

p -ADIC L -FUNCTIONS FOR GALOIS DEFORMATIONS AND RELATED PROBLEMS ON PERIODS

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ABSTRACT. At the first half of this article, we present a conjecture (cf. Conjecture 1.10) to associate “the p -adic L -function” to a family of Galois representation. In recent years, we have plenty of examples for families of Galois representations of the absolute Galois group of \mathbb{Q} . However, because of lack of examples of explicit constructions, formulating a conjecture was difficult. We give conjectures in §1.4 based on our detailed study [O3] of the Iwasawa theory for the two-variable Hida deformation and various other examples coming from Hida theory on higher dimensional modular forms, Coleman theory, convolution product, etc.

At the later half of this article, we explain the example of the two-variable Hida deformation by showing that it will be the first evidence (except the cyclotomic deformations) which satisfies our generalized Iwasawa main conjecture (cf. Conjecture 1.12).

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1. OVERVIEW OF OUR PROGRAM

For a “pure motive” M defined over the rational number field \mathbb{Q} with coefficients in a number field K , we define the Hasse-Weil L -function as follows:

$$L(M, s) = \prod_{l \in \{\text{primes}\}} \frac{1}{\det(1 - \text{Frob}_l t; H_{\text{et}}(M_{\overline{\mathbb{Q}}}, K_{\lambda})^{I_l})} \Big|_{t=l^{-s}},$$

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where λ is a prime of K . The L -function $L(M, s)$ is convergent at the right half-plane $\operatorname{Re}(s) > \frac{\operatorname{wt}(M)}{2} + 1$ and is conjectured to be meromorphically continued to the whole complex plane \mathbb{C} . One of the simplest examples of pure motives is the Tate motive $\mathbb{Q}(n)$. The L -function $L(\mathbb{Q}(n), s)$ is nothing but $\zeta(s+n)$ where $\zeta(s)$ is Riemann's zeta function. $L(M, s)$ is conjectured to be holomorphic when M contains no component isomorphic to a Tate motive $\mathbb{Q}(n)$.

1.1. Complex and p -adic periods. We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ throughout the paper. Let us assume that M is critical in the sense of Deligne (see [De] for the definition of critical motives). Then, we have the following isomorphism induced from the de Rham's theorem:

$$\operatorname{Per}_{M,\infty}^\pm : H_{\operatorname{Betti}}(M)^\pm \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\operatorname{dR}}(M)^\pm \otimes_{\mathbb{Q}} \mathbb{C},$$

where $H_{\operatorname{Betti}}(M)^\pm$ is the \pm -eigen spaces with respect to the action of the complex conjugate and $H_{\operatorname{dR}}(M)^+$ (resp. $H_{\operatorname{dR}}(M)^-$) is $\operatorname{Fil}^0 H_{\operatorname{dR}}(M)$ (resp. $H_{\operatorname{dR}}(M)/\operatorname{Fil}^0 H_{\operatorname{dR}}(M)$). The following definition of period as well as the notion of “critical” was given in [De]:

Definition 1.1. The complex period $\Omega_{M,\infty}^\pm \in (K \otimes_{\mathbb{Q}} \mathbb{C})^\times$ is defined to be $\det(\operatorname{Per}_{M,\infty}^\pm)$ with respect to a fixed K -basis of $H_{\operatorname{Betti}}(M)^\pm$ and $H_{\operatorname{dR}}(M)^\pm$.

Via the fixed embedding $K \hookrightarrow \mathbb{C}$, we identify $\Omega_{M,\infty}^\pm$ as an element in \mathbb{C} .

Remark 1.2. 1. Let M be $H^d(X)(r)$ for a certain projective smooth variety X over \mathbb{Q} . Then, $H_{\operatorname{dR}}(M)^\pm \otimes_{\mathbb{Q}} \mathbb{C}$ is given as follows (we have $p+q=d$ below):

$$\begin{aligned} H_{\operatorname{dR}}(M)^+ \otimes_{\mathbb{Q}} \mathbb{C} &= \bigoplus_{p \geq r} H(X(\mathbb{C}))^{p,q}, \\ H_{\operatorname{dR}}(M)^- \otimes_{\mathbb{Q}} \mathbb{C} &= \bigoplus_{p < r} H(X(\mathbb{C}))^{p,q}, \end{aligned}$$

We fix a \mathbb{Q} -basis $\{b_i^\pm\}$ of $H_d^{\operatorname{Betti}}(X(\mathbb{C}), \mathbb{Q})^\pm \cong \operatorname{Hom}_{\mathbb{Q}}(H_{\operatorname{Betti}}^d(X(\mathbb{C}), \mathbb{Q})^\pm, \mathbb{Q})$ and a \mathbb{Q} -basis $\{\omega_j^\pm\}$ of $H^{\operatorname{dR}}(M)^\pm$. The complex period $\Omega_{M,\infty}^\pm \in \mathbb{C}$ is the determinant of the matrix $(\int_{b_i^\pm} \omega_j^\pm)_{i,j}$ of period integrals.

2. $\Omega_{M,\infty}^\pm$ depends on the choice of K -bases of $H_{\operatorname{Betti}}(M)^\pm$ and $H_{\operatorname{dR}}(M)^\pm$. However it is independent as an element in $\mathbb{C}^\times/K^\times$.

The following conjecture was formulated in [De]:

Conjecture 1.3. *Let M be a critical motive in the sense of Deligne.*

1. *There exist complex numbers $C_{M,\infty}^\pm \in \mathbb{C}$ such that $\frac{L(M, \chi, r)}{(2\pi\sqrt{-1})^{r-1} C_{M,\infty}^{(-1)^r \chi(-1)}} \in \overline{\mathbb{Q}}$ for every integers r and for every Dirichlet characters χ making $M(r) \otimes \chi$ a critical motive, where $L(M, \chi, s)$ is the twist $\sum_{n \geq 1} \frac{\chi(n) a_n(M)}{n^s}$ of $L(M, s) = \sum_{n \geq 1} \frac{a_n(M)}{n^s}$ and*

$M(r)$ is the r -th Tate twist of M . Especially, we have $\frac{L(M, 0)}{C_{M,\infty}^+} \in \overline{\mathbb{Q}}$.

2. *$C_{M,\infty}^\pm$ is equal to $\Omega_{M,\infty}^\pm$ in $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$.*

Let us fix an odd prime number p and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ throughout the paper. At the same time, we fix a norm-compatible sequence of p -power roots of unity $\{\zeta_{p^n}\}_{n \geq 1}$. We introduce a p -adic period associated to a motive M , which is a p -adic counter part of the complex period given above. We denote by B_{HT} the ring of Hodge-Tate period's. The fixed norm-compatible sequence induces the following isomorphism:

$$B_{\text{HT}} = \mathbb{C}_p[t, t^{-1}]$$

where t is an element of B_{HT} on which $G_{\mathbb{Q}_p} = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts via the cyclotomic character and \mathbb{C}_p is the completion of $\overline{\mathbb{Q}}_p$ with respect to the p -adic absolute value. By the comparison theorem of p -adic Hodge theory proved by Faltings and Tsuji independently, we have the following isomorphism:

$$(1) \quad H_{\text{Betti}}(M) \otimes_{\mathbb{Q}} B_{\text{HT}} \xrightarrow{\sim} H_{\text{dR}}(M) \otimes_{\mathbb{Q}} B_{\text{HT}}.$$

Recall that we can discuss the following

$$\text{Per}_{M,p}^{\pm} : H_{\text{Betti}}(M)^{\pm} \otimes_{\mathbb{Q}} B_{\text{HT}} \longrightarrow H_{\text{dR}}(M)^{\pm} \otimes_{\mathbb{Q}} B_{\text{HT}},$$

by decomposing the above isomorphism (1).

Definition 1.4. The determinant $\det(\text{Per}_{M,p}^{\pm}) \in K \otimes_{\mathbb{Q}} B_{\text{HT}}$ with respect to a fixed K -basis of $H_{\text{Betti}}(M)^{\pm}$ and $H_{\text{dR}}(M)^{\pm}$ can be regarded as an element of B_{HT} via the morphism $K \otimes_{\mathbb{Q}} B_{\text{HT}} \longrightarrow B_{\text{HT}}$ induced by the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow B_{\text{HT}}$. Further, it is not difficult to see that $\det(\text{Per}_{M,p}^{\pm})$ is supported on a certain single component $\mathbb{C}_p t^m \subset B_{\text{HT}}$. The p -adic period $\Omega_{M,p}^{\pm} \in \mathbb{C}_p$ is defined to be the coefficient of this monomial in t^m .

Remark 1.5. 1. As is also discussed in Proposition 1.6 in detail, the p -adic period $\Omega_{M,p}^{\pm} \in \mathbb{C}_p$ is dependent on the choice of the fixed embeddings $\iota_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. For certain choice of ι_{∞} , $\Omega_{M,p}^{\pm} \in \mathbb{C}_p$ can be zero. (cf. a vanishing example by Y. André explained in [H3, §3.3])

2. The comparison isomorphisms (1) are also proved by Niziol by using K-theory. Thus, there are three major different proofs for the comparison isomorphisms of p -adic Hodge theory (Faltings, Tsuji and Niziol). In the case of curves or abelian varieties, several other constructions are obtained by Tate, Fontaine and Coleman etc. As already remarked in [H3](and also remarked by Illusie), comparison isomorphisms proved by these different methods are not known to be equivalent to each other. Hence, we have an ambiguity of the definition of p -adic periods according to which comparison theorem to choose.

The following type of non-vanishing of p -adic periods are given in [H3, Theorem 3.4.1]. A result with more general situation is stated in the above reference. Since, the author could not follow the argument in [H3, Theorem 3.4.1] when he prepared this article, we give here a result with a slightly special condition different from the formulation there and we try to prove it in a very elementary way:

Proposition 1.6. *Assume that M is a critical motive of rank-two with coefficient in a number field K . We denote by $H_{p,\text{ét}}(M)$ the p -adic étale realization of M , which is a*

two-dimensional $\overline{\mathbb{Q}}_p$ -vector space on which $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts continuously. Let us assume the following conditions:

1. The diagonal component of the Hodge structure for $M_{\mathbb{C}}$ is trivial.
2. The Zariski closure of the image of the representation $\rho_{M,p} : G_{\mathbb{Q}} \longrightarrow \text{Aut}(H_{p,\text{ét}}(M))$ contains $SL_2(\overline{\mathbb{Q}}_p)$.

Then, by replacing ι_{∞} by $\iota_{\infty} \circ \sigma$ for certain $\sigma \in G_{\mathbb{Q}}$ if necessary, $\Omega_{M,p}^{\pm} \in \mathbb{C}_p$ is a non-zero element in \mathbb{C}_p .

Proof. For simplicity, we assume that the coefficient field K of M is \mathbb{Q} . By the first condition, the \pm -eigen space for complex conjugation $H_{\text{Betti}}(M)^{\pm}$ has dimension-one over \mathbb{Q} for each of $+$ and $-$. We will only show the assertion for $\Omega_{M,p}^{+}$ since the proof for $\Omega_{M,p}^{-}$ is done in the same manner. The comparison map (1) is decomposed as follows:

$$(2) \quad H_{\text{Betti}}(M)^+ \otimes_{\mathbb{Q}} B_{\text{HT}} \hookrightarrow H_{\text{Betti}}^*(M_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{\text{HT}} \xrightarrow{\sim} H_{p,\text{ét}}^*(M_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}} \\ \xrightarrow{\sim} H_{\text{dR}}(M_{\overline{\mathbb{Q}}_p}) \otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}} \xrightarrow{\sim} H_{\text{dR}}(M) \otimes_{\mathbb{Q}} B_{\text{HT}} \twoheadrightarrow H_{\text{dR}}(M)^+ \otimes_{\mathbb{Q}} B_{\text{HT}}$$

The isomorphism $H_{\text{Betti}}^*(M_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{\text{HT}} \xrightarrow{\sim} H_{p,\text{ét}}^*(M_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}}$ is an extension of the comparison map:

$$(3) \quad H_{\text{Betti}}^*(M_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p \xrightarrow{\sim} H_{p,\text{ét}}^*(M_{\mathbb{C}}, \overline{\mathbb{Q}}_p) \xrightarrow[\iota_{\infty}]{\sim} H_{p,\text{ét}}^*(M_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p),$$

where the first isomorphism is canonical and the second one is induced by the fixed embedding $\iota_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Note that $\Omega_{M,p}^{\pm}$ can vanish only when the one-dimensional subspace:

$$\text{Ker} \left[H_{p,\text{ét}}^*(M_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}} \longrightarrow H_{\text{dR}}(M)^+ \otimes_{\mathbb{Q}} B_{\text{HT}} \right]$$

in $H_{p,\text{ét}}^*(M_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}}$ is equal to the image of $H_{\text{Betti}}(M)^+ \otimes_{\mathbb{Q}} B_{\text{HT}} \subset H_{\text{Betti}}^*(M_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{\text{HT}}$ via the base extension $\otimes_{\overline{\mathbb{Q}}_p} B_{\text{HT}}$ of the isomorphism (3). Since $H_{\text{Betti}}(M)^+$ is independent of the choice of ι_{∞} , we can replace ι_{∞} by $\iota_{\infty} \circ \sigma$ with certain $\sigma \in G_{\mathbb{Q}}$ if necessary so that this coincidence does not happen (We used the second assumption of the proposition here). Thus, we prove that there exists $\sigma \in G_{\mathbb{Q}}$ such that $\Omega_{M,p}^{+} \in \mathbb{C}_p$ is non-zero if we replace our fixed embedding $\iota_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ by $\iota_{\infty} \circ \sigma$. \square

Both a complex period $\Omega_{M,\infty}^{\pm} \in \mathbb{C}$ and a p -adic period $\Omega_{M,p}^{\pm} \in \mathbb{C}_p$ depends on the choice of a K -basis $\boldsymbol{\eta} = (\eta_1^{\pm}, \dots, \eta_{r^{\pm}}^{\pm})$ of $H_{\text{Betti}}^{\pm}(M)$ and a K -basis $\boldsymbol{\delta} = (\delta_1^{\pm}, \dots, \delta_{r^{\pm}}^{\pm})$ of $H_{\text{dR}}^{\pm}(M)$. They should have been denoted by $\Omega_{M,\infty}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})$ and $\Omega_{M,p}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})$ to show that they are dependent on choice of $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$. The following observation is important for the interpolation property of p -adic L -function discussed later (cf. §1.3 and §1.4).

Proposition 1.7 (Blasius, Hida). *Suppose that $\Omega_{M,p}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})$ is not zero. Then for any other choice of basis $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$, $\Omega_{M,p}^{\pm}(\boldsymbol{\eta}'; \boldsymbol{\delta}')$ is non-zero element in \mathbb{C}_p such that the ratio $\Omega_{M,p}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})/\Omega_{M,p}^{\pm}(\boldsymbol{\eta}'; \boldsymbol{\delta}')$ is an algebraic number. Further we have an equality*

$$\Omega_{M,\infty}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})/\Omega_{M,\infty}^{\pm}(\boldsymbol{\eta}'; \boldsymbol{\delta}') = \Omega_{M,p}^{\pm}(\boldsymbol{\eta}; \boldsymbol{\delta})/\Omega_{M,p}^{\pm}(\boldsymbol{\eta}'; \boldsymbol{\delta}')$$

in $\overline{\mathbb{Q}}$.

We omit the proof of this proposition since it is straight-forward from the definition of these periods.

1.2. p -adic Galois deformations. For each i with $1 \leq i \leq d$, let $\Gamma^{(i)}$ be a group with canonical isomorphism $\chi^{(i)} : \Gamma^{(i)} \xrightarrow{\sim} \mathbb{Z}_p$. We denote by $\Lambda^{(i)}$ the Iwasawa algebra $\mathbb{Z}_p[[\Gamma^{(i)}]]$. Let us denote by $\mathfrak{X}(\Lambda^{(i)})$ the set of continuous homomorphisms of \mathbb{Z}_p -algebras from $\Lambda^{(i)}$ to $\overline{\mathbb{Q}}_p$. Obviously, $\mathfrak{X}(\Lambda^{(i)})$ is identified with the following set:

$$\text{Char}(\Gamma^{(i)}) = \{\kappa : \Gamma^{(i)} \longrightarrow \overline{\mathbb{Q}}_p \mid \kappa \text{ is continuous}\},$$

which is regarded as an open unit ball around $1 \in \overline{\mathbb{Q}}_p$ in $\overline{\mathbb{Q}}_p$ with radius 1. Let $\Lambda = \Lambda^{(1)} \widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} \Lambda^{(d)} = \mathbb{Z}_p[[\Gamma^{(1)} \times \cdots \times \Gamma^{(d)}]]$. Then we have the following isomorphism similarly as above:

$$\mathfrak{X}(\Lambda) \cong \text{Char}(\Gamma^{(1)} \times \cdots \times \Gamma^{(d)}) \cong \text{Char}(\Gamma^{(1)}) \times \cdots \times \text{Char}(\Gamma^{(d)}) \cong \mathfrak{X}(\Lambda^{(1)}) \times \cdots \times \mathfrak{X}(\Lambda^{(d)}).$$

For a local domain \mathcal{R} which is finite flat over Λ , we denote by $\mathfrak{X}(\mathcal{R})$ the set of continuous homomorphisms of \mathbb{Z}_p -algebras from \mathcal{R} to $\overline{\mathbb{Q}}_p$. $\mathfrak{X}(\mathcal{R})$ is naturally regarded as a finite cover of $\mathfrak{X}(\Lambda)$. The following notation of arithmetic points, which are used frequently for a long time, is useful to describe interpolation properties of Galois representations:

Definition 1.8. An element $\kappa \in \mathfrak{X}(\Lambda)$ is called an arithmetic point (of weight (w_1, \dots, w_d)) if, for each i , there exists an open subgroup U_i of $\Gamma^{(i)}$ and an integer w_i such that $\kappa|_{U_i}$ is equal to $(\chi^{(i)})^{w_i}|_{U_i}$. More generally, an element $\kappa \in \mathfrak{X}(\mathcal{R})$ is called an arithmetic point (of weight (w_1, \dots, w_d)) if the image via the finite cover $\mathfrak{X}(\mathcal{R}) \longrightarrow \mathfrak{X}(\Lambda)$ is an arithmetic point (of weight (w_1, \dots, w_d)) in the above sense. We denote by $\mathfrak{X}_{\text{arith}}(\mathcal{R})$ the subset of $\mathfrak{X}(\mathcal{R})$ which consists of all arithmetic points.

Definition 1.9. Let \mathcal{T} be a free module of finite rank over \mathcal{R} with continuous action of $G_{\mathbb{Q}}$ and let $P \subset \mathfrak{X}_{\text{arith}}(\mathcal{R})$ be a subset which is dense in $\mathfrak{X}(\mathcal{R})$. A pair (\mathcal{T}, P) is called a geometric pair if it satisfies the following conditions:

1. There exists a finite number of primes Σ containing $\{\infty\}$ such that the action of $G_{\mathbb{Q}}$ on \mathcal{T} is unramified outside Σ .
2. The specialization of \mathcal{T} at $\kappa \in P$ is a $G_{\mathbb{Q}}$ -stable lattice of the p -adic étale realization of M_{κ} for a critical motive M_{κ} .
3. At each point $\kappa \in P$, M_{κ} does not have a component isomorphic to a Tate-twist of a Dirichlet motive.

1.3. “Periods” + “Galois deformations” \implies “ p -adic L -functions”. Suppose that we are given a geometric pair (\mathcal{T}, P) over \mathcal{R} . We expect that there exists a p -adic L -function which interpolates the special value of Hasse-Weil L -function at each arithmetic point divided by a complex period. In a special case, where our p -adic family is the cyclotomic deformation of a certain p -ordinary motive M over \mathbb{Q} (cf. Remark 3.3), such conjecture was formulated by [CP].

However, general deformations other than the cyclotomic deformations of p -ordinary motives are much more difficult to treat. Several important philosophical contributions

are given by Greenberg, Hida and Panchishkin.¹

Previous Contributions

1. Hida[H3] studied the interpolation properties characterizing the analytic p -adic L -function $L_p(\mathcal{T})$ when \mathcal{T} is an “admissible” Galois deformation, which is equipped with a certain local filtration with respect to the action of the decomposition group at p . In [H3], the p -adic periods are introduced and the idea of “balanced interpolation properties” (cf. Proposition 1.7 and Remark 1.11) are introduced in order to overcome the difficulties that we have no canonical choice of complex periods $\Omega_{M,\infty}^\pm$.
2. Greenberg [Gr2], instead of the characterization of the analytic p -adic L -function itself, discuss the relation with the algebraic p -adic L -function which should be called the “Iwasawa Main Conjecture”. This is the first reference which insists on the importance of such generalization of the Iwasawa Main Conjecture.
3. The above two contributions are devoted only to “admissible” Galois deformations. Panchishkin [P1] discusses p -adic L -functions for various Galois deformations. His formulation as well as several other work by Perrin-Riou give us a perspective on a well-formulated conjecture for the cyclotomic analytic p -adic L -function of a critical motive which is not necessarily ordinary. Note that non-ordinary motive such as supersingular elliptic curves are not “admissible”. Panchishkin’s important idea in [P1] is to define “Hasse invariant of a critical motive M ” to be the difference of the Hodge polygon and the Newton polygon of M . The p -adic L -function for M is an element in a certain extension of the cyclotomic Iwasawa algebra which allows certain logarithmic growth of denominators according to Hasse invariant of M .

Let us introduce the sub-ring \mathcal{H}_r of $\mathbb{Q}_p[[T]]$ defined to be

$$\mathcal{H}_r := \{f(T) = \sum a_n(T-1)^n \mid \lim_{n \rightarrow \infty} |a_n|n^{-r} = 0\}.$$

We have the inclusion

$$\mathbb{Z}_p[[\Gamma]] \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \cdots \subset \mathcal{H}_r \subset \cdots,$$

where $\Gamma = \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and $\mathbb{Z}_p[[\Gamma]]$ is embedded in \mathcal{H}_1 via the isomorphism $\mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbb{Z}_p[[T]]$ which sends a topological generator γ of Γ to $1+T$.

Naive Expectation . *Let (\mathcal{T}, P) be a geometric pair over \mathcal{R} . Assume that we have d^+ -functions $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)} \in \mathcal{R}$, a character $\tilde{\eta} : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \rightarrow \mathbb{Z}_p[[\Gamma]]^\times$, a free \mathcal{R} -module \tilde{T} with continuous $G_\mathbb{Q}$ -action and $p : \mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]] \rightarrow \mathcal{R}$ so that we have the following conditions:*

1. *We have $\mathcal{T} \cong (\tilde{T} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]])(\tilde{\eta}) \otimes_{\mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]]} \mathcal{R}$.*
2. *For each $\kappa \in P$, \tilde{T}_κ is isomorphic to the p -adic étale realization of M_κ^* such that $\det(t - \varphi; D_{\text{crys}}(M_\kappa^*))$ is divisible by $\prod_{1 \leq i \leq d^+} (t - \alpha_\kappa^{(i)})$, where $\alpha_\kappa^{(i)} = \kappa(A^{(i)})$.*

¹During the preparation of this article, I learned from there is a paper by Fukaya-Kato [FK] for the formulation of the framework of the p -adic L -functions from the viewpoint of non-commutative Iwasawa theory.

3. *There exists a set of d^+ non-negative rational numbers $\mathbf{e} = \{e_1, \dots, e_{d^+}\}$ such that $\text{ord}_p(\alpha_\kappa^{(i)}) = e_i$ at every non-trivial specializations $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}}_p$.*

Let $\mathcal{R}_{p,\mathbf{e}}$ be the extension of \mathcal{R} obtained by the specialization of $\mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{H}_{[\mathbf{e}]}$ via the above map p , where $[\mathbf{e}]$ is the smallest integer which is greater than or equal to $e_1 + \dots + e_{d^+}$. Then, there exists $L_p(\mathcal{T}) \in \mathcal{R}_{p,\mathbf{e}}$ such that we have:

$$\begin{aligned} \frac{L_p(\mathcal{T})(\kappa)}{C_{\kappa,p}^+} &= \prod_{1 \leq i \leq d^+} \left(\frac{p^{w^*(\kappa)-1}}{\alpha_\kappa^{(i)}} \right)^{c^*(\kappa)} \prod_{d^+ < i \leq d_p} \left(1 - \frac{\eta_\kappa(p) \alpha_\kappa^{(i)}}{p^{w^*(\kappa)-1}} \right) \\ &\quad \times \prod_{1 \leq i \leq d^+} \left(1 - \frac{\eta_\kappa^{-1}(p) p^{w^*(\kappa)}}{\alpha_\kappa^{(i)}} \right) \frac{L(M_\kappa, 0)}{\Omega_{\kappa,\infty}^+} \end{aligned}$$

for each $\kappa \in P$, where $d_p \leq d$ is the rank of $D_{\text{crys}}(M_\kappa^*)$ and $\alpha^{(d^++1)}, \dots, \alpha^{(d_p)}$ is the other Frobenius eigenvalues on $D_{\text{crys}}(M_\kappa^*)$, $w^*(\kappa)$ is the weight of the character $\kappa \circ \tilde{\eta} : \Gamma \hookrightarrow \mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]] \xrightarrow{p} \mathcal{R} \xrightarrow{\kappa} \overline{\mathbb{Q}}_p$ and $c^*(\kappa)$ is the p -order of the conductor of $(\kappa \circ \tilde{\eta})/\chi^{w^*(\kappa)}$, $C_{\kappa,p}^+ \in \overline{\mathbb{Q}}_p$ is an error term which is “naturally” arise from the construction.

In the above “Naive Expectation”, the formulation are detailed enough, but we left ambiguity in the definition of $C_{\kappa,p}^+$ and $\Omega_{\kappa,\infty}^+$. The normalized factors $C_{\kappa,p}^+$ are ad hoc and not enough to characterize the p -adic L -function. On the other hand, the complex period $\Omega_{\kappa,\infty}^+$ for the motive M_κ depends on the choice of the rational bases of the Betti realizations and the de Rham realizations of M_κ for which canonical choice of the optimal period. For example, in the case of Hida’s nearly ordinary deformation discussed in a later section of this paper, we have $C_{\kappa,p}^+ = 1$ in the Rankin-Selberg type construction (Panchishkin, Fukaya, Ochiai). In the modular symbol construction for Hida family, $C_{\kappa,p}^+$ seems to be non-trivial, but is not a p -adic period in the sense of Definition 1.4. Later in Section 4, I will discuss the relation between such different constructions and I explain about a modification [O3] my previous construction [O1].

1.4. Conjectures. Based on the preparation and the observation given in the previous subsection, we try to present conjectures on p -adic L -functions and the Iwasawa Main Conjecture.

Conjecture 1.10. *Assume that we have d^+ -functions $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)} \in \mathcal{R}$, a character $\tilde{\eta} : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \longrightarrow \mathbb{Z}_p[[\Gamma]]^\times$, a free \mathcal{R} -module \tilde{T} with continuous $G_{\mathbb{Q}}$ -action and $p : \mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]] \rightarrow \mathcal{R}$ so that we have the following conditions:*

1. *We have $\mathcal{T} \cong (\tilde{T} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]])(\tilde{\eta}) \otimes_{\mathcal{R} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]]} \mathcal{R}$.*
2. *For each $\kappa \in P$, \tilde{T}_κ is isomorphic to the p -adic étale realization of M_κ^* such that $\det(t - \varphi; D_{\text{crys}}(M_\kappa^*))$ is divisible by $\prod_{1 \leq i \leq d^+} (t - \alpha_\kappa^{(i)})$, where $\alpha_\kappa^{(i)} = \kappa(A^{(i)})$.*
3. *We have $L(M_\kappa, 0) \neq 0$ for every $\kappa \in P$ except those contained in a certain Zariski closed subset of $\mathfrak{X}(\mathcal{R})$.*

4. There exists a set of d^+ non-negative rational numbers $\mathbf{e} = \{e_1, \dots, e_{d^+}\}$ such that $\text{ord}_p(\alpha_\kappa^{(i)}) = e_i$ at every non-trivial specializations $\kappa : \mathcal{R} \longrightarrow \overline{\mathbb{Q}_p}$ ².

Then, there exists $L_p^{\text{anal}}(\mathcal{T}) \in \mathcal{R}_{p,\mathbf{e}} \widehat{\otimes} \mathcal{O}_{\mathbb{C}_p}$ such that we have:

$$\begin{aligned} \frac{L_p^{\text{anal}}(\mathcal{T})(\kappa)}{\Omega_{M_\kappa^*,p}^+} &= \prod_{1 \leq i \leq d^+} \left(\frac{p^{w^*(\kappa)-1}}{\alpha_\kappa^{(i)}} \right)^{c^*(\kappa)} \prod_{d^+ < i \leq d_p} \left(1 - \frac{\eta_\kappa(p) \alpha_\kappa^{(i)}}{p^{w^*(\kappa)}} \right) \\ &\quad \times \prod_{1 \leq i \leq d^+} \left(1 - \frac{\eta_\kappa^{-1}(p) p^{w^*(\kappa)-1}}{\alpha_\kappa^{(i)}} \right) \frac{L(M_\kappa, 0)}{\Omega_{M_\kappa,\infty}^+} \end{aligned}$$

for each $\kappa \in P$, where the basic notations are the same as those in “Naive expectation” at the end of the previous subsection.

I remark about the contribution of Conjecture 1.10 on the existence of the p -adic L -function.

Remark 1.11.

1. The introduction of the universal character $\tilde{\eta}$ to give a precise formulation interpolation is not found in any previous reference. As we remark below, this formulation seems to be useful to answer the questions:

When p -adic L -function exist for a given family of Galois representation \mathcal{T} ?

What is the natural algebra where the p -adic L -function for \mathcal{T} (if exists) should live?

How is a precise interpolation of the p -adic L -function?

2. The interpolation property in Conjecture 1.10 is independent of the choice of rational bases on $H_{\text{Betti}}^\pm(M_\kappa)$ and $H_{\text{dR}}^\pm(M_\kappa)$. In fact, we remarked in Proposition 1.7 that $\Omega_{M_\kappa,p}^\pm$ and $\Omega_{M_\kappa,\infty}^\pm$ changes by the same difference when we replace rational bases on $H_{\text{Betti}}^\pm(M_\kappa)$ and $H_{\text{dR}}^\pm(M_\kappa)$ to another ones. This is an idea of “balanced interpolation property” due to Blasius and Hida as discussed in [H3]. However, I believe that we need to introduce $\Omega_{M_\kappa^*,p}^\pm$ and $\Omega_{M_\kappa,\infty}^\pm$ to be consistent with known construction including the cyclotomic deformation of an ordinary cusp form and Hida families associated to an ordinary cusp form. This idea on “modified balanced interpolation property” is one of a new point in our formulation.
3. The rank of $D_{\text{crys}}(M_\kappa^*)$ and $D_{\text{crys}}(M_\kappa)$ is not the same in general. To be precise for this difference, it is also necessary to introduce the character $\tilde{\eta}$ at the beginning.
4. For most of the case, the choice of the set $\{\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}\}$ of d^+ -functions in \mathcal{R} is automatically determined from \mathcal{T} . However, there are certain cases where we have ambiguity in the choice of the set $\{\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}\}$. One of the most basic example for such case is the cyclotomic deformation of an elliptic curve E which has supersingular reduction at p . We also remark that the p -adic L -function $L_p^{\text{anal}}(\mathcal{T})$ does depend on the choice of the set $\{\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}\}$ when we have several choices.

²It seems that we need a slight modification on the choice of \mathbf{e} to exclude the choice of the non-unit root of Euler p -polynomial for the cyclotomic deformation of an ordinary cuspform, where the p -adic L -function is not associated. However, we leave such problem at the moment.

5. Our formulation will include the case of Coleman's family \mathcal{T} of modular forms with a fixed slope (Note that the p -adic L -function does not live in the p -adic Hecke algebra \mathcal{R} in this case).

In Iwasawa theory, it is important to discuss the relation between the analytic p -adic L -function $L_p^{\text{anal}}(\mathcal{T})$ studied above and the another object so called “the algebraic p -adic L -function $L_p^{\text{alg}}(\mathcal{T})$ ”, which is defined to be the characteristic ideal of the Pontrjagin dual $(\text{Sel}_{\mathcal{T}})^{\vee}$ of “ $\text{Sel}_{\mathcal{T}}$ ”.

Naive Expectation . *Let \mathcal{T} be a Galois representation over \mathcal{R} , which makes a geometric pair (\mathcal{T}, P) with a certain choice of $P \subset \mathfrak{X}_{\text{arith}}(\mathcal{R})$. Then the following conditions are equivalent:*

1. *We have a choice of $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}$ which are all units in \mathcal{R}^{\times} .*
2. *$(\text{Sel}_{\mathcal{T}})^{\vee}$ is a finitely generated torsion \mathcal{R} -module.*

From this expectation, the algebraic p -adic L -function $L_p^{\text{alg}}(\mathcal{T})$ (if suitably defined) is non-trivial if and only if we have $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)} \in \mathcal{R}^{\times}$ satisfying the condition of Conjecture 1.10. On the other hand, if we have such $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}$, they determine $G_{\mathbb{Q}_p}$ -stable subspace $F^+\mathcal{T} \subset \mathcal{T}$ of rank d^+ over \mathcal{R} . We define a two-variable Selmer group $\text{Sel}_{\mathcal{T}}$ to be

$$\text{Sel}_{\mathcal{T}} = \text{Ker} \left[H^1(\mathbb{Q}, \mathcal{A}) \longrightarrow H^1(I_p, F^-\mathcal{A}) \times \prod_{l \neq p} H^1(I_l, \mathcal{A}) \right],$$

where $\mathcal{A} = \mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}^{\vee}$ and $F^-\mathcal{A} = \mathcal{A}/F^+\mathcal{A}$, I_v is the inertia subgroup of $G_{\mathbb{Q}}$ at any finite prime v . If we assume that \mathcal{R} is integrally closed in the fraction field $\text{Frac}(\mathcal{R})$, we define $L_p^{\text{alg}}(\mathcal{T})$ to be a generator of the characteristic ideal $\text{Char}_{\mathcal{R}}(\text{Sel}_{\mathcal{T}})^{\vee}$.

Conjecture 1.12 (Iwasawa Main conjecture). *Suppose that \mathcal{R} is integrally closed in $\text{Frac}(\mathcal{R})$ and that we have a choice of $\tilde{A}^{(1)}, \dots, \tilde{A}^{(d^+)}$ which are all units in \mathcal{R}^{\times} . Then we have:*

1. *$L_p^{\text{alg}}(\mathcal{T})$ is non-zero, or equivalently, $(\text{Sel}_{\mathcal{T}})^{\vee}$ is a torsion \mathcal{R} -module.*
2. *We have the equality $(L_p^{\text{anal}}(\mathcal{T})) = (L_p^{\text{alg}}(\mathcal{T}))$ in $\mathcal{R} \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}$.*

At the end of this section, we remark that the Iwasawa Main conjecture is a very slight modification of the one proposed by Greenberg [Gr2].

2. COMPLEX PERIODS FOR ELLIPTIC CUSPFORMS

Let N be a fixed natural number. Let $f = \sum a_n(f)q^n \in S_k(\Gamma_1(N))$ be a normalized eigen cuspform.

Theorem 2.1 (Shimura). *Let us fix a Dirichlet character ψ with $\psi(-1) = (-1)^{k-1}$ such that $L(f, \psi, k-1) \neq 0$. Define complex periods $\Omega_{f, \infty}^{\pm, (\psi)}$ by*

$$\Omega_{f, \infty}^{+, (\psi)} = \frac{\langle f, f \rangle}{L(f, \psi, k-1)}, \quad \Omega_{f, \infty}^{-, (\psi)} = L(f, \psi, 1)$$

where $\langle f, g \rangle$ is the Peterson inner product $\frac{1}{\text{vol}(\mathfrak{H}/\Gamma_1(N))} \int_{\mathfrak{H}/\Gamma_1(N)} \overline{f(z)}g(z)y^{k-2}dx dy$. Then, we have

$$\frac{L(f, \chi, j)}{\pi^{j-1}\Omega_{f,\infty}^{\pm,(\psi)}} \in \overline{\mathbb{Q}}$$

for any Dirichlet character χ and for any integer j with $1 \leq j \leq k-1$, where $\pm = (-1)^{j-1}\chi(-1)$.

We denote by $L_n(\mathbb{Z})$ the space of homogeneous polynomials in variables s, t with coefficients in \mathbb{Z} , which is a free \mathbb{Z} -module of rank $n+1$ generated by $s^n, s^{n-1}t, \dots, st^{n-1}, t^n$. For any \mathbb{Z} -algebra A , we define $L_n(A)$ to be $L_n(\mathbb{Z}) \otimes_{\mathbb{Z}} A$. Let D_0 be the group of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ which is regarded to be on the boundary of the upper half plane \mathfrak{H} , on which the group $SL_2(\mathbb{Z})$ acts via natural linear transformation. Recall that the module of modular symbols $MS(\Gamma_1(N), L_n(A))$ is defined as follows:

$$MS(\Gamma_1(N), L_n(A)) := \text{Hom}_{\Gamma_1(N)}(D_0, L_n(A)) \cong H_c^1(Y_1(N)(\mathbb{C}), F_n(A)),$$

We have natural decomposition

$$MS(\Gamma_1(N), L_n(A)) = MS(\Gamma_1(N), L_n(A))^+ \oplus MS(\Gamma_1(N), L_n(A))^-$$

under complex conjugate when 2 is invertible in A . Let $\mathcal{H}_k(\Gamma_1(N); A) \subset \text{End}(M_k(\Gamma_1(N); A))$ be the Hecke algebra. $MS(\Gamma_1(N), L_n(A))$ is naturally a finitely generated module over $\mathcal{H}_k(\Gamma_1(N))$. Let \mathcal{O}_f be the ring of integers of $K_f = \mathbb{Q}(\{a_f(n)\})$. By the ring homomorphism $\lambda_f : \mathcal{H}_k(\Gamma_1(N)) \rightarrow \mathcal{O}_f$, $T_n \mapsto a_n(f)$, we define

$$MS(f)^\pm = MS(\Gamma_1(N), L_n(\mathcal{O}_f[1/2]))[\text{Ker}(\lambda_f)].$$

The modules $MS(f)^\pm$ are free of rank one over $\mathcal{O}_f[1/2]$.

Definition 2.2. Let η^\pm be a basis of $MS(f)^\pm$ over $\mathcal{O}_f[1/2]$. Note that the module $MS(f)^\pm$ is identified with an $\mathcal{O}_f[1/2]$ -lattice of the Betti realization $H_{\text{Betti}}(M_f)$. When the notation $\Omega_{M_f,\infty}^\pm(\eta^\pm, f)$ seems to be complicated, we denote it by $\Omega_{f,\infty}^{\pm,\text{MS}}$ forgetting the dependence on η^\pm . Thus, $\Omega_{f,\infty}^{\pm,\text{MS}}$ not defined as an element in \mathbb{C} , but it is a well-defined element in $\mathbb{C}^\times/\mathcal{O}_f[1/2]^\times$.

We close this section with the following remark.

Remark 2.3. Let f be an eigen cuspform of weight $k \geq 2$ and let j be an integer satisfying the inequality $1 \leq j \leq k-1$. The p -order of an algebraic number $L(f, j)/(2\pi\sqrt{-1})^{j-1}\Omega_{f,\infty}^{(-1)^{j-1},\text{MS}}$ is independent of the choice of the basis of $MS(f)^\pm$, since the change of the basis of $MS(f)^\pm$ induces multiplication by a unit of $\mathcal{O}_f[1/2]$. By the Tamagawa number conjecture formulated by Bloch-Kato, the special value modulo period is an algebraic number satisfying the equality:

$$\text{ord}_p \left(\frac{L(f, j)}{(2\pi\sqrt{-1})^{j-1}\Omega_{f,\infty}^{(-1)^{j-1},\text{MS}}} \right) = \#(\text{Sel}(f, j)\{p^\infty\})$$

most of the time except certain special cases.

2. The p -order $\text{ord}_p \left(\Omega_{f,\infty}^{\pm, \text{MS}} / \Omega_{f,\infty}^{\pm, (\psi)} \right)$ of the ratio of two periods is not necessarily trivial for a Dirichlet character ψ . It seems to be not known if there exists a finite even Dirichlet character ψ so that this becomes zero.
3. Both cases, we have no canonical choice of complex periods. The period $\Omega_{M_f, \infty}^{\pm}$ is dependent on the choice of a basis η^{\pm} of $H_{\text{Betti}}(M_f)^{\pm}$. For this reason, the conjecture on the existence of p -adic L -functions is difficult to formulate.

3. CYCLOTOMIC ONE-VARIABLE p -ADIC DEFORMATION

Let $\Gamma^{(1)} = \text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})$, where $\mathbb{Q}_{\infty} \subset \mathbb{Q}(\mu_{p^{\infty}})$ be the cyclotomic \mathbb{Z}_p -extension. We have the canonical isomorphism $\chi^{(1)} : \Gamma^{(1)} \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ via the cyclotomic character $\chi^{(1)}$. Let $\widehat{\mathcal{O}}_f$ be the p -adic completion of \mathcal{O}_f . Let f be a p -stabilized eigen cuspform of weight $k \geq 2$ and level Np . By Deligne, we have a continuous irreducible Galois representation $T_f \cong (\widehat{\mathcal{O}}_f)^{\oplus 2} \curvearrowright^{\rho_f} G_{\mathbb{Q}}$ satisfying the following properties:

1. The representation ρ_f is unramified at every finite primes not dividing Np .
2. For each prime $l \nmid Np$, we have $\text{Tr}(\rho_f(\text{Frob}_l)) = a_l(f)$, where $a_l(f)$ is the l -th Fourier coefficient of f .

Let $\Lambda^{(1)}(\widetilde{\chi})$ be a free rank-one $\Lambda^{(1)}$ -module on which $G_{\mathbb{Q}}$ acts via the character $\widetilde{\chi} : G_{\mathbb{Q}} \rightarrow \text{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q}) \hookrightarrow \Lambda^{(1)}$. In this section, we define \mathcal{T} to be $\mathcal{T} := T_f \otimes_{\mathbb{Z}_p} \Lambda^{(1)}(\widetilde{\chi})$ on which $G_{\mathbb{Q}}$ acts diagonally. The representation \mathcal{T} is free of rank two over $\mathcal{R} := \widehat{\mathcal{O}}_f[[\Gamma^{(1)}]]$. In this case, the construction of the p -adic L -function is known for long time (cf. [MS] and [MTT]) as follows:

Theorem 3.1 (Mazur/Swinnerton-Dyer, Manin, Amice-Velu, Mazur/Tate/Teitelbaum). *Assume that $a_p(f)$ is a p -adic unit. Then, we have a p -adic L -function $L_p^{\text{MTT}}(f) \in \mathcal{R}$ associated to the above \mathcal{T} and \mathcal{R} which satisfies the following interpolation property:*

$$L_p^{\text{MTT}}(f)(\kappa) = \left(\frac{p^{(j-1)}}{a_p(f)} \right)^{c(\epsilon)} \left(1 - \frac{\epsilon^{-1}(p)p^{c(\epsilon)}}{a_p(f)} \right) G(\epsilon^{-1}) \frac{L(f, \epsilon, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{f,\infty}^{\pm, \text{MS}}}$$

at each arithmetic point $\kappa \in \mathfrak{X}_{\text{arith}}(\mathcal{R})$ of weight j with $1 \leq j \leq k-1$, where $\epsilon = \omega^{1-j}\kappa/\chi^j$ is a finite order character of $\Gamma^{(1)}$ and $c(\epsilon)$ is the p -order of the conductor of ϵ .

The construction was done by the method of modular symbol. We have another construction as follows:

Theorem 3.2 (Kato, Panchishkin). *Assume that $a_p(f)$ is a p -adic unit. Suppose that we have a Dirichlet character ψ with $\psi(-1) = (-1)^{k-1}$ such that $L(f, \psi, k-1) \neq 0$. Then, we have a p -adic L -function $L_p^{(\psi)}(f) \in \mathcal{R}$ associated to the above \mathcal{T} and \mathcal{R} which satisfies the following interpolation property:*

$$L_p^{(\psi)}(f)(\kappa) = \left(\frac{p^{(j-1)}}{a_p(f)} \right)^{c(\epsilon)} \left(1 - \frac{\epsilon^{-1}(p)p^{c(\epsilon)}}{a_p(f)} \right) G(\epsilon^{-1}) \frac{L(f, \epsilon, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{f,\infty}^{\pm, (\psi)}}$$

at each arithmetic point $\kappa \in \mathfrak{X}_{\text{arith}}(\mathcal{R})$ of weight j with $1 \leq j \leq k-1$.

The construction by Kato factors through the Galois cohomology and Perrin-Riou's theory of the interpolation of exponential maps of local Galois cohomology in the cyclotomic tower. Panchishkin's theory are more direct method using a family of Eisenstein series. However, both are based on the Shimura's method of algebraicity of critical values of Hecke L -function of modular forms. Kato's method has advantage that the p -adic L -function is related to the Selmer group of f .

Remark 3.3.

1. By the interpolation property of the p -adic L -function, the ratio $L_p^{(\psi)}(f)/L_p^{\text{MTT}}(f)$ is nothing but the constant $\Omega_{f,\infty}^{\pm,\text{MS}}/\Omega_{f,\infty}^{\pm,(\psi)}$.
2. In general, the deformation $\mathcal{T} := T \otimes_{\mathbb{Z}_p} \Lambda^{(1)}(\tilde{\chi})$ constructed from the p -adic realization T of a certain motive M is called a ‘‘cyclotomic deformation’’. As Theorem 3.1 shows, there appears only one motive M in the cyclotomic deformation.

4. TWO-VARIABLE p -ADIC DEFORMATIONS

We consider the tower of modular curves with level $\Gamma_1(p^t)$ -structures $\{Y_1(p^t)\}_{t \geq 1}$. Since $Y_1(p^t)$ parametrizes pairs (E, e) of an elliptic curve E with a point of order p^t on E , we have the group of diamond operators

$$\mathbb{Z}_p^\times \subset \text{Aut}(\{Y_1(p^t)\}_{t \geq 1}),$$

where the diamond operator $\langle a \rangle$ corresponding to $a \in \mathbb{Z}_p^\times$ sends (E, e) to $(E, a \cdot e)$. Let $\Gamma^{(2)}$ be the p -Sylow subgroup of the group of diamond operators. We have the canonical isomorphism:

$$\chi^{(2)} : \Gamma^{(2)} \xrightarrow{\sim} 1 + p\mathbb{Z}_p$$

By this canonical character $\chi^{(2)}$, we have arithmetic points on $\mathfrak{X}(\Lambda^{(2)})$ as defined in §1.2. Let N_0 be a natural number prime to p .

Theorem 4.1 (Hida). *Let $f_0 \in S_{k_0}(\Gamma_1(N_0p))$ be a p -stabilized eigen cuspform of weight $k_0 \geq 2$ such that $a_p(f_0)$ is a p -unit. We have a finite extension $\mathcal{R}^{(2)}$ over $\Lambda^{(2)} = \mathbb{Z}_p[[\Gamma^{(2)}]]$ and an $\mathcal{R}^{(2)}$ -adic eigen cuspform $\mathcal{F} = \sum_{n \geq 0} A_n(\mathcal{F})q^n$ which satisfies the following properties.*

1. *There exists an arithmetic point $\kappa_0^{(2)}$ of weight $k_0 - 2$, such that the specialization of \mathcal{F} at $\kappa_0^{(2)}$ is equal to f_0 .*
2. *For each arithmetic point $\kappa^{(2)}$ of weight $w(\kappa^{(2)}) \geq 2$, $f_{\kappa^{(2)}} \in S_k(\Gamma_1(Np^*))$ is an eigen cuspform with $k = w(\kappa^{(2)}) + 2$.*

For a family of cuspforms \mathcal{F} as above, we associate a family of Galois representation as follows:

Theorem 4.2 (Hida, Wiles). *Let $\mathcal{F} = \sum_{n \geq 0} A_n(\mathcal{F})q^n$ be an $\mathcal{R}^{(2)}$ -adic eigen cusp form which satisfies the properties as in Theorem 4.1. Then, we have a continuous irreducible representation $\rho_{\mathcal{F}}$ of $G_{\mathbb{Q}}$ on a two-dimensional vector space \mathcal{V} over $\text{Frac}(\mathcal{R}^{(2)})$ satisfying the following properties:*

1. *The representation is unramified at prime not dividing N_0p .*
2. *We have $\text{Tr}(\rho_{\mathcal{F}}(\text{Frob}_l)) = A_l(\mathcal{F})$ for each prime number $l \nmid N_0p$.*

Assume further the following conditions:

1. $\mathcal{R}^{(2)}$ is Gorenstein algebra.
2. The formal q -expansion obtained by reduction modulo the maximal ideal of $\mathcal{R}^{(2)}$ is not congruent to an Eisenstein series.

Then, the representation $\rho_{\mathcal{F}}$ on \mathcal{V} has a $G_{\mathbb{Q}}$ -stable lattice $\tilde{T}_{\mathcal{F}} \subset \mathcal{V}$ which is free of rank-two over $\mathcal{R}^{(2)}$.

Following the spirit of Conjecture 1.10 and Conjecture 1.12, we expect to have the two-variable p -adic L -function associated to \mathcal{T} . Let us recall a result of Kitagawa whose main ingredient is the construction of Λ -adic modular symbols³. Kitagawa studied the following inverse limit

$$\mathcal{UM}_{N_0}^{\pm} = \varprojlim_{r,s} H_1(X_1(N_0 p^r)(\mathbb{C}), \partial X_1(N_0 p^r)(\mathbb{C}); \mathbb{Z}/(p^s)\mathbb{Z})^{\pm},$$

where $H_1(X_1(N_0 p^r)(\mathbb{C}), \partial X_1(N_0 p^r)(\mathbb{C}); \mathbb{Z}/(p^s)\mathbb{Z})$ is the Homology group with support at the boundary $\partial X_1(N_0 p^r)(\mathbb{C})$ consisting of finite numbers of cusps. This $\mathcal{UM}_{N_0}^{\pm}$ is naturally endowed with a structure of a $\Lambda^{(2)}$ -module. By using $\mathcal{UM}_{N_0}^{\pm}$, Kitagawa defines the module of Λ -adic modular symbols $\mathcal{MS}_{N_0}^{\pm}$ by $\mathcal{MS}_{N_0}^{\pm} = \text{Hom}_{\Lambda^{(2)}}(\mathcal{UM}_{N_0}^{\pm}, \Lambda^{(2)})$. We define the \mathcal{F} -component $\mathcal{MS}(\mathcal{F})$ of $\mathcal{MS}_{N_0}^{\pm}$ as follows:

$$\mathcal{MS}(\mathcal{F})^{\pm} = \mathcal{MS}_{N_0}^{\pm} \otimes_{\Lambda^{(2)}} \mathcal{R}^{(2)}[\lambda_{\mathcal{F}}]$$

Proposition 4.3. Assume the following conditions

1. $\mathcal{R}^{(2)}$ is Gorenstein.
2. The p -tame character of the action of $G_{\mathbb{Q}_p}$ on the residual representation $\mathcal{T}/\mathfrak{MT}$ is ω^i -twist of ordinary representation with $i \not\equiv 2 \pmod{p}$.

Then, $\mathcal{MS}(\mathcal{F})^{\pm}$ is free of rank-one over $\mathcal{R}^{(2)}$ and the natural map:

$$\mathcal{MS}(\mathcal{F})^{\pm} / \text{Ker}(\kappa^{(2)}) \mathcal{MS}(\mathcal{F})^{\pm} \longrightarrow \mathcal{MS}(f_{\kappa^{(2)}})^{\pm} \otimes_{\mathcal{O}_{f_{\kappa^{(2)}}}} \hat{\mathcal{O}}_{f_{\kappa^{(2)}}}$$

is isomorphism for every arithmetic point $\kappa^{(2)} \in \mathfrak{X}(\mathcal{R}^{(2)})$ with $w(\kappa^{(2)}) \geq 0$.

Let us fix a basis b^{\pm} of $\mathcal{MS}(\mathcal{F})^{\pm}$. For each arithmetic point $\kappa^{(2)} \in \mathfrak{X}(\mathcal{R}^{(2)})$ with $w(\kappa^{(2)}) \geq 0$, we denote by $b_{\kappa^{(2)}}^{\pm} \in \mathcal{MS}(f_{\kappa^{(2)}})^{\pm} \otimes_{\mathcal{O}_{f_{\kappa^{(2)}}}} \hat{\mathcal{O}}_{f_{\kappa^{(2)}}}$ the image of b^{\pm} .

Theorem 4.4. We have $L_p^{\text{Ki}}(\mathcal{F}) = L_p^{\text{Ki}, b^+}(\mathcal{F})$ in \mathcal{R} which satisfies the following interpolation property at each arithmetic point $\kappa = (\kappa^{(1)}, \kappa^{(2)})$ in $\mathfrak{X}(\mathcal{R}) = \mathfrak{X}(\Lambda^{(1)}) \times \mathfrak{X}(\mathcal{R}^{(2)})$ of weight $(j, k-2)$ with condition $1 \leq j \leq k-1$:

$$\frac{L_p^{\text{Ki}}(\mathcal{T})(\kappa)}{C_{\kappa^{(2)}, p}^+} = \left(\frac{p^{(j-1)}}{a_p(f_{\kappa^{(2)}})} \right)^{c(\epsilon^{(1)})} \left(1 - \frac{(\epsilon^{(1)})^{-1}(p)p^{c(\epsilon^{(1)})}}{a_p(f_{\kappa^{(2)}})} \right) G((\epsilon^{(1)})^{-1}) \frac{L(f_{\kappa^{(2)}}, \epsilon^{(1)}, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{\kappa^{(2)}, \infty}^{+, \text{MS}}},$$

where $\epsilon^{(1)}$ is the Dirichlet character $\omega^{1-j}\kappa^{(1)}/\chi^j$ and $C_{\kappa^{(2)}, p}^+ \in \hat{\mathcal{O}}_{f_{\kappa^{(2)}}} \subset \overline{\mathbb{Q}_p}$ is an error term defined to be $b_{\kappa^{(2)}}^+ = C_{\kappa^{(2)}, p}^+ \eta_{\kappa^{(2)}}^+$ with respect to the basis $\eta_{\kappa^{(2)}}^+$ which is involved in the definition of complex period $f = \Omega_{\kappa^{(2)}, \infty}^{+, \text{MS}} \eta_{\kappa^{(2)}}^+$.

³Greenberg-Stevens[GS] also constructs a two-variable p -adic L -function by a similar method. (cf. Remark 4.8)

Among various construction of two-variable p -adic L -functions, we show that Kitagawa's one has better property for the analytic p -adic functions. The following proposition is the first evidence for Conjecture 1.10 except cyclotomic deformations:

Proposition 4.5. *Assume that the image of $G_{\mathbb{Q}} \longrightarrow \text{Aut}_{\mathcal{R}^{(2)}}(\tilde{T}_{\mathcal{F}}) \cong GL_2(\mathcal{R}^{(2)})$ contains a subgroup $SL_2(\mathcal{R}^{(2)})$. Then, by replacing the fixed embedding $\iota_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ if necessary, there exists a unit $U \in (\mathcal{R}^{(2)} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}})^{\times} \subset (\mathcal{R} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}})^{\times}$ such that $L_p^{\text{anal}}(T) = L_p^{\text{Ki}}(T) \cdot U \in \mathcal{R} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}}$ satisfies the characterization of the Conjecture 1.10.*

Proof. By an important property of Kitagawa's construction, we have

$$(4) \quad \mathcal{MS}(\mathcal{F})^+ / \text{Ker}(\kappa^{(2)}) \mathcal{MS}(\mathcal{F})^+ \xrightarrow{\sim} \mathcal{MS}(f_{\kappa^{(2)}})^+ \otimes \widehat{\mathcal{O}}_{f_{\kappa^{(2)}}},$$

at each arithmetic point $\kappa^{(2)} \in \mathfrak{X}_{\text{arith}}(\mathcal{R}^{(2)})$ with $w(\kappa^{(2)}) \geq 0$. Note that $\mathcal{MS}(f_{\kappa^{(2)}})^+$ is a lattice of the Betti realization $H_{\text{Betti}}(M_{\kappa^{(2)}})^+$. By the same argument as the proof of Proposition 1.6, we prove that $H_{\text{Betti}}(M_{\kappa^{(2)}}) \otimes \overline{\mathbb{Q}}_p \cong H_{p,\text{ét}}((M_{\kappa^{(2)}})_{\overline{\mathbb{Q}}})$ induces the isomorphism from $H_{\text{Betti}}(M_{\kappa^{(2)}})^+ \otimes \overline{\mathbb{Q}}_p$ to the one-dimensional subspace of $H_{p,\text{ét}}((M_{\kappa^{(2)}})_{\overline{\mathbb{Q}}})$ unramified under the action of $G_{\mathbb{Q}_p}$ by replacing $\iota_{\infty} : \overline{\mathbb{Q}} \longrightarrow \mathbb{C}$ if necessary. Thus, we have an isomorphism $H_{\text{Betti}}(M_{\kappa^{(2)}})^+ \otimes B_{\text{HT}} \longrightarrow H_{\text{dR}}(M_{\kappa^{(2)}})^+ \otimes B_{\text{HT}}$ which is the base extension of

$$(5) \quad H_{\text{Betti}}(M_{\kappa^{(2)}})^+ \otimes \widehat{\mathbb{Q}}_p^{\text{ur}} \cong H_{\text{dR}}(M_{\kappa^{(2)}})^+ \otimes \widehat{\mathbb{Q}}_p^{\text{ur}}.$$

Now, we suppose that $w(\kappa^{(2)}) = 0$ (hence, the weight of $f_{\kappa^{(2)}}$ is two). By a result in [O3], the integral structure associated to the (plus-part of) integral Betti cohomology on the lefthand side of (5) is equal to the integral structure on the righthand side generated by $f_{\kappa^{(2)}} \in H_{\text{dR}}(M_{\kappa^{(2)}})^+$. Let us take $u_{\kappa^{(2)}} \in (\mathcal{O}_K \widehat{\mathbb{Z}}_p^{\text{ur}})^{\times}$ such that $f_{\kappa^{(2)}} = u_{\kappa^{(2)}} b_{\kappa^{(2)}}^+$. Let U be an element in $(\mathcal{R} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}})^{\times}$ such that $U(\kappa^{(2)}) = u_{\kappa^{(2)}}$. Note that $u_{\kappa^{(2)}} \cdot C_{\kappa^{(2)},p}^+ = \Omega_{\kappa^{(2)},p}^+$ with $f_{\kappa^{(2)}} = \Omega_{\kappa^{(2)},p}^+ \eta_{\kappa^{(2)}}^+$. Thus, $L_p^{\text{anal}}(T) = L_p^{\text{Ki}}(T) \cdot U$ satisfies the interpolation property:

$$\frac{L_p^{\text{anal}}(T)(\kappa)}{\Omega_{\kappa^{(2)},p}^+} = a_p(f_{\kappa^{(2)}})^{-c(\epsilon^{(1)})} \left(1 - \frac{(\epsilon^{(1)})^{-1}(p)p^{c(\epsilon^{(1)})}}{a_p(f_{\kappa^{(2)}})} \right) G((\epsilon^{(1)})^{-1}) \frac{L(f_{\kappa^{(2)}}, \epsilon^{(1)}, 1)}{\Omega_{\kappa^{(2)},\infty}^{+, \text{MS}}},$$

at every arithmetic point $\kappa^{(1)} \in \mathfrak{X}_{\text{arith}}(\Lambda^{(1)})$ with $w(\kappa^{(1)}) = 1$, where we have and $f_{\kappa^{(2)}} = \Omega_{\kappa^{(2)},\infty}^{+, \text{MS}} \eta_{\kappa^{(2)}}^+$ respectively in $H_{\text{dR}}(M_{\kappa^{(2)}})^+ \otimes \widehat{\mathbb{Q}}_p^{\text{ur}}$ and $H_{\text{dR}}(M_{\kappa^{(2)}})^+ \otimes \mathbb{C}$. Though U depends on $\kappa^{(2)}$ and the point $\kappa^{(2)}$ is fixed in the above interpolation, it is not difficult to have $U \in (\mathcal{R}^{(2)} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}})^{\times}$ so that $u_{\kappa^{(2)}} = U(\kappa^{(2)})$ is related to the p -adic period $\Omega_{\kappa^{(2)},p}^+$ at every $\kappa^{(2)} \in \mathfrak{X}_{\text{arith}}(\mathcal{R}^{(2)})$ with $w(\kappa^{(2)}) \geq 0$. We do not continue the similar argument for the existence of $U \in (\mathcal{R}^{(2)} \hat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}})^{\times}$ which covers every arithmetic point in $\mathfrak{X}_{\text{arith}}(\mathcal{R}^{(2)})$ of weight ≥ 0 . We only remark that the key of our proof is that the exact control as in (4) are satisfied by every ideal $I \subset \mathcal{R}^{(2)}$ of the form $I = \cap \text{Ker}(\kappa_i)$ where κ_i runs finite number of elements in $\mathfrak{X}_{\text{arith}}(\mathcal{R}^{(2)})$ with a fixed weight $w = w(\kappa_i)$. \square

On the other hand, we have the following result on the algebraic side:

Proposition 4.6. *The Pontrjagin dual $(\text{Sel}_{\mathcal{T}})^{\vee}$ of $\text{Sel}_{\mathcal{T}}$ is a finitely generated torsion \mathcal{R} -module and it is identified with the other construction using Bloch-Kato's local condition.*

We define an algebraic p -adic L -function $L_p^{\text{alg}}(\mathcal{T}) \in \mathcal{R}$ to be the characteristic power series of $(\text{Sel}_{\mathcal{T}})^{\vee}$. Thus, the Iwasawa Main Conjecture in this case is formulated as follows (cf. Conjecture 1.12):

Conjecture 4.7 (Iwasawa Main Conjecture). *Let \mathcal{T} be the two-variable Galois deformation associated to a certain Λ -adic cuspform $\mathcal{F} \in \mathcal{R}^{(2)}[[q]]$. We assume technical conditions such as Gorenstein property of $\mathcal{R}^{(2)}$ as well as the irreducibility of the residual representation of $G_{\mathbb{Q}}$ for \mathcal{T} . Then, we have the equality:*

$$(L_p^{\text{alg}}(\mathcal{T})) = (L_p^{\text{Ki}}(\mathcal{T}))$$

between ideals in $\mathcal{R} \widehat{\otimes} \widehat{\mathbb{Z}}^{\text{ur}}$.

Remark 4.8. Conjecture 4.7 is a refinement of the two-variable Iwasawa Main conjecture proposed by Greenberg [Gr2]. In the analytic side, the Iwasawa Main Conjecture depends on which two-variable analytic p -adic L -function to choose among several constructions containing [GS], [F], [O1] and [P2]. For example, [GS] gives a two variable p -adic L -function $L_p^{\text{Gr}}(\mathcal{T}) \in \text{Frac}(\mathcal{R})$. by a similar method of “Module of Λ -adic modular symbols”. However, the method is slightly different from the one by Kitagawa and their construction does not give a priori the property (4) at every $\kappa^{(2)}$ simultaneously as they remark in their paper. Thus, we are not sure if $L_p^{\text{Gr}}(\mathcal{T})$ satisfies the characterization as in Proposition 4.5 and we are not sure if the ideal $(L_p^{\text{Gr}}(\mathcal{T})) \subset \mathcal{R}$ is equal to $(L_p^{\text{Ki}}(\mathcal{T}))$. The constructions [F], [O1] and [P2] are based on Shimura’s theory of the Rankin-Selberg integral and the periods of modular forms. Thus, the ideal defined by these ones are not equal to $(L_p^{\text{Ki}}(\mathcal{T}))$ in general. In [O3], we also discuss different candidates for the definition of Selmer groups of \mathcal{T} as well as several properties on the behavior of the Selmer group with respect to the specializations of Galois representations at ideals of the ring of coefficients. Since we have few examples for Iwasawa theory of Galois representations other than cyclotomic deformations, we believe that such detailed study is important to justify our formulation of the conjecture.

4.1. Comparison. We gave the following analogue of Perrin-Riou map, which give an interpolation of dual exponential map on local Galois cohomologies in the case of two-variable nearly ordinary Galois deformations.

Theorem 4.9. [O1] *There exists*

$$\Xi : H_{/f}^1(\mathbb{Q}_p, \mathcal{T}) \longrightarrow \mathcal{R}$$

which satisfies the following properties for each element $\mathcal{C} \in H_{/f}^1(\mathbb{Q}_p, \mathcal{T})$. For any arithmetic point $\kappa = (\kappa^{(1)}, \kappa^{(2)})$ of weight (w_1, w_2) with $1 \leq w_1 \leq w_2$, we have

$$\kappa(\Xi(\mathcal{C})) = \langle \exp^*(c_{\kappa}), f_{\kappa^{(2)}} \rangle,$$

where $c_{\kappa} \in H_{/f}^1(\mathbb{Q}_p, \mathcal{T}/\text{Ker}(\kappa)\mathcal{T})$ is the specialization of \mathcal{C} under the map $H_{/f}^1(\mathbb{Q}_p, \mathcal{T}) \longrightarrow H_{/f}^1(\mathbb{Q}_p, \mathcal{T}/\text{Ker}(\kappa)\mathcal{T})$.

On the other hand, by taking the projective limit with respect to m, n of elements in $H^1(\mathbb{Q}, H_{\text{ét}}^1(Y_1(N_0 p^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \otimes \mathbb{Z}_p[\Gamma/\Gamma^{p^m}](\tilde{\chi}))$ constructed by Kato and sending it via the

natural finite map

$$H^1(\mathbb{Q}, \varprojlim_{m,n} H_{\text{ét}}^1(Y_1(N_0 p^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \otimes \mathbb{Z}_p[\Gamma/\Gamma^{p^m}](\tilde{\chi})) \longrightarrow H^1(\mathbb{Q}, T),$$

we have an element $\mathcal{Z}^{(\psi)} \in H^1(\mathbb{Q}, T)$. The element $\mathcal{Z}^{(\psi)}$ has the property that

$$\langle \exp^*(z_{\kappa}^{(\psi)}), f_{\kappa(2)} \rangle = \frac{L_{(p)}(f_{\kappa(2)}, \epsilon^{(1)} \omega^{1-j}, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{\kappa(2), \infty}^{+, (\psi)}}$$

at each arithmetic point $\kappa = (\kappa^{(1)}, \kappa^{(2)})$ in $\mathfrak{X}(\mathcal{R}) = \mathfrak{X}(\Lambda^{(1)}) \times \mathfrak{X}(\mathcal{R}^{(2)})$ of weight $(j, k-2)$ with the condition $1 \leq j \leq k-1$. Here, $L_{(p)}(f_{\kappa(2)}, \epsilon^{(1)} \omega^{1-j}, s)$ is the Hecke L -function with the p -factor removed.

Corollary 4.10. [O1] $\Xi(\mathcal{Z}^{(\psi)}) \in \mathcal{R}$ satisfies the following interpolation property at each arithmetic point $\kappa = (\kappa^{(1)}, \kappa^{(2)})$ in $\mathfrak{X}(\mathcal{R}) = \mathfrak{X}(\Lambda^{(1)}) \times \mathfrak{X}(\mathcal{R}^{(2)})$ of weight $(j, k-2)$ with the condition $1 \leq j \leq k-1$:

$$\Xi(\mathcal{Z}^{(\psi)})(\kappa) = \left(\frac{p^{(j-1)}}{a_p(f_{\kappa(2)})} \right)^{c(\epsilon^{(1)})} \left(1 - \frac{\epsilon^{-1}(p)p^{c(\epsilon)}}{a_p(f_{\kappa(2)})} \right) G(\epsilon^{-1}) \frac{L(f_{\kappa(2)}, \epsilon^{(1)} \omega^{1-j}, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{\kappa(2), \infty}^{+, (\psi)}},$$

We remark that the above element $\Xi(\mathcal{Z}^{(\psi)}) \in \mathcal{R}$ satisfies the two-variable interpolation property which resemble to the one for the ideal two-variable p -adic L -function $L_p^{\text{anal}}(T)$ except the difference on the complex period and the p -adic period. There appears no p -adic periods on the right hand side and the complex period $\Omega_{\kappa(2), \infty}^{+, (\psi)}$ might not be well-optimized.

Theorem 4.11. [O3] *There exists an Euler system $\mathcal{Z}^{\text{Ki}} \in H_{/f}^1(\mathbb{Q}_p, T)$ such that*

$$\langle \exp^*(z_{\kappa}^{\text{Ki}}), f_{\kappa(2)} \rangle = C_{\kappa(2), p}^+ \cdot \frac{L_{(p)}(f_{\kappa(2)}, \epsilon^{(1)} \omega^{1-j}, j)}{(2\pi\sqrt{-1})^{j-1} \Omega_{\kappa(2), \infty}^{+, \text{MS}}}$$

at each arithmetic point $\kappa = (\kappa^{(1)}, \kappa^{(2)})$ in $\mathfrak{X}(\mathcal{R}) = \mathfrak{X}(\Lambda^{(1)}) \times \mathfrak{X}(\mathcal{R}^{(2)})$ of weight $(j, k-2)$ with the condition $1 \leq j \leq k-1$, where $C_{\kappa(2), p}^+$ is an error term which appeared in Kitagawa's construction (cf. Proposition 4.4).

Corollary 4.12. *We have $\Xi(\mathcal{Z}^{\text{Ki}}) = L_p^{\text{Ki}}(T)$.*

4.2. Application to Iwasawa theory. We established in [O2] “the Euler system theory for Galois deformations” which works for certain general non-cyclotomic Galois deformations. According to this theory, the algebraic and analytic p -adic L -functions are related to each other once we have constructed the analytic p -adic L -function via an Euler system as in Theorem 4.11. Thus, we have:

Theorem 4.13. *We have the following inequality:*

$$(L_p^{\text{Ki}}(T)) \subset (L_p^{\text{alg}}(T))$$

As an application of this inequality, we have:

Corollary 4.14. *The following statements are equivalent:*

1. *There exists an arithmetic point $\kappa^{(2)}$ with $w(\kappa^{(2)}) \geq 0$ such that the cyclotomic Iwasawa main conjecture for $f_{\kappa^{(2)}}$ holds.*
2. *At every arithmetic point $\kappa^{(2)}$ with $w(\kappa^{(2)}) \geq 0$, the cyclotomic Iwasawa main conjecture for $f_{\kappa^{(2)}}$ holds.*
3. *The two-variable Iwasawa Main Conjecture 4.7 holds.*

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