On the two-variable Iwasawa theory

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Main Reference

[1] "A generalization of the Coleman map for Hida deformation", the American Journal of Mathematics, 2003.
[2] "Euler system for Galois deformation", preprint, 2002, submitted.
[3] "On the two-variable Iwasawa Main conjecture for Hida deformations", in preparation.

Contents of the talk

★ Family of Galois representations
★ Two-variable p-adic L-function
★ Selmer groups
★ Two-variable Iwasawa Main conjecture
★ Examples, applications and problems (specializations to various one-variable deformations etc)

Situation

p : prime number,

$\mathcal{R}$: finite over $\mathbb{Z}_p[[X_1, \cdots, X_g]]$

$\mathcal{T} \cong \mathcal{R}^{\oplus d} \curvearrowleft G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

How much such big rep's $\mathcal{T}$?

- Example from Hida deformation (since early 80's)
- "Deformation theory" by Mazur (since late 80's)

Study of Arithmetic for $(\mathcal{R}, \mathcal{T})$?

$\bigstar g = 0$ case (over C.D.V.R) $\Rightarrow$ (Tamagawa Number Conjecture)

$\bigstar$cyclotomic deformation

$\Gamma := \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow[\chi]{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$

$\widetilde{\chi} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathcal{O}[[\Gamma]]^\times$

$T \cong \mathcal{O}^{\oplus d} \curvearrowleft G_{\mathbb{Q}}$

$\mathcal{T} = T \otimes \mathcal{O}[[\Gamma]](\widetilde{\chi})$, $\mathcal{R} = \mathcal{O}[[\Gamma]]$

(Mazur, Greenberg, $\cdots$,Perrin-Riou, Kato, $\cdots$)

$\bigstar$"$R = T$ Theorem" by Wiles etc

$\Downarrow$ (Greenberg)

Study of arithmetic for other $\mathcal{T}$?

(Iwasawa theory for $\mathcal{T}$?)

Hida's deformation

- Γ': the group of diamond operators on $\{Y_1(p^t)\}_{t\geq 1}$
- $\chi' : \Gamma' \xrightarrow{\sim} 1 + p\mathbb{Z}_p \subset (\mathbb{Z}_p)^\times$
- $\mathcal{F} = \sum_{n\geq 1} A_n q^n$: ordinary Λ-adic newform of tame conductor N
- $\mathbb{H} = \mathbb{Z}_p[[\Gamma']]\left[\{A_n\}_{n\geq 1}\right]$: finite extension of $\mathbb{Z}_p[[\Gamma']]$

Interpolation property of $\mathcal{F}$

$\mathbb{H} \xrightarrow{\kappa} \overline{\mathbb{Q}}_p$: arithmetic character of weight $k \geq 2$ (i.e. $\exists U \underset{\text{open}}{\subset} \Gamma'$, $\exists k \in \mathbb{Z}$ such that $\kappa|_U = {\chi'}^k$) $\Rightarrow$ $f_\kappa = \sum_{n\geq 1} a_n(\kappa) q^n \in S_k(\Gamma_1(Np^*))$ $(a_n(\kappa) = \kappa(A_n))$

$\bullet\mathfrak{M}_{\mathbb{H}}$: the maximal ideal of $\mathbb{H}$

$\bullet\mathbb{F} = \mathbb{H}/\mathfrak{M}_{\mathbb{H}}$

$\bullet\overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$ semi-simple with $\text{Trace}(\overline{\rho}(\text{Fr}_l)) \equiv A_l$ modulo $\mathfrak{M}_{\mathbb{H}}$ for almost all l (such $\overline{\rho}$ always exists)

Assume the following condition:

(**Ir**) $\overline{\rho}$ is irreducible.

Then we have

$\exists\mathbb{T} \cong \mathbb{H}^{\oplus 2}$ & $\rho : G_{\mathbb{Q}} \longrightarrow GL(\mathbb{T})$

$\text{Trace}(\rho(\text{Fr}_l)) = A_l$ for almost all l

$$\left(\varprojlim_t H^1_{\text{ét}}(X_1(Np^t)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}} \underset{\text{fin. degree}}{\twoheadrightarrow} \mathbb{T}\right)$$

$\mathcal{T} = \left(\mathbb{T}\widehat{\otimes}\mathbb{Z}_p[[\Gamma]](\widetilde{\chi})\right) \otimes \omega^i$

$\mathcal{R} = \mathbb{H}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[\Gamma]] = \mathbb{H}[[\Gamma]]$

<u>Our questions on $\mathcal{T}$?</u>

★ p-adic L-function for $\mathcal{T}$

★ Selmer group for $\mathcal{T}$

★ Iwasawa Main Conj. for $\mathcal{T}$

★ Similar studies as above on $\mathcal{T}/I\mathcal{T}$ for various ideals $I \subset \mathcal{R}$

($\mathfrak{X}(\mathcal{R}) := \{\text{characters on } \mathcal{R}\}$

$\mathfrak{X}(\mathcal{R}) \underset{\text{finite cover}}{\longrightarrow} D(0;1) \subset \overline{\mathbb{Q}}_p^{\oplus 2}$

$\mathcal{T}$ is a family of Galois representations over $\mathfrak{X}(\mathcal{R})$

$V(I) \subset \mathfrak{X}(\mathcal{R})$ closed subspace)

Two-variable p-adic L function

$\mathcal{L}_p(\mathcal{T}) \in \mathcal{R}$ with the interpolation property:

$(\chi^{j-1} \circ \kappa)(\mathcal{L}_p(\mathcal{T}))/C_{p,\kappa} =$

$$(-1)^{j-1}(j-1)! \left(1 - \frac{\omega^{i-j}(p)p^{j-1}}{a_p(f_\kappa)}\right) \times \left(\frac{p^{j-1}}{a_p(f_\kappa)}\right)^{q(i,j)} G(\omega^{j-i}) \times \frac{L(f_\kappa, \omega^{i-j}, j)}{(2\pi\sqrt{-1})^{j-1}C_{\infty,\kappa}}$$

for $1 \leq j \leq k-1$ ($k =$weight of κ)

$q(i,j) = \mathrm{ord}_p(\mathrm{Cond}(\omega^{i-j}))$

$G(\omega^{j-i})$: Gauss sum for ω^{j-i}

$C_{p,\kappa}, C_{\infty,\kappa}$: p-adic and archimedian period for f_κ

(Mazur, Kitagawa, Greenberg-Stevens, Ohta, $\cdots$, Panchishkin, Fukaya, Delbourgo, $\cdots$)

Two-variable Selmer groups

$\mathcal{A} = \mathcal{T} \otimes_{\mathcal{R}} \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \mathbb{Q}/\mathbb{Z}_p) \curvearrowleft G_{\mathbb{Q}}$

$\mathrm{Sel}_{\mathcal{T}} \subset H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}, \mathcal{A})$

unramified condition at $l \nmid p$

condition at p by $G_{\mathbb{Q}_p}$-filtration:

$0 \longrightarrow \mathrm{F}^+\mathcal{A} \longrightarrow \mathcal{A} \longrightarrow \mathrm{F}^-\mathcal{A} \longrightarrow 0$

The Pontrjagin dual $(\mathrm{Sel}_{\mathcal{T}})^\vee$ is a finitely generated torsion $\mathcal{R}$-module (Rubin-Kato + control theorem).

Conjecture(Two-variable I.M.C)

$\mathrm{char}_{\mathcal{R}}(\mathrm{Sel}_{\mathcal{T}})^\vee = (\mathcal{L}_p(\mathcal{T}))$

Theorem A ([1]) We have

$\Xi : H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1)) \longrightarrow \mathcal{R}$

with the following properties:

- Ξ is an $\mathcal{R}$-linear pseudo-isom.
- $\chi^j \circ \kappa(\Xi(\mathcal{C}))$ is equal to

$$\left(1-\frac{\omega^{i-j}(p)p^{j-1}}{a_p(f_\kappa)}\right)\left(\frac{p^{j-1}}{a_p(f_\kappa)}\right)^{q(i,j)}$$

$$\times\left(1-\frac{\omega^{i-j}(p)a_p(f_\kappa)}{p^j}\right)^{-1}\exp^*(c^{(j,\kappa)})$$

for each $\mathcal{C} \in H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1))$ and for each (j,κ) with $1 \le j \le k-1$ ($k =$weight of κ).

$\big(c^{(j,\kappa)} \in H^1(\mathbb{Q}_p, \mathsf{F}^-T^*_{f_\kappa}(1-j) \otimes \omega^{j-i})$ means $\chi^j \circ \kappa(\mathcal{C})\big)$

Theorem A is true for more general nearly ordinary deformations (modularity condition is not necessary!).

Idea of the construction of Ξ

Classical Coleman theory on $\mathbb{Q}_p^{\mathrm{ur}}(\mu_{p^\infty})/\mathbb{Q}_p^{\mathrm{ur}}$ + nearly ordinary
$\Longrightarrow$ the interpolation property

for $\exp^*(c^{(j,\kappa)})$

weight-monodromy condition on the "Frobenius" action on D_{pst} of $\kappa(\mathbb{T})$
$\Longrightarrow$ Ξ is a pseudo-isom.

we have a Beilinson-Kato element $\mathcal{Z} \in H^1(\mathbb{Q}_{\Sigma_p}/\mathbb{Q}, \mathcal{T}^*(1))$

$$
\begin{array}{ccc}
\mathcal{Z} & \in & H^1(\mathbb{Q}_{\Sigma_p}/\mathbb{Q}, \mathcal{T}^*(1)) \\
& & \downarrow \text{localization to } p \\
\mathcal{Z} & \in & H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1)) \\
& & \downarrow \chi^j \circ \kappa \\
z^{(j,\kappa)} & \in & H^1(\mathbb{Q}_p, \mathsf{F}^- T^*_{f_\kappa}(1-j) \otimes \omega^{j-i}) \\
& & \downarrow \exp^* \\
\exp^*(z^{(j,\kappa)}) & \in & \overline{\mathbb{Q}}_p
\end{array}
$$

$$\exp^*(z^{(j,\kappa)}) = C_{p,\kappa} \cdot \frac{L_{(p)}(f_\kappa, \omega^{i-j}, j)}{(2\pi\sqrt{-1})^{j-1} C_{\infty,\kappa}}$$

($L_{(p)}(f_\kappa, \omega^{i-j}, s)$ is the Hecke L-function with removed p-factor)

Corollary A

$$\mathrm{char}_{\mathcal{R}}(H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1))/\mathcal{Z}\mathcal{R}) = (\mathcal{L}_p(\mathcal{T}))$$

Theorem B ([2]) Assume

(**H**) $\mathcal{R} \cong \mathcal{O}[[X_1, X_2]]$

(**G**) $\exists \tau \in G_{\mathbb{Q}}$ which is conjugate to $\begin{pmatrix} 1 & P_\tau \\ 0 & 1 \end{pmatrix}$ in $GL(\mathcal{T}) \cong GL_2(\mathcal{R})$ where $P_\tau \in \mathcal{R}^\times$. $\exists \tau' \in G_{\mathbb{Q}}$ acting on $\mathcal{T}/\mathfrak{M}\mathcal{T}$ via the multiplication by -1.

Then we have:

$$\mathrm{char}_{\mathcal{R}}(\mathrm{Sel}_{\mathcal{T}})^\vee \supset \mathrm{char}_{\mathcal{R}}(H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1))/\mathcal{Z}\mathcal{R}).$$

For "general" $\eta : \mathcal{R} \to \overline{\mathbb{Q}}_p$

$$\sharp Ш(\eta(\mathcal{T}^*(1))) \leq \sharp H^1(\mathbb{Q}_{\Sigma_p}/\mathbb{Q}, \eta(\mathcal{T}^*(1)))/\eta(\mathcal{Z})$$

(by (**G**) + E.S theory over D.V.R)

$\Downarrow$ (**H**) + "specialization method"

$$\mathrm{char}_{\mathcal{R}}(Ш(\mathcal{T}^*(1))) \supset \mathrm{char}_{\mathcal{R}}(H^1(\mathbb{Q}_{\Sigma_p}/\mathbb{Q}, \mathcal{T}^*(1))/\mathcal{Z}\mathcal{R}))$$

$\Downarrow$

Theorem B follows from:

$$H^1(\mathbb{Q}_{\Sigma_p}/\mathbb{Q}, \mathcal{T}^*(1)) \to H^1(\mathbb{Q}_p, \mathsf{F}^-\mathcal{T}^*(1)) \to (\mathrm{Sel}_{\mathcal{T}})^\vee \to Ш(\mathcal{T}^*(1))$$

Corollary B Assume the conditions (**H**) and (**G**). Then we have the inequality:

$$\mathrm{char}_{\mathcal{R}}(\mathrm{Sel}_{\mathcal{T}})^{\vee} \supset (\mathcal{L}_p(\mathcal{T})).$$

Corollary B
$\Rightarrow$ applications to "Iwasawa theory" on $(\mathcal{T}/J\mathcal{T}, \mathcal{R}/J)$ for height one ideals $J \subset \mathcal{R}$ (reference [3])

<u>Example when $J = (\mathrm{Ker}(\kappa))$</u>

•$\mathcal{T}/J\mathcal{T} = T_{f_\kappa} \otimes \mathcal{O}[[\Gamma]](\widetilde{\chi}) \otimes \omega^i$
•$\mathcal{R}/J = \mathcal{O}[[\Gamma]]$

•$(\mathrm{Sel}_{T_{f_\kappa}\otimes\mathcal{O}[[\Gamma]](\tilde{\chi})\otimes\omega^i})^\vee$ is a torsion $\mathcal{R}/J$-module (Kato-Rubin)

•$\exists L_p^{\mathrm{MTT}}(f_\kappa \otimes \omega^i) \in \mathcal{O}[[\Gamma]]$

with the interpolation property:

$\chi\eta(L_p^{\mathrm{MTT}}(f_\kappa \otimes \omega^i)) =$

$$\left(1 - \frac{\omega^{i-1}\eta(p)}{a_p(f_\kappa)}\right)\left(\frac{1}{a_p(f_\kappa)}\right)^{q(i,1,\eta)} G(\omega^{1-i}\eta^{-1}) \times \frac{L(f_\kappa, \omega^{i-1}\eta, 1)}{C_{\infty,\kappa}}$$

for finite order characters η of Γ.

Conjecture (cyclotomic I.M.C)

$$\mathrm{char}_{\mathcal{O}[[\Gamma]]}(\mathrm{Sel}_{T_{f_\kappa}\otimes\mathcal{O}[[\Gamma]](\tilde{\chi})\otimes\omega^i})^\vee = (L_p^{\mathrm{MTT}}(f_\kappa \otimes \omega^i))$$

Proposition. (i) Assume (**G**) and (**H**). Then the cyclotomic I.M.C for one of $f_\kappa \otimes \omega^i$ implies the two-variable I.M.C for $\mathcal{T}$.
(ii) The I.M.C for $\mathcal{T}$ implies the cyclotomic I.M.C of $f_\kappa \otimes \omega^i$ for every f_κ in the Hida family $\mathcal{F}$.

$\rightsquigarrow$ an example of the equality of I.M.C for infinite family $\{f_\kappa\}$

No assumption on μ-invariant.
(cf. Emerton-Pollack-Weston)

<u>p-adic L-function side</u>

$\bullet(\kappa(\mathcal{L}_p(\mathcal{L}))) = (L_p^{\mathsf{MTT}}(f_\kappa))$

<u>Selmer group side</u>

$\bullet(\mathrm{Sel}_{\mathcal{T}})^\vee/\mathrm{Ker}(\kappa)(\mathrm{Sel}_{\mathcal{T}})^\vee$

$$\to (\mathrm{Sel}_{T_{f_\kappa}\otimes\mathcal{O}[[\Gamma]](\tilde{\chi})\otimes\omega^i})^\vee$$

is pseudo-isom as $\mathcal{O}[[\Gamma]]$-module.

$\bullet(\mathrm{Sel}_{\mathcal{T}})^\vee_{\text{p-null}}/\mathrm{Ker}(\kappa)(\mathrm{Sel}_{\mathcal{T}})^\vee_{\text{p-null}}$ is a pseudo-null $\mathcal{O}[[\Gamma]]$-module for every arithmetic specialization κ of weight ≥ 2.

Example (reference [3])

$\Delta = q \prod_{n \geq 1} (1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z}))$

$\mathcal{R} = \mathbb{Z}_p[[\Gamma \times \Gamma']]$. $0 \leq i \leq 10$.

$p \leq 10,000$: ordinary prime for Δ $(p \geq 11, p \neq 2411)$,

$\mathcal{T} = \mathbb{T} \widehat{\otimes} \mathbb{Z}_p[[\Gamma]](\widetilde{\chi}) \otimes \omega^i$.

Proposition.

(i) Except for $(p,i) = (11,1)$, $(23,1)$ and $(691,1)$, we have $\mathrm{Sel}_{\mathcal{T}} = 0$ and $\mathcal{L}_p(\mathcal{T})$ is a unit.

(ii) $(p,i) = (11,1) \Rightarrow$

$(\mathrm{Sel}_T)^{\vee} \cong \mathbb{Z}_p[[\Gamma \times \Gamma']]/(\gamma^2 - \gamma'^{-1})$

$(\gamma^2 - \gamma'^{-1}) = (\mathcal{L}_p(T))$ (I.M.C.)

((23,1), (691,1): work in preparation)