^-\overline{T}_{s,t}^{(w,w')}$) the specialization of $\mathcal{T}$ (resp. $\mathrm{F}^+\mathcal{T}$, $\mathrm{F}^-\mathcal{T}$, $\overline{\mathcal{T}}$, $\mathrm{F}^+\overline{\mathcal{T}}$, $\mathrm{F}^-\overline{\mathcal{T}}$) obtained by applying $\otimes_\mathcal{R}\mathcal{R}/\Phi_{s,t}^{(w,w')}$, where $\Phi_{s,t}^{(w,w')}$ is the height two ideal defined in §1. These representations are free $\mathbb{Z}_p$-modules of finite rank on which $G_\mathbb{Q}$ acts continuously. We denote by $V_{s,t}^{(w,w')}$ (resp. $\mathrm{F}^+V_{s,t}^{(w,w')}$, $\mathrm{F}^-V_{s,t}^{(w,w')}$, $\overline{V}_{s,t}^{(w,w')}$, $\mathrm{F}^+\overline{V}_{s,t}^{(w,w')}$, $\mathrm{F}^-\overline{V}_{s,t}^{(w,w')}$) the extension of $T_{s,t}^{(w,w')}$ (resp. $\mathrm{F}^+T_{s,t}^{(w,w')}$, $\mathrm{F}^-T_{s,t}^{(w,w')}$, $\overline{T}_{s,t}^{(w,w')}$, $\mathrm{F}^+\overline{T}_{s,t}^{(w,w')}$, $\mathrm{F}^-\overline{T}_{s,t}^{(w,w')}$) by applying $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. For each $(\eta,\mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty)\times\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$, we denote by $T_{\eta,\mathfrak{p}}$ (resp. $\mathrm{F}^+T_{\eta,\mathfrak{p}}$, $\mathrm{F}^-T_{\eta,\mathfrak{p}}$, $\overline{T}_{\eta,\mathfrak{p}}$, $\mathrm{F}^+\overline{T}_{\eta,\mathfrak{p}}$, $\mathrm{F}^-\overline{T}_{\eta,\mathfrak{p}}$) the specialization of $\mathcal{T}$ (resp. $\mathrm{F}^+\mathcal{T}$, $\mathrm{F}^-\mathcal{T}$, $\overline{\mathcal{T}}$, $\mathrm{F}^+\overline{\mathcal{T}}$, $\mathrm{F}^-\overline{\mathcal{T}}$) via $\mathcal{R} \longrightarrow \eta\circ\mathfrak{p}(\mathcal{R})$, where $\eta\circ\mathfrak{p}(\mathcal{R})$ is finite flat over $\mathbb{Z}_p$. Similarly, we define $V_{\eta,\mathfrak{p}}$ (resp. $\mathrm{F}^+V_{\eta,\mathfrak{p}}$, $\mathrm{F}^-V_{\eta,\mathfrak{p}}$, $\overline{V}_{\eta,\mathfrak{p}}$, $\mathrm{F}^+\overline{V}_{\eta,\mathfrak{p}}$, $\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}}$) by applying $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ to the above representations. For later use in this section and the next, we summarize basic facts on these specializations:

1. For $(\eta,\mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty)\times\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$, $T_{\eta,\mathfrak{p}}$ (resp. $\overline{T}_{\eta,\mathfrak{p}}$) is a quotient of $T_{s(\eta),s(\mathfrak{p})}^{(w,w')}$ (resp. $\overline{T}_{s(\eta),s(\mathfrak{p})}^{(w,w')}$), where $w = w(\eta)$ and $w' = w(\mathfrak{p})$.
2. $\overline{T}_{\eta,\mathfrak{p}}$ is the Kummer dual $\mathrm{Hom}_{\mathbb{Z}_p}(T_{\eta,\mathfrak{p}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$ of $T_{\eta,\mathfrak{p}}$.

3. $V_{s,t}^{(w,w')}$ (resp. $\overline{V}_{s,t}^{(w,w')}$) is isomorphic to $\underset{\eta,\mathfrak{p}}{\oplus} V_{\eta,\mathfrak{p}}$ (resp. $\underset{\eta,\mathfrak{p}}{\oplus}\overline{V}_{\eta,\mathfrak{p}}$) where η (resp. $\mathfrak{p}$) runs arithmetic characters of G_∞ (resp. arithmetic points of $\mathbb{H}$) satisfying $w(\eta) = w$ and $s(\eta) \leq s$ (resp. $w(\mathfrak{p}) = w'$ and $s(\mathfrak{p}) \leq t$).

Fontaine defines the rings of p-adic periods $B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$. The rings B_{crys} and B_{dR} have continuous $G_{\mathbb{Q}_p}$-action and B_{dR} is a complete discrete valuation field. We denote by B_{dR}^+ the valuation ring of B_{dR} and denote by $\mathrm{Fil}^i B_{\mathrm{dR}}$ for each $i \in \mathbb{Z}$ the decreasing filtration of B_{dR} defined by $u^i B_{\mathrm{dR}}^+$ where u is a uniformizer of B_{dR}^+. For a p-adic representation V of $G_{\mathbb{Q}_p}$, we denote by $\mathrm{D}_{\mathrm{crys}}(V)$ (resp. $\mathrm{D}_{\mathrm{dR}}(V)$) the Fontaine's module defined by $(V \otimes B_{\mathrm{crys}})^{G_{\mathbb{Q}_p}}$ (resp. $(V \otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$). The module $\mathrm{D}_{\mathrm{crys}}(V)$ (resp. $\mathrm{D}_{\mathrm{dR}}(V)$) is a finite dimensional $\mathbb{Q}_p$-vector space such that $\dim_{\mathbb{Q}_p}\mathrm{D}_{\mathrm{crys}}(V) \leq \dim_{\mathbb{Q}_p} V$ (resp. $\dim_{\mathbb{Q}_p}\mathrm{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$). A p-adic representation V is called a crystalline representation (resp. de Rham representation) if $\dim_{\mathbb{Q}_p}\mathrm{D}_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V$ (resp. $\dim_{\mathbb{Q}_p}\mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$). The module $\mathrm{D}_{\mathrm{dR}}(V)$ has a separated and exhausted decreasing filtration $\mathrm{Fil}^i\mathrm{D}_{\mathrm{dR}}(V) := (V \otimes \mathrm{Fil}^i B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$. We refer the reader to [Bu] for Fontaine's theory of p-adic representations. For each $\eta \in \mathfrak{X}_{\mathrm{arith}}(G_\infty)$ (resp. $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$), we denote by K_η (resp. $K_{\mathfrak{p}}$, $K_{\eta,\mathfrak{p}}$) the fraction field of $\eta(\mathbb{Z}_p[[G_\infty]])$ (resp. $\mathfrak{p}(\mathbb{H})$, $\eta \circ \mathfrak{p}(\mathbb{H}[[G_\infty]])$), which is a finite extension of $\mathbb{Q}_p$. We have the following lemma:

Lemma 3.2. *Let $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ satisfying $0 \leq w(\eta) - 1 \leq w(\mathfrak{p})$. Then, $V_{\eta,\mathfrak{p}}$ is a de Rham representation of $G_{\mathbb{Q}_p}$ such that $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ is canonically isomorphic to $\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})$. $\overline{V}_{\eta,\mathfrak{p}}$ is also a de Rham representation such that $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})$ is canonically isomorphic to $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}})$.*

Proof. By the definition of $\mathcal{T}$, there exists a finite extention K of $\mathbb{Q}_p$ such that $V_{\eta,\mathfrak{p}}$ is an ordinary representation of G_K. By a result of Perrin-Riou [P3], an ordinary representation is semi-stable in the sense of Fontaine. Especially, $V_{\eta,\mathfrak{p}}$ is a de Rham representation of G_K. Since a potentially de Rham representation is a de Rham representation (cf. [Bu]), $V_{\eta,\mathfrak{p}}$ is a de Rham representation of $G_{\mathbb{Q}_p}$. We have $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}) = (\mathrm{F}^+V_{\eta,\mathfrak{p}} \otimes B_{\mathrm{dR}}^+)^{G_{\mathbb{Q}_p}} = 0$ since the action of a sufficiently small open subgroup of the inertia subgroup I_p on $\mathrm{F}^+V_{\eta,\mathfrak{p}}$ is given by χ^w with $w > 0$. Thus we have a $K_{\eta,\mathfrak{p}}$-linear injection $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})$. Similarly we have a $K_{\eta,\mathfrak{p}}$-linear surjection $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}}) \longrightarrow \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-V_{\eta,\mathfrak{p}})$ and $\dim_{K_{\eta,\mathfrak{p}}}\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}((V_{\eta,\mathfrak{p}}/\mathrm{F}^+V_{\eta,\mathfrak{p}})) = 1$. We have $\dim_{K_{\eta,\mathfrak{p}}}\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}}) = 2$ since $V_{\eta,\mathfrak{p}}$ is a de Rham representation and we have $\dim_{K_{\eta,\mathfrak{p}}}\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}) = 1$ since a sub-representation of a de Rham representation is also a de Rham representation. In conclusion, we have $\dim_{K_{\eta,\mathfrak{p}}}\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}}) \leq 1$ and the above mentioned injection $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}) \hookrightarrow \mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})$ must be an isomorphism. The assertion for $\overline{V}_{\eta,\mathfrak{p}}$ is shown in the same way. □

Lemma 3.3. *Let M be a free $\mathbb{H}$-module of finite rank e endowed with unramified $G_{\mathbb{Q}_p}$-action. Then $(M\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ is a free $\mathbb{H}$-module of finite rank e.*

Proof. Let I be a height two ideal of $\mathbb{H}$. Then we have

$$(M\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} = (\varprojlim_n (M/I^n)\otimes_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} = \varprojlim_n (M/I^n \otimes_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}.$$

M/I^n is a free module of rank one over the ring $\mathbb{H}/I^n$ with finite number of elements. It suffices to show that $(M/I^n \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ is free of rank one over $\mathbb{H}/I^n$. Clearly the proof follows from the following claim:

Claim 3.4. Let R be a $\mathbb{Z}_p$-algebra with finite number of elements. For a free R-module M of finite rank e endowed with unramified $G_{\mathbb{Q}_p}$-action, $(M \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ is free of finite rank e over R.

We prove the claim in the rest. Let p^m be the characteristic of the ring R. Then $M \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is isomorphic to $M \otimes_{\mathbb{Z}/p^m\mathbb{Z}} W_m(\overline{\mathbb{F}}_p)$, where $W_m(\overline{\mathbb{F}}_p)$ is the ring of Witt vectors of length m (cf. [Se, Chap. II, §6]). Since M is finite, there is an open subgroup H of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ such that H acts trivially on M. We have $(M \otimes_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} = (M \otimes_{\mathbb{Z}/p^m\mathbb{Z}} W_m(\mathbb{F}))^G$, where $\mathbb{F}$ is the fixed field $\overline{\mathbb{F}}_p^H$ and $G = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)/H$. Since $M \otimes_{\mathbb{Z}/p^m\mathbb{Z}} W_m(\mathbb{F})$ is isomorphic to $M \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z}[G]$ as an $R[G]$-module, $(M \otimes_{\mathbb{Z}/p^m\mathbb{Z}} W_m(\mathbb{F}))^G$ is free of rank e over R. □

By Lemma 3.3, we give the following definition.

Definition 3.5. Let $\mathbb{T}$ be an ordinary deformation. We define a free $\mathbb{H}$-module $\mathcal{D}$ of rank one by $\mathcal{D} = (\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ where the Galois action of $g \in G_{\mathbb{Q}_p}$ on $\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is the diagonal action $g \otimes g$.

Lemma 3.6. *Let $\mathcal{T} = \mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$ be a nearly ordinary deformation and let $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ satisfying $0 \leq w(\eta) - 1 \leq w(\mathfrak{p})$. Then we have the canonical isomorphism:*

$$\mathrm{D}_{\mathrm{dR}}(K_\eta(\eta)) \otimes_{\mathcal{O}_{K_\eta \cap K_\mathfrak{p}}} \mathcal{D}_\mathfrak{p} \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}),$$

where $\mathcal{D}_\mathfrak{p}$ is the specialization $\mathcal{D} \otimes_{\mathbb{H}} \mathfrak{p}(\mathbb{H})$ of $\mathcal{D}$ and $K_\eta(\eta)$ is the one dimensional Galois representation over K_η on which $G_{\mathbb{Q}_p}$ acts via η.

Proof. Let $V_\mathfrak{p}$ be the representation $(\mathbb{T} \otimes_{\mathbb{H}} \mathfrak{p}(\mathbb{H})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Since $\mathrm{F}^+V_\mathfrak{p}$ is unramified, $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_\mathfrak{p})$ is canonically isomorphic to $(\mathrm{F}^+V_\mathfrak{p} \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$. Hence $\mathcal{D} \otimes_{\mathbb{H}} K_\mathfrak{p} = (\mathrm{F}^+V_\mathfrak{p} \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$ is canonically isomorphic to $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_\mathfrak{p})$. Recall that $\mathrm{F}^+V_{\eta,\mathfrak{p}} = K_\eta(\eta) \otimes_{K_\eta \cap K_\mathfrak{p}} \mathrm{F}^+V_\mathfrak{p}$. Since the functor D_{dR} is compatible with a tensor product of two de Rham representations, we have the canonical isomorphism $\mathrm{D}_{\mathrm{dR}}(K_\eta(\eta)) \otimes_{K_\eta \cap K_\mathfrak{p}} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_\mathfrak{p}) \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$. This completes the proof of the lemma. □

Before giving the main result, we prepare some general definitions. Let B_{st} be the ring of p-adic periods for semi-stable Galois representations, which is a subring of B_{dR} equipped with continuous $G_{\mathbb{Q}_p}$-action (cf. [Bu]). For a representation V of $G_{\mathbb{Q}_p}$, we denote by $\mathrm{D}_{\mathrm{pst}}(V)$ the inductive limit $\varinjlim_{J \subset I_p} (V \otimes B_{\mathrm{st}})^J$ where J runs through open subgroups of the inertia subgroup I_p of $G_{\mathbb{Q}_p}$. Let σ be an arithmetic Frobenius element in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$. The module $\mathrm{D}_{\mathrm{pst}}(V)$ is a finite dimensional $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$-vector space with the following properties:

1. We have the inequality $\dim_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathrm{D}_{\mathrm{pst}}(V) \leq \dim_{\mathbb{Q}_p}(V)$.
2. $\mathrm{D}_{\mathrm{pst}}(V)$ is endowed with the monodromy operator N, which is a $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$-linear nilpotent endomorphism on $\mathrm{D}_{\mathrm{pst}}(V)$ and is induced from the monodromy operator of B_{st}.

3. We have a σ-semilinear $G_{\mathbb{Q}_p}$-action on $\mathrm{D}_{\mathrm{pst}}(V)$ and the action of I_p factors through a finite quotient of I_p.
4. The module $\mathrm{D}_{\mathrm{pst}}(V)$ has the Frobenius operator f, which is σ-semilinear and is induced from the Frobenius operator of B_{st}.

The restriction of the action of $G_{\mathbb{Q}_p}$ on $\mathrm{D}_{\mathrm{pst}}(V)$ to the Weil group $W_p \subset G_{\mathbb{Q}_p}$ gives us a σ-semilinear action of W_p on $\mathrm{D}_{\mathrm{pst}}(V)$. We denote by $u : W_p \twoheadrightarrow W_p/I_p \xrightarrow{\sim} \mathbb{Z}$ the natural map which sends σ to 1. By the twist of the W_p-action which replaces the action of $g \in W_p$ with the action of $g \cdot f^{-u(g)}$, we obtain a $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$-linear action of W_p on $\mathrm{D}_{\mathrm{pst}}(V)$. Since the inertia subgroup $I_p \subset W_p$ acts through a finite quotient of I_p, the complex absolute values of the eigenvalues of the Frobenius element of W_p are well-defined if they are algebraic numbers.

Definition 3.7. For a nearly ordinary deformation $\mathcal{T}$, we consider the following condition:
(**MW**) Every eigenvalue α of the action of a lift of a geometric Frobenius Frob_p on $\mathrm{D}_{\mathrm{pst}}(V_{\eta,\mathfrak{p}})$ is an algebraic number whose complex absolute value is

$$\begin{cases} p^{\frac{w'+1}{2}-w} & \text{if the monodromy } N \text{ is zero on } \mathrm{D}_{\mathrm{pst}}(V_{\eta,\mathfrak{p}}), \\ p^{\frac{w'}{2}-w+1} & \text{if } N \text{ is non zero on } \mathrm{D}_{\mathrm{pst}}(V_{\eta,\mathfrak{p}}) \text{ and the eigen vector of } \alpha \text{ is in } \mathrm{Coker}(N), \\ p^{\frac{w'}{2}-w} & \text{if } N \text{ is non zero on } \mathrm{D}_{\mathrm{pst}}(V_{\eta,\mathfrak{p}}) \text{ and the eigen vector of } \alpha \text{ is in } \mathrm{Ker}(N). \end{cases}$$

for each $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \geq 0$, where $w = w(\eta)$ and $w' = w(\mathfrak{p})$.

Remark 3.8. Assume that $V_{\mathfrak{p}}$ is the p-adic realization of a certain pure motive of weight $w(\mathfrak{p}) + 1$ for each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \geq 0$. Then the above assertion (**MW**) is conjectured to be true. By a result of T. Saito [Sa] for the monodromy-weight conjecture for elliptic modular forms, the above assertion on the complex eigenvalues of the lift of the Frobenius on $\mathrm{D}_{\mathrm{pst}}(V_{\eta,\mathfrak{p}})$ is true if $V_{\mathfrak{p}}$ is the p-adic representation associated to an elliptic cusp form of weight $w(\mathfrak{p})+2$. Hence (**MW**) is true if $\mathcal{T}$ is associated to a certain Λ-adic cusp form.

Let us recall the definition of the dual exponential map $\exp^*$. We denote by $\log(\chi) \in H^1(\mathbb{Q}_p, \mathbb{Q}_p) = \mathrm{Hom}(G_{\mathbb{Q}_p}, \mathbb{Q}_p)$ the homomorphism $G_{\mathbb{Q}_p} \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{\log} \mathbb{Q}_p$ defined by the cyclotomic character χ. Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. Let us consider the map:

$$H^0(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}^+) \xrightarrow{\cup \log(\chi)} H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}^+),$$

obtained by the cup product with $\log(\chi) \in H^1(\mathbb{Q}_p, \mathbb{Q}_p)$. By [Ka1] Chap. II, Proposition 1.2.3, the above map $\cup \log(\chi)$ is an isomorphism.

Definition 3.9. The dual exponential map $\exp^*$ is defined to be the composite:

$$H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}^+) \xrightarrow[(\cup \log(\chi))^{-1}]{\sim} \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) = H^0(\mathbb{Q}_p, V \otimes B_{\mathrm{dR}}^+).$$

Bloch-Kato [BK] defines a subgroup $H_f^1(\mathbb{Q}_p, V) \subset H^1(\mathbb{Q}_p, V)$ called the finite part as follows:

$$H_f^1(\mathbb{Q}_p, V) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, V) \longrightarrow H^1(\mathbb{Q}_p, V \otimes B_{\mathrm{crys}})\right]$$

Let T (resp. A) be a $G_{\mathbb{Q}_p}$-stable lattice of V (resp. a discrete Galois module $T\otimes\mathbb{Q}_p/\mathbb{Z}_p$). We have the following exact sequence:

$$H^1(\mathbb{Q}_p,T)\xrightarrow{i}H^1(\mathbb{Q}_p,V)\xrightarrow{p}H^1(\mathbb{Q}_p,A).$$

We define $H^1_f(\mathbb{Q}_p,T)\subset H^1(\mathbb{Q}_p,T)$ (resp. $H^1_f(\mathbb{Q}_p,A)\subset H^1(\mathbb{Q}_p,A)$) to be the pullback $i^{-1}H^1_f(\mathbb{Q}_p,V)$ (resp. the push-forward $p_*H^1_f(\mathbb{Q}_p,V)$. The dual exponential map is known to factor as:

$$H^1(\mathbb{Q}_p,V)\longrightarrow H^1_{/f}(\mathbb{Q}_p,V)\xrightarrow{\exp^*}\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V),$$

where $H^1_{/f}(\mathbb{Q}_p,V)$ is $\dfrac{H^1(\mathbb{Q}_p,V)}{H^1_f(\mathbb{Q}_p,V)}$. We prove the following lemma in §4 (see Corollary 4.13):

Lemma 3.10. *Assume that $\mathcal{T}$ satisfies the condition* (**MW**). *For each pair $(w,w')\in\mathbb{Z}\times\mathbb{Z}$ satisfying $0\leq w-1\leq w'$, $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(w,w')}_{s,t})$ is canonically isomorphic to $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(1,0)}_{s,t})$. Especially, $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(w,w')}_{s,t})$ is independent of the choice of (w,w').*

We denote the module $\varprojlim_{s,t}H^1_{/f}(\mathbb{Q}_p,\overline{T}^{(w,w')}_{s,t})$ by $H^1_{/f}(\mathbb{Q}_p,\overline{\mathcal{T}})$. Since $\mathcal{R}$ is finite flat over $\mathbb{Z}_p[[G_\infty\times D_\infty]]$, $\mathrm{Hom}_{\mathbb{Z}_p[[G_\infty\times D_\infty]]}(\mathcal{R},\mathbb{Z}_p[[G_\infty\times D_\infty]])$ is finitely generated $\mathcal{R}$-module by $(r\cdot f)(x)=f(r\cdot x)$ for $r\in\mathcal{R}$ and $f\in\mathrm{Hom}_{\mathbb{Z}_p[[G_\infty\times D_\infty]]}(\mathcal{R},\mathbb{Z}_p[[G_\infty\times D_\infty]])$.

Definition 3.11. The $\mathbb{Z}_p[[G_\infty\times D_\infty]]$-algebra $\mathcal{R}=\mathbb{H}[[G_\infty]]$ is called *a Gorenstein ring* if $\mathrm{Hom}_{\mathbb{Z}_p[[G_\infty\times D_\infty]]}(\mathcal{R},\mathbb{Z}_p[[G_\infty\times D_\infty]])$ is free of rank one over $\mathcal{R}$.

Definition 3.12.

1. For each $\mathfrak{p}\in\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$, we denote by $\mathrm{Sp}_{\mathfrak{p}}:\mathcal{D}\longrightarrow\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\mathfrak{p}})$ the map induced by the $G_{\mathbb{Q}_p}$-invariant of the map $\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}\longrightarrow\mathrm{F}^+T_{\mathfrak{p}}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$, where $T_{\mathfrak{p}}=\mathbb{T}\otimes_{\mathbb{H}}\mathfrak{p}(\mathbb{H})$ (note that $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\mathfrak{p}})=\mathrm{D}_{\mathrm{crys}}(\mathrm{F}^+V_{\mathfrak{p}})=(\mathrm{F}^+T_{\mathfrak{p}}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ since $\mathrm{F}^+T_{\mathfrak{p}}$ is unramified).
2. Let $\eta\in\mathfrak{X}_{\mathrm{arith}}(G_\infty)$ and let ϕ be the finite order character $\eta\chi^{-w(\eta)}$. We denote by $\mathcal{O}$ a finite flat extension of $\mathbb{Z}_p$ whose fraction field is K. We define $\mathrm{Sp}_\eta:\mathcal{O}[[G_\infty]]\longrightarrow\mathrm{D}_{\mathrm{dR}}((K\otimes_{K\cap K_\eta}K_\eta)(\eta))$ to be the $\mathbb{Z}_p$-linear homomorphism:

$$\begin{aligned}&\mathcal{O}[[G_\infty]]\longrightarrow\mathrm{D}_{\mathrm{dR}}(K(\chi^w))\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(\mu_{p^s})\cong\mathrm{D}_{\mathrm{dR}}(K(\chi^w)\otimes_{\mathbb{Q}_p}\mathbb{Q}_p[G_s])\\&\twoheadrightarrow\mathrm{D}_{\mathrm{dR}}(K(\chi^w)\otimes_{\mathbb{Q}_p}K_\eta(\phi))\twoheadrightarrow\mathrm{D}_{\mathrm{dR}}(K(\chi^w)\otimes_{K\cap K_\eta}K_\eta(\phi))=\mathrm{D}_{\mathrm{dR}}((K\otimes_{K\cap K_\eta}K_\eta)(\eta)),\end{aligned}$$

   where the first map is the $\mathcal{O}[[G_\infty]]$-linear map which sends $g\in G_\infty$ to $\delta^{\otimes w}_{\mathbb{Q}_p(1)}\otimes\zeta^g_{p^s}$. The isomorphism in the upper line is nothing but the isomorphism $(K(\chi^w)\otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p(\mu_{p^s})}}\cong(K(\chi^w)\otimes\mathbb{Q}_p[G_s]\otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$ by Shapiro's lemma.

3. Let $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$. Then we denote by $\mathrm{Sp}_{\eta,\mathfrak{p}} : \mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] \longrightarrow \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ the composite:

$$\mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] \overset{\mathrm{Sp}_{\mathfrak{p}}\otimes 1}{\longrightarrow} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\mathfrak{p}}) \otimes \mathcal{O}_{\mathfrak{p}}[[G_\infty]] \overset{1\otimes \mathrm{Sp}_\eta}{\longrightarrow} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\mathfrak{p}}) \otimes_{K_{\mathfrak{p}}\cap K_\eta} \mathrm{D}_{\mathrm{dR}}(K_\eta(\eta)) \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$$

Let Δ be the largest finite subgroup of G_∞ and let $\mathcal{R}_\Delta$ be $\mathcal{R} \otimes_{\mathbb{Z}_p[\Delta]} \mathbb{Z}_p$. The ring $\mathcal{R}_\Delta$ is an integral domain and we have an isomorphism $\mathcal{R} \cong \prod_{1\le i\le p-1} \mathcal{R}_\Delta$. Our main result in this paper is the following theorem:

Theorem 3.13. *Let $\mathcal{T}$ be a nearly ordinary deformation in the sense of Definition* 3.1. *Assume that $\mathcal{R}$ is Gorenstein, $\mathcal{R}_\Delta$ is a normal domain and that $\mathcal{T}$ satisfies the condition* (**MW**). *Let us fix a basis d of the $\mathbb{H}$-module $\mathcal{D} = (\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$. Then we have an $\mathcal{R}$-linear homomorphism $\overline{\Xi}_d : H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow \mathcal{R}$ with the following properties:*

1. *The map $\overline{\Xi}_d$ is an injective $\mathcal{R}$-homomorphism whose cokernel is a pseudo-null $\mathcal{R}$-module.*
2. *Let $\mathcal{C}$ be an element of $H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ and let $c_{\eta,\mathfrak{p}} \in H^1_{/f}(\mathbb{Q}_p, \overline{T}_{\eta,\mathfrak{p}})$ be the specialization of $\mathcal{C}$ at $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$. Assume that $0 \le w-1 \le w'$ for $w = w(\eta)$ and $w' = w(\mathfrak{p})$. Then, $\overline{\Xi}_d(\mathcal{C})_{\eta,\mathfrak{p}}$ is given by :*

$$(-1)^{w-1}(w-1)!\left(\frac{a_{\mathfrak{p}}}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_{\mathfrak{p}}}\right)\left(1-\frac{a_{\mathfrak{p}}\phi(p)}{p^w}\right)^{-1}\langle \exp^*(c_{\eta,\mathfrak{p}}), d_{\eta,\mathfrak{p}}\rangle,$$

where $a_{\mathfrak{p}}$ is the value of the action of the geometric Frobenius Frob_p *on $\mathrm{F}^+V_{\mathfrak{p}}$, $\langle\ ,\ \rangle$ is the pairing :*

$$\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}}) \times \mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}}) \longrightarrow \mathrm{D}_{\mathrm{dR}}(K_{\eta,\mathfrak{p}}(1)) \cong K_{\eta,\mathfrak{p}},$$

ϕ is the finite order character $\eta\chi^{-w}$ of G_∞ and s is the p-order of the conductor of ϕ.

Now, we apply our main theorem to a two-variable modular deformation explained in §2. From now on throughout the section, we take $\mathcal{T}$ to be the nearly ordinary deformation associated to a certain Λ-adic new form $\mathcal{F}$.

In order to introduce Beilinson-Kato elements, we prepare notations. For each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) \ge 0$, we denote by $\overline{f}_{\mathfrak{p}} = \sum_{n>0} a_n(f_{\mathfrak{p}})^c q^n$ the dual modular form of $f_{\mathfrak{p}} = \sum_{n>0} a_n(f_{\mathfrak{p}})q^n$ where c is the complex conjugate. The dual modular form $\overline{f}_{\mathfrak{p}}$ is known to be a Hecke cuspidal eigen form of weight $w(\mathfrak{p})+2$ with Neben character dual of that of $f_{\mathfrak{p}}$. We denote by $\mathbb{Q}_{\overline{f}_{\mathfrak{p}}}$ the finite extension of $\mathbb{Q}$ obtained by adjoining Fourier coefficients of $\overline{f}_{\mathfrak{p}}$ to $\mathbb{Q}$. For a Dirichlet character ϕ of p-power conductor, let $\mathbb{Q}_{\overline{f}_{\mathfrak{p}},\phi}$ be the finite extension of $\mathbb{Q}_{\overline{f}_{\mathfrak{p}}}$ obtained by adjoining the values of ϕ and let $V_{\mathrm{dR}}(\phi)$ be the de Rham realization of the Dirichlet motive for ϕ, which is a one dimensional vector space over

K_ϕ. We associate the de Rham representation $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ to $\overline{f}_\mathfrak{p}$. The de Rham realization $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ has the following properties:

1. $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ is a two dimensional vector space over $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$ and is equipped with a de Rham filtration $\mathrm{Fil}^i V_{\mathrm{dR}}(\overline{f}_\mathfrak{p}) \subset V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$, which is a decreasing filtration of $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$-vector spaces.
2. We have $\mathrm{Fil}^0 V_{\mathrm{dR}}(\overline{f}_\mathfrak{p}) = V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ and $\mathrm{Fil}^{w(\mathfrak{p})+2} V_{\mathrm{dR}}(\overline{f}_\mathfrak{p}) = \{0\}$. For each w such that $0 \leq w-1 \leq w(\mathfrak{p})$, $\mathrm{Fil}^w V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ is naturally identified with the one-dimensional $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$-vector space $\mathbb{Q}_{\overline{f}_\mathfrak{p}} \cdot \overline{f}_\mathfrak{p}$.
3. For a Dirichlet character ϕ of p-power conductor, we define the ϕ-twist $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})(\phi)$ of $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ to be $V_{\mathrm{dR}}(\overline{f}_\mathfrak{p}) \otimes_{\mathbb{Q}_{\overline{f}_\mathfrak{p}} \cap K_\phi} V_{\mathrm{dR}}(\phi)$, which has dimension two over $\mathbb{Q}_{\overline{f}_\mathfrak{p},\phi}$. If ϕ is the finite order character $\eta\chi^{-w(\eta)}$ for $\eta \in \mathfrak{X}_{\mathrm{arith}}(G_\infty)$ satisfying $1 \leq w(\eta) \leq w(\mathfrak{p})+1$, $\mathrm{Fil}^{w(\mathfrak{p})-w+2} V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})(\phi^{-1}) \otimes_{\mathbb{Q}_{\overline{f}_\mathfrak{p},\phi}} K_{\eta,\mathfrak{p}}$ is naturally identified with $\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}})$.

We also associate the Betti representation $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})$ to $\overline{f}_\mathfrak{p}$. The Betti realization $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})$ has the following properties:

1. $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})$ is a two dimensional vector space over $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$ and is equipped with the action of the complex conjugate c on $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})$.
2. Each ± 1-eigen space $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})^{\pm}$ for the action of c is one dimensional over $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$.

Let $\mathcal{E}_{Y_1(Np^t)} \xrightarrow{\lambda_t} Y_1(Np^t)$ be the universal elliptic curve and let $\nu : \mathfrak{H} \longrightarrow Y_1(Np^t)(\mathbb{C}) = \mathfrak{H}/\Gamma_1(Np^t)$ be the uniformization map, where $\mathfrak{H}$ is the complex upper half plane. Consider the continuous map $\tau : (0,\infty) \longrightarrow Y_1(Np^t)(\mathbb{C}),\ y \longmapsto \nu(y\sqrt{-1})$. Let $\underline{H}^1$ be the higher direct image $R^1\lambda_{t,*}(\mathbb{Z})$, which is a locally free sheaf on $Y_1(Np^t)(\mathbb{C})$ of $\mathbb{Z}$-rank two. We denote by $\underline{H}_1$ the dual sheaf $\underline{\mathrm{Hom}}(\underline{H}^1, \mathbb{Z})$. The stalk of $\tau^{-1}(\underline{H}_1)$ at $y \in (0,\infty)$ is identified with $H_1(\mathbb{C}/y\sqrt{-1}\mathbb{Z}+\mathbb{Z}, \mathbb{Z}) = y\sqrt{-1}\mathbb{Z}+\mathbb{Z}$. The sheaf $\tau^{-1}(\underline{H}_1)$ is a constant free sheaf of $\mathbb{Z}$-rank two with basis $e_1 = (y\sqrt{-1}, 0)$, $e_2 = (0,1)$.

Definition 3.14. Let $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ be an arithmetic point of weight w'.

1. For each integer w satisfying $0 \leq w-1 \leq w'$, we denote by $\overline{\delta}_\mathfrak{p}^{\mathrm{dR}}$ the $\mathbb{Q}_{\overline{f}_\mathfrak{p}}$-basis of $\mathrm{Fil}^{w'-w+2} V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})$ coming from $\overline{f}_\mathfrak{p}$.
2. We define a basis $\overline{\delta}_\mathfrak{p}^{\mathrm{B},\pm}$ of $V_{\mathrm{B}}(\overline{f}_\mathfrak{p})^{\pm}$ to be the image of the class $(\tau, e_1^{w'})$ in the cohomology $H_1(X_1(Np^{s(\mathfrak{p})}), X_1(Np^{s(\mathfrak{p})}) \setminus Y_1(Np^{s(\mathfrak{p})}); \mathrm{Sym}^{w'}(\underline{H}_1))$ under the following composite map:

$$\begin{aligned} H_1(X_1(Np^{s(\mathfrak{p})}), X_1(Np^{s(\mathfrak{p})}) \setminus Y_1(Np^{s(\mathfrak{p})}); \mathrm{Sym}^{w'}(\underline{H}_1)) \\ \xrightarrow{\sim} H_1(X_1(Np^{s(\mathfrak{p})}), X_1(Np^{s(\mathfrak{p})}) \setminus Y_1(Np^{s(\mathfrak{p})}); \mathrm{Sym}^{w'}(\underline{H}^1)) \\ \xrightarrow{\sim} H^1(Y_1(Np^{s(\mathfrak{p})}); \mathrm{Sym}^{w'}(\underline{H}^1)) \longrightarrow V_{\mathrm{B}}(\overline{f}_\mathfrak{p})^{\pm}. \end{aligned}$$

We have the Eichler-Shimura isomorphism:

$$\overline{I}_{\infty,\mathfrak{p}}^{\pm} :\ \mathrm{Fil}^{w'-w+2} V_{\mathrm{dR}}(\overline{f}_\mathfrak{p})(\phi^{-1}) \otimes_{\mathbb{Q}_{\overline{f}_\mathfrak{p},\phi}} \mathbb{C} \xrightarrow{\sim} V_{\mathrm{B}}(\overline{f}_\mathfrak{p})^{\pm(-1)^{w'-w+1}\phi(-1)} \otimes_{\mathbb{Q}_{\overline{f}_\mathfrak{p}}} \mathbb{C}.$$

Kato constructs elements in the K_2 of modular curves [Ka3]. By using his elements, we have the following system of elements in Galois cohomology.

Proposition 3.15. [Ka3] *Let $\mathcal{T}$ be the nearly ordinary representation associated to a certain Λ-adic newform $\mathcal{F}$ satisfying the condition* (**Ir**) *stated in §2 for* (**Ir**). *There exists an element $\mathcal{Z} \in H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ satisfying the following properties:*

1. *Let $z_{\eta,\mathfrak{p}} \in H^1_{/f}(\mathbb{Q}_p, \overline{T}_{\eta,\mathfrak{p}})$ be the specialization of $\mathcal{Z}$ at $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ satisfying $0 \leq w - 1 \leq w'$ for $w = w(\eta)$ and $w' = w(\mathfrak{p})$. Then $\exp^*(z_{\eta,\mathfrak{p}}) \in \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}})$ is contained in $\mathrm{Fil}^{w'-w+2}V_{\mathrm{dR}}(\overline{f}_{\mathfrak{p}})(\phi^{-1}) \subset \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}})$.*
2. *The image of $\exp^*(z_{\eta,\mathfrak{p}}) \in \mathrm{Fil}^{w'-w+2}V_{\mathrm{dR}}(\overline{f}_{\mathfrak{p}})(\phi^{-1})$ under the map*

$$\overline{I}^+_{\infty,\mathfrak{p}} : \ \mathrm{Fil}^{w'-w+2}V_{\mathrm{dR}}(\overline{f}_{\mathfrak{p}})(\phi^{-1}) \longrightarrow V_{\mathrm{B}}(\overline{f}_{\mathfrak{p}})^{(-1)^{w'-w+1}\phi(-1)} \otimes_{\mathbb{Q}_{\overline{f}_{\mathfrak{p}}}} \mathbb{C}$$

is equal to $G(\phi^{-1}, \zeta_{p^s})(2\pi\sqrt{-1})^{w'-w+1}L_{(p)}(f_{\mathfrak{p}}, \phi, w) \cdot \overline{\delta}_{\mathfrak{p}}^{\mathrm{B},(-1)^{w'-w+1}\phi(-1)}$.

We define a complex period and a p-adic period at each arithmetic point $\mathfrak{p}$ as follows:

Definition 3.16. Let the notations be as defined in Definition 3.14.

1. A complex period $\overline{C}^{\pm}_{\infty,\mathfrak{p}}$ is the complex number given by $\overline{I}_{\infty,\mathfrak{p}}(\overline{\delta}_{\mathfrak{p}}^{\mathrm{dR}}) = \overline{C}^{\pm}_{\infty,\mathfrak{p}}\overline{\delta}_{\mathfrak{p}}^{\mathrm{B},\pm}$.
2. Fix a basis d of the $\mathbb{H}$-module $\mathcal{D}$. For each arithmetic point $\mathfrak{p} : \mathbb{H} \longrightarrow \overline{\mathbb{Q}}_p$ of weight w' and each integer w such that $0 \leq w - 1 \leq w'$, we denote by $d_{\eta,\mathfrak{p}}$ the basis of rank one $K_{\eta,\mathfrak{p}}$-vector space $\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})$ defined as the image of $d\widehat{\otimes}1$ via the map $\mathrm{Sp}_{\eta,\mathfrak{p}} : \mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]] \longrightarrow \mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})$ of Definition 3.12. We define a p-adic period $C_{p,\mathfrak{p}} = C_{p,\mathfrak{p},d}$ (depending on the choices of d) to be the value $\langle \overline{\delta}_{\mathfrak{p}}^{\mathrm{dR}}, d_{\eta,\mathfrak{p}} \rangle$ where $\langle\ ,\ \rangle$ is the pairing :

$$\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}}) \times \mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}})/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V_{\eta,\mathfrak{p}}) \longrightarrow \mathrm{D}_{\mathrm{dR}}(K_{\eta,\mathfrak{p}}(1)) \cong K_{\eta,\mathfrak{p}}.$$

The p-adic period $C_{p,\mathfrak{p},d}$ does not depend on w and depends only on d.

We fix a basis d of the $\mathbb{H}$-module $\mathcal{D}$ from now on. Let $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $0 \leq w - 1 \leq w'$ for $w = w(\eta), w' = w(\mathfrak{p})$. Then $\exp^*(z_{\eta,\mathfrak{p}}) \in \mathrm{Fil}^{w'-w+2}V_{\mathrm{dR}}(\overline{f}_{\mathfrak{p}})(\phi^{-1})$ is equal to $\dfrac{(2\pi\sqrt{-1})^{w'-w+1}L_{(p)}(f_{\mathfrak{p}}, \phi, w)}{\overline{C}^{(-1)^{w'-w+1}\phi(-1)}_{\infty,\mathfrak{p}}} \cdot \overline{\delta}_{\mathfrak{p}}^{\mathrm{dR}}$, where $\phi = \eta\chi^{-w}$. Hence $\exp^*(z_{\eta,\mathfrak{p}})/\,C_{p,\mathfrak{p},d} \in \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}}) \otimes_{K_{\eta,\mathfrak{p}}} \overline{\mathbb{Q}}_p$ is sent to the L-value $G(\phi^{-1}, \zeta_{p^s})\dfrac{(2\pi\sqrt{-1})^{w'-w+1}L_{(p)}(f_{\mathfrak{p}}, \phi, w)}{\overline{C}^{(-1)^{w'-w+1}\phi(-1)}_{\infty,\mathfrak{p}}} \in \overline{\mathbb{Q}}$ under the pairing $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}}) \otimes_{K_{\eta,\mathfrak{p}}} \overline{\mathbb{Q}}_p \xrightarrow{\langle\ ,d_{\eta,\mathfrak{p}}\rangle} \overline{\mathbb{Q}}_p$.

As stated in Remark 3.8, a Hida deformation $\mathcal{T}$ satisfies the condition (**MW**). Hence we have the interpolation map $\overline{\Xi}_d$ by Theorem 3.13. By the interpolation property of the

map $\overline{\overline{\Xi}}_d$, $\overline{\overline{\Xi}}_d(\mathcal{Z})_{\eta,\mathfrak{p}}/\,C_{p,\mathfrak{p},d}$ is given by:

$$(-1)^{w-1}(w-1)!\left(\frac{a_p(f_\mathfrak{p})}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_p(f_\mathfrak{p})}\right)\left(1-\frac{a_p(f_\mathfrak{p})\phi(p)}{p^w}\right)^{-1}\frac{\langle \exp^*(z_{\eta,\mathfrak{p}}), d_{\eta,\mathfrak{p}}\rangle}{C_{p,\mathfrak{p},d}}$$
$$=(-1)^{w-1}(w-1)!\left(\frac{a_p(f_\mathfrak{p})}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_p(f_\mathfrak{p})}\right)\frac{(2\pi\sqrt{-1})^{w'-w+1}L_{(p)}(f_\mathfrak{p},\phi,w)}{G(\phi^{-1},\zeta_{p^s})^{-1}\overline{C}_{\infty,\mathfrak{p}}^{(-1)^{w'-w+1}\phi(-1)}}.$$

From the above argument, we obtain the following theorem by applying Theorem 3.13:

Theorem 3.17. *Assume the condition* (**Ir**) *for* $\mathcal{F}$. *Assume that* $\mathcal{R}$ *is Gorenstein and integrally closed in* $\mathrm{Frac}(\mathcal{R})$. *Let us fix a basis* d *of the* $\mathbb{H}$*-module* $\mathcal{D}$. *Then* $\overline{\overline{\Xi}}_d(\mathcal{Z})_{\eta,\mathfrak{p}}/\,C_{p,\mathfrak{p},d}$ *is given by* :

$$(-1)^{w-1}\frac{(w-1)!G(\phi^{-1},\zeta_{p^s})(2\pi\sqrt{-1})^{w'-w+1}}{\overline{C}_{\infty,\mathfrak{p}}^{(-1)^{w'-w+1}\phi(-1)}}\left(\frac{a_p(f_\mathfrak{p})}{p^{w-1}}\right)^{-s}\left(1-\frac{\phi(p)p^{w-1}}{a_p(f_\mathfrak{p})}\right)L(f_\mathfrak{p},\phi,w),$$

at each $(\eta,\mathfrak{p})\in\mathfrak{X}_{\mathrm{arith}}(G_\infty)\times\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ *satisfying* $0\le w-1\le w'$ *with* $w=w(\eta)$ *and* $w'=w(\mathfrak{p})$, *where* ϕ *is the finite order character* $\eta\chi^{-w}$ *of* G_∞ *and* s *is the* p*-order of the conductor of* ϕ.

Remark 3.18. A two-variable p-adic L-function for a Hida deformation is also constructed by Greenberg-Stevens, Kitagawa and Ohta independently. The main ingredient of their work is a construction of the $\mathbb{H}$-adic modular symbol $\mathcal{B}^\pm$, which is a free $\mathbb{H}$-module of rank one. The module $\mathcal{B}^\pm$ has an interpolation property that $\mathcal{B}^\pm/\mathfrak{p}\mathcal{B}^\pm$ is canonically identified with the p-adic completion of the Betti realization $H_{\mathrm{B}}(f_\mathfrak{p})^{\pm w(\mathfrak{p})}$ for each arithmetic point $\mathfrak{p}$ of $\mathbb{H}$ with $w(\mathfrak{p})\ge 0$. They define their p-adic period $C^\pm_{p,\mathfrak{p},b^\pm}\in\overline{\mathbb{Q}}_p$ to be the error term $\delta_\mathfrak{p}^{\mathrm{B},\pm w(\mathfrak{p})}=C^\pm_{p,\mathfrak{p},b^\pm}b^\pm_\mathfrak{p}$ where $b^\pm_\mathfrak{p}$ is a basis of $\mathcal{B}^\pm/\mathfrak{p}\mathcal{B}^\pm$ coming from a fixed $\mathbb{H}$-basis $b^\pm$ of $\mathcal{B}^\pm$. On the other hand, our p-adic period $C_{p,\mathfrak{p},d}$ is defined to be the error term on the de Rham side. Fukaya [Fu] announces another construction of the p-adic L-functions as an application of her theory of K_2-version of the theory of Coleman power series.

4. Calculation of local Iwasawa modules

In this section, we calculate projective limits of various local Galois cohomology groups. The local calculation given in this section immediately implies the coincidence of two-variable Selmer groups of Greenberg type and of Bloch-Kato type for Hida's nearly ordinary deformations $\mathcal{T}$ (see [O]). For the proof of the main theorem (Theorem 3.13), we need only Corollary 4.13.

For a p-adic representation V of $G_{\mathbb{Q}_p}$, a subspace $H^1_g(\mathbb{Q}_p,V)$ (resp. $H^1_e(\mathbb{Q}_p,V)$) of $H^1(\mathbb{Q}_p,V)$ is defined as follows (see [BK, §3]):

$$H^1_g(\mathbb{Q}_p,V)=\mathrm{Ker}\left[H^1(\mathbb{Q}_p,V)\longrightarrow H^1(\mathbb{Q}_p,V\otimes_{\mathbb{Q}_p}B_{\mathrm{dR}})\right],$$
$$H^1_e(\mathbb{Q}_p,V)=\mathrm{Ker}\left[H^1(\mathbb{Q}_p,V)\longrightarrow H^1(\mathbb{Q}_p,V\otimes_{\mathbb{Q}_p}B_{\mathrm{crys}}^{f=1})\right].$$

We have

$$H^1_e(\mathbb{Q}_p, V) \subset H^1_f(\mathbb{Q}_p, V) \subset H^1_g(\mathbb{Q}_p, V) \subset H^1(\mathbb{Q}_p, V)$$

by definition. Let $\mathcal{T}$ be a nearly ordinary deformation. For a pair (w, w') of integers and a pair (s, t) of non-negative integers, we define the specialization $T^{(w,w')}_{s,t}$ (resp. $V^{(w,w')}_{s,t}$) of $\mathcal{T}$ and the specialization $\overline{T}^{(w,w')}_{s,t}$ (resp. $\overline{V}^{(w,w')}_{s,t}$) of $\overline{\mathcal{T}}$ as in the beginning of §3. We define a subspace $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ of $H^1(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ to be:

$$H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, V^{(w,w')}_{s,t}) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, V^{(w,w')}_{s,t}) \longrightarrow H^1(\mathbb{Q}^{\mathrm{ur}}_p, \mathrm{F}^- V^{(w,w')}_{s,t})\right]$$

By a result of Flach [Fl], we have the following lemma:

Lemma 4.1. *For integers w, w' such that $0 \leq w - 1 \leq w'$ and integers $s, t \geq 0$, the subspace $H^1_g(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ of $H^1(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ is equal to $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, V^{(w,w')}_{s,t})$.*

We prove the following lemma:

Lemma 4.2. *Assume that the nearly ordinary deformation $\mathcal{T}$ satisfies the condition* (**MW**) (*cf. Definition* 3.7). *Then we have the equality $H^1_f(\mathbb{Q}_p, V^{(w,w')}_{s,t}) = H^1_g(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ for integers w, w' such that $0 \leq w - 1 \leq w'$ and integers $s, t \geq 0$.*

Proof. By Proposition 3.8 and Corollary 3.8.4 of [BK], we see:

$$\begin{aligned}\frac{H^1_g(\mathbb{Q}_p, V^{(w,w')}_{s,t})}{H^1_f(\mathbb{Q}_p, V^{(w,w')}_{s,t})} &\cong \left(H^1_f(\mathbb{Q}_p, \overline{V}^{(w,w')}_{s,t})/H^1_e(\mathbb{Q}_p, \overline{V}^{(w,w')}_{s,t})\right)^* \\ &\cong \left(\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(w,w')}_{s,t})/(1-f)\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(w,w')}_{s,t})\right)^*,\end{aligned}$$

where $(\ )^*$ means a $\mathbb{Q}_p$-linear dual. Since a slope of $\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(w,w')}_{s,t})$ is $w - w' - 2$ or $w - 1$, this implies that $H^1_f(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ is equal to $H^1_g(\mathbb{Q}_p, V^{(w,w')}_{s,t})$ when $w \neq 1$ (Note that $w - w' - 2$ can not be zero by the assumption of the lemma).

Let us discuss the case $w = 1$ in the rest. To see that $\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(1,w')}_{s,t})/(1-f)\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(1,w')}_{s,t})$ is zero in this case, we study the complex absolutes of the eigenvalues of the Frobenius f on $\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(1,w')}_{s,t}) = \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})_N, \mathbb{Q}_p)^{G_{\mathbb{Q}_p}}$, where $\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})_N$ is the cokernel of the monodromy operator N acting on $\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})$. The set of eigenvalues of f on $\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(1,w')}_{s,t})$ is equal to the set of the eigenvalues of the inverse of f on $\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})_N$ and hence is equal to the set of the eigenvalues of the inverse of the geometric Frobenius element Frob_p on $\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})_N$. By the assumption (**MW**), the complex absolute values of the eigenvalues of Frob_p on $\mathrm{D}_{\mathrm{pst}}(V^{(0,w')}_{s,t})$ are equal to $p^{\frac{w'+1}{2}}$ or $p^{\frac{w'}{2}+1}$. Thus, the eigenvalues of f on $\mathrm{D}_{\mathrm{crys}}(\overline{V}^{(1,w')}_{s,t})$ can not be trivial for any $w' \geq 0$. This completes the proof. □

For each pair (s, t) of non-negative integers and each pair (w, w') of integers such that $0 \leq w - 1 \leq w'$, we define $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A^{(w,w')}_{s,t})$ to be:

$$H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A^{(w,w')}_{s,t}) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, A^{(w,w')}_{s,t}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^- A^{(w,w')}_{s,t})\right],$$

where $A_{s,t}^{(w,w')}$ is the discrete Galois representation $T_{s,t}^{(w,w')} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. The inductive limit $\varinjlim_{s,t} H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ is equal to $\mathrm{Ker}\left[H^1(\mathbb{Q}_p, \mathcal{A}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})\right]$. Since it is independent of the choice of (w, w') with $0 \leq w - 1 \leq w'$, we denote it by $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$. We have the following proposition:

Proposition 4.3. *Assume the condition* (**MW**) *for* $\mathcal{T}$. *Let* (w, w') *be a pair of integers such that* $0 \leq w - 1 \leq w'$. *Then the following statements hold:*

1. *The group* $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ *is a quotient of the Pontryagin dual of the group* $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$.
2. *Assume that* $w \neq 1$. *Then the group* $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ *is the Pontryagin dual of* $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$.

Remark 4.4. We will eliminate the assumption $w \neq 1$ later and prove that the group $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ is the Pontryagin dual of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$ for any pair of integers (w, w') such that $0 \leq w - 1 \leq w'$ (see Corollary 4.13).

First, we have the following lemma:

Lemma 4.5. *Let* $\mathcal{T} = \mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi})$ *be a nearly ordinary deformation. We assume the condition* (**MW**) *for* $\mathcal{T}$. *Then the value* $\widetilde{\alpha}(\mathrm{Frob}_p) \in \mathbb{H}$ *at* Frob_p *is not a root of unity for the unramified character* $\widetilde{\alpha}$ *associated to the unramified representation* $\mathrm{F}^+\mathbb{T}$.

Proof. The specialization $\widetilde{\alpha}(\mathrm{Frob}_p)_{\mathfrak{p}} \in \overline{\mathbb{Q}}_p$ of $\widetilde{\alpha}(\mathrm{Frob}_p)$ at an arithmetic point $\mathfrak{p}$ is equal to the eigenvalue of the action of Frob_p on $\mathrm{F}^+V_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}}_p^{\mathrm{ur}} = \mathrm{D}_{\mathrm{crys}}(\mathrm{F}^+V_{\mathfrak{p}}) \otimes_{\mathbb{Q}_p} \widehat{\mathbb{Q}}_p^{\mathrm{ur}} = \mathrm{D}_{\mathrm{pst}}(\mathrm{F}^+V_{\mathfrak{p}}) \subset \mathrm{D}_{\mathrm{pst}}(V_{\mathfrak{p}})$. On the other hand, the eigenvalues of every lift of Frob_p on $\mathrm{D}_{\mathrm{pst}}(V_{\mathfrak{p}})$ are algebraic integers with the complex eigenvalues $\geq p^{\frac{w(\mathfrak{p})}{2}}$ by the assumption (**MW**). Hence $\widetilde{\alpha}(\mathrm{Frob}_p)_{\mathfrak{p}}$ is not a root of unity for each $\mathfrak{p} \in \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ with $w(\mathfrak{p}) > 0$. This completes the proof. □

For the proof of Proposition 4.3, we introduce other subgroups of $H^1(\mathbb{Q}_p, \mathcal{A})$. Define $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ by:

$$H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(w,w')}) = \mathrm{Ker}\left[H^1(\mathbb{Q}_p, A_{s,t}^{(w,w')}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^- A_{s,t}^{(w,w')})\right].$$

The inductive limit $\varinjlim_{s,t} H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ is equal to $\mathrm{Ker}\left[H^1(\mathbb{Q}_p, \mathcal{A}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\mathcal{A})\right]$. Since it is also independent of the choice of (w, w') with $0 \leq w - 1 \leq w'$, we denote it by $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \mathcal{A})$. Let $H^1_f(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ be as given after Definition 3.9. By Lemma 4.1 and Lemma 4.2, $H^1_f(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ is the maximal divisible subgroup of $H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(w,w')})$. Taking inductive limit with respect to s, t, we define

$$H^1_f(\mathbb{Q}_p, \mathcal{A})^{(w,w')} = \varinjlim_{s,t} H^1_f(\mathbb{Q}_p, A_{s,t}^{(w,w')}).$$

By [BK, Proposition 3.8], the group $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ is the Pontryagin dual of $H^1_f(\mathbb{Q}_p, \mathcal{A})^{(w,w')}$. Hence Proposition 4.3 is equivalent to the following proposition:

Proposition 4.6. *Assume the condition* (**MW**) *for* $\mathcal{T}$. *Let* (w, w') *be a pair of integers such that* $0 \leq w - 1 \leq w'$. *Then the following statements hold:*

1. *The group* $H^1_f(\mathbb{Q}_p, \mathcal{A})^{(w,w')}$ *is a subgroup of* $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$.
2. *If further* $w \neq 1$, $H^1_f(\mathbb{Q}_p, \mathcal{A})^{(w,w')}$ *is equal to* $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$.

For a finitely generated $\mathcal{R}$-module M, we denote the specialization $M/\Phi_{s,t}^{(w,w')}M$ by $M_{s,t}^{(w,w')}$. For $(s', t') \geq (s, t)$, we have a natural surjection $M_{s',t'}^{(w,w')} \twoheadrightarrow M_{s,t}^{(w,w')}$. We define the augmentation map $M_{s,t}^{(w,w')} \longrightarrow M_{s',t'}^{(w,w')}$ by $x \mapsto \sum_{g \in G_\infty^{p^s}/G_\infty^{p^{s'}}, g' \in D_\infty^{p^t}/D_\infty^{p^{t'}}} gg'\widetilde{x}$, where $\widetilde{x}$ is a lift of x. The augmentation map is well-defined and is independent of the choice of a lift $\widetilde{x}$. We have the following lemma:

Lemma 4.7. *Let* M *be a finitely generated torsion* $\mathcal{R}$*-module whose ideal of support has height at least two. Let* (w, w') *be a pair of integers. For any* $t \geq 0$, *assume that* $M_{s,t}^{(w,w')}$ *is a finite group whose order is bounded when* $s \geq 0$ *varies. Then the limit* $\varinjlim_{s,t} M_{s,t}^{(w,w')}$ *with respect to the augmentation maps above is equal to zero.*

Proof. It suffices to show that $\varinjlim_s M_{s,t}^{(w,w')} = 0$ for each $t \geq 0$. By the assumption of the lemma, $\varprojlim_s M_{s,t}^{(w,w')}$ is finite for any $t \geq 0$. Hence there exists a sufficiently large natural number such that $G_\infty^{s_0}$ acts trivially on $\varprojlim_s M_{s,t}^{(w,w')}$. For $s' > s \geq s_0$, the augmentation map $M_{s,t}^{(w,w')} \longrightarrow M_{s',t}^{(w,w')}$ is the multiplication by $p^{s'-s}$. Hence, by taking $s' - s$ greater than the p-order of $\sharp M_{s,t}^{(w,w')}$, $M_{s,t}^{(w,w')} \longrightarrow M_{s',t}^{(w,w')}$ is the zero map. This completes the proof. □

Let $\Phi_s^{(w)}$ (resp. $\Psi_t^{(w')}$) be the height one ideal of $\mathbb{Z}_p[[G_\infty]]$ (resp. $\mathbb{Z}_p[[D_\infty]]$) defined to be the kernel of the algebra homomorphism $\chi^{wp^s} : \mathbb{Z}_p[[G_\infty]] \longrightarrow \overline{\mathbb{Q}}_p$ (resp. $\kappa^{w'p^t} : \mathbb{Z}_p[[D_\infty]] \longrightarrow \overline{\mathbb{Q}}_p$). We also denote by $\Phi_s^{(w)}$ and $\Psi_t^{(w')}$ the height one ideals of $\mathbb{Z}_p[[G_\infty \times D_\infty]]$ through the inclusions $\mathbb{Z}_p[[G_\infty]] \hookrightarrow \mathbb{Z}_p[[G_\infty \times D_\infty]]$ and $\mathbb{Z}_p[[D_\infty]] \hookrightarrow \mathbb{Z}_p[[G_\infty \times D_\infty]]$.

Lemma 4.8. *Assume the condition* (**MW**) *for* $\mathcal{T}$. *Let* (w, w') *be a pair of integers such that* $w' > 0$. *Then, for any* $t \geq 0$, $(T_{s,t}^{(w,w')})_{G_{\mathbb{Q}_p}} = (\mathcal{T}_{G_{\mathbb{Q}_p}})_{s,t}^{(w,w')}$ (*resp.* $(\mathrm{F}^+T_{s,t}^{(w,w')})_{G_{\mathbb{Q}_p}} = (\mathrm{F}^+\mathcal{T}_{G_{\mathbb{Q}_p}})_{s,t}^{(w,w')}$) *is a finite group whose order is bounded when* $s \geq 0$ *varies.*

Proof. As a $G_{\mathbb{Q}_p}$-module, we have the following exact sequence (see Definition 3.1):

$$0 \longrightarrow \mathbb{H}(\widetilde{\alpha})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow \mathcal{T} \longrightarrow \mathbb{H}(\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0)\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow 0,$$

where $\mathbb{H}(\widetilde{\alpha})$ (resp. $\mathbb{H}(\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0)$) is the free $\mathbb{H}$-module of rank one on which $G_{\mathbb{Q}_p}$ acts via the character $\widetilde{\alpha}$ (resp. $\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0$). Hence we have the following exact sequence for each $t \geq 0$:

$$0 \longrightarrow \mathbb{H}(\widetilde{\alpha})_{\Psi_t^{(w')}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow \mathcal{T}_{\Psi_t^{(w')}} \longrightarrow \mathbb{H}(\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0)_{\Psi_t^{(w')}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow 0,$$

where $M_{\Psi_t^{(w')}}$ means the quotient $M/\Psi_t^{(w')}M$. By the assumption that $w' > 0$, the largest coinvariant quotient by the action of $G_{\mathbb{Q}_p(\mu_{p^\infty})}$ on $\mathbb{H}(\widetilde{\alpha})_{\Psi_t^{(w')}}$ and $\mathbb{H}(\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0)_{\Psi_t^{(w')}}$ is finite group. We have the following exact sequence:

$$(\mathbb{H}(\widetilde{\alpha})_{\Psi_t^{(w')}})_{G_{\mathbb{Q}_p(\mu_{p^\infty})}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow (\mathcal{T}_{\Psi_t^{(w')}})_{G_{\mathbb{Q}_p(\mu_{p^\infty})}}$$
$$\longrightarrow (\mathbb{H}(\widetilde{\alpha}^{-1}\chi^{-1}\widetilde{\kappa}^{-1}\psi_0)_{\Psi_t^{(w')}})_{G_{\mathbb{Q}_p(\mu_{p^\infty})}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow 0.$$

It is easy to see that the coinvariant quotients by the action of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ in the right hand term and the left hand term of the above sequence are finite groups. Hence $(\mathcal{T}_{\Psi_t^{(w')}})_{G_{\mathbb{Q}_p}} = (\mathcal{T}_{G_{\mathbb{Q}_p}})_{\Psi_t^{(w')}}$ is a finite group. The assertion for $(\mathrm{F}^+T_{s,t}^{(w,w')})_{G_{\mathbb{Q}_p}}$ is proved in the same way. □

Proof of Proposition 4.3 *and Proposition* 4.6. Since Proposition 4.3 and Proposition 4.6 are equivalent to each other, it suffices to prove Proposition 4.6. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{A}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \\ & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \mathcal{A}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathcal{A}) & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\mathcal{A}). \end{array}$$

The cokernel of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}) \hookrightarrow H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, \mathcal{A})$ is a sub-quotient of $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathrm{F}^-\mathcal{A})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}) = \mathrm{Ker}[H^1(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\mathcal{A})$. Recall that $\mathrm{F}^-\mathcal{A}$ is isomorphic to

$$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}(\widetilde{\alpha}\chi\widetilde{\kappa}\psi_0^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}), \mathbb{Q}_p/\mathbb{Z}_p).$$

Let us decompose the Dirichlet character ψ_0 as $\psi_0 = \psi_0'\psi_0''$ where the conductor of ψ_0' is prime to p and the conductor of ψ_0'' is of p-power order. Then $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathrm{F}^-\mathcal{A})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}})$ is the Pontryagin dual of the representation:

$$((\mathbb{H}(\widetilde{\alpha}\chi\widetilde{\kappa}\psi_0^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}))_{G_{\mathbb{Q}_p^{\mathrm{ur}}}})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} = \mathbb{H}(\widetilde{\alpha}(\psi_0')^{-1})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}.$$

By Lemma 4.5, $\mathbb{H}(\widetilde{\alpha}(\psi_0')^{-1})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is zero. This completes the proof of the assertion 1.

Let us now prove the assertion 2 in the rest. Since $w \neq 1$, we have the following commutative diagram for $s, t \geq 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V_{s,t}^{(w,w')}) & \longrightarrow & H^1(\mathbb{Q}_p, V_{s,t}^{(w,w')}) & \xrightarrow{\underline{d}_{s,t}} & H^1(\mathbb{Q}_p, \mathrm{F}^-V_{s,t}^{(w,w')}) & \longrightarrow & 0 \\ & & \underline{a}_{s,t}\downarrow & & \downarrow\underline{b}_{s,t} & & \downarrow\underline{c}_{s,t} & & \\ 0 & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{s,t}^{(w,w')}) & \longrightarrow & H^1(\mathbb{Q}_p, A_{s,t}^{(w,w')}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^-A_{s,t}^{(w,w')}). & & \end{array}$$

Note that $H^1(\mathbb{Q}_p, V_{s,t}^{(w,w')}) \xrightarrow{\underline{d}_{s,t}} H^1(\mathbb{Q}_p, \mathrm{F}^-V_{s,t}^{(w,w')})$ is surjective since $H^2(\mathbb{Q}_{p,s}, \mathrm{F}^+V_t^{(w,w')}) \cong H^0(\mathbb{Q}_p, (\mathrm{F}^+V_{s,t}^{(w-1,w')})^*)^* = 0$ by the assumption that $w \neq 1$, where $(\ )^*$ means the $\mathbb{Q}_p$-linear dual here. By definition, $H^1_f(\mathbb{Q}_p, A_{s,t}^{(w,w')})$ is equal to the image of $\underline{a}_{s,t}$. We have

the following exact sequence by applying the snake lemma to the above commutative diagram:

$$\varinjlim_{s,t} \frac{H^1(\mathbb{Q}_p, T_{s,t}^{(w,w')})}{H^1(\mathbb{Q}_p, T_{s,t}^{(w,w')})_{\mathrm{tor}}} \longrightarrow \varinjlim_{s,t} \frac{H^1(\mathbb{Q}_p, \mathrm{F}^- T_{s,t}^{(w,w')})}{H^1(\mathbb{Q}_p, \mathrm{F}^- T_{s,t}^{(w,w')})_{\mathrm{tor}}}$$
$$\longrightarrow \frac{H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})}{H^1_f(\mathbb{Q}_p, \mathcal{A})^{(w,w')}} \longrightarrow \varinjlim_{s,t} H^2(\mathbb{Q}_p, T_{s,t}^{(w,w')})_{\mathrm{tor}},$$

where tor means the $\mathbb{Z}_p$-torsion part. We have:

$$\varinjlim_{s,t} H^2(\mathbb{Q}_p, T_{s,t}^{(w,w')}) \cong \varinjlim_{s,t} H^0(\mathbb{Q}_p, \mathrm{Hom}(T_{s,t}^{(w,w')}, \mathbb{Q}_p/\mathbb{Z}_p(1)))^\vee$$
$$\cong \varinjlim_{s,t} (T_{s,t}^{(w-1,w')})_{G_{\mathbb{Q}_p}} \cong \varinjlim_{s,t} (\mathcal{T}_{G_{\mathbb{Q}_p}})_{s,t}^{(w-1,w')}.$$

Since $w \neq 1$, we have $w' > 0$. Hence $\varinjlim_{s,t} H^2(\mathbb{Q}_p, T_{s,t}^{(w,w')})$ must be zero by Lemma 4.7 and Lemma 4.8. On the other hand, the group:

$$\varinjlim_{s,t} \mathrm{Coker} \left[\frac{H^1(\mathbb{Q}_p, T_{s,t}^{(w,w')})}{H^1(\mathbb{Q}_p, T_{s,t}^{(w,w')})_{\mathrm{tor}}} \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathrm{F}^- T_{s,t}^{(w,w')})}{H^1(\mathbb{Q}_p, \mathrm{F}^- T_{s,t}^{(w,w')})_{\mathrm{tor}}} \right]$$

is a quotient of

$$\varinjlim_{s,t} \mathrm{Coker} \left[H^1(\mathbb{Q}_p, T_{s,t}^{(w,w')}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^- T_{s,t}^{(w,w')}) \right],$$

which is a subgroup of $\varinjlim_{s,t} H^2(\mathbb{Q}_p, \mathrm{F}^+ T_{s,t}^{(w,w')})$. By the same argument as above, we prove that $\varinjlim_{s,t} H^2(\mathbb{Q}_p, \mathrm{F}^+ T_{s,t}^{(w,w')}) = 0$ by Lemma 4.7 and Lemma 4.8. This completes the proof. □

Let Δ, $\mathcal{R}_\Delta$ be as defined before Theorem 3.13. For each integer i such that $0 \leq i \leq p-2$, we define the idempotent $e_i \in \mathbb{Z}_p[\Delta]$ to be $e_i = \dfrac{1}{p-1} \sum_{g \in \Delta} \omega^{-i}(g) g$. For a finitely generated $\mathcal{R}$-module M and each i, $e_i M$ is equal to the submodule of M on which Δ acts by the character ω^i. Each $e_i M$ is an $\mathcal{R}_\Delta$-module and we have the decomposition $M = \bigoplus_{0 \leq i \leq p-2} e_i M$ as an $\mathcal{R}$-module.

Lemma 4.9. *Assume the condition* (**MW**) *for* $\mathcal{T}$. *For each* $0 \leq i \leq p-2$, *we have the following statements.*

1. *The Pontryagin dual* $e_i(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee)$ *of* $e_{p-2-i}(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}))$ *is a finitely generated* $\mathcal{R}_\Delta$*-module such that the extension* $e_i(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ *is a one dimensional vector space over* $\mathrm{Frac}(\mathcal{R}_\Delta)$.

2. *For a pair* (w, w') *of integers such that* $0 \leq w-1 \leq w'$, $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right)$ *is a finitely generated* $\mathcal{R}_\Delta$*-module such that* $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ *is a one dimensional vector space over* $\mathrm{Frac}(\mathcal{R}_\Delta)$.

Proof. First, we calculate the group $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$. By definition, we have the following exact sequence:

$$H^0(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{A}) \longrightarrow H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A}) \longrightarrow 0.$$

By definition, $e_i\left(H^0(\mathbb{Q}_p, \mathrm{F}^-\mathcal{A})\right)$ is a torsion $\mathcal{R}_\Delta$-module for each $0 \leq i \leq p-2$. By this, it suffices to show that the Pontryagin dual $e_i\left(H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}})\right)$ of $e_{p-2-i}\left(H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{A})\right)$ is a finitely generated $\mathcal{R}_\Delta$-module of rank one for each $0 \leq i \leq p-2$. Consider the inflation-restriction sequence of continuous Galois cohomology:

$$0 \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{\mathcal{T}})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} \longrightarrow 0.$$

Here, the last map is surjective since $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ has cohomological dimension one. Since $\mathrm{F}^-\overline{\mathcal{T}}^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}$ is zero, $H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}})$ is isomorphic to $H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{\mathcal{T}})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$. Now, we have the following isomorphism:

$$\begin{aligned} H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{\mathcal{T}}) &\cong H^1\big(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{H}(\widetilde{\alpha}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}\chi)\big) \\ &= H^1\big(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}\chi))\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1}). \end{aligned}$$

By Shapiro's lemma, $H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}\chi))$ is isomorphic to $\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(1))$. On the other hand, the cohomology $H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(1))$ is isomorphic to the p-adic completion of $(\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s}))^\times$ by the Kummer theory where $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ is the p-adic completion of $\mathbb{Q}_p^{\mathrm{ur}}$. Hence, we have $H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(1)) \cong \pi_s^{\mathbb{Z}_p} \times U_s^1$ where U_s^1 is the group of principal units of $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})$ and π_s is a uniformizer of the complete discrete valuation field $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})$. Thus, $H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}^{-1}\chi))$ is isomorphic to $\mathbb{Z}_p \times \varprojlim_s U_s^1$. Recall that we have the following lemma by the theory of Coleman power series (see Lemma 5.8 and Lemma 5.9 for the theory of Coleman power series) :

Lemma 4.10. *We have the following short exact sequence:*

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \varprojlim_s U_s^1 \longrightarrow \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \longrightarrow 0.$$

By the above lemma, we have the following exact sequence:

$$\begin{aligned} 0 \longrightarrow (\mathbb{Z}_p \times \mathbb{Z}_p(1))\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{\mathcal{T}}) \\ \longrightarrow (\mathbb{Z}_p \times \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]])\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1}) \longrightarrow 0. \end{aligned}$$

Since $\widetilde{\alpha}$ is non-trivial by the condition (**MW**) and by Lemma 4.5, $\mathbb{H}(\widetilde{\alpha}^{-1})^{G_{\mathbb{Q}_p}}$ is zero. By Lemma 3.3, $\left(\mathbb{H}(\widetilde{\alpha}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\right)^{G_{\mathbb{Q}_p}} = \varprojlim_s (\mathbb{H}(\widetilde{\alpha}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_s]$ is free of rank one over $\mathcal{R}$. Thus the $G_{\mathbb{Q}_p}$-invariant of $(\mathbb{Z}_p \times \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]])\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1})$ is a free

$\mathcal{R}$-module of rank one. By taking Galois cohomology for $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ of the above short exact sequence, we have

$$0 \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \longrightarrow \mathcal{R} \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathbb{Z}_p \times \mathbb{Z}_p(1))\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1})).$$

Since $(\mathbb{Z}_p \times \mathbb{Z}_p(1))\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1})$ is torsion over $\mathcal{R}_\Delta$, $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathbb{Z}_p \times \mathbb{Z}_p(1))\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{H}(\widetilde{\alpha}^{-1}))$ is a torsion $\mathcal{R}_\Delta$-module. Thus, $e_i\left(H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}})\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ is a one dimensional $\mathrm{Frac}(\mathcal{R}_\Delta)$-vector space for each $0 \leq i \leq p-2$. This completes the proof of the assertion 1.

By Proposition 4.3, $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ is isomorphic to $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee$ for $w \neq 1$. Hence the assertion 2 follows from the assertion 1 if $w \neq 1$. In the rest, we treat the case, $w = 1$. The group $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(1,w')})$ is the Pontryagin dual of $H^1_f(\mathbb{Q}_p, \mathcal{A})$. Since $H^1_f(\mathbb{Q}_p, \mathcal{A})^{(1,w')}$ is a subgroup of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$ by Proposition 4.6, it suffices to show that $e_i(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})/H^1_f(\mathbb{Q}_p, \mathcal{A})^{(1,w')})$ is a cotorsion $\mathcal{R}_\Delta$-module for each $1 \leq i \leq p-2$. Consider the following commutative diagram for each $s, t \geq 0$:

$$\begin{array}{ccccccc}
H^0(\mathbb{Q}_p, \mathrm{F}^-V_{s,t}^{(1,w')}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^+V_{s,t}^{(1,w')}) & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, V_{s,t}^{(1,w')}) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle a_{s,t}} & & \downarrow{\scriptstyle b_{s,t}} & & \\
H^0(\mathbb{Q}_p, \mathrm{F}^-A_{s,t}^{(1,w')}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^+A_{s,t}^{(1,w')}) & \longrightarrow & H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{s,t}^{(1,w')}) & \longrightarrow & 0.
\end{array}$$

By Lemma 4.1 and Lemma 4.2, we have:

$$\mathrm{Image}(b_{s,t}) \subset H^1_f(\mathbb{Q}_p, A_{s,t}^{(1,w')}) \subset H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(1,w')}).$$

Since $\varinjlim_{s,t} H^1_{\mathrm{Gr}}(\mathbb{Q}_p, A_{s,t}^{(1,w')})$ is equal to $\varinjlim_{s,t} H^1_{\mathrm{Gr}'}(\mathbb{Q}_p, A_{s,t}^{(1,w')})$ by the proof of Proposition 4.3, it suffices to show that $e_i\left(\varinjlim_{s,t}\mathrm{Coker}(b_{s,t})\right)$ is a cotorsion $\mathcal{R}_\Delta$-module for each $1 \leq i \leq p-2$. In the above commutative diagram, we have $\mathrm{Coker}(a_{s,t})$ is a subgroup of $H^2(\mathbb{Q}_p, \mathrm{F}^+T_{s,t}^{(w,w')}) = (\mathrm{F}^+T_{s,t}^{(w-1,w')})_{G_{\mathbb{Q}_p}}$. The module $e_i\left(\varinjlim_{s,t}(\mathrm{F}^+T_{s,t}^{(w-1,w')})_{G_{\mathbb{Q}_p}}\right)$ is a cotorsion $\mathcal{R}_\Delta$-module for each $0 \leq i \leq p-2$. Since $e_i\left(\varinjlim_{s,t}\mathrm{Coker}(b_{s,t})\right)$ is a subquotient of $e_i\left(\varinjlim_{s,t}(\mathrm{F}^+T_{s,t}^{(w-1,w')})_{G_{\mathbb{Q}_p}}\right)$, this completes the proof. □

Proposition 4.11. *Let (w, w') be a pair of integers such that $0 < w - 1 < w'$. Then the group $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')}\right)$ is a torsion-free $\mathcal{R}_\Delta$-module for each $0 \leq i \leq p-2$.*

Proof. For the proof, it suffices to give an injective $\mathcal{R}$-module homomorphism from $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')})$ into a free $\mathcal{R}$-module of rank one. Consider the following short exact sequence:

$$H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^+\overline{T}_{s,t}^{(w,w')}) \longrightarrow H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t}^{(w,w')}) \longrightarrow H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^-\overline{T}_{s,t}^{(w,w')}).$$

By Proposition 3.8 of [BK], $H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^+\overline{T}^{(w,w')}_{s,t})$ is the Pontryagin dual of the group $H^1_f(\mathbb{Q}_p, \mathrm{F}^- A^{(w,w')}_{s,t})$. On the representation $\mathrm{F}^- V^{(w,w')}_{s,t}$, a sufficiently small open subgroup of $G_{\mathbb{Q}_p}$ acts via the character $\chi^{w-w'-1}$ modulo twist by an unramified character for all sufficiently large s. Since we assume that $w - w' - 1 < 0$, $H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^+\overline{T}^{(w,w')}_{s,t}) = H^1_f(\mathbb{Q}_p, \mathrm{F}^- A^{(w,w')}_{s,t})^\vee$ is zero by [BK, Corollary 3.8.4]. This gives us an injection:

$$H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t}) \hookrightarrow H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}).$$

Consider the commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t} \otimes B_{\mathrm{crys}}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t} \otimes B_{\mathrm{crys}}). \end{array}$$

Since the kernel of the restriction map

$$H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t} \otimes B_{\mathrm{crys}}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t} \otimes B_{\mathrm{crys}})$$

is $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \mathrm{D}_{\mathrm{crys}}(\mathrm{F}^-\overline{V}^{(w,w')}_{s,t}) \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}) = 0$, we have an injection:

$$\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) \hookrightarrow \left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t})\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$$

The representation $\mathrm{F}^-\overline{T}^{(w,w')}_{s,t}$ is isomorphic to $\mathbb{Z}_p[[G_\infty]](\chi\widetilde{\chi}^{-1})_{\Phi_s^{(w)}} \otimes_{\mathbb{Z}_p} \mathbb{H}(\widetilde{\alpha}^{-1})_{\Psi_t^{(w')}}$, where $\Phi_s^{(w)}$ and $\Psi_t^{(w')}$ is as defined before Lemma 4.8. Hence, we have the following isomorphisms:

$$\begin{aligned} H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) &\cong H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_\infty]](\chi\widetilde{\chi}^{-1})_{\Phi_s^{(w)}}) \otimes_{\mathbb{Z}_p} \mathbb{H}(\widetilde{\alpha}^{-1})_{\Psi_t^{(w')}} \\ &\cong H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^{1-w})) \otimes_{\mathbb{Z}_p} \mathbb{H}(\widetilde{\alpha}^{-1})_{\Psi_t^{(w')}} \\ &\cong \frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^{1-w}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^{1-w}))} \otimes_{\mathbb{Z}_p} \mathbb{H}(\widetilde{\alpha}^{-1})_{\Psi_t^{(w')}}, \end{aligned}$$

where the second isomorphism is obtained by Shapiro's lemma and the third isomorphism is due to [BK, Corollary 3.8.4]. Hence we have

$$\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) \cong \left(\varprojlim_{s} \frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^{1-w}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^{1-w}))}\right) \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{H}(\widetilde{\alpha}^{-1}).$$

We have the following claim:

Claim 4.12. The inverse limit $\varprojlim_{s} \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^{1-w}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^{1-w}))}$ is isomorphic to the limit $\varprojlim_{s} \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi))}$.

By the same argument as that of the proof of Lemma 4.9, we have:

$$\varprojlim_s H^1_{/f}(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^-\overline{T}^{(w,w')}_{s,t}) \cong \mathbb{H}(\widetilde{\alpha}^{-1}) \times \left(\mathbb{H}(\widetilde{\alpha}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\right).$$

As shown in the proof of Lemma 4.9, the $G_{\mathbb{Q}_p}$-invariant of $\mathbb{H}(\widetilde{\alpha}^{-1})\times\left(\mathbb{H}(\widetilde{\alpha}^{-1})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\right)$ is a free $\mathcal{R}$-module of rank one. This completes the proof assuming the above claim. Finally, we prove Claim 4.12. Consider the following short exact sequence of $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-modules:

$$0 \longrightarrow \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \overset{\times(\gamma-1)}{\longrightarrow} \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}) \longrightarrow \mathbb{Z}_p \longrightarrow 0, \tag{1}$$

where γ is a topological generator of G_∞. By the connecting homomorphism of the Galois cohomology of this sequence for $G_{\mathbb{Q}_p^{\mathrm{ur}}}$, we have the following isomorphism:

$$\mathbb{Z}_p \overset{\sim}{\longrightarrow} H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))[(\gamma-1)] \tag{2}$$

Let $r \neq 0$ be an integer. For each $s \geq 0$, we have the following short exact sequence by applying the functor $\otimes_{\mathbb{Z}_p[[G_\infty]]}\mathbb{Z}_p[[G_\infty]]/(\gamma^{p^s} - \chi^r(\gamma^{p^s}))$ of the sequence (1):

$$0 \longrightarrow \mathbb{Z}_p(\chi^r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_s] \overset{\times(\gamma-1)}{\longrightarrow} \mathbb{Z}_p(\chi^r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_s] \longrightarrow \mathbb{Z}_p/(\chi^r(\gamma^{p^s})-1)\mathbb{Z}_p \longrightarrow 0, \tag{3}$$

where $G_s = G_\infty/G_\infty^{p^s}$. By taking the Galois cohomology of this sequence for $G_{\mathbb{Q}_p^{\mathrm{ur}}}$, we have the following isomorphism:

$$\mathbb{Z}_p/(\chi^r(\gamma^{p^s})-1)\mathbb{Z}_p \overset{\sim}{\longrightarrow} H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p(\chi^r) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_s])[(\gamma-1)] \tag{4}$$

By taking projective limit of (4) with respect to $s \geq 0$, we have the isomorphism of (2). On the other hand, the left hand side (resp. the right hand side) of (4) is isomorphic to $H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^r))$ (resp. $H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^r))[(\gamma-1)]$) Hence $\varprojlim_s \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(\chi^r))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^r))}$ is isomorphic to $\dfrac{H^1(\mathbb{Q}_p, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))}{H^1(\mathbb{Q}_p, \mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))[\gamma-1]}$. Especially, $\varprojlim_s \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathbb{Z}_p(\chi^r))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p/\mathbb{Z}_p(\chi^r))}$ is isomorphic to each other for every $r \neq 0$. □

Corollary 4.13. *Let $\mathcal{T}$ be a nearly ordinary deformation. Assume the condition* (**MW**) *for $\mathcal{T}$. Let (w, w') be a pair of integers such that $0 \leq w - 1 \leq w'$.*

1. *Then the group $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}_{s,t})$ is the Pontryagin dual of $H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})$. Especially, the $\mathcal{R}$-module $\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})$ is independent of the choice of (w, w').*
2. *For each integer i such that $0 \leq i \leq p-2$, $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right)$ is a torsion free $\mathcal{R}_\Delta$-module such that $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ is a one dimensional vector space over $\mathrm{Frac}(\mathcal{R}_\Delta)$.*

Proof. For any pair (w, w') such that $0 < w - 1 < w'$, the assertions are already shown by Proposition 4.3, Lemma 4.5 and Proposition 4.11. Assume now that $w = 1$ or $w = w' + 1$. We already know that $e_i(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee)$ is a torsion free $\mathcal{R}_\Delta$-module such that $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ is a rank one vector space over $\mathrm{Frac}(\mathcal{R}_\Delta)$. By Proposition 4.3, $e_i\left(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ is a quotient of $e_i(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee)$. Since

$e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right) \otimes_{\mathcal{R}_\Delta} \mathrm{Frac}(\mathcal{R}_\Delta)$ is also a rank one vector space over $\mathrm{Frac}(\mathcal{R}_\Delta)$, $e_i\left(H^1_{\mathrm{Gr}}(\mathbb{Q}_p, \mathcal{A})^\vee\right)$ must be equal to $e_i\left(\varprojlim_{s,t} H^1_{/f}(\mathbb{Q}_p, \overline{T}^{(w,w')}_{s,t})\right)$ for each $0 \leq i \leq p-2$. This completes the proof. □

5. Proof of Theorem 3.13

In this section, we prove Theorem 3.13. Throughout the section, we assume that $\mathcal{T}$ is a nearly ordinary deformation in the sense of Definition 3.1. First, we give the interpolation of exponential maps for $\mathcal{T}$ (Theorem 5.3) by using the inverse of the Coleman power series map for the cyclotomic tower $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^\infty})/\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. Then we obtain the desired result (Theorem 3.13) on the interpolation of the dual exponential maps for $\overline{\mathcal{T}}$ by taking the $\mathcal{R}$-linear dual. We will keep the notation of the previous sections.

Proposition 5.1. *Let the assumptions and the notations be as in Theorem* 3.13. *Assume that $\mathcal{R}$ is Gorenstein, $\mathcal{R}_\Delta$ is a normal domain and that $\mathcal{T}$ satisfies the condition* (**MW**). *Let us fix a basis d of the $\mathbb{H}$-module $\mathcal{D}$ defined in Definition* 3.5. *Then, we have an $\mathcal{R}$-linear homomorphism*

$$\overline{\Xi}_{d,+} : H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \longrightarrow \mathcal{R}$$

with the following properties:

1. *The kernel and the cokernel of $\overline{\Xi}_{d,+}$ are pseudo-null $\mathcal{R}$-modules.*
2. *Let $\mathcal{C}$ be an element of $H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}})$ and let $c_{\eta,\mathfrak{p}} \in H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{T}_{\eta,\mathfrak{p}})$ be the specialization of $\mathcal{C}$ at $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ satisfying $0 \leq w-1 \leq w'$ for $w = w(\eta)$ and $w' = w(\mathfrak{p})$. Then, $\overline{\Xi}_{d,+}(\mathcal{C})_{\eta,\mathfrak{p}}$ is given as follows*:

$$(-1)^{w-1}(w-1)!\left(\frac{a_\mathfrak{p}}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_\mathfrak{p}}\right)\left(1-\frac{a_\mathfrak{p}\phi(p)}{p^w}\right)^{-1}\langle \exp^*(c_{\eta,\mathfrak{p}}), d_{\eta,\mathfrak{p}}\rangle,$$

where $\langle\ ,\ \rangle$ is the de Rham pairing

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}}) \times \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}) \longrightarrow \mathrm{D}_{\mathrm{dR}}(K_{\eta,\mathfrak{p}}(1)) \cong K_{\eta,\mathfrak{p}},$$

ϕ is the finite order character $\eta\chi^{-w}$ and s is the p-order of the conductor of ϕ.

Let us show that Theorem 3.13 is deduced from this proposition. In fact, the de Rham module $\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(\overline{V}_{\eta,\mathfrak{p}})$ is canonically isomorphic to $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})$ by Lemma 3.2 and the dual exponential map $H^1(\mathbb{Q}_p, \overline{V}_{\eta,\mathfrak{p}}) \xrightarrow{\exp^*} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})$ factors through:

$$H^1(\mathbb{Q}_p, \overline{V}_{\eta,\mathfrak{p}}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}}) \xrightarrow{\exp^*} \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}}).$$

We define $\overline{\Xi}'_d$ to be the following composite map:

$$H^1(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \xrightarrow{\overline{\Xi}_{d,+}} \mathcal{R}.$$

The map $\overline{\Xi}'_d$ satisfies the desired interpolation property. The cokernel of the natural map $H^1(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}})$ is a submodule of $H^2(\mathbb{Q}_p, \mathrm{F}^+\overline{\mathcal{T}}) \cong \mathrm{F}^+\overline{\mathcal{T}}(-1)_{G_{\mathbb{Q}_p}}$. Since

$\mathrm{F}^+\overline{\mathcal{T}}(-1)_{G_{\mathbb{Q}_p}}$ is pseudo-null by Lemma 4.8, the cokernel of $\overline{\Xi}'_d$ is a pseudo-null $\mathcal{R}$-module. As explained in §3, the map $\overline{\Xi}'_d$ factors through:

$$H^1(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}}) \xrightarrow{\overline{\Xi}_d} \mathcal{R}.$$

Since $H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}})$ is a torsion free $\mathcal{R}$-module of rank one by Corollary 4.13, the map $\overline{\Xi}_d : H^1_{/f}(\mathbb{Q}_p, \overline{\mathcal{T}}) \longrightarrow \mathcal{R}$ is an injective $\mathcal{R}$-homomorphism whose cokernel is a pseudo-null $\mathcal{R}$-module. Thus, we deduce Theorem 3.13 from Proposition 5.1.

Let us recall the definition of the exponential map of Bloch-Kato.

Definition 5.2. Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. We have the following short exact sequence of $G_{\mathbb{Q}_p}$-modules (see (3.8.4) of [BK]):

$$0 \longrightarrow V \longrightarrow (B_{\mathrm{crys}}^{f=1} \oplus B_{\mathrm{dR}}^+) \otimes V \longrightarrow B_{\mathrm{dR}} \otimes V \longrightarrow 0.$$

The exponential map $\mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V) \xrightarrow{\exp} H^1(\mathbb{Q}_p, V)$ is the map induced by the connecting homomorphism of the long exact sequence of the Galois cohomology of $G_{\mathbb{Q}_p}$:

$$0 \longrightarrow H^0(\mathbb{Q}_p, V) \longrightarrow \mathrm{D}_{\mathrm{crys}}^{f=1}(V) \oplus \mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}(V) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V) \longrightarrow H^1(\mathbb{Q}_p, V).$$

Let us denote by $\mathcal{I}$ the height two ideal $(\widetilde{\alpha}(\mathrm{Frob}_p) - 1, \gamma - 1)$ of $\mathcal{R}$, where γ is a topological generator of G_∞. We have the following proposition:

Proposition 5.3. *Assume the condition* (**MW**) *for* $\mathcal{T}$. *Let* $\mathcal{D}$ *be as in Definition* 3.5. *Then we have an injective* $\mathcal{R}$*-linear homomorphism*

$$\Xi_+ : \mathcal{I}(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]]) \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T})$$

with the following properties :

1. *The cokernel of* Ξ_+ *is a pseudo-null* $\mathcal{R}$*-module.*
2. *Let* $(\eta, \mathfrak{p}) \in \mathfrak{X}_{\mathrm{arith}}(G_\infty) \times \mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ *satisfying* $0 \leq w - 1 \leq w'$ *for* $w = w(\eta)$ *and* $w' = w(\mathfrak{p})$. *Then, we have the following commutative diagram* :

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T}) & \xleftarrow{\Xi_+} & \mathcal{I}(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]]) \\ {\scriptstyle \mathrm{Sp}_{\eta,\mathfrak{p}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sp}_{\eta,\mathfrak{p}}} \\ H^1(\mathbb{Q}_p, \mathrm{F}^+V_{\eta,\mathfrak{p}}) & \xleftarrow{m_{\eta,\mathfrak{p}}} & \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}), \end{array}$$

where $m_{\eta,\mathfrak{p}}$ *is the map*

$$(-1)^{w-1}(w-1)!\left(\frac{a_\mathfrak{p}}{p^{w-1}}\right)^{-s}\left(1 - \frac{p^{w-1}\phi(p)}{a_\mathfrak{p}}\right)\left(1 - \frac{a_\mathfrak{p}\phi(p)}{p^w}\right)^{-1}\exp.$$

Let us fix a pair of integers (w, w'). Let $\Phi_{s,t,u}^{(w,w')}$ be the height three ideal $(\Phi_{s,t}^{(w,w')}, p^u)$ of $\mathcal{R}$ (see §1 for the definition of $\Phi_{s,t}^{(w,w')}$) for positive integers s, t, u and let us denote by $\mathcal{R}_{s,t,u}$ the ring $\mathcal{R}/\Phi_{s,t,u}^{(w,w')}$ for short. Since we assume that $\mathcal{R}$ is a Gorenstein $\mathbb{Z}_p[[G_\infty \times D_\infty]]$-algebra, $\mathrm{Hom}_{\mathbb{Z}_p[[G_\infty \times D_\infty]]}(\mathcal{R}, \mathbb{Z}_p[[G_\infty \times D_\infty]])$ is free of rank one as an $\mathcal{R}$-module. Hence

$\mathrm{Hom}_{(\mathbb{Z}/p^u\mathbb{Z})[G_s\times D_t]}(\mathcal{R}_{s,t,u},(\mathbb{Z}/p^u\mathbb{Z})[G_s\times D_t])$ is free $\mathcal{R}_{s,t,u}$-module of rank one. On the other hand, we have

$$\mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(\mathcal{R}_{s,t,u},\mathbb{Z}/p^u\mathbb{Z})\xrightarrow{\sim}\mathrm{Hom}_{(\mathbb{Z}/p^u\mathbb{Z})[G_s\times D_t]}(\mathcal{R}_{s,t,u},(\mathbb{Z}/p^u\mathbb{Z})[G_s\times D_t])$$

defined by $f\mapsto\sum_{g\in G_s\times D_s}gf(g^{-1}x)$. Thus $\mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(\mathcal{R}_{s,t,u},\mathbb{Z}/p^u\mathbb{Z})$ is a free $\mathcal{R}_{s,t,u}$-module of rank one, where G_s (resp. D_t) is the group $G_\infty/G_\infty^{p^s}$ (resp. $D_\infty/D_\infty^{p^t}$). Hence $\mathcal{R}_{s,t,u}$ is a zero dimensional Gorenstein ring in the sense of [E, §21]. We have the following lemma:

Lemma 5.4. *Let S be a cofinitely generated $\mathcal{R}$-module and let $S^\vee$ be the Pontryagin dual of S. Assume that $\mathcal{R}$ is Gorenstein. Then there exists an isomorphism $\varprojlim_{s,t,u}S[\Phi_{s,t,u}^{(w,w')}]\cong\mathrm{Hom}_{\mathcal{R}}(S^\vee,\mathcal{R})$ for each fixed pair of integers (w,w').*

Proof. For the proof of this lemma, it suffices to give the following isomorphism for each $s,t,u\geq 0$:

$$S[\Phi_{s,t,u}^{(w,w')}]=\mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(S^\vee/\Phi_{s,t,u}^{(w,w')}S^\vee,\mathbb{Z}/p^u\mathbb{Z})\cong\mathrm{Hom}_{\mathcal{R}_{s,t,u}}(S^\vee/\Phi_{s,t,u}^{(w,w')}S^\vee,\mathcal{R}_{s,t,u}).\tag{5}$$

Since $\mathrm{Hom}_{\mathcal{R}_{s,t,u}}(S^\vee/\Phi_{s,t,u}^{(w,w')}S^\vee,\mathcal{R}_{s,t,u})$ is isomorphic to $\mathrm{Hom}_{\mathcal{R}}(S^\vee,\mathcal{R})\otimes_{\mathcal{R}}\mathcal{R}_{s,t,u}$, the lemma is proved by taking the projective limit with respect to s,t,u once we have (5). The first equality of (5) is nothing but the definition of the Pontryagin dual. In fact, since S is equal to $(S^\vee)^\vee$, we have

$$S[\Phi_{s,t,u}^{(w,w')}]=\mathrm{Hom}_{\mathbb{Z}_p}(S^\vee,\mathbb{Q}_p/\mathbb{Z}_p)[\Phi_{s,t,u}^{(w,w')}]=\mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(S^\vee/\Phi_{s,t,u}^{(w,w')}S^\vee,\mathbb{Z}/p^u\mathbb{Z}).$$

The last isomorphism of (5) is due to the fact that $\mathcal{R}_{s,t,u}$ is a zero dimensional Gorenstein ring. For the fundamental properties of zero dimensional Gorenstein rings and the modules over zero dimensional Gorenstein rings, we refer the reader to [E, §21]. □

Now, we deduce Proposition 5.1 from Proposition 5.3.

Proof of Proposition 5.1. Let $\mathrm{F}^-\overline{\mathcal{A}}$ be $\mathrm{F}^-\overline{\mathcal{T}}\otimes_{\mathcal{R}}\mathcal{R}^\vee$ where $\mathcal{R}^\vee$ is the Pontryagin dual of $\mathcal{R}$. We have the following map:

$$H^1(\mathbb{Q}_p,\mathrm{F}^-\overline{\mathcal{T}})\xrightarrow{\sim}\varprojlim_{s,t,u}H^1\big(\mathbb{Q}_p,\mathrm{F}^-\overline{\mathcal{T}}/\Phi_{s,t,u}^{(w,w')}\mathrm{F}^-\overline{\mathcal{T}}\big)\xrightarrow{\xi}\varprojlim_{s,t,u}H^1\big(\mathbb{Q}_p,\mathrm{F}^-\overline{\mathcal{A}}\big)[\Phi_{s,t,u}^{(w,w')}]\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{R}}\big(H^1(\mathbb{Q}_p,\mathrm{F}^+\mathcal{T}),\mathcal{R}\big)$$

The last isomorphism is due to Lemma 5.4 and the fact that $H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T})$ is the Pontryagin dual of $H^1\big(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}}\big)$. The map ξ is defined to be the composite:

$$\begin{aligned}\varprojlim_{s,t,u} H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}/\Phi_{s,t,u}^{(w,w')}\mathrm{F}^-\overline{\mathcal{T}}) &\xrightarrow{\sim} \varprojlim_{s,t,u} H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}}[\Phi_{s,t,u}^{(w,w')}]) \\ &\xrightarrow{\xi_1} \varprojlim_{s,t,u} H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}}[\Phi_s^{(w)}, \Psi_t^{(w')}])[p^u] \\ &\xrightarrow{\xi_2} \varprojlim_{s,t,u} H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}}[\Phi_s^{(w)}])[\Psi_t^{(w')}, p^u] \\ &\xrightarrow{\xi_3} \varprojlim_{s,t,u} H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}})[\Phi_{s,t,u}^{(w,w')}],\end{aligned}$$

where $\Phi_s^{(w)}$ and $\Psi_t^{(w')}$ are as defined before 4.8. The first isomorphism is obtained by the assumption that $\mathcal{R}$ is Gorenstein and is induced by an isomorphism $\mathcal{R}_{s,t,u} \cong \mathrm{Hom}_{\mathbb{Z}/p^u\mathbb{Z}}(\mathcal{R}_{s,t,u}, \mathbb{Z}/p^u\mathbb{Z})$ as an $\mathcal{R}$-module. Note that the kernel and the cokernel of each ξ_i are annihilated by the annihilator of the module $H^0(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{A}})$, which is a height two ideal of $\mathcal{R}$. Thus, we obtained an $\mathcal{R}$-linear homomorphism:

$$H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \longrightarrow \mathrm{Hom}_{\mathcal{R}}\Big(H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T}), \mathcal{R}\Big)$$

whose kernel and cokernel are pseudo-null $\mathcal{R}$-modules. We define an $\mathcal{R}$-linear map $\overline{\Xi}_+$ to be the composite map:

$$\begin{aligned}(6)\quad H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) &\longrightarrow \mathrm{Hom}_{\mathcal{R}}\big(H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T}), \mathcal{R}\big) \\ &\longrightarrow \mathrm{Hom}_{\mathcal{R}}\big(\mathcal{I}(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]]), \mathcal{R}\big) \cong \mathrm{Hom}_{\mathcal{R}}\big(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]], \mathcal{R}\big),\end{aligned}$$

where the first map is the one constructed above, the second one is the dual of Ξ_+.

Let us show the last isomorphism. By applying e_i, it suffices to show that the natural injection

$$\alpha_i : \mathrm{Hom}_{\mathcal{R}_\Delta}\big(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty/\Delta]], \mathcal{R}_\Delta\big) \hookrightarrow \mathrm{Hom}_{\mathcal{R}_\Delta}\big(e_i(\mathcal{I})(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty/\Delta]]), \mathcal{R}_\Delta\big)$$

is an isomorphism for each $0 \le i \le p-2$, where $e_i(\mathcal{I}) \subset \mathcal{R}_\Delta$ is the projection to $e_i(\mathcal{R}) \cong \mathcal{R}_\Delta$. Since $e_i(\mathcal{I})$ is a height two ideal of $\mathcal{R}_\Delta$, the localization $(\alpha_i)_{\mathfrak{q}}$ of α_i is an $(\mathcal{R}_\Delta)_{\mathfrak{q}}$-linear isomorphism for every height one prime $\mathfrak{q}$ of $\mathcal{R}_\Delta$. Since we assume that $\mathcal{R}_\Delta$ is normal, this implies that α_i is an isomorphism for each $0 \le i \le p-2$. Hence the last isomorphism of (6) is given by the inverse of the map $\bigoplus_{0\le i\le p-2} \alpha_i$.

Let us fix a basis d of the $\mathbb{H}$-module $\mathcal{D}$. We define an $\mathcal{R}$-linear map $\overline{\Xi}_{d,+}$ to be the composite map:

$$H^1(\mathbb{Q}_p, \mathrm{F}^-\overline{\mathcal{T}}) \xrightarrow{\overline{\Xi}_+} \mathrm{Hom}_{\mathcal{R}}\big(\mathcal{D}\widehat{\otimes}\mathbb{Z}_p[[G_\infty]], \mathcal{R}\big) \xrightarrow{f\mapsto f(d\otimes 1,\ )} \mathcal{R}.$$

By the above argument, the kernel and the cokernel of $\overline{\Xi}_{d,+}$ are pseudo-null $\mathcal{R}$-modules.

Before describing the interpolation property of the map $\overline{\Xi}_{d,+}$, we recall the following lemma:

Lemma 5.5 ([Ka1], Chap. II, Theorem 1.4.1)**.** *For each* $(\eta,\mathfrak{p})\in\mathfrak{X}_{\mathrm{arith}}(G_\infty)\times\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ *with* $0\leq w(\eta)-1\leq w(\mathfrak{p})$*, the dual exponential map* $H^1(\mathbb{Q}_p,\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})\longrightarrow\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})$ *coincides with the composite map*:

$$H^1(\mathbb{Q}_p,\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})\xrightarrow{\sim}\mathrm{Hom}_{K_{\eta,\mathfrak{p}}}(H^1(\mathbb{Q}_p,\mathrm{F}^+V_{\eta,\mathfrak{p}}),K_{\eta,\mathfrak{p}})\\ \longrightarrow\mathrm{Hom}_{K_{\eta,\mathfrak{p}}}(\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}),K_{\eta,\mathfrak{p}})\cong\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}}),$$

where the first isomorphism is the local Tate duality for the Galois cohomology, the second map is the $K_{\eta,\mathfrak{p}}$*-linear dual of the exponential map of* $\mathrm{F}^+V_{\eta,\mathfrak{p}}$*, and the third map is induced by the pairing*:

$$\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})\times\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^-\overline{V}_{\eta,\mathfrak{p}})\longrightarrow\mathrm{D}_{\mathrm{dR}}(K_{\eta,\mathfrak{p}}(1))\cong K_{\eta,\mathfrak{p}}.$$

Let us denote by $[\ ,\ ]$ the $\mathcal{R}$-linear paring:

$$H^1(\mathbb{Q}_p,\mathrm{F}^-\overline{\mathcal{T}})\times H^1(\mathbb{Q}_p,\mathrm{F}^+\mathcal{T})\longrightarrow\mathcal{R}$$

given by (1). For an arithmetic character η (resp. arithmetic point $\mathfrak{p}$) of G_∞ (resp. $\mathbb{H}$), we have:

$$\begin{aligned}\overline{\Xi}_{d,+}(\mathcal{C})_{\eta,\mathfrak{p}}&=\left[\mathcal{C},\Xi_+(d\widehat{\otimes}1)\right]_{\eta,\mathfrak{p}}\\&=(-1)^{w-1}(w-1)!\left(\frac{a_{\mathfrak{p}}}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_{\mathfrak{p}}}\right)\left(1-\frac{a_{\mathfrak{p}}\phi(p)}{p^w}\right)^{-1}(c_{\eta,\mathfrak{p}},\exp(d_{\eta,\mathfrak{p}}))\\&=(-1)^{w-1}(w-1)!\left(\frac{a_{\mathfrak{p}}}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{a_{\mathfrak{p}}}\right)\left(1-\frac{a_{\mathfrak{p}}\phi(p)}{p^w}\right)^{-1}\langle\exp^*(c_{\eta,\mathfrak{p}}),d_{\eta,\mathfrak{p}}\rangle\end{aligned}$$

Thus Proposition 5.1 is reduced to Proposition 5.3. □

Before giving the proof of Proposition 5.3, we review classical results by Coleman and Perrin-Riou. As in the definition given in §2, the finite part $H^1_f(\mathbb{Q}_p^{\mathrm{ur}},V)$ is defined by the kernel of the map $H^1(\mathbb{Q}_p^{\mathrm{ur}},V)\longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}},V\otimes B_{\mathrm{crys}})$.

Definition 5.6. We define a subring $\mathcal{H}_\infty$ of $\mathbb{Q}_p[[X]]$ to be:

$$\left\{g=\sum_{i\geq0}a_iX^i\in\mathbb{Q}_p[[X]]\ \middle|\ \exists h_g\in\mathbb{N}\text{ such that }\lim_{n\to\infty}n^{-h_g}|a_n|_p=0\right\}$$

where $|\ |_p$ is the p-adic absolute value normalized so that $|p|_p=\frac{1}{p}$. Recall that we have an isomorphism $\mathbb{Z}_p[[G_\infty/\Delta]]\xrightarrow[\gamma\mapsto X+1]{\sim}\mathbb{Z}_p[[X]]$ once we fix a topological generator γ of G_∞/Δ. Let $\mathrm{Func}(\mathfrak{X}(G_\infty/\Delta),\mathbb{Q}_p)$ be the ring of $\mathbb{Q}_p$-valued functions on the space $\mathfrak{X}(G_\infty/\Delta)$ of continuous characters from G_∞/Δ into $\mathbb{Q}_p^\times$. Since we fix a topological generator γ of G_∞/Δ, this gives us an injection $\mathcal{H}_\infty\hookrightarrow\mathrm{Func}(\mathfrak{X}(G_\infty/\Delta),\mathbb{Q}_p)$ by $g=\sum_{i\geq0}a_iX^i\longmapsto\{\rho\in\mathfrak{X}(G_\infty/\Delta)\mapsto\sum_{i\geq0}a_i(\rho(\gamma)-1)^i\}$. We also check that the subalgebra $\mathcal{H}_\infty$ of $\mathrm{Func}(\mathfrak{X}(G_\infty/\Delta),\mathbb{Q}_p)$ does not depend on the choice of the topological generator γ. We denote by $\mathcal{H}_\infty(G_\infty)$, the extension $\mathcal{H}_\infty\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Delta]$. $\mathcal{H}_\infty(G_\infty)$ is a subring of $\mathrm{Func}(\mathfrak{X}(G_\infty/\Delta),\mathbb{Q}_p)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p[\Delta]$ which contains $\mathbb{Z}_p[[G_\infty]]$.

Perrin-Riou [P2] interpolates the Bloch-Kato exponential maps for crystalline representations T of the Galois group of an absolute unramified complete discrete valuation field of mixed characteristic in the cyclotomic tower. Let $w \geq 1$. For each $s \geq 0$, we denote by P_s the natural projection map $\dfrac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}), \mathbb{Z}_p(w))} \underset{\mathbb{Z}_p[[G_\infty]]}{\otimes} \mathcal{H}_\infty(G_\infty) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p(w))$. Let $\mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}}$ be the canonical integral lattice of $\mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))$. By abuse of notation, we denote by the same symbol P_s, the $\mathbb{Z}_p[[G_\infty]]$-linear homomorphism $\mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}} \otimes \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \longrightarrow \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w)) \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})$ induced by $x \otimes g \mapsto x \otimes \zeta_{p^s}^g$, where $x \in \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}}$ and $g \in G_\infty$. For the trivial representation $T = \mathbb{Z}_p$ of $G_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} = G_{\mathbb{Q}_p^{\mathrm{ur}}}$, her result is stated as follows:

Proposition 5.7. [P2, Theorem 3.2.3] *For each integer $w \geq 1$, we have a $\mathbb{Z}_p[[G_\infty]]$-linear map:*

$$\Xi^{\mathrm{ur},(w)} : \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}} \otimes \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \longrightarrow \frac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}), \mathbb{Z}_p(w))} \underset{\mathbb{Z}_p[[G_\infty]]}{\otimes} \mathcal{H}_\infty(G_\infty),$$

with the following properties:

1. *We have the following commutative diagram for each integer $s \geq 0$:*

$$\begin{array}{ccc} \dfrac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}), \mathbb{Z}_p(w))} \underset{\mathbb{Z}_p[[G_\infty]]}{\otimes} \mathcal{H}_\infty(G_\infty) & \xleftarrow{\Xi^{\mathrm{ur},(w)}} & \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}} \otimes \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \\ \Big\downarrow{\scriptstyle \mathrm{P}_s} & & \Big\downarrow{\scriptstyle \mathrm{P}_s} \\ H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p(w)) & \xleftarrow{m_{w,s}} & \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w)) \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s}), \end{array}$$

   where $m_{w,s}$ is the map:

$$\begin{cases} (-1)^{w-1}(w-1)!\exp \circ \left(\dfrac{\sigma}{p^{w-1}}\right)^{-s} & \text{if } s > 0, \\ (-1)^{w-1}(w-1)!\exp \circ \left(1 - \dfrac{p^{w-1}}{\sigma}\right)\left(1 - \dfrac{\sigma}{p^w}\right)^{-1} & \text{if } s = 0. \end{cases}$$

2. *For each $w \geq 1$, we have*

$$\Xi^{\mathrm{ur},(w+1)} = (-1)(\otimes\{\zeta_{p^s}\}_{s\geq 0}) \circ \Xi^{\mathrm{ur},(w)} \circ (\otimes\delta^{-1}_{\mathbb{Q}_p(1)}).$$

If $w = 1$, the above map $\Xi^{\mathrm{ur},(1)}$ is given by the theory of Coleman power series. The cohomology group $H_f^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(1))$ is identified with the group of principal units U_s^1 of $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})$ in the group $H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Z}_p(1)) = \varprojlim_n (\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})^\times / p^n)$ and the Bloch-Kato exponential map $\mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(1)) \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s}) \xrightarrow{\exp} H_f^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}), \mathbb{Q}_p(1))$ is identified with the classical p-adic exponential map $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s}) \xrightarrow{\exp} U_s^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let φ be the operator on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$ which maps $(1+Z)$ to $(1+Z)^p$ and which acts on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ by (arithmetic) Frobenius automorphism σ. Via $\varphi : \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]] \longrightarrow \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$, $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$ is free of rank p over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$. We denote by T_φ (resp. N_φ) its trace (resp. norm).

Lemma 5.8 (Coleman)**.** *There is an isomorphism of groups:*

$$\mathrm{Col} : \varprojlim_s U_s^1 \xrightarrow{\sim} \left(1+(p,Z)\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]\right)^{N_\varphi=\sigma},$$

which sends a norm compatible system $u=(u_s)_{s\geq 1} \in \varprojlim_s U_s^1$ *to a power series* $g_u \in \left(1+(p,Z)\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]\right)^{N_\varphi=\sigma} \subset (\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]])^\times$ *such that* $g_u(\zeta_{p^s}-1)^{\sigma^{-s}}=u_s$ *for each* $s\geq 1$.

The group G_∞ acts on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$ so that $g\cdot(1+Z)=(1+Z)^{\chi(g)}$ for $g\in G_\infty$. We denote by $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]](1+Z)$ the rank one $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$-module generated by $(1+Z)$.

Lemma 5.9. [P1, Theorem 2.3′]

(1) *The image of the map*

$$(1-\frac{\varphi}{p})\log : \left(1+(p,Z)\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]\right)^{N_\varphi=\sigma} \longrightarrow \widehat{\mathbb{Q}}_p^{\mathrm{ur}}[[Z]]$$

is equal to $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]^{\mathrm{T}_\varphi=0}$.

(2) *The* $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$*-submodule* $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]^{\mathrm{T}_\varphi=0}$ *of* $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[Z]]$ *is identified with* $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]](1+Z)$ *by* $g\cdot(1+Z)=(1+Z)^{\chi(g)}$.

(3) *We have an exact sequence:*

$$0\longrightarrow \mathbb{Z}_p(1)\longrightarrow \varprojlim_s U_s^1 \xrightarrow{(1-\frac{\varphi}{p})\circ\log\circ\mathrm{Col}} \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]](1+Z)\longrightarrow 0.$$

This gives us a commutative diagram of the interpolation of the exponential maps when $T=\mathbb{Z}_p$ and $w=1$ by putting $\Xi^{\mathrm{ur},(1)}$ to be the inverse map of $(1-\frac{\varphi}{p})\circ\log\circ\mathrm{Col}$:

$$\begin{array}{ccc}
\dfrac{\varprojlim_s H_f^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(1))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(1))} & \xleftarrow[\mathrm{Inv}((1-\frac{\varphi}{p})\circ\log\circ\mathrm{Col})]{\sim} & \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(1))_{\mathrm{Int}}\otimes\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \\
{\scriptstyle \mathrm{P}_s}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{P}_s} \\
H_f^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Q}_p(1)) & \xleftarrow{m_s} & \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(1))\otimes\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s}),
\end{array}$$

where the map m_s is equal to $\begin{cases} \exp\circ\,\sigma^{-s} & \text{if } s>0, \\ \exp\circ\left(1-\sigma^{-1}\right)\left(1-\dfrac{\sigma}{p}\right)^{-1} & \text{if } s=0. \end{cases}$

Since the twist operators

$$\frac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(w))} \xrightarrow{\otimes\{\zeta_{p^s}\}_{s\geq 0}} \frac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(w+1))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(w+1))}$$

and

$$\mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w))_{\mathrm{Int}}\otimes\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]] \xrightarrow{\otimes\delta_{\mathbb{Q}_p(1)}} \mathrm{D}_{\mathrm{crys}}^{\mathrm{ur}}(\mathbb{Q}_p(w+1))_{\mathrm{Int}}\otimes\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$$

are isomorphisms, the image of $\Xi^{\mathrm{ur},(w)}$ is contained in $\dfrac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(w))}$ and we do not have to extend $\dfrac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(w))}$ to $\dfrac{\varprojlim_s H^1(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^s}),\mathbb{Z}_p(w))}{H^0(\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty}),\mathbb{Z}_p(w))}\underset{\mathbb{Z}_p[[G_\infty]]}{\otimes}$

$\mathcal{H}_\infty(G_\infty)$ when $T=\mathbb{Z}_p$. Let $\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(V)=(V\otimes B_{\mathrm{dR}})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}$ for a p-adic representation V of $G_{\mathbb{Q}_p}$. If V is a crystalline representation of $G_{\mathbb{Q}_p}$, we have the canonical isomorphisms $\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(V)\cong\mathrm{D}^{\mathrm{ur}}_{\mathrm{crys}}(V)$ and $\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(V\otimes\mathbb{Q}_p[G_s])\cong\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(V)\otimes\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})\cong\mathrm{D}^{\mathrm{ur}}_{\mathrm{crys}}(V)\otimes\widehat{\mathbb{Q}}_p^{\mathrm{ur}}(\mu_{p^s})$. Thus in the case where $T=\mathbb{Z}_p$, Proposition 5.7 gives the following proposition:

Proposition 5.10. *We have a $\mathbb{Z}_p[[G_\infty]]$-linear isomorphism:*

$$\Xi^{\mathrm{ur}}:\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\longrightarrow\frac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p)},$$

with the following commutative diagram for each $\eta\in\mathfrak{X}_{\mathrm{arith}}(G_\infty)$ with $w(\eta)\geq 1$:

$$\begin{array}{ccc}\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p)} & \xleftarrow{\Xi^{\mathrm{ur}}} & \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\\ \Big\downarrow{\scriptstyle \mathrm{Sp}_\eta} & & \Big\downarrow{\scriptstyle \mathrm{Sp}_\eta}\\ H^1(\mathbb{Q}_p^{\mathrm{ur}},K_\eta(\eta)) & \xleftarrow{m_\eta} & \mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(K_\eta(\eta)).\end{array}$$

In the above diagram, m_η is the map

$$(-1)^{w-1}(w-1)!\exp\left(\frac{\sigma}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{\sigma}\right)\left(1-\frac{\sigma\phi(p)}{p^w}\right)^{-1},$$

where ϕ is the finite order character $\eta\chi^{-w}$ of G_∞ and s is the p-order of the conductor of ϕ.

Let $\mathcal{T}$ be a nearly ordinary deformation. For each η and each $\mathfrak{p}$, we define a specialization map $\mathrm{Sp}_{\eta,\mathfrak{p}}:\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{F}^+\mathbb{T}\longrightarrow\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ to be the composite:

$$\begin{aligned}\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]&\xrightarrow{\mathrm{Sp}_{\mathfrak{p}}\otimes 1}\mathrm{F}^+V_{\mathfrak{p}}\otimes_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\\&\xrightarrow{1\otimes\mathrm{Sp}_\eta}\mathrm{F}^+V_{\mathfrak{p}}\otimes_{K_\eta\cap K_{\mathfrak{p}}}\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(K_\eta(\eta))=\mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})\end{aligned}$$

By taking the formal tensor product $\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{F}^+\mathbb{T}$ of the diagram obtained in Proposition 5.10, we have the following proposition:

Proposition 5.11. *Let $\mathcal{T}$ be a nearly ordinary deformation. We have an $\mathcal{R}$-linear isomorphism:*

$$\Xi^{\mathrm{ur}}_+:\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{F}^+\mathbb{T}\longrightarrow\frac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathbb{T})}$$

with the following commutative diagram for each η and each $\mathfrak{p}$ satisfying $0\leq w(\eta)-1\leq w(\mathfrak{p})$:

$$\begin{array}{ccc}\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathbb{T})} & \xleftarrow{\Xi^{\mathrm{ur}}_+} & \widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{F}^+\mathbb{T}\\ \Big\downarrow{\scriptstyle \mathrm{Sp}_{\eta,\mathfrak{p}}} & & \Big\downarrow{\scriptstyle \mathrm{Sp}_{\eta,\mathfrak{p}}}\\ H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+V_{\eta,\mathfrak{p}}) & \xleftarrow{m_{\eta,\mathfrak{p}}} & \mathrm{D}^{\mathrm{ur}}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}).\end{array}$$

In the above diagram, $m_{\eta,\mathfrak{p}}$ is the map

$$(-1)^{w-1}(w-1)!\exp\circ\left(\frac{\sigma}{p^{w-1}}\right)^{-s}\left(1-\frac{p^{w-1}\phi(p)}{\sigma}\right)\left(1-\frac{\phi(p)\sigma}{p^{w}}\right)^{-1},$$

where ϕ is the finite order character $\eta\chi^{-w(\eta)}$ of G_∞ and s is the p-order of the conductor of ϕ.

Proof. In fact, $\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathbb{T})}$ is isomorphic to $\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p[[G_\infty]](\widetilde{\chi}))}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathbb{Z}_p)}\widehat{\otimes}_{\mathbb{Z}_p}\mathrm{F}^+\mathbb{T}$ since $\mathrm{F}^+\mathbb{T}$ is an unramified representation of $G_{\mathbb{Q}_p}$. We define the desired map Ξ_+^{ur} to be $\Xi^{\mathrm{ur}}\widehat{\otimes}\mathrm{Id}_{\mathrm{F}^+\mathbb{T}}$. The map Ξ_+^{ur} is an $\mathcal{R}$-linear isomorphism because Ξ^{ur} is a $\mathbb{Z}_p[[G_\infty]]$-linear isomorphism. Since the exponential map for $\mathrm{F}^+V_{\eta,\mathfrak{p}}$ is the connecting homomorphism of the cohomology of $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-modules associated to the short exact sequence:

$$0\longrightarrow\mathrm{F}^+V_{\eta,\mathfrak{p}}\longrightarrow(B_{\mathrm{crys}}^{f=1}\oplus B_{\mathrm{dR}}^+)\otimes\mathrm{F}^+V_{\eta,\mathfrak{p}}\longrightarrow B_{\mathrm{dR}}\otimes\mathrm{F}^+V_{\eta,\mathfrak{p}}\longrightarrow 0,$$

we have the following commutative diagram :

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p^{\mathrm{ur}},K_\eta(\eta))\otimes_{\mathcal{O}_{K_\eta\cap K_\mathfrak{p}}}\mathrm{F}^+T_\mathfrak{p} & \xleftarrow{\exp\otimes\mathrm{Id}} & \mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(K_\eta(\eta))\otimes_{\mathcal{O}_{K_\eta\cap K_\mathfrak{p}}}\mathrm{F}^+T_\mathfrak{p}\\
\| & & \|\\
H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+V_{\eta,\mathfrak{p}}) & \xleftarrow{\exp} & \mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}),
\end{array}$$

where the map exp in the upper line is the exponential map of a $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module $K_\eta(\eta)$ and the map exp in the lower line is the exponential map of a $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module $\mathrm{F}^+V_{\eta,\mathfrak{p}}$. Hence we obtain the desired commutative diagram. □

To deduce Proposition 5.3 from Proposition 5.11, we prepare the following lemma:

Lemma 5.12. *Assume the condition* (**MW**) *for $\mathcal{T}$. Let $(\eta,\mathfrak{p})\in\mathfrak{X}_{\mathrm{arith}}(G_\infty)\times\mathfrak{X}_{\mathrm{arith}}(\mathbb{H})$ be a pair satisfying $0\le w(\eta)-1\le w(\mathfrak{p})$. Then the following statements hold:*

1. *The $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-invariant part of $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ is equal to $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$.*
2. *The operator σ on $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ induces the multiplication by $a_\mathfrak{p}$ on the $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-invariant part $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$.*
3. *The restriction map $H^1(\mathbb{Q}_p,\mathrm{F}^+V_{\eta,\mathfrak{p}})\longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+V_{\eta,\mathfrak{p}})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is an isomorphism.*
4. *We have an exact sequence:*

$$0\longrightarrow H^1(\mathbb{Q}_p,\mathrm{F}^+\mathcal{T})\longrightarrow\left(\frac{H^1(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}},\mathrm{F}^+\mathbb{T})}\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}\longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p,\mathrm{F}^+\mathbb{T}).$$

5. *The module $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p,\mathrm{F}^+\mathbb{T})$ is annihilated by the height two ideal $\mathcal{I}=(\gamma-1,\widetilde{\alpha}(\mathrm{Frob}_p)-1)\subset\mathcal{R}$.*

Proof. The assertion (1) is nothing but the definition of $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$ and $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})$. Since we have $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})\cong\mathrm{D}_{\mathrm{dR}}(K_\eta(\eta))\otimes_{K_\eta\cap K_\mathfrak{p}}\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_\mathfrak{p})$ and $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_\mathfrak{p})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}=\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_\mathfrak{p})$, it suffices to show that the arithmetic Frobenius σ on $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_\mathfrak{p})$ induces the multiplication by $a_\mathfrak{p}$ on $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_\mathfrak{p})$. Since $\mathrm{F}^+V_\mathfrak{p}$ is an unramified representation of $G_{\mathbb{Q}_p}$,

$\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\mathfrak{p}})$ is isomorphic to $\mathrm{F}^+V_{\mathfrak{p}} \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ and the operator σ on $\mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\mathfrak{p}})$ is identified with the operator $1 \otimes \sigma$ on $\mathrm{F}^+V_{\mathfrak{p}} \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$, which is equal to $\sigma^{-1} \otimes 1 = \mathrm{Frob}_p \otimes 1$ on the invariant part $\mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\mathfrak{p}}) = (\mathrm{F}^+V_{\mathfrak{p}} \otimes \widehat{\mathbb{Q}}_p^{\mathrm{ur}})^{\sigma\otimes\sigma}$. This completes the proof of (2). For the proof of (3), the restriction map $H^1(\mathbb{Q}_p, \mathrm{F}^+V_{\eta,\mathfrak{p}}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+V_{\eta,\mathfrak{p}})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is surjective since $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ has cohomological dimension one. The kernel of the restriction map is isomorphic to $H^1\bigl(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, (\mathrm{F}^+V_{\eta,\mathfrak{p}})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}}\bigr)$, which is zero since $(\mathrm{F}^+V_{\eta,\mathfrak{p}})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}} = 0$. For the assertion (4), let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T}) & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T}) & \longrightarrow & \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T})} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1-\mathrm{Frob}_p} & & \downarrow{\scriptstyle 1-\mathrm{Frob}_p} & & \downarrow{\scriptstyle 1-\mathrm{Frob}_p} & & \\
0 & \longrightarrow & H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T}) & \longrightarrow & H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T}) & \longrightarrow & \dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T})} & \longrightarrow & 0.
\end{array}$$

By the condition (**MW**), we have $(\mathrm{F}^+\mathbb{T})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)} = 0$. Since $(\mathrm{F}^+\mathcal{T})^{G_{\mathbb{Q}_p^{\mathrm{ur}}}} = 0$, the restriction map $H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T}) \longrightarrow H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T})^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$ is an isomorphism. By applying the snake lemma to the above commutative diagram, we obtain the desired short exact sequence. This completes the proof of (4). For (5), note that $H^1(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p, \mathrm{F}^+\mathbb{T})$ is annihilated by the ideal $(\gamma - 1)$ since $\mathrm{F}^+\mathbb{T} = \mathrm{F}^+\mathcal{T}/(\gamma-1)\mathrm{F}^+\mathcal{T}$ by definition. On the other hand, the Galois group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$ acts on $\mathrm{F}^+\mathbb{T}$ via the unramified character $\widetilde{\alpha}$. This completes the proof of (5). □

Let us return to the proof of Proposition 5.3

Proof of Proposition 5.3. Recall that $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-invariant part of the $\mathbb{Z}_p[[G_\infty \times D_\infty]]$-module $\mathrm{F}^+\mathbb{T}\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}}[[G_\infty]]$ is $\mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]]$ by definition. By Lemma 5.12, the image of the restriction of the map:

$$\mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] \xrightarrow{\Xi_+^{\mathrm{ur}}} \left(\frac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T})}\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$$

to $\mathcal{I}\mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]]$ is contained in $H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T}) \subset \left(\dfrac{H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathcal{T})}{H^0(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+\mathbb{T})}\right)^{\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)}$. We denote by Ξ_+ the map $\mathcal{I}\mathcal{D}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[G_\infty]] \longrightarrow H^1(\mathbb{Q}_p, \mathrm{F}^+\mathcal{T})$ thus obtained. For each η and each $\mathfrak{p}$ satisfying $0 \leq w(\eta) - 1 \leq w(\mathfrak{p})$, we have the following commutative diagram:

$$\begin{array}{ccc}
H^1(\mathbb{Q}_p^{\mathrm{ur}}, \mathrm{F}^+V_{\eta,\mathfrak{p}})^{G_{\mathbb{Q}_p}} & \xleftarrow{\exp} & \mathrm{D}_{\mathrm{dR}}^{\mathrm{ur}}(\mathrm{F}^+V_{\eta,\mathfrak{p}})^{G_{\mathbb{Q}_p}} \\
\| & & \| \\
H^1(\mathbb{Q}_p, \mathrm{F}^+V_{\eta,\mathfrak{p}}) & \xleftarrow{\exp} & \mathrm{D}_{\mathrm{dR}}(\mathrm{F}^+V_{\eta,\mathfrak{p}}),
\end{array}$$

where the map exp in the upper (resp. lower) line is the Bloch-Kato exponential map of $\mathrm{F}^+V_{\eta,\mathfrak{p}}$ as a $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module (resp. $G_{\mathbb{Q}_p}$-module). The equalities in the diagram are obtained by Lemma 5.12. The commutativity of the diagram is due to the fact that

the exponential map of $\mathrm{F}^+V_{\eta,\mathfrak{p}}$ as a $G_{\mathbb{Q}_p^{\mathrm{ur}}}$-module (resp. $G_{\mathbb{Q}_p}$-module) is the connecting homomorphism of the Galois cohomology for the short exact sequence:

$$0 \longrightarrow \mathrm{F}^+V_{\eta,\mathfrak{p}} \longrightarrow (B_{\mathrm{crys}}^{f=1} \oplus B_{\mathrm{dR}}^+) \otimes \mathrm{F}^+V_{\eta,\mathfrak{p}} \longrightarrow B_{\mathrm{dR}} \otimes \mathrm{F}^+V_{\eta,\mathfrak{p}} \longrightarrow 0,$$

of $G_{\mathbb{Q}_p}$-modules. Hence we have the commutative diagram of Proposition 5.3. □

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