Greenberg's view on generalizing Iwasawa theory via Galois deformations

Tadashi Ochiai (Osaka University)

June 14th, 2005

$\begin{cases} \text{Hida} \cdots \text{ deformations of modular forms} \\ \text{Mazur} \cdots \text{ deformations of Galois rep's} \end{cases}$

$$+$$

Iwasawa theory of <u>cyclotomic</u> deformation of motives(Mazur, Greenberg, etc)

$\Longrightarrow$ Iwasawa theory for <u>general</u> Galois deformations (Greenberg)

<u>Related articles by Greenberg</u>

[1] "p-adic L-functions and p-adic periods of modular forms" (with Stevens),(1993)
[2] " Iwasawa theory and p-adic deformations of motives "(1994)
[3] "Elliptic curves and p-adic deformations " (1994)

p : prime number fixed,
$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, fixed

$\mathcal{R}$: finite flat over $\mathbb{Z}_p[[X_1, \cdots, X_g]]$
$\mathcal{T} \cong \mathcal{R}^{\oplus d} \curvearrowleft^{\rho} G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
nearly ordinary filtration $\mathrm{F}^{+}\mathcal{T} \subset \mathcal{T}$

analytic p-adic L
$L_p^{\mathrm{anal}}(\mathcal{T})$

algebraic p-adic L
$L_p^{\mathrm{alg}}(\mathcal{T})$

General open problems

★ Construction of analytic p-adic L-function $L_p^{\mathsf{anal}}(\mathcal{T}) \in \mathcal{R}$ for various $\mathcal{T}$

★ Iwasawa Main conjecture $(L_p^{\mathsf{anal}}(\mathcal{T})) = (L_p^{\mathsf{alg}}(\mathcal{T}))$ for various $\mathcal{T}$

★ Zero and pole of $L_p^{\mathsf{anal}}(\mathcal{T})$ in Spec$(\mathcal{R})$

• Trivial zero of $L_p^{\mathsf{MTT}}(E, s)$ for the cyclotomic deformation of an elliptic curve split multiplicative at p (Greenberg-Stevens[1])

• Order of the two-variable p-adic L-function $L_p(f_k, j)$ along diagonal divisor $j = \frac{k}{2}$ in the case of nearly ordinary Hida deformation (Conjecture by Greenberg).

★ Iwasawa theory non nearly ordinary Galois deformations

• Iwasawa theory for Coleman family for modular forms with positive slope?

etc.

Hida's deformation

$\bullet \Gamma'$: the group of diamond operators on $\{Y_1(p^t)\}_{t\geq 1}$

$\bullet \chi' : \Gamma' \xrightarrow{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$

$\bullet \mathcal{F} = \sum\limits_{n\geq 1} A_n q^n$: ordinary Λ-adic newform of tame conductor N

$\bullet \mathbb{H} = \mathbb{Z}_p[[\Gamma']] \left[\{A_n\}_{n\geq 1}\right]$: finite extension of $\mathbb{Z}_p[[\Gamma']]$

Interpolation property of $\mathcal{F}$

$\mathbb{H} \xrightarrow{\kappa} \overline{\mathbb{Q}}_p$: arithmetic character of weight $k \geq 2$ (i.e. $\exists U \underset{\text{open}}{\subset} \Gamma'$, $\exists k \in \mathbb{Z}$ such that $\kappa|_U = {\chi'}^k$)

$\Rightarrow f_\kappa = \sum\limits_{n\geq 1} a_n(f_\kappa) q^n \in S_k(\Gamma_1(Np^*))$ $(a_n(f_\kappa) = \kappa(A_n))$

$\bullet\mathfrak{M}_{\mathbb{H}}$: the maximal ideal of $\mathbb{H}$, $\quad \mathbb{F} = \mathbb{H}/\mathfrak{M}_{\mathbb{H}}$
$\bullet\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$ semi-simple with
$\mathrm{Trace}(\overline{\rho}(\mathrm{Fr}_l)) \equiv A_l$ modulo $\mathfrak{M}_{\mathbb{H}}$ for almost all
l ("residual representation" $\overline{\rho}$ always exists)

Assume the following conditions:
(**Ir**) $\overline{\rho}$ is irreducible.
(**Fr**) $\exists \mathbb{T} \cong \mathbb{H}^{\oplus 2}$ & $\rho : G_{\mathbb{Q}} \longrightarrow GL(\mathbb{T})$
$\mathrm{Trace}(\rho(\mathrm{Fr}_l)) = A_l$ for almost all l

We study the two-variable representation:
$\mathcal{T} = \left(\mathbb{T}\widehat{\otimes}\mathbb{Z}_p[[\Gamma]](\tilde{\chi})\right) \otimes \omega^i$
(i is a fixed integer with $0 \leq i \leq p-2$)
$\mathcal{R} = \mathbb{H}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\Gamma]] = \mathbb{H}[[\Gamma]]$
where
$\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow[\chi]{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$
$\tilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathbb{Z}_p[[\Gamma]]^\times$

Algebraic p-adic L-function $L_p^{\mathrm{alg}}(\mathcal{T})$

$\mathcal{A} = \mathcal{T} \otimes_{\mathcal{R}} \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \mathbb{Q}/\mathbb{Z}_p) \curvearrowleft G_{\mathbb{Q}}$

$$\mathrm{Sel}_{\mathcal{T}} := \mathrm{Ker}\Big[H^1(\mathbb{Q}, \mathcal{A}) \to \prod_{l \neq p} H^1(I_l, \mathcal{A}) \times H^1(I_p, \mathcal{A}/\mathrm{F}^+\mathcal{A})\Big]$$

(Definition given by Greenberg in [2])

$L_p^{\mathrm{alg}}(\mathcal{T}) := \mathrm{char}_{\mathcal{R}}(\mathrm{Sel}_{\mathcal{T}})^{\vee}$

Analytic p-adic L function $L_p^{\mathrm{anal}}(\mathcal{T})$

We want $L_p^{\mathrm{anal}}(\mathcal{T}) \in \mathcal{R}$ with the interpolation:

$$(\chi^{j-1} \circ \kappa)(\mathcal{L}_p(\mathcal{T}))/C_{p,\kappa} = (-1)^{j-1}(j-1)!\left(1 - \frac{p^{j-1}}{a_p(f_\kappa)}\right) \times \frac{L(f_\kappa, j)}{(2\pi\sqrt{-1})^{j-1} C_{\infty,\kappa}}$$

for each arithmetic point $\kappa \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{H}, \overline{\mathbb{Q}}_p)$ of weight $k \geq 2$ and for each $1 \leq j \leq k-1$ ($i \equiv j$ modulo $p-1$)

$C_{p,\kappa}, C_{\infty,\kappa}$: p-adic and archimedean period for f_κ

Known results for $L_p^{\mathrm{anal}}(\mathcal{T})$

- Construction via Modular symbol:
Mazur, Kitagawa, Greenberg-Stevens, etc
- Construction via Rankin-Selberg integral:
Panchishkin, Ochiai, Fukaya, etc

Remark

1. The relation between different constructions are not clear. (In each construction, the definitions of $C_{p,\kappa}$ is slightly different)
2. Kitagawa's $L_p^{\mathrm{Ki}}(\mathcal{T})$ is globally defined over Spec($\mathcal{R}$).
3. In Kitagawa's $L_p^{\mathrm{Ki}}(\mathcal{T})$, the p-adic period $C_{p,\kappa}$ is a p-adic unit at each κ.

$$\Downarrow$$

Two-variable Iwasawa Main Conjecture

Under the condition (**Ir**), $(L_p^{\mathrm{alg}}(\mathcal{T})) = (L_p^{\mathrm{Ki}}(\mathcal{T}))$

Under the conditions (**Ir**) and (**Fr**), we have the following result:

Theorem. (Ochiai) Assume

(**H**) $\mathcal{R} \cong \mathcal{O}[[Y_1, Y_2]]$

(**G**) $\exists \tau \in G_{\mathbb{Q}}$ which is conjugate to $\begin{pmatrix} 1 & P_\tau \\ 0 & 1 \end{pmatrix}$ in $GL(\mathcal{T}) \cong GL_2(\mathcal{R})$ where $P_\tau \in \mathcal{R}^\times$. $\exists \tau' \in G_{\mathbb{Q}}$ acting on $\mathcal{T}/\mathfrak{M}\mathcal{T}$ via the multiplication by -1.

Then we have:

$$(L_p^{\mathrm{alg}}(\mathcal{T})) \supset (L_p^{\mathrm{Ki}}(\mathcal{T})).$$

Strategy of the proof

Theorem A (Ochiai, [2003] + preprint)

The condition (**Ir**) $\Longrightarrow$ an ideal of Euler system (ES) $\subset \mathcal{R}$ obtained by modifying (twisting) Beilinson-Kato elements

Further, we have (ES) $= (L_p^{\mathrm{Ki}}(\mathcal{T}))$

Theorem B (Ochiai, [2005])

Assume the condition (**H**) and (**G**). Then we have the following inequality:

$(L_p^{\mathrm{alg}}(\mathcal{T})) \supset$ (ES)

Further progress

1. Study of $\mathrm{Sel}_{\mathcal{T}/J\mathcal{T}} \longrightarrow \mathrm{Sel}_{\mathcal{T}}[J]$ (preprint)

Proposition

For each height-one prime $J \subset \mathcal{R}$,
(One-variable) Iwasawa Main Conjecture for $\mathcal{T}/J\mathcal{T} \Longleftrightarrow$ (Two-variable) Iwasawa Main Conjecture for $\mathcal{T}$

$J = (\gamma - \chi'(\gamma')\gamma')$ (interpolating $L(f_k, k-1)$)

- $\mathrm{Sel}(\mathcal{T}/J\mathcal{T})$ defined by Greenberg's method
- $L_p^{\mathrm{anal}}(\mathcal{T}/J\mathcal{T})$ defined by $L_p^{\mathrm{Ki}}(\mathcal{T})$ modulo J
- Iwasawa Main Conjecture for $\mathcal{T}/J\mathcal{T}$

$$(L_p^{\mathrm{anal}}(\mathcal{T}/J\mathcal{T})) = E \cdot (L_p^{\mathrm{alg}}(\mathcal{T}/J\mathcal{T}))$$

where $E = (1 - A_p(\mathcal{F}))$ or (1) (extra factor).

$J = (\gamma' - \kappa(\gamma'))$ (cyclo. deformation of f_κ)

Corollary of Proposition

Cyclo. I.M.C. for $f_\kappa \Longleftrightarrow$ Cyclo. I.M.C. for $f_{\kappa'}$
(for every arithmetic specilizations f_κ, $f_{\kappa'}$ of $\mathcal{F}$)
(without assuming $\mu = 0$)

2. Residually reducible cases (in preparation)
When (**Ir**) or (**Fr**) is false,
• $\mathcal{T} \sim \mathcal{T}'$ isogenious $\longrightarrow$ $\mathrm{Sel}_{\mathcal{T}}/\mathrm{Sel}_{\mathcal{T}'}$ is studied. (generalization of Perrin-Riou's result for variation of μ-invariant under isogeny in the cyclotomic deformation case)
• Iwasawa Main conjecture in the case without (**Ir**).

3. Examples (preprint + in preparation)
$\Delta = q \prod_{n \geq 1} (1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z}))$
p :ordinary prime for Δ $\quad (p \geq 11, p \neq 2411, \cdots)$
R $= \mathrm{Z}_p[[\Gamma \times \Gamma']]$.
$\mathcal{T} = \mathbb{T} \widehat{\otimes} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \omega^i \quad (0 \leq i \leq 10)$
such that $\mathbb{T}/(\gamma' - \chi'(\gamma')^{12}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V_{\Delta}$.

Lemma

Except for $(p, 1)$ with anomalous primes p of Δ, we have $\mathrm{Sel}_{\mathcal{T}} = 0$ and $L_p^{\mathrm{Ki}}(\mathcal{T})$ is a unit.
$p = 11, 23, 691, \cdots$ are anomalous primes. $(p, i) = (11, 1) \Rightarrow$
$(\mathrm{Sel}_{\mathcal{T}})^{\vee} \cong \mathbb{Z}_p[[\Gamma \times \Gamma']]/(\gamma^2 - \gamma'^{-1})$
$(L_p^{\mathrm{alg}}(\mathcal{T})) = (L_p^{\mathrm{Ki}}(\mathcal{T})) = (\gamma^2 - \gamma'^{-1})$ (I.M.C. is true).

$(p,i)=(691,1) \Rightarrow$
Lattices $\mathcal{T}$ are not unique. There is a minimal lattice $\mathcal{T}_0$ so that $(L_p^{\mathrm{alg}}(\mathcal{T}_0)) = (L_p^{\mathrm{Ki}}(\mathcal{T}_0)) = \mathcal{R}$.
For other lattices $\mathcal{T} \sim \mathcal{T}_0$, $L_p^{\mathrm{alg}}(\mathcal{T})/L_p^{\mathrm{alg}}(\mathcal{T}_0)$ and $L_p^{\mathrm{anal}}(\mathcal{T})/L_p^{\mathrm{anal}}(\mathcal{T}_0)$ are compared.
$(p,i)=(23,1) \Rightarrow$
The condition (**G**) is not satisfied.
(The residual representation is dihedral)

At the end, we will propose several problems related to results mentioned above...

<u>Problem 1</u> Weaken the condition (**H**) in Theorem B.
Greenberg-Stevens[1] $\Longrightarrow$ Example of the cases where Hecke algebra might have singularity.

Problem 2 Weaken the condition (**G**) in Theorem B. (Euler system should work for every "non-CM" deformations!)

Related to Problem 2, we are led to the following conjecture:

Conjecture

Let $\mathcal{T} \cong \mathbb{Z}_p[[X_1, \cdots, X_g]]^{\oplus d}$ and let $G \subset GL(\mathcal{T})$ be an analytic subgroup. Assume that
$\exists \phi \in \mathrm{Hom}(\mathbb{Z}_p[[X_1, \cdots, X_g]], \overline{\mathbb{Q}}_p)$
such that $G_\phi \subset GL_d(\overline{\mathbb{Q}}_p)$ is a reductive p-adic Lie group.

Then, $H^i(G, \mathcal{A})^\vee$ is a finitely generated torsion $\mathbb{Z}_p[[X_1, \cdots, X_g]]$-module for every i.

Remark

1. Conjecture in the case $g = 0$ is Lazard's result in his paper in 1965.
2. Example of G for $g > 0$ is nearly ordinary Hida deformation $\mathcal{T}$ and $G = \mathrm{Image}[G_{\mathbb{Q}} \to GL(\mathcal{T})]$.
3. Condition (**G**) implies $H^1(G, \mathcal{A}) = 0$ in the case of Hida deformation.