

BERTINI THEOREM FOR NORMALITY ON LOCAL RINGS IN MIXED CHARACTERISTIC (APPLICATIONS TO CHARACTERISTIC IDEALS)

TADASHI OCHIAI AND KAZUMA SHIMOMOTO

ABSTRACT. In this article, we prove a strong version of local Bertini theorem for normality on local rings in mixed characteristic. The main result asserts that a generic hyperplane section of a normal, Cohen-Macaulay, and complete local domain of dimension at least 3 is normal. Applications include the study of characteristic ideals attached to torsion modules over normal domains, which is fundamental in the study of Euler system theory and Iwasawa main conjectures.

1. INTRODUCTION

The classical Bertini theorem says that a generic hyperplane section of a smooth complex projective variety is smooth. A (local) Bertini theorem may be stated for a local ring $(R, \mathfrak{m}, \mathbf{k})$ as follows. Let \mathbf{P} be a ring-theoretic property (e.g. regular, reduced, normal, seminormal and so on). Then if R is \mathbf{P} and $x \in \mathfrak{m}$ is a non-zero divisor, then is it true that R/xR is so for a generic choice of x ? A local Bertini theorem (in a slightly weak form) was first raised by Grothendieck ([6], Exposé XIII, Conjecture 2.6) and was proved by Flenner [2] and Trivedi [15]. In this article, for an ideal $I \subseteq R$, we denote by $D(I)$ the set of all primes of R which do not contain I , and by $V(I)$ the complement of $D(I)$ in $\text{Spec } R$. Before stating our main theorems, let us recall the following result from [2]:

Theorem 1.1 (Flenner-Trivedi). *Let (R, \mathfrak{m}) be a local Noetherian ring and let $I \subseteq \mathfrak{m}$ be an ideal. Assume that Q is a finite subset of $D(I)$. Then there exists an element $x \in I$ such that:*

- (1) $x \notin \mathfrak{p}^{(2)}$ for all $\mathfrak{p} \in D(I)$;
- (2) $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in Q$;

for $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$, the n -th symbolic power ideal of \mathfrak{p} .

We make a remark on the second symbolic power of ideals. Let (R, \mathfrak{m}) be a local ring, let $x \in \mathfrak{p}$ be a non-zero divisor and let $x \notin \mathfrak{p}^{(2)}$ for a prime ideal $\mathfrak{p} \subseteq R$. Then $R_{\mathfrak{p}}$ is regular if and only if $R_{\mathfrak{p}}/xR_{\mathfrak{p}}$ is regular. Many ring-theoretic properties such as regular, normal, reduced can be verified at the localization $R_{\mathfrak{p}}$, which is the reason why we require $x \notin \mathfrak{p}^{(2)}$ (but not merely $x \notin \mathfrak{p}^2$), which is equivalent to the condition: $x \notin \mathfrak{p}^2 R_{\mathfrak{p}} \cap R$. A strong

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version of local Bertini theorem similar to our main theorem below was already proved for local rings containing a field in [2]. To extend his result to the mixed characteristic case, we need to introduce some new ideas. Theorem 1.1 does not suffice for our purpose, since it only assures us that there is an element $x \in \mathfrak{m}$ for which R/xR is normal.

Here is our notation which will be used throughout: We denote by $\mathbb{P}^n(S)$ (resp. $\mathbb{A}^n(S)$) the projective space (resp. an affine space) for a commutative ring S . In case when S is a field or a discrete valuation ring, we will consider these spaces as sets of points in the ring S endowed with some topology (see discussions in § 2). In these cases, every point $a = (a_0 : \cdots : a_n) \in \mathbb{P}^n(S)$ is normalized so that a_i is a unit for some i . This implies that, for a complete discrete valuation ring A with residue field \mathbf{k} , we have a *specialization map* $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$, which is well-defined. Let us now state one of our main theorems (see Theorem 4.3 together with Theorem 4.2 in § 4).

Main Theorem A (Local Bertini Theorem). Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete local domain of mixed characteristic $p > 0$ and suppose the following conditions:

- (1) let $A \rightarrow R$ be a coefficient ring map for a complete discrete valuation ring (A, π_A) ;
- (2) let x_0, x_1, \dots, x_d be a fixed set of minimal generators of \mathfrak{m} ;
- (3) R is normal, of depth $R \geq 3$, and the residue field \mathbf{k} is infinite.

Then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$ satisfying the following properties. For any $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U)$, the quotient $R/\mathbf{x}_a R$ is a normal domain of mixed characteristic $p > 0$, where we put

$$\mathbf{x}_a := \sum_{i=0}^d a_i x_i.$$

We will also discuss a version of the above theorem for the case when the residue field is finite at the end of § 4. This theorem allows us to find sufficiently many normal local domains of mixed characteristic as specializations. The above theorem does not tell us how to find U , but we will show how to find U in Example 4.8. It is worth pointing out that if $\dim R = 2$, the local Bertini theorem fails due to a simple reason. In fact, if the quotient R/yR is normal, then it is a discrete valuation ring, so R must be regular. By Cohen structure theorem, there is a surjection $A[[z_0, \dots, z_d]] \twoheadrightarrow R$, where $d+1$ is the number of minimal generators of \mathfrak{m} , so that the minimal generators of \mathfrak{m} are just the image of z_0, z_1, \dots, z_d under this surjection. Related to our main result, if (R, \mathfrak{m}) is a local ring and $y \in \mathfrak{m}$ is a non-zero divisor such that R/yR is normal, then R is normal ([5] 5.12.7).

In § 7, we prove some basic results on characteristic ideals over general normal domains. As an application, we prove another main theorem (see Theorem 8.7 in § 8) in our paper.

Main Theorem B (Control Theorem for Characteristic Ideals). Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete local domain of mixed characteristic $p > 0$ satisfying the conditions (1), (2) and (3) of Main Theorem A. Assume that M and N are finitely generated torsion R -modules. Then, for certain infinite subsets $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}})$ and $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}})$ of the projective space $\mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$ defined in a natural way (cf. Definition 8.1), the following statements are equivalent:

- (1) $\text{char}_R(M) \subseteq \text{char}_R(N)$, where $\text{char}_R(-)$ means the characteristic ideal (cf. Definition 7.1).
- (2) For all but finitely many height-one primes:

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

there exists a finite étale extension of discrete valuation rings $\mathcal{O} \rightarrow \mathcal{O}'$ such that we have $\mathbf{x}_a \in R_{\mathcal{O}'}$ and

$$\text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(M_{\mathcal{O}'}/\mathbf{x}_a M_{\mathcal{O}'}) \subseteq \text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(N_{\mathcal{O}'}/\mathbf{x}_a N_{\mathcal{O}'}).$$

- (3) For all but finitely many height-one primes:

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

we have

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Main Theorem B will be crucial in a forthcoming paper [11], where we plan to compare characteristic ideals of certain torsion modules arising from Iwasawa theory as developed in [10] (for example, those torsion modules arising as the Pontryagin dual of Selmer groups associated with two-dimensional Galois representations of certain type with values in a complete local ring with finite residue field).

2. SPECIALIZATION MAP AND THE TOPOLOGY FOR BERTINI-TYPE THEOREMS

Let us discuss the specialization map. Let A be a complete discrete valuation ring and let $a := (a_0 : \cdots : a_n) \in \mathbb{P}^n(A)$. Then we may normalize a so that $a = (b_0 : \cdots : b_n) \in \mathbb{P}^n(A)$ and some b_i is a unit of A . So we may think of the projective space as

$$\mathbb{P}^n(A) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n(A) \mid |a_i| = 1 \text{ for some } i\}$$

for the normalized valuation $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$. Let \mathbf{k} be the residue field of A . Then reducing a to a point $\bar{a} = (\bar{b}_0 : \cdots : \bar{b}_n) \in \mathbb{P}^n(\mathbf{k})$, we have constructed the map:

$$\text{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k}),$$

called the *specialization map*. This map does not depend on the choice of the representative of $a \in \mathbb{P}^n(A)$ as above. The set $\mathbb{P}^n(\mathbf{k})$ is endowed with the Zariski topology, while $\mathbb{P}^n(A)$ is endowed with the topology induced by the valuation on A . Hence, we simply regard

$\mathbb{P}^n(A)$ as a set of points equipped with this topology. For more details, see the remark below. We refer the reader to [3] for the rigid geometry and the following fact.

Lemma 2.1. *The specialization map $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$ is continuous and surjective.*

We begin to pin down the suitable topology for formulating Bertini-type theorems in mixed characteristic. Let $(R, \mathfrak{m}, \mathbf{k})$ be a local Noetherian ring. Then we say that a reduced local ring $(R, \mathfrak{m}, \mathbf{k})$ is of *mixed characteristic* $p > 0$, if every component of the total ring of fractions of R is of characteristic zero and the residue field \mathbf{k} is of characteristic p .

Now we assume that (A, π_A, \mathbf{k}) is a complete discrete valuation ring such that $\pi_A A = pA$ and there is an injection $A \hookrightarrow R$ of rings, which induces an isomorphism on residue fields, say $\mathbf{k} = A/\pi_A A \simeq R/\mathfrak{m}$. Such A is called a *coefficient ring* of R .

Example 2.2. Let $R := \mathbb{Z}_p[[x, y]]/(p - xy)$. Then R is a finite extension of $\mathbb{Z}_p[[x + y]]$ by the Eisenstein equation $t^2 - (x + y)t + p = 0$ and \mathbb{Z}_p is a coefficient ring of R .

In what follows, a coefficient ring will be considered as a fixed one. We denote by $\mathbf{Loc.alg}_{/A}$ the category of local A -algebras that are obtained as quotients of all local rings having A as their fixed coefficient ring. Note that objects of $\mathbf{Loc.alg}_{/A}$ also include local \mathbf{k} -algebras.

Definition 2.3. With the notation as above, assume that \mathbf{P} is a ring-theoretic property on Noetherian rings and fix $(R, \mathfrak{m}, \mathbf{k}) \in \mathbf{Loc.alg}_{/A}$. We say that the “local Bertini theorem” holds for \mathbf{P} , if a set of minimal generators x_0, \dots, x_n of \mathfrak{m} is given, then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^n(\mathbf{k})$ such that $R/\mathbf{x}_a R$ has \mathbf{P} for all $a = (a_0, \dots, a_n) \in \mathrm{Sp}_A^{-1}(U) \subseteq \mathbb{P}^n(A)$, where we put

$$\mathbf{x}_a = \sum_{i=0}^n a_i x_i.$$

The “local Bertini theorem” can be formulated in a different way. For example, the maximal ideal \mathfrak{m} may be replaced with a smaller ideal. But we adopt the above definition.

Remark 2.4. The naturality of the above formalism is explained as follows. We endow $\mathbb{P}^n(\mathbf{k})$ with the Zariski topology. Let $\bar{f} \in \mathbf{k}[x_0, \dots, x_n]$ be a non-zero homogeneous polynomial, let $f \in A[x_0, \dots, x_n]$ be any fixed homogeneous lifting of \bar{f} , and let $U_{\bar{f}} \subseteq \mathbb{P}^n(\mathbf{k})$ be an open subset defined by $\bar{f} \neq 0$. Then the inverse image of the open subset $U_{\bar{f}}$ under the map $\mathrm{Sp}_A : \mathbb{P}^n(A) \rightarrow \mathbb{P}^n(\mathbf{k})$ can be described as follows. We have

$$\mathrm{Sp}_A^{-1}(U_{\bar{f}}) = \{a = (a_0 : \dots : a_n) \in \mathbb{P}^n(A) \mid |f(a)| = 1\}$$

for the normalized valuation $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$, which is an admissible open subset of $\mathbb{P}^n(A)$. Our objective is to show that this topology is adequate in formulating the local Bertini theorem in the mixed characteristic case.

The following proposition is indispensable for the proof of Theorem 4.2 and Theorem 4.3. However, the proposition fails for a finite field.

Proposition 2.5 ([14]; Proposition 3.3). *Let $U \subseteq \mathbb{A}^n(L)$ be any non-empty Zariski open subset for an infinite field L . Then U is dense. Furthermore, if $K \rightarrow L$ is any field extension such that K is infinite, the intersection $U \cap \mathbb{A}^n(K)$ is also a Zariski dense open subset. The above assertions hold over the projective space as well.*

3. DISCUSSION ON BASIC ELEMENTS

Let (R, \mathfrak{m}) be a local Noetherian ring and let M be a finitely generated R -module. The depth of M , denoted by $\text{depth}_R M$, is the maximal length of M -regular sequences. For $\mathfrak{p} \in \text{Spec } R$, $\mu_{\mathfrak{p}}(M)$ stands for the number of minimal generators of the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$. The following notion is due to Swan.

Definition 3.1 (Swan). Let M be a module over a ring A and let \mathfrak{p} be its prime ideal. An element $m \in M$ is called *basic* at \mathfrak{p} , if $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/A \cdot m) = 1$. More generally, a set of elements m_1, \dots, m_n of M is called *k-fold basic* at \mathfrak{p} , if $\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/\sum_{i=1}^n A \cdot m_i) \geq k$; that is, $N = A \cdot m_1 + \dots + A \cdot m_n$ contains at least k minimal generators at \mathfrak{p} .

Let $M^{(r)} := M/\sum_{i=1}^r A \cdot m_i$ for a set of elements m_1, \dots, m_k of M and r satisfying $0 \leq r \leq k-1$ and pick a prime ideal \mathfrak{p} of A . Then

$$\mu_{\mathfrak{p}}(M^{(r)}) - \mu_{\mathfrak{p}}(M^{(r)}/A \cdot m_{r+1}) = 1 \iff m_{r+1} \notin \mathfrak{p}M_{\mathfrak{p}}^{(r)}$$

for $0 \leq r \leq k-1$ by Nakayama's lemma. In other words, m_1, \dots, m_k form partial generators of the $k(\mathfrak{p})$ -vector space $M \otimes_A k(\mathfrak{p})$.

We shall use (finite) Kähler differentials. For a complete local ring (R, \mathfrak{m}) with its coefficient ring A , the usual module of Kähler differentials $\Omega_{R/A}$ is not a finite R -module. Instead, one uses the completed module $\widehat{\Omega}_{R/A}$. This is the \mathfrak{m} -adic completion of $\Omega_{R/A}$ and it is a finite R -module. It can be also defined as follows. Let I denote the kernel of the map $\mu : R \widehat{\otimes}_A R \rightarrow R$ defined by $\mu(a \otimes b) = ab$. Then $\widehat{\Omega}_{R/A} := I/I^2$. The connection of Kähler differential modules with the symbolic power ideals is expressed by the following simple fact (see [2], Lemma 2.2 for its proof).

Lemma 3.2. *Let M be a module over a ring R , let \mathfrak{p} be a prime of R , and let $d : R \rightarrow M$ be a derivation. If for $x \in R$, $dx \in M$ is basic at \mathfrak{p} , then $x \notin \mathfrak{p}^{(2)}$.*

For a ring A and an open subset $U \subseteq \text{Spec } A$, we set

$$\dim_U(R/I) := \dim(V(I) \cap U)$$

for an ideal I of A . If M is a finite projective R -module, an element $m \in M$ is basic at all primes of R if and only if $R \cdot m \subseteq M$ is a direct summand. The property that $m \in M$ is

basic at $\mathfrak{p} \in \text{Spec } A$ is stable under taking quotient and localization of A . We denote by $\text{Min}_R(I)$ the set of all minimal prime divisors of an ideal $I \subseteq R$. The authors are grateful to Prof. V. Trivedi for explaining the proof of following lemma ([2], Lemma 1.2).

Lemma 3.3 (Flenner). *Suppose that R is a Noetherian ring, M is a finite R -module, $U \subseteq \text{Spec } R$ is a Zariski open subset, and $\{m_1, \dots, m_n\}$ is a set of elements of M , which generates the submodule $N \subseteq M$. Suppose that we have $t \in \mathbb{Z}$ (which can be negative) such that*

$$\mu_{\mathfrak{p}}(M) - \mu_{\mathfrak{p}}(M/N) \geq \dim_U(R/\mathfrak{p}) - t$$

for every $\mathfrak{p} \in U$. Let $\phi^{-1}(U)$ be the inverse image of U under the map $\phi : \text{Spec } R[X_1, \dots, X_n] \rightarrow \text{Spec } R$. Then there exists an ideal $(F_1, \dots, F_r) \subseteq R[X_1, \dots, X_n]$ such that

$$\dim(\phi^{-1}(U) \cap V(F_1, \dots, F_r)) \leq n + t$$

and the element

$$\sum_{i=1}^n m_i \otimes X_i \in M \otimes_R R[X_1, \dots, X_n]$$

is basic on $D(F_1, \dots, F_r) \cap \phi^{-1}(U)$.

Corollary 3.4. *Let R be a Noetherian ring and suppose M is a finite R -module. Fix $x \in M$ and define U_x as the subset of $\text{Spec } R$ such that $x \in M$ is basic at every point of U_x . Then U_x is a (possibly empty) constructible subset of $\text{Spec } R$.*

Proof. Recall that for a finite projective R -module N , $x \in N$ is basic at $\mathfrak{p} \in \text{Spec } R$ if and only if $R_{\mathfrak{p}} \cdot x$ spans a direct summand of $N_{\mathfrak{p}}$, which is an open property. We may assume that R is reduced, so that there exists $f \in R$ for which $M[f^{-1}]$ is a free $R[f^{-1}]$ -module. Hence there exists an open subset $V \subseteq \text{Spec } R[f^{-1}]$ such that $x \in M$ is basic at $\mathfrak{p} \in \text{Spec } R \iff \mathfrak{p} \in V$. By Noetherian induction applied to $V(f)$, we find a maximal constructible subset $Z \subseteq V(f)$ on which $x \in M$ is basic. So $U_x := V \cup Z$ is the sought one. \square

4. MAIN THEOREMS

In this section, we establish main theorems. Let us collect some facts for later use. A local domain S is *catenary*, if and only if $\text{ht } \mathfrak{p} + \dim S/\mathfrak{p} = \dim S$ for all $\mathfrak{p} \in \text{Spec } S$.

Lemma 4.1. *Let (A, π_A, \mathbf{k}) be a discrete valuation ring and let $f \in A[y_1, \dots, y_d]$ be a non-zero (possibly constant) polynomial. Then there exists $t \in \mathbb{Z}_{\geq 0}$ such that $\pi_A^{-t} f \in A[y_1, \dots, y_d]$ and the reduction of $\pi_A^{-t} f$ modulo π_A is a non-zero (possibly constant) polynomial in $\mathbf{k}[y_1, \dots, y_d]$.*

Proof. The proof goes by induction on d . If $d = 1$, we write $f = a_m y_1^m + a_{m-1} y_1^{m-1} + \dots + a_0$ with $a_i \in A$. Let $0 \leq h \leq m$ be such that the valuation $v(a_h)$ is the smallest in the set $\{v(a_i) \mid 0 \leq i \leq m\}$ and write $a_h = (\text{unit}) \cdot \pi_A^t$. Dividing f by π_A^t , we get the desired polynomial.

In general, write $f = b_n y_d^n + b_{n-1} y_d^{n-1} + \cdots + b_0$ for $b_i \in A[y_1, \dots, y_{d-1}]$. Applying the induction hypothesis to every b_i , we may find $t_i \in \mathbb{Z}_{\geq 0}$ such that $\pi_A^{-t_i} b_i$ has the desired property. Let $t_s := \min\{t_i \mid 0 \leq i \leq n\}$. Then the term $\pi_A^{-t_s} b_s y^s$ modulo π_A is non-zero and it is clear that $\pi_A^{-t_s} f$ is contained in $A[y_1, \dots, y_d]$. Hence it suffices to put $t := t_s$. \square

Theorem 4.2. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete local domain of mixed characteristic $p > 0$ and suppose the following conditions:*

- (1) *let $A \rightarrow R$ be a coefficient ring map for a complete discrete valuation ring (A, π_A) ;*
- (2) *let x_0, x_1, \dots, x_d be a fixed set of minimal generators of \mathfrak{m} ;*
- (3) *the residue field of R is infinite.*

Then there exists a Zariski dense open subset $V \subseteq \mathbb{P}^d(\mathbf{k})$ such that we have

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}^{(2)}$$

for every prime \mathfrak{p} of R and $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(V)$.

It is noted that x_0, x_1, \dots, x_d are not a system of parameters of R unless it is regular.

Proof. It is clear that for $a \in \mathbb{A}^{d+1}(A)$, $\mathfrak{p} \in \mathrm{Spec} R$ and $u \in A^\times$,

$$\sum_{i=0}^d a_i x_i \notin \mathfrak{p}^{(2)} \iff u \left(\sum_{i=0}^d a_i x_i \right) \notin \mathfrak{p}^{(2)}.$$

This implies that our statement for x_a holds on $\mathbb{P}^d(A)$. Note that the R -module $\widehat{\Omega}_{R/A}$ is generated by dx_0, \dots, dx_d and $\mathrm{Supp} \widehat{\Omega}_{R/A} = \mathrm{Spec} R$. We first establish the theorem on $D(\mathfrak{m})$. It follows from ([2], Lemma 2.6, or [15], Lemma 2) that for $\mathfrak{p} \in D(\mathfrak{m})$

$$\mu_{\mathfrak{p}}(\widehat{\Omega}_{R/A}) \geq \dim(R/\mathfrak{p}) - 1 = \dim_{D(\mathfrak{m})}(R/\mathfrak{p}).$$

Under the notation as above and applying Lemma 3.3 for the R -module $\widehat{\Omega}_{R/A}$, there is an ideal $(F_1, \dots, F_r) \subseteq R[X_0, \dots, X_d]$ such that

$$\dim_{\phi^{-1}(D(\mathfrak{m}))} R[X_0, \dots, X_d] / (F_1, \dots, F_r) \leq d + 1 \cdots (*),$$

where $\phi^{-1}(D(\mathfrak{m}))$ is the inverse image of $D(\mathfrak{m})$ under the map $\phi : \mathrm{Spec} R[X_0, \dots, X_d] \rightarrow \mathrm{Spec} R$ and

$$\sum_{k=0}^d dx_k \otimes X_k \in \widehat{\Omega}_{R/A} \otimes_R R[X_0, \dots, X_d]$$

is basic on $D(F_1, \dots, F_r) \cap \phi^{-1}(D(\mathfrak{m}))$. Let T denote the localization of $R[X_0, \dots, X_d]$ at the prime ideal $\mathfrak{m}R[X_0, \dots, X_d]$. Then we have a flat local map of local rings: $(R, \mathfrak{m}) \rightarrow (T, \mathfrak{m}_T)$ in which all rings are catenary local domains, and $\dim R = \dim T$.

In what follows, it suffices to consider the case that $(F_1, \dots, F_r)T$ is a proper ideal. Let $P \subseteq T$ be any minimal prime divisor of $(F_1, \dots, F_r)T$. Then combining the dimension

equality for the catenary local domain T , together with $(*)$ above, it follows that $\text{ht } P = \dim T - 1$. Since T is local, $Z := V(F_1, \dots, F_r) \subseteq \text{Spec } T$ is a finite set, and

$$\sum_{k=0}^d dx_k \otimes X_k$$

is basic on $U := \text{Spec } T \setminus Z$ (the complement of $\phi(U)$ is a finite set of primes). By Noetherian induction, it suffices to consider the proof on $D(g) \subseteq \text{Spec } T$ for some $g \in R$ such that $\widehat{\Omega}_{R/A} \otimes_R T[g^{-1}]$ is a $T[g^{-1}]$ -free module. Then there exists a $G \in R[X_0, \dots, X_d] \setminus \mathfrak{m}R[X_0, \dots, X_d]$ such that

$$\sum_{k=0}^d dx_k \otimes X_k$$

is basic on $D(g \cdot G) \subseteq \text{Spec } R[X_0, \dots, X_d]$. Assume that $G(a) \notin \mathfrak{m}$ for $a = (a_0, \dots, a_d) \in \mathbb{A}^{d+1}(A)$ (such a point exists, because $\#\mathbf{k} = \infty$). Then

$$\sum_{k=0}^d a_k dx_k \in \widehat{\Omega}_{R/A}$$

is basic on $\phi(U) \subseteq \text{Spec } R$ in view of ([2], Lemma 1.1). Now we get the following implication. Take $G(a) \notin \mathfrak{m}$. Then for all $\mathfrak{p} \in \phi(U) = \text{Spec } R \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{m}\}$, it follows from Lemma 3.2 that

$$\mathbf{x}_a = \sum_{k=0}^d a_k x_k \notin \mathfrak{p}^{(2)}.$$

So it remains to deal with the issue on a finite set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{m}\}$. First, if $a_i \in A^\times$ for some i , then $\mathbf{x}_a \notin \mathfrak{m}^2$ by the minimality of x_0, \dots, x_d . So modifying G , we may assume henceforth that $\mathbf{x}_a \notin \mathfrak{p}^2$ is satisfied for all $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, whenever $G(a) \notin \mathfrak{m}$.

Put $\overline{V}_0 := D(\overline{G}(\overline{x})) \subseteq \mathbb{A}^{d+1}(\mathbf{k}) \setminus \{0\}$ (the origin of $\mathbb{A}^{d+1}(\mathbf{k})$ is excluded). Then define an open subset $V_0 \subseteq \mathbb{P}^d(\mathbf{k})$ as the image of \overline{V}_0 under the geometric quotient: $\mathbb{A}^{d+1}(\mathbf{k}) \setminus \{0\} \rightarrow \mathbb{P}^d(\mathbf{k})$. To deal with the issue on $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, take the homogeneous polynomial:

$$F(X_0, \dots, X_d) := \sum_{i=0}^d x_i X_i \in R[X_0, \dots, X_d].$$

It suffices to force $\sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j$ for all $1 \leq j \leq r$. Then for each $1 \leq j \leq r$, we have

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j \iff \overline{F}(\overline{a}_0, \dots, \overline{a}_d) \neq 0 \text{ in } R/\mathfrak{p}_j \cdots (**),$$

which is stable under taking multiplication by elements of A^\times .

Our final goal is to identify the set of points of $\mathbb{P}^d(A)$ satisfying the condition $(**)$ and describe it as the inverse image of an open subset under the map $\text{Sp}_A : \mathbb{P}^d(A) \rightarrow \mathbb{P}^d(\mathbf{k})$. It is clear that the ideal of R generated by the set: $\{F(a_0, \dots, a_d) \mid (a_0, \dots, a_d) \in \mathbb{P}^d(A)\}$

is the maximal ideal \mathfrak{m} . Since the union of all primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ is strictly contained in \mathfrak{m} , there exists $(a_0, \dots, a_d) \in \mathbb{P}^d(A)$ for which $F(a_0, \dots, a_d) \notin \mathfrak{p}_i$ for all i .

Case1: Assume that $\pi_A \in \mathfrak{p}_j$ for some j . Let R_j be the localization of R at \mathfrak{p}_j . Then $A \rightarrow R_j$ is a flat local map of local rings, say R_j is of mixed characteristic. Let \mathfrak{m}_j be the maximal ideal of R_j and let $\mathbf{k}_j := R_j/\mathfrak{m}_j$. Then we have a mapping:

$$\mathbb{P}^d(\mathbf{k}) \rightarrow \mathbb{P}^d(\mathbf{k}_j).$$

The condition that the reduced polynomial $\bar{F}(\bar{a}_0, \dots, \bar{a}_d)$ is non-zero on the projective space $\mathbb{P}^d(\mathbf{k}_j)$ defines an open subset $U_j \subseteq \mathbb{P}^d(\mathbf{k}_j)$. By Proposition 2.5, $V_j := U_j \cap \mathbb{P}^d(\mathbf{k})$ is a dense open subset, as the field \mathbf{k} is infinite.

Case2: Assume that $\pi_A \notin \mathfrak{p}_j$ for some j . In this case, notations being as in (i), we have a mapping:

$$\mathbb{P}^d(\text{Frac}(A)) \rightarrow \mathbb{P}^d(\mathbf{k}_j).$$

The condition $\bar{F}(\bar{a}_0, \dots, \bar{a}_d) \neq 0$ on the projective space $\mathbb{P}^d(\mathbf{k}_j)$ defines a Zariski open subset $U_j \subseteq \mathbb{P}^d(\mathbf{k}_j)$.

Proposition 2.5 implies that $U_j \cap \mathbb{P}^d(\text{Frac}(A))$ is a dense open subset which is covered by basic open subsets. Let U_f be one of those basic open subsets for a homogeneous polynomial f , where U_f has the usual meaning. We may assume that $f \in A[X_0, \dots, X_d]$. Furthermore, by applying Lemma 4.1 to f , it can be normalized into g so that $g \in A[X_0, \dots, X_d]$ and $0 \neq \bar{g} \in \mathbf{k}[X_0, \dots, X_d]$. Consider the subset:

$$U_{|g|=1} := \{a = (a_0 : \dots : a_d) \in \mathbb{P}^d(A) \mid |g(a)| = 1\} \subseteq \mathbb{P}^d(A)$$

for any fixed valuation $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$. Then we see that $F(a_0, \dots, a_d) \notin \mathfrak{p}_j$ for all $(a_0 : \dots : a_d) \in U_{|g|=1}$ and $U_{|g|=1}$ is non-empty, since $|g(a)| = 1$ if and only if $\bar{g}(\text{Sp}_A(a)) \neq 0$. Moreover, this implies that $\text{Sp}_A^{-1}(U_{\bar{g}}) = U_{|g|=1}$. By taking the union of all affine pieces $U_{\bar{g}}$ coming from a basic open covering of $U_j \cap \mathbb{P}^d(\text{Frac}(A))$, we obtain a desired open subset $V_j \subseteq \mathbb{P}^d(\mathbf{k})$, as in **Case1**.

Combining both **Case1** and **Case2** together, $V_1 \cap \dots \cap V_r$ is a non-empty open subset of $\mathbb{P}^d(\mathbf{k})$. Hence V is defined as the intersection $V_0 \cap V_1 \cap \dots \cap V_r$, where V_0 is defined by $G(x)$ as previously. This completes the proof of the theorem. \square

The techniques used in the proof of the above theorem will play an important role in the next main theorem.

Theorem 4.3 (Local Bertini Theorem). *Let $(R, \mathfrak{m}, \mathbf{k})$ be a complete local domain of mixed characteristic $p > 0$ and suppose the following conditions:*

- (1) *let $A \rightarrow R$ be a coefficient ring map for a complete discrete valuation ring (A, π_A) ;*
- (2) *let x_0, x_1, \dots, x_d be a fixed set of minimal generators of \mathfrak{m} ;*
- (3) *R is normal, of depth $R \geq 3$, and the residue field \mathbf{k} is infinite.*

Then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$ satisfying the following properties. For any $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U)$, the quotient $R/\mathbf{x}_a R$ is a normal domain of mixed characteristic $p > 0$, where we put

$$\mathbf{x}_a := \sum_{i=0}^d a_i x_i.$$

Proof. The first step of the proof of the theorem has been completed in Theorem 4.2. Taking $V \subseteq \mathbb{P}^d(\mathbf{k})$ as given in Theorem 4.2 and denoting by $\mathrm{Reg}(R)$ the regular locus of R , we find that

$$\mathrm{Reg}(R) \cap V(\mathbf{x}_a) \subseteq \mathrm{Reg}(R/\mathbf{x}_a R) \cdots (1)$$

for all $a = (a_0, \dots, a_d) \in \mathrm{Sp}_A^{-1}(V)$. Let us explain some basic ideas. In this proof, after some preparations, we give a candidate of $U \subseteq \mathbb{P}^d(\mathbf{k})$ for our theorem as a Zariski open subset $U \subseteq V$. In **Step1** below we prove that U is non-empty. Then in **Step2**, we will show that the quotients $R/\mathbf{x}_a R$ with $a \in \mathrm{Sp}_A^{-1}(U)$ satisfy the well-known conditions (R_1) and (S_2) and thus are normal. Finally in **Step3**, we will show that the quotients $R/\mathbf{x}_a R$ with $a \in \mathrm{Sp}_A^{-1}(U)$ are of mixed characteristic.

Step1: Let $X := \mathrm{Spec} R - V(\mathfrak{m})$. Then since R is a complete local domain, the singular locus $\mathrm{Sing}(X)$ is a proper closed subset. Hence the set of minimal primes in $\mathrm{Sing}(X)$ is finite, and let

$$Q_1 := \{\mathfrak{p} \in X \mid \mathfrak{p} \text{ is a minimal prime in } \mathrm{Sing}(X)\}.$$

Note that every prime in Q_1 has height ≥ 2 (due to the (S_2) condition on R). On the other hand,

$$Q_2 := \{\mathfrak{p} \in X \mid \mathrm{depth} R_{\mathfrak{p}} = 2 \text{ and } \dim R_{\mathfrak{p}} > 2\}$$

is also a finite set ([2], Lemma 3.2 and the (S_2) condition on R). Now let $Q_1 \cup Q_2 := \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and let $F(X_0, \dots, X_d) = \sum_{i=0}^d x_i X_i \in R[X_0, \dots, X_d]$ as previously. Then for each $1 \leq j \leq m$, it follows that

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i \notin \mathfrak{p}_j \iff \bar{F}(\bar{a}_0, \dots, \bar{a}_d) \neq 0 \text{ in } R/\mathfrak{p}_j.$$

Then as argued in **Case1** and **Case2** in the proof of Theorem 4.2, we obtain Z_1, \dots, Z_m , which are non-empty open subsets of $\mathbb{P}^d(\mathbf{k})$. Put $Z := Z_1 \cap \cdots \cap Z_m \cdots (2)$.

Next pick $\mathfrak{p} \in X \cap V(\mathbf{x}_a)$ such that $\mathrm{ht} \mathfrak{p} \geq 2$ and assume that $\mathbf{x}_a = \sum_{i=0}^d a_i x_i$ satisfies (1) and \mathbf{x}_a avoids all primes in $Q_1 \cup Q_2$.

Step2: (i) If $\mathrm{ht} \mathfrak{p} > 2$, then since \mathbf{x}_a avoids all primes in Q_2 , it follows that $\dim(R/\mathbf{x}_a R)_{\mathfrak{p}} \geq 2$ and $\mathrm{depth}(R/\mathbf{x}_a R)_{\mathfrak{p}} \geq 2$.

(ii) If $\mathrm{ht} \mathfrak{p} = 2$, then since \mathbf{x}_a avoids all primes ideal in Q_1 and the height of every prime in Q_1 is at least 2, it follows that $R_{\mathfrak{p}}$ is regular. By (1), one finds that $(R/\mathbf{x}_a R)_{\mathfrak{p}}$ is a discrete valuation ring. On the other hand, the hypothesis that $\mathrm{depth}(R) \geq 3$ implies

that $\text{depth}(R/\mathfrak{x}_a R) \geq 2$. Hence $R/\mathfrak{x}_a R$ is a normal domain in view of Serre's normality criterion.

Finally, it remains to make the quotient $R/\mathfrak{x}_a R$ into a local ring of mixed characteristic $p > 0$.

Step3: Let $\{q_1, \dots, q_n\}$ be a set of all height-one primes of R lying above π_A . Then again, applying the discussion of **Case1** and **Case2** in the proof of Theorem 4.2 to $\{q_1, \dots, q_n\}$, we find a non-empty open subset $W \subseteq \mathbb{P}^d(\mathbf{k}) \cdots (3)$, which avoids the union $q_1 \cup \dots \cup q_n$.

Combining (1), (2), together with (3), and taking a non-empty open subset $U := V \cap W \cap Z \subseteq \mathbb{P}^d(\mathbf{k})$, it turns out that $\text{Sp}_A^{-1}(U) \subseteq \mathbb{P}^d(A)$ has the required property. \square

Remark 4.4. In particular, if R is a Cohen-Macaulay normal local domain, then $R/\mathfrak{x}_a R$ is Cohen-Macaulay and normal. One can continue this process until $\dim R = 2$ is attained. Note that the module-finite extension of local rings $A[[z_1, \dots, z_n]] \rightarrow R$ is flat if and only if R is Cohen-Macaulay, by the Auslander-Buchsbaum formula.

Next, let us consider the case when the residue field is finite. Let $(R, \mathfrak{m}, \mathbb{F})$ be a complete normal local domain of mixed characteristic $p > 0$ with finite residue field \mathbb{F} . In other words, R is a finite extension of $W(\mathbb{F})[[z_1, \dots, z_n]]$, where $W(\mathbb{F})$ is the ring of Witt vectors of \mathbb{F} . Let $\mathcal{O} := W(\mathbb{F})$ for simplicity and let $\widehat{\mathcal{O}^{\text{ur}}}$ be the completion of the maximal unramified extension of \mathcal{O} . Then $W(\overline{\mathbb{F}}) = \widehat{\mathcal{O}^{\text{ur}}}$. Put

$$R_{\widehat{\mathcal{O}^{\text{ur}}}} := R \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{O}^{\text{ur}}} \text{ (resp. } R_{\mathcal{O}^{\text{ur}}} := \text{strict henselization of } R).$$

Then $R_{\mathcal{O}^{\text{ur}}}$ is local Noetherian, but not complete. By the main result of [4], $R_{\widehat{\mathcal{O}^{\text{ur}}}}$ is the completion of $R_{\mathcal{O}^{\text{ur}}}$ and a normal local domain. From algebraic number theory, \mathcal{O}^{ur} is obtained from \mathcal{O} by adjoining all n -th roots of unity for $(n, p) = 1$. There is a structure map $\widehat{\mathcal{O}^{\text{ur}}} \rightarrow R_{\widehat{\mathcal{O}^{\text{ur}}}}$. We define the multiplicative map (not additive)

$$\theta_{\widehat{\mathcal{O}^{\text{ur}}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}^{\text{ur}}}$$

as just the Teichmüller map $\overline{\mathbb{F}} \rightarrow W(\overline{\mathbb{F}})$. In particular, we have $q \circ \theta_{\widehat{\mathcal{O}^{\text{ur}}}} = \text{Id}_{\overline{\mathbb{F}}}$, where $q : \widehat{\mathcal{O}^{\text{ur}}} \rightarrow \overline{\mathbb{F}}$ is the residue field map. There is a set-theoretic mapping:

$$\langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}^{\text{ur}}})$$

defined by $\langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle(a) := (\theta_{\widehat{\mathcal{O}^{\text{ur}}}}(a_0) : \dots : \theta_{\widehat{\mathcal{O}^{\text{ur}}}}(a_d))$. Note that $\langle q \rangle = \text{Sp}_{\widehat{\mathcal{O}^{\text{ur}}}}$ and $\langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle$ is well-defined, since $\theta_{\widehat{\mathcal{O}^{\text{ur}}}}$ is multiplicative. Furthermore, since the composite $\langle q \rangle \circ \langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle$ is an identity map, $\langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle$ is injective.

Corollary 4.5 (Finite Residue Field Case). *Let the hypothesis be as in Theorem 4.3 for $(R, \mathfrak{m}, \mathbb{F})$, except that the residue field \mathbb{F} is finite. Then there exists a non-empty open subset $U \subseteq \mathbb{P}^d(\overline{\mathbb{F}})$ such that for any fixed $a \in \langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle(U)$, there is a finite étale extension $\mathcal{O} \rightarrow \mathcal{O}'$ of*

discrete valuation rings such that $\mathbf{x}_a \in R_{\mathcal{O}'} := R \otimes_{\mathcal{O}} \mathcal{O}'$ and $R_{\mathcal{O}'} / \mathbf{x}_a R_{\mathcal{O}'}$ is a normal domain of mixed characteristic $p > 0$.

The proof will be done by constructing a multiplicative map $\tilde{\theta}_{\widehat{\mathcal{O}^{\text{ur}}}} : \overline{\mathbb{F}} \rightarrow \mathcal{O}^{\text{ur}}$, which extends to the map $\theta_{\widehat{\mathcal{O}^{\text{ur}}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}^{\text{ur}}}$.

Proof. We keep the notation as in Theorem 4.3. First, note that $\langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle(U) \subseteq \mathbb{P}^d(\mathcal{O}^{\text{ur}})$ and that x_0, x_1, \dots, x_d are the minimal generators of the maximal ideal of $R_{\widehat{\mathcal{O}^{\text{ur}}}}$. Then the hypotheses of Theorem 4.3 are fulfilled for the complete local domain $R_{\widehat{\mathcal{O}^{\text{ur}}}}$. First, let us construct a multiplicative map $\tilde{\theta}_{\mathcal{O}^{\text{ur}}} : \overline{\mathbb{F}} \rightarrow \mathcal{O}^{\text{ur}}$ which extends to the map $\theta_{\widehat{\mathcal{O}^{\text{ur}}}}$. The local ring \mathcal{O}^{ur} is constructed as the direct limit of finite unramified extensions of \mathcal{O} ; $\mathcal{O}^{\text{ur}} = \varinjlim_{\lambda \in \Lambda} \mathcal{O}_{\lambda}$ with $\mathcal{O}_{\lambda} = W(\mathbb{F}_{\lambda})$ and we have the Teichmüller mapping $\mathbb{F}_{\lambda} \rightarrow \mathcal{O}_{\lambda}$ for the residue field \mathbb{F}_{λ} of \mathcal{O}_{λ} . Then we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{F}_{\lambda'} & \xrightarrow{\text{Teich}} & \mathcal{O}_{\lambda'} \\ \uparrow & & \uparrow \\ \mathbb{F}_{\lambda} & \xrightarrow{\text{Teich}} & \mathcal{O}_{\lambda} \end{array}$$

which naturally forms a direct system, so the desired map $\tilde{\theta}_{\widehat{\mathcal{O}^{\text{ur}}}}$ is given by its direct limit. On the other hand, it is easy to see that the map $\theta_{\widehat{\mathcal{O}^{\text{ur}}}} : \overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}^{\text{ur}}}$ factors as

$$\overline{\mathbb{F}} \xrightarrow{\tilde{\theta}_{\mathcal{O}^{\text{ur}}}} \mathcal{O}^{\text{ur}} \longrightarrow \widehat{\mathcal{O}^{\text{ur}}},$$

and thus we have $\mathbf{x}_a = \sum_{i=0}^d a_i x_i \in R_{\mathcal{O}^{\text{ur}}}$ for any $a \in \langle \theta_{\widehat{\mathcal{O}^{\text{ur}}}} \rangle(U)$. Since the map

$$R_{\mathcal{O}^{\text{ur}}} / \mathbf{x}_a R_{\mathcal{O}^{\text{ur}}} \rightarrow R_{\widehat{\mathcal{O}^{\text{ur}}}} / \mathbf{x}_a R_{\widehat{\mathcal{O}^{\text{ur}}}}$$

is local flat, $R_{\mathcal{O}^{\text{ur}}} / \mathbf{x}_a R_{\mathcal{O}^{\text{ur}}}$ is a normal local domain.

By what we have said above, all the coefficients of a linear form $\mathbf{x}_a = \sum_{i=0}^d a_i x_i$ are contained in some finite subextension $\mathcal{O} \rightarrow \mathcal{O}' \rightarrow \mathcal{O}^{\text{ur}}$. In other words, for a finite étale extension $R \rightarrow R_{\mathcal{O}'}$ of normal domains, the quotient $R_{\mathcal{O}'} / \mathbf{x}_a R_{\mathcal{O}'}$ is normal. \square

Remark 4.6. Let $\phi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat local map of local rings. Then one might think of the relationship between R/xR and S/xS for a non-zero divisor $x \in \mathfrak{m}$. In fact, in order to use the local Bertini theorem for S in terms of R , for example, assume that R and all fibers of ϕ are normal. Then for any x such that R/xR is normal, S/xS is so.

It is important and necessary to answer the following question:

Question 4.7. Resume the hypothesis of Theorem 4.3 and assume that $\mathbf{x}_a = u \mathbf{x}_{a'}$ for a unit $u \in R^{\times}$. Then is it true that $u \in A^{\times}$?

This question asks the following: Which subset of $\mathbb{P}^d(A)$ does parametrize the set of height-one primes $\{\mathbf{x}_a R\}$. In other words, does it give mutually distinct height-one primes of R ? In the next section, we will answer the above question. In fact, we need to restrict to the set of those points which are in the image of the Teichmüller mapping. This fact will be important in the proof of the control theorem, which will be discussed later. We end this section with an example, which applies Theorem 4.3 and its proof for a given normal domain R .

Example 4.8. This example deals with the finite residue field case, but the result is valid for any discrete valuation coefficient ring. Suppose that $p \geq 3$ and

$$R := \mathbb{Z}_p[[x_1, x_2, x_3]] / (x_1^2 + x_2^2 + x_3^2),$$

which is a three-dimensional Cohen-Macaulay normal local domain. By Theorem 4.3, there exists an open set $U \subseteq \mathbb{P}^3(\mathbb{F}_p)$ which does the job. Now we keep track of the following steps to find U explicitly.

- (i) Need to have $d\mathbf{x}_a \in \widehat{\Omega}_{R/\mathbb{Z}_p}$ basic at every $\mathfrak{p} \in \text{Spec } R$.
- (ii) Need to determine two finite sets of primes Q_1 and Q_2 in Theorem 4.3.
- (iii) Need to avoid Q_1 and Q_2 as above.

Then we know

$$\widehat{\Omega}_{R/\mathbb{Z}_p} \simeq \frac{Rdx_1 \oplus Rdx_2 \oplus Rdx_3}{R(x_1dx_1 + x_2dx_2 + x_3dx_3)},$$

the singular locus of R is defined by the ideal (x_1, x_2, x_3) , $Q_1 = \{(x_1, x_2, x_3)\}$ and $Q_2 = \emptyset$. To get a normal ring of mixed characteristic, take $\mathbf{x}_a := a_0p + \sum_{i=1}^3 a_i x_i$ such that

$$\bar{a} = (\bar{a}_0 : \bar{a}_1 : \bar{a}_2 : \bar{a}_3) \in U := U(z_0) \cap \left(\bigcup_{i=1}^3 U(z_i) \right) \subseteq \mathbb{P}^3(\mathbb{F}_p)$$

for the homogeneous coordinate $(z_0 : z_1 : z_2 : z_3)$. Then $\mathbf{x}_a \notin Q_1$. If we assume a_1 is a unit for simplicity, we see that

$$\widehat{\Omega}_{R/\mathbb{Z}_p} \left[\frac{1}{x_3} \right] \simeq R \left[\frac{1}{x_3} \right] dx_1 \oplus R \left[\frac{1}{x_3} \right] dx_2$$

is a free module, in which the image of $d\mathbf{x}_a$ spans a direct summand. On the other hand, for $\bar{R} := R/(x_3)$,

$$\widehat{\Omega}_{R/\mathbb{Z}_p} / x_3 \cdot \widehat{\Omega}_{R/\mathbb{Z}_p} \simeq \frac{\bar{R}dx_1 \oplus \bar{R}dx_2 \oplus \bar{R}dx_3}{\bar{R}(x_1dx_1 + x_2dx_2)}.$$

To show that the image of $d\mathbf{x}_a$ is basic on $\widehat{\Omega}_{R/\mathbb{Z}_p} / x_3 \cdot \widehat{\Omega}_{R/\mathbb{Z}_p}$, keep track of the same steps as above by inverting and killing first x_2 , and then x_1 .

Finally, if one takes $R := \mathbb{Z}_p[[x]]$, then $p \notin \mathfrak{p}^{(2)}$ for every prime \mathfrak{p} of R , since p is a regular parameter. But $dp = 0$ and so the notion of basic elements is stronger than that of non-containment in the second symbolic power of a prime ideal.

5. DISTINCT HYPERPLANE SECTIONS IN LOCAL BERTINI THEOREM

In this section, we give an answer to Question 4.7. Assume that $(R, \mathfrak{m}, \mathbf{k})$ is a complete local normal domain with perfect residue field of characteristic $p > 0$ with its coefficient ring $W(\mathbf{k})$, the ring of Witt vectors. Then as previously, we have the mapping: $\langle \theta_{W(\mathbf{k})} \rangle : \mathbb{P}^d(\mathbf{k}) \rightarrow \mathbb{P}^d(W(\mathbf{k}))$. The following proposition asserts that the parameter set of specializations in the local Bertini theorem may be identified with an open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$. In the case $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$, we put

$$\mathbf{x}_a = a_0 \pi_{W(\mathbf{k})} + \sum_{i=1}^d a_i x_i.$$

Proposition 5.1. *Let the notation and the hypothesis be as above, and let $U \subseteq \mathbb{P}^d(\mathbf{k})$ be a non-empty subset. Moreover, put $x_0 = \pi_{W(\mathbf{k})}$ in the case $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$. Suppose that $\mathbf{x}_a = u \mathbf{x}_{a'}$ for $a, a' \in \langle \theta_{W(\mathbf{k})} \rangle(U)$ and $u \in R^\times$. Then we have $u \in W(\mathbf{k})^\times$.*

Proof. We need to divide the proof, according to the case $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$ or $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$.

Case1: Assume $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$ and denote by a_i the image of $a_i \in W(\mathbf{k})$ under the surjection $W(\mathbf{k})[[X_0, \dots, X_d]] \twoheadrightarrow R$ ($X_i \mapsto x_i$). Let P be its kernel and let $\mathbf{k} \rightarrow W(\mathbf{k})$ be the Teichmüller mapping. Then we prove that

$$P \subseteq \pi_{W(\mathbf{k})} W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2,$$

where $\mathfrak{M} := (X_0, \dots, X_d)$. Let $f \in P$ be a non-zero element. Assume that

$$f \notin \pi_{W(\mathbf{k})} W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$$

and derive a contradiction. For the proof of the claim, we may reduce $W(\mathbf{k})[[X_0, \dots, X_d]]$ by $\pi_{W(\mathbf{k})}$. Then we find that $\bar{f} \notin \bar{\mathfrak{M}}^2$ in $\mathbf{k}[[X_0, \dots, X_d]]$. The number of minimal generators of the maximal ideal of $R/\pi_{W(\mathbf{k})}R$ is equal to

$$\dim_{\mathbf{k}} \mathfrak{m} / (\pi_{W(\mathbf{k})}R + \mathfrak{m}^2) = \dim_{\mathbf{k}} \mathfrak{m} / \mathfrak{m}^2,$$

due to $\pi_{W(\mathbf{k})} \in \mathfrak{m}^2$. So $\bar{x}_0, \dots, \bar{x}_d$ are the minimal generators of the maximal ideal of $R/\pi_{W(\mathbf{k})}R$. Then there is a surjection

$$\mathbf{k}[[X_0, \dots, X_d]] \twoheadrightarrow R/\pi_{W(\mathbf{k})}R.$$

Now choose s such that

$$\bar{f} = \sum_{i=0}^{\infty} h_i X_s^i \in \mathbf{k}[[X_0, \dots, X_d]],$$

$h_i \in \mathbf{k}[[X_1, \dots, X_{s-1}, X_{s+1}, \dots, X_d]]$ for all $i \geq 0$, and h_1 is a unit. Indeed, if h_1 is not a unit, then since $\bar{f} \notin \bar{\mathfrak{M}}^2$, h_0 contains a non-zero linear term after presenting it as an (infinite) sum of homogeneous polynomials. Then replacing s by a suitable one, we can assume that h_1 is a unit.

Taking reduction of $\bar{f} = \sum_{i=0}^{\infty} h_i X_s^i$ by \bar{P} , we get $\sum_{i=1}^{\infty} \bar{h}_i \bar{x}_s^i = -\bar{h}_0$ in $R/\pi_{W(\mathbf{k})}R$, and \bar{h}_1 is a unit of $R/\pi_{W(\mathbf{k})}R$. Thus,

$$\bar{x}_s \cdot (\text{unit}) = -\bar{h}_0.$$

But this gives a contradiction to the fact that $\bar{x}_0, \dots, \bar{x}_d$ are the minimal generators of the maximal ideal of $R/\pi_{W(\mathbf{k})}R$ and that $-\bar{h}_0 \in (\bar{x}_0, \dots, \bar{x}_{s-1}, \bar{x}_{s+1}, \dots, \bar{x}_d)$. Hence, $P \subseteq \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$.

In the next place, fix an arbitrary lifting $\tilde{u} \in W(\mathbf{k})[[X_0, \dots, X_d]]$ of u under the map $W(\mathbf{k})[[X_0, \dots, X_d]] \rightarrow R$. We write:

$$\tilde{u} = \sum_{s_0, \dots, s_d} \left(\sum_{r=0}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d},$$

where (s_0, \dots, s_d) signifies the multi-index, and all the elements $b_r^{(s_0, \dots, s_d)}$ are the Teichmüller lifts. Since \tilde{u} is a unit, $b_0^{(0, \dots, 0)} \neq 0$. By lifting the relation $\mathbf{x}_a = u\mathbf{x}_{a'}$ to $W(\mathbf{k})[[X_0, \dots, X_d]]$, we have

$$\sum_{i=0}^d a_i X_i \equiv \left(\sum_{s_0, \dots, s_d} \left(\sum_{r=0}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left(\sum_{i=0}^d a'_i X_i \right) \pmod{P}.$$

Rewrite the above presentation as:

$$\begin{aligned} \sum_{i=0}^d (a_i - a'_i b_0^{(0, \dots, 0)}) X_i &\equiv \left(\sum_{s_0, \dots, s_d} \left(\sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left(\sum_{i=0}^d a'_i X_i \right) \\ &+ \left(\sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \left(\sum_{i=0}^d a'_i X_i \right) \pmod{P}. \end{aligned}$$

Then by mapping the above formula to the quotient $\mathbf{k}[[X_0, \dots, X_d]]$, comparing the degrees on both sides, and then using the fact $P \subseteq \pi_{W(\mathbf{k})}W(\mathbf{k})[[X_0, \dots, X_d]] + \mathfrak{M}^2$, we find that

$$a_i = a'_i b_0^{(0, \dots, 0)} + \pi_{W(\mathbf{k})} \cdot v_i$$

for some $v_i \in W(\mathbf{k})$. However if $v_i \neq 0$, this implies that a_i is not a Teichmüller lift, which is false. So we have $v_i = 0$ for all i and the following relation holds:

$$\begin{aligned} &\left(\sum_{s_0, \dots, s_d} \left(\sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} \right) \left(\sum_{i=0}^d a'_i X_i \right) \\ &+ \left(\sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \left(\sum_{i=0}^d a'_i X_i \right) \in P. \end{aligned}$$

Since P is a prime ideal, we deduce that

$$\left(\sum_{s_0, \dots, s_d} \left(\sum_{r=1}^{\infty} b_r^{(s_0, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_0^{s_0} \cdots X_d^{s_d} + \sum_{\substack{(s_0, \dots, s_d) \\ \neq (0, \dots, 0)}} b_0^{(s_0, \dots, s_d)} X_0^{s_0} \cdots X_d^{s_d} \right) \in P$$

and thus $\tilde{u} \equiv b_0^{(0, \dots, 0)} \pmod{P}$ and $u \in W(\mathbf{k})$. Hence $u \in W(\mathbf{k})^\times$, as desired.

Case2: Assume $\pi_{W(\mathbf{k})} \notin \mathfrak{m}^2$. Then taking $x_0 = \pi_{W(\mathbf{k})}$, we may consider the surjection $W(\mathbf{k})[[X_1, \dots, X_d]] \twoheadrightarrow R$ ($X_i \mapsto x_i$) and let P be its kernel. In this case, we claim that

$$P \subseteq \mathfrak{M}^2,$$

where $\mathfrak{M} := (\pi_{W(\mathbf{k})}, X_1, \dots, X_d)$. Indeed, any non-zero element $f \in P$ can be presented in the form:

$$\sum_{s_1, \dots, s_d} \left(\sum_{r=0}^{\infty} b_r^{(s_1, \dots, s_d)} \pi_{W(\mathbf{k})}^r \right) X_1^{s_1} \cdots X_d^{s_d}.$$

Assume that $f \notin \mathfrak{M}^2$. Then we have $\bar{f} \notin \overline{\mathfrak{M}}^2$ in R and the calculation of $\bar{f} = 0$ in R as in **Case 1** shows that f contains a linear term with respect to either $\pi_{W(\mathbf{k})}$, or some X_i , which leads to a contradiction to the minimality of $\pi_{W(\mathbf{k})}, x_1, \dots, x_d$, as discussed in **Case 1**. Assume that $\mathbf{x}_a = u\mathbf{x}_{a'}$ for some $u \in R^\times$. Then by applying the final step of **Case 1** together with the fact that $P \subseteq \mathfrak{M}^2$, we conclude that $u \in W(\mathbf{k})^\times$, as desired. \square

Now Proposition 5.1 assures us that there are sufficiently many normal hyperplane sections for a normal local domain, which is as follows.

Corollary 5.2. *In addition to the hypothesis of Proposition 5.1, assume that \mathbf{k} is an infinite perfect field of characteristic $p > 0$ and $U \subseteq \mathbb{P}^d(\mathbf{k})$ is an infinite subset. Then $\{\mathbf{x}_a R \mid a \in \langle \theta_{W(\mathbf{k})} \rangle(U)\}$ is an infinite set of mutually distinct height-one primes. In particular, we have*

$$\bigcap_{a \in \langle \theta_{W(\mathbf{k})} \rangle(U)} \mathbf{x}_a R = 0.$$

Proof. By the above proposition, the subset $\langle \theta_{W(\mathbf{k})} \rangle(U) \subseteq \mathbb{P}^d(W(\mathbf{k}))$ parametrizes the set of height-one primes $\{\mathbf{x}_a R\}$ and it is infinite, since U is so by assumption. Since R is a Noetherian domain, every non-zero nonunit element of R has only finitely many prime divisors (of height one). Therefore, we have the claim. \square

The authors believe that Proposition 5.1 holds for any infinite residue field. Via the proof of the proposition, we obtain the following corollary, which is necessary in dealing with only the unramified case.

Corollary 5.3. *In addition to the notation of Theorem 4.3, assume that π_A is part of minimal generators of \mathfrak{m} and $x_0 = \pi_A$. Then π_A is part of minimal generators of the maximal ideal of $R/\mathbf{x}_a R$ for every $a \in \mathbb{P}^d(A)$.*

Proof. It is not necessary to assume that \mathbf{k} is a perfect field. Let $A[[X_1, \dots, X_d]] \twoheadrightarrow R$ be a surjection with its kernel P , which is so constructed due to the assumption: $\pi_A \notin \mathfrak{m}^2$. Then $P \subseteq (\pi_A, X_1, \dots, X_d)^2$, as was seen in the proof of Proposition 5.1. Let $\tilde{\mathbf{x}}_a = a_0\pi_A + \sum_{i=1}^d a_i X_i$ be a lifting of \mathbf{x}_a . Since every point of $\mathbb{P}^d(A)$ is normalized, $a_i \in A$ is a unit for some $1 \leq i \leq d$, we have an isomorphism: $A[[X_1, \dots, X_d]]/(\tilde{\mathbf{x}}_a) \simeq A[[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d]]$ together with $\pi_A \notin (\pi_A, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d)^2$.

But since $\bar{P} \subseteq (\pi_A, X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_d)^2$, where \bar{P} is the image of P in the quotient $A[[X_1, \dots, X_d]]/(\tilde{\mathbf{x}}_a)$, it follows that π_A is part of minimal generators of the maximal ideal of $R/\mathbf{x}_a R$, as required. \square

6. SERRE'S CONDITIONS (R_n) AND (S_n)

Let us briefly state Bertini theorems for the punctured spectra of local rings in the case when R satisfies Serre's conditions. The essence for these cases already appears in the proof of the Bertini theorem for normal rings. In the following corollary, the quotient $R/\mathbf{x}_a R$ may fail to be reduced. As usual, we put $X = \text{Spec } R - V(\mathfrak{m})$ and

$$\mathbf{x}_a = \sum_{i=0}^d a_i x_i$$

for $a = (a_0, \dots, a_d) \in \mathbb{A}^{d+1}(A)$.

Corollary 6.1. *Suppose that $(R, \mathfrak{m}, \mathbf{k})$ is a complete local reduced ring of mixed characteristic $p > 0$, that conditions (1) and (2) of Theorem 4.3 hold, and that the residue field \mathbf{k} is infinite. If $R_{\mathfrak{p}}$ has Serre's (R_n) (resp. (S_n)) for all $\mathfrak{p} \in X$, then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$ such that for every*

$$a = (a_0 : \dots : a_d) \in \text{Sp}_A^{-1}(U),$$

the quotient $R_{\mathfrak{p}}/\mathbf{x}_a R_{\mathfrak{p}}$ has (R_n) (resp. (S_n)) for all $\mathfrak{p} \in X \cap V(\mathbf{x}_a)$.

Proof. We briefly sketch the proof of the corollary. Since R is complete local and reduced, the singular locus of R is non-empty. Thus, the set

$$Q_1 := \{\mathfrak{p} \in X \mid \mathfrak{p} \text{ is a minimal prime in } \text{Sing}(X)\}$$

is finite. Let

$$Q_2 := \{\mathfrak{p} \in X \mid \text{depth } R_{\mathfrak{p}} = n \text{ and } \dim R_{\mathfrak{p}} > n\},$$

which is also finite by ([2], lemma 3.2). The proof is similar to that of Theorem 4.3. So it suffices to avoid the union of finite set of prime ideals in $Q_1 \cup Q_2$. Namely, for any $a = (a_0 : \dots : a_d) \in \text{Sp}_A^{-1}(U)$, the localization $R_{\mathfrak{p}}/\mathbf{x}_a R_{\mathfrak{p}}$ has (R_n) (resp. (S_n)) for all $\mathfrak{p} \in X \cap V(\mathbf{x}_a)$ and $\text{Reg}(R) \cap V(\mathbf{x}_a) \subseteq \text{Reg}(R/\mathbf{x}_a R)$. \square

The above proof also shows that the Bertini theorem holds for mixed Serre's conditions. That is, if R has $(R_s) + (S_r)$, then so does $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$ for all $a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U)$ and all $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$. For instance, we obtain the Bertini theorem for reduced rings, since R is reduced $\iff R$ has $(R_0) + (S_1)$. To be precise, we have the following version of Bertini theorem:

Corollary 6.2. *Suppose that $(R, \mathfrak{m}, \mathbf{k})$ is a complete local normal domain of dimension 2 in mixed characteristic $p > 0$, that conditions (1) and (2) of Theorem 4.3 hold, and that the residue field \mathbf{k} is infinite. Then there exists a Zariski dense open subset $U \subseteq \mathbb{P}^d(\mathbf{k})$ such that for every*

$$a = (a_0 : \cdots : a_d) \in \mathrm{Sp}_A^{-1}(U),$$

the quotient $R/\mathfrak{x}_a R$ is a reduced ring of mixed characteristic $p > 0$.

Proof. By the previous corollary, $R_{\mathfrak{p}}/\mathfrak{x}_a R_{\mathfrak{p}}$ is reduced for all $\mathfrak{p} \in X \cap V(\mathfrak{x}_a)$. Since R is a local normal domain of dimension 2, we have $\mathrm{depth} R = 2$. Hence $R/\mathfrak{x}_a R$ is reduced. To make the quotient $R/\mathfrak{x}_a R$ into a ring of mixed characteristic, it suffices to choose U such that π_A is not a zero divisor of $R/\mathfrak{x}_a R$. In fact, we may take \mathfrak{x}_a so that it avoids the union of finitely many minimal prime divisors of $\pi_A R$, since every system of parameters of R is a regular sequence. So the rest of the proof is similar to that of Theorem 4.3. \square

Remark 6.3. In place of the hypothesis of Corollary 6.2, assume that R is only a domain. Then can one find \mathfrak{x}_a such that $R/\mathfrak{x}_a R$ is a domain as well? In the mixed characteristic case, the answer to this question is not clear yet. But there is a two-dimensional complete normal local domain over \mathbb{C} without principal prime ideals at all (such an example is due to Laufer, as mentioned in [2]. However, an explicit example is not given there). In light of this, both Corollary 6.1 and Corollary 6.2 seem to be the best results.

7. CHARACTERISTIC IDEALS OF TORSION MODULES OVER NORMAL DOMAINS

Throughout this section, we assume that R is a Noetherian normal domain and M is a finitely generated torsion R -module. Then the localization of R at every height-one prime is a discrete valuation ring. We introduce an invariant of the module M . For an ideal I of R , let $M[I]$ denote the maximal submodule of M which is annihilated by I . We follow the definition of characteristic ideals by Skinner-Urban as in [13]. For more results and properties on characteristic ideals with its relation to the reflexive closure and the Fitting ideal, see § 9.

Definition 7.1. Let the notation be as above. Then the *characteristic ideal* is an ideal of R defined by

$$\mathrm{char}_R(M) = \{x \in R \mid v_P(x) \geq \ell_{R_P}(M_P) \text{ for any height-one prime } P\},$$

where $v_P(-)$ is the normalized valuation of R_P .

Since M is torsion, $\ell_{R_P}(M_P) = 0$ for all but finitely many height-one primes \mathfrak{p} of R and it suffices to take only height-one primes in the support of M in the definition. When M is not a torsion R -module, we put $\text{char}_R(M) = 0$.

Remark 7.2. The formation of characteristic ideals does not commute with base change in general. For example, let $R = \mathbb{Z}_p[[x, y]]$ and let $M = R/xR$. Then M is a torsion module and $\text{char}_R(M) = xA$. However, the R/xR -module M/xM is not torsion. Therefore,

$$0 = \text{char}_{R/xR}(M/xM) \subsetneq xR = \text{char}_R(M)R/xR$$

in this case. In general, even when M/xM is a torsion R/xR -module, it may happen that $\text{char}_{R/xR}(M/xM) \neq \text{char}_R(M)R/xR$, which is caused by the presence of pseudo-null submodules (see the definition below).

Definition 7.3. A finitely generated module M over a Noetherian normal domain R is *pseudo-null*, if $M_P = 0$ for all height-one primes $\mathfrak{p} \in \text{Spec } R$. A homomorphism of R -modules $f : M \rightarrow N$ is a *pseudo-isomorphism*, if both $\ker(f)$ and $\text{coker}(f)$ are pseudo-null modules.

The proof of the next lemma is found in ([8], Proposition 5.1.7).

Lemma 7.4 (Structure Theorem). *Let M be a finitely generated torsion module over a Noetherian normal domain R . Then there exist a finite set of height-one primes $\{P_i\}_{i \in I}$ (which is not necessarily a redundant set of height-one primes) and a set of natural numbers $\{e_i\}_{i \in I}$ such that there is a homomorphism:*

$$f : M \rightarrow \bigoplus_{i \in I} R/P_i^{e_i}$$

that is a pseudo-isomorphism. Moreover, both $\{P_i\}_{i \in I}$ and $\{e_i\}_{i \in I}$ are uniquely determined.

Henceforth, we use the notation $M \approx N$ to indicate that there is a pseudo-isomorphism between M and N . As mentioned before, the formation of characteristic ideals does not commute with base change in general, which can produce extra error terms.

Proposition 7.5. *Let M be a finitely generated torsion module over a Noetherian normal domain R . Let $x \in R$ be an element which satisfies the following conditions:*

- (1) R/xR is a normal domain (which implies that xR is a prime ideal);
- (2) x is contained in no prime ideal of height one in the support of M (which implies that M/xM is a torsion R/xR -module).

Then we have:

$$\text{char}_{R/xR}(M/xM) = \left(\text{char}_{R/xR}(M[x]) \cdot \prod_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} \mathfrak{p}(R/xR)^{\ell_{R_P}(M_P)} \right)^{**}.$$

Proof. Recall from Proposition 9.6 in Appendix, that if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of torsion R -modules, then $\text{char}_R(M) = (\text{char}_R(L) \cdot \text{char}_R(N))^{**}$, where $(-)^{**}$ denotes the reflexive closure. The condition stated in the proposition implies that all relevant modules are torsion. First off, we claim that

$$\text{char}_{R/xR} \left(\bigoplus_{\text{ht } \mathfrak{p}=1} R/(\mathfrak{p} + xR)^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} \right) = \left(\prod_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} \mathfrak{p}(R/xR)^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} \right)^{**}.$$

For the proof of this equality, let $\{P_1, \dots, P_m\}$ be a set of all height-one primes of R/xR which contain $\mathfrak{p}(R/xR)$. Then the localization $(R/xR)_{P_i}$ is a discrete valuation ring and

$$\mathfrak{p}(R/xR)_{P_i} = P_i^{\ell_{(R/xR)_{P_i}}((R/(\mathfrak{p}+xR))_{P_i})} (R/xR)_{P_i}$$

for all i . Then the above equality immediately follows from this. Thus, it suffices to prove the following equality:

$$\begin{aligned} & \left(\text{char}_{R/xR}(M/xM) \cdot \text{char}_{R/xR}(M[x])^{-1} \right)^{**} \\ &= \text{char}_{R/xR} \left(\bigoplus_{\substack{\text{ht } \mathfrak{p}=1, \\ \mathfrak{p} \in \text{Spec } R}} R/(\mathfrak{p} + xR)^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} \right), \end{aligned}$$

which is an element in the group of all reflexive fractional ideals of R . By considering all torsion R -modules such that $x \in R$ is contained in no height-one prime ideal in their support, we show that both sides of the above formula is multiplicative on short exact sequences. Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be a short exact sequence of such R -modules. Then since the function $\ell(-)$ is additive, it follows that the right hand side of the above formula is multiplicative. On the other hand, there follows the exact sequence:

$$0 \rightarrow L[x] \rightarrow M[x] \rightarrow N[x] \rightarrow L/xL \rightarrow M/xM \rightarrow N/xN \rightarrow 0$$

by the snake lemma. If \mathfrak{p} is a height-two prime ideal of R containing $x \in R$, we may localize the above exact sequence at \mathfrak{p} , so it follows that the left hand side of the above formula is multiplicative as well.

Hence we are reduced to the case that $M = R/\mathfrak{q}$ for a prime ideal \mathfrak{q} by the prime filtration argument. Assume first that $\text{ht } \mathfrak{q} = 1$. Then we clearly have $\text{Supp}(M) = \{\mathfrak{q}\}$, $M[x] = 0$, and $\ell_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = 1$, because $M_{\mathfrak{q}}$ is a field. Now the formula is obviously true. Next assume that $\text{ht } \mathfrak{q} > 1$. Then it is easy to see that both sides of the formula are equal to a unit ideal, which completes the proof. \square

8. APPLICATIONS TO CHARACTERISTIC IDEALS

Our final goal is to prove Theorem 8.7. The aim of the main theorem is to establish some techniques, which enables us to study Iwasawa Main Conjecture (see [11]), so we need to deal with local rings with finite residue field. Let $(R, \mathfrak{m}, \mathbb{F})$ be a complete normal local domain of mixed characteristic with finite residue field. In other words, R is the integral closure of $\mathbb{Z}_p[[z_1, \dots, z_n]]$ in a finite field extension of the field of fractions of $\mathbb{Z}_p[[z_1, \dots, z_n]]$. Let us recall the set-up of Corollary 4.5 and prove some preliminary results.

Recall that $R_{\widehat{\mathcal{O}}^{\text{ur}}} := R \widehat{\otimes}_{\mathcal{O}} \widehat{\mathcal{O}}^{\text{ur}}$ with its coefficient ring $\widehat{\mathcal{O}}^{\text{ur}}$. Then if $\text{depth } R \geq 3$, the complete local ring $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ fits into the hypothesis of Theorem 4.3. The Teichmüller mapping $\overline{\mathbb{F}} \rightarrow \widehat{\mathcal{O}}^{\text{ur}}$ induces $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}^{\text{ur}})$ (there is an inclusion $\mathbb{P}^d(\mathcal{O}^{\text{ur}}) \subseteq \mathbb{P}^d(\widehat{\mathcal{O}}^{\text{ur}})$).

To establish the fundamental theorem for characteristic ideals, we need to relate the torsion R -modules to the $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ -modules and then descend to R by faithful flatness. The advantage of working with $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ is that the residue field is infinite and perfect. We introduce some notation. Denote by $\text{Fitt}_A(M)$ the Fitting ideal of the A -module M . We make free use of results and notation from Appendix.

Definition 8.1. Under the notation as above, fix a set of minimal generators x_1, \dots, x_d of the unique maximal ideal of R . Then we set

$$\mathcal{L}_{\widehat{\mathcal{O}}^{\text{ur}}} := \{ \mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}} \mid a = (a_0 : \dots : a_d) \in \langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle(U) \}$$

for the mapping $\langle \theta_{\widehat{\mathcal{O}}^{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}^{\text{ur}})$. For a finitely generated torsion $R_{\widehat{\mathcal{O}}^{\text{ur}}}$ -module M , we define a subset $\mathcal{L}_{\widehat{\mathcal{O}}^{\text{ur}}}(M) \subseteq \mathcal{L}_{\widehat{\mathcal{O}}^{\text{ur}}}$ which consists of all height-one primes $\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}^{\text{ur}}}$ such that the following conditions are satisfied:

- (A) $M/\mathbf{x}_a M$ is a torsion $R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}}$ -module.
- (B) The following equalities of ideals hold in $R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}}$:

$$\begin{aligned} \text{char}_{R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}}}(M/\mathbf{x}_a M) &= (\text{char}_{R_{\widehat{\mathcal{O}}^{\text{ur}}}}(M) R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}})^{**} \\ &= (\text{Fitt}_{R_{\widehat{\mathcal{O}}^{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}^{\text{ur}}}/P_i^{e_i}) R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}})^{**}, \end{aligned}$$

where $(-)^* = \text{Hom}_{R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}}}(-, R_{\widehat{\mathcal{O}}^{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}^{\text{ur}}})$ and $M \approx \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}^{\text{ur}}}/P_i^{e_i}$ is a fundamental pseudo-isomorphism.

When the base ring R is complete regular, this definition actually coincides with the format as given in ([10], Definition 3.2.). The way of interpreting the condition (B) is that one wants to see the characteristic ideals through Fitting ideals as an intermediate invariant. Note that all of three ideals appearing in (B) may differ in general.

Lemma 8.2. *Let the notation and the hypothesis be as in Corollary 4.5 and let $I \subseteq R_{\widehat{\mathcal{O}}_{\text{ur}}}$ be an ideal and consider the natural injection $I \rightarrow I^{**}$ with its cokernel N . Then there exists a finite set $\{Q_i\}_{1 \leq i \leq \ell}$ consisting of height-two primes of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ such that for*

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i)$$

and for all $P \in \text{Spec } R_{\widehat{\mathcal{O}}_{\text{ur}}}$ with the property that $\mathbf{x}_a \in P$ and $\text{ht } P \leq 2$, we have $N_P = 0$.

Proof. By definition, the module N is supported on a closed subset of codimension two in $\text{Spec } \widehat{R}^{\text{sh}}$. So there are only finitely many height-two prime ideals contained in $\text{Supp } N$. Hence it is sufficient to choose Q_1, \dots, Q_ℓ as height-two prime ideals in $\text{Supp } N$. \square

In the following discussion, we examine when equalities occur between various ideals in the condition **(B)**.

Discussion 8.3. Suppose M is a finitely generated torsion A -module. Then we have

$$\text{Fitt}_A(M)^{**} = \text{char}_A(M)$$

(see Proposition 9.6 in Appendix). Take a fundamental pseudo-isomorphism:

$$M \approx \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}$$

for a (not necessarily redundant) finite set of height-one primes $\{P_i\}$ of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$. Choose $\mathbf{x}_a \in R_{\widehat{\mathcal{O}}_{\text{ur}}}$ such that $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ is normal and $M/\mathbf{x}_a M$ is a torsion $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module. In particular, the multiplication map:

$$\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} \xrightarrow{\mathbf{x}_a} \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}$$

is injective. Then since $\text{Fitt}_B(M \otimes_A B) = \text{Fitt}_A(M)B$ for any Noetherian A -algebra B , letting $B = R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$, it follows that

$$\begin{aligned} (\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M) R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} &= (\text{Fitt}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}) R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} \\ &= \text{Fitt}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i}))^{**} \end{aligned}$$

as long as

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i),$$

where $\{Q_i\}_{1 \leq i \leq \ell}$ is specified as in Lemma 8.2 (of course, we have assumed the normality of $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ as well as the torsion property of $M/\mathbf{x}_a M$). It also yields that

$$(\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M) R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})),$$

which we will use below.

Some preliminary results from [10] may be proved in a similar way over general normal domains with necessary modifications.

Lemma 8.4. *Under the notation and the hypothesis as in Corollary 4.5, assume that M is a finitely generated torsion $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module. Then the following assertions hold:*

(1) *The set $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$ is identified with the intersection:*

$$\{\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \mid M/\mathbf{x}_a M \text{ is a torsion } R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}R_{\widehat{\mathcal{O}}_{\text{ur}}}\text{-module}\} \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}}) \cap \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i),$$

where M_{null} is the maximal pseudo-null submodule of M and $\{Q_i\}_{1 \leq i \leq \ell}$ is a set of height-two primes as stated in Lemma 8.2.

(2) *Assume that N is a finitely generated pseudo-null $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module and $\{Q'_i\}_{1 \leq i \leq k}$ is a set of all associated prime ideals of height two for the module N . Then we have:*

$$\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N) = \bigcap_{1 \leq i \leq k} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q'_i).$$

Proof. (1): This is taken from ([10], Lemma 3.4.), but we give its proof, as it requires some modifications. Let M_{null} be the maximal pseudo-null submodule of M . Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M/M_{\text{null}} & \longrightarrow & \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} & \longrightarrow & N \longrightarrow 0 \\ & & \mathbf{x}_a \downarrow & & \mathbf{x}_a \downarrow & & \mathbf{x}_a \downarrow \\ 0 & \longrightarrow & M/M_{\text{null}} & \longrightarrow & \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} & \longrightarrow & N \longrightarrow 0 \end{array}$$

for a (not necessarily redundant) set of height-one primes $\{P_i\}_{1 \leq i \leq m}$ of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ and N is a pseudo-null module.

Assume that $M/\mathbf{x}_a M$ is a torsion $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module for $\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \bigcap_{1 \leq i \leq \ell} \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_i)$ with the notation as in Lemma 8.2. Then the map

$$\bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} \xrightarrow{\mathbf{x}_a} \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i}$$

is injective. So the snake lemma yields the following exact sequence:

$$0 \rightarrow N[\mathbf{x}_a] \rightarrow M/(\mathbf{x}_a M + M_{\text{null}}) \rightarrow \bigoplus_{i=1}^m R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i}) \rightarrow N/\mathbf{x}_a N \rightarrow 0.$$

There is a short exact sequence:

$$0 \longrightarrow N[\mathbf{x}_a] \longrightarrow N \xrightarrow{\mathbf{x}_a} N \longrightarrow N/\mathbf{x}_a N \longrightarrow 0$$

of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules, whereas both $N[\mathbf{x}_a]$ and $N/\mathbf{x}_a N$ are naturally $R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules. Localizing this sequence at all height-two primes P of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ containing \mathbf{x}_a , a length computation for the sequence localized at P reveals that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M/(\mathbf{x}_a M + M_{\text{null}})) = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}\left(\bigoplus_{1 \leq i \leq m} R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})\right).$$

On the other hand, Discussion 8.3 shows that

$$(\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M)R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}})^{**} = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}\left(\bigoplus_{1 \leq i \leq m} R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} + P_i^{e_i})\right).$$

Finally, since the multiplication on M/M_{null} by \mathbf{x}_a is injective, we have an exact sequence:

$$0 \rightarrow M_{\text{null}}/\mathbf{x}_a M_{\text{null}} \rightarrow M/\mathbf{x}_a M \rightarrow M/(\mathbf{x}_a M + M_{\text{null}}) \rightarrow 0.$$

Taking characteristic ideals to this exact sequence, we find that

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M) \iff \mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}}),$$

because of the condition **(B)**. This completes the proof of (1).

(2): This part is done in ([10], Lemma 3.5 together with Lemma 3.1) in the case $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ is regular, so we leave the proof with necessary modifications to the reader. \square

The next objective is to identify the set $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$ for a finitely generated torsion R -module M . For the proof of the main theorem, we need to make sure that there are sufficiently many specializations that are normal, and then study characteristic ideals by combining Proposition 5.1, Lemma 8.2, and Lemma 8.4.

Lemma 8.5. *Under the notation and the hypothesis as in Corollary 4.5, assume that M is a finitely generated torsion $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -module. Then we have the following assertions:*

- (1) *The subset $\mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M) \subseteq \langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle(\mathcal{U})$ may be identified with a non-empty open subset of $\mathbb{P}^d(\overline{\mathbb{F}})$ under the mapping $\langle \theta_{\widehat{\mathcal{O}}_{\text{ur}}} \rangle : \mathbb{P}^d(\overline{\mathbb{F}}) \rightarrow \mathbb{P}^d(\widehat{\mathcal{O}}_{\text{ur}})$. In particular, it is infinite.*
- (2) *There exists an infinite sequence $\{\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}\}_{i \in \mathbb{N}} \subseteq \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M)$ such that the union of the set of all minimal prime divisors:*

$$\bigcup_{i \in \mathbb{N}} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(P + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}})$$

is infinite, where P appears as one of the components in $\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M)$.

Proof. (1): By Lemma 8.4, it follows that $\mathbf{x}_a \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\text{null}})$ if and only if $\mathbf{x}_a \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}$ avoids the union of all height-two primes Q'_1, \dots, Q'_k contained in $\text{Ass}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\text{null}})$. Let Q_1, \dots, Q_ℓ be as given in Lemma 8.2. By the assumption that $\dim R \geq 3$, all those primes together with the set of height-one primes P_1, \dots, P_m in $\text{Supp } M$ are strictly contained in the maximal ideal of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$. So it remains to determine the non-empty Zariski open subset of $\mathbb{P}^d(\overline{\mathbb{F}})$ with the required properties. In other words, it suffices to choose \mathbf{x}_a such that

$\mathbf{x}_a \notin (\bigcup_{1 \leq i \leq m} P_i) \cup (\bigcup_{1 \leq i \leq k} Q'_i) \cup (\bigcup_{1 \leq i \leq \ell} Q_i)$. However, this part may be carried out in the same way as the final step of Theorem 4.2, so we leave the detail to the reader.

(2): We shall be done by completing inductive steps using (1) as follows. Let P be a fixed divisor of $\text{char}_{\widehat{R^{\text{sh}}}}(M)$. Since P is a non-maximal prime ideal of $R_{\widehat{\mathcal{O}^{\text{ur}}}}$, it corresponds to a non-empty Zariski open subset of $\mathbb{P}(\overline{\mathbb{F}})$ and we may find $\mathbf{x}_{a_0} \in \mathcal{L}_{\widehat{\mathcal{O}^{\text{sh}}}}(M)$, which is not contained in P . Then this initial choice satisfies our requirement.

Choose \mathbf{x}_{a_1} so that it is not contained in a finite set of prime divisors $\text{Min}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(P + \mathbf{x}_{a_0} R_{\widehat{\mathcal{O}^{\text{ur}}}})$ (which is possible due to $\dim R \geq 3$). Next, choose \mathbf{x}_{a_2} so that it is not contained in the union of the finite set of prime divisors:

$$\text{Min}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(P + \mathbf{x}_{a_0} R_{\widehat{\mathcal{O}^{\text{ur}}}}) \cup \text{Min}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(P + \mathbf{x}_{a_1} R_{\widehat{\mathcal{O}^{\text{ur}}}})$$

Hence we may keep this process and get a sequence $\mathbf{x}_{a_0}, \mathbf{x}_{a_1}, \mathbf{x}_{a_2}, \dots$ satisfying the required properties. \square

Let M be a module over a complete local domain R with \mathcal{O} its coefficient ring and let $\mathcal{O} \rightarrow \mathcal{O}'$ be a torsion-free extension of complete discrete valuation rings. Then we set $M_{\mathcal{O}'} := M \widehat{\otimes}_{\mathcal{O}} \mathcal{O}'$, which coincides with our previous notation. Let M be a finitely generated torsion R -module. Then

$$\text{char}_{R_{\mathcal{O}'}}(M_{\mathcal{O}'}) = \text{char}_R(M) R_{\mathcal{O}'},$$

which may be verified directly from the definition.

Lemma 8.6. *Let M, N be finitely generated torsion R -modules. Then we have*

$$\text{char}_R(M) \subseteq \text{char}_R(N) \iff \text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(M_{\widehat{\mathcal{O}^{\text{ur}}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(N_{\widehat{\mathcal{O}^{\text{ur}}}}).$$

Proof. The implication \Rightarrow is obvious. So let us prove the other implication. Since $R \rightarrow R_{\widehat{\mathcal{O}^{\text{ur}}}}$ is ind-étale, it suffices to show that

$$\text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(M) \subseteq \text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(N) \iff \text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(M_{\widehat{\mathcal{O}^{\text{ur}}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}^{\text{ur}}}}}(N_{\widehat{\mathcal{O}^{\text{ur}}}}).$$

Note that $R_{\widehat{\mathcal{O}^{\text{ur}}}} \rightarrow R_{\widehat{\mathcal{O}^{\text{ur}}}}$ is faithfully flat, with trivial residue field extension. Then the claim follows from this. \square

Let us prove the following main theorem on the inclusion of characteristic ideals, which is fundamental for the study of Iwasawa theory and Euler system theory over general normal domains.

Theorem 8.7 (Control Theorem for Characteristic Ideals). *With the notation and the hypothesis as in Corollary 4.5, assume that M and N are finitely generated torsion R -modules. Then the following statements are equivalent:*

- (1) $\text{char}_R(M) \subseteq \text{char}_R(N)$.

(2) For all but finitely many height-one primes:

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

there exists a finite étale extension of discrete valuation rings $\mathcal{O} \rightarrow \mathcal{O}'$ such that we have $\mathbf{x}_a \in R_{\mathcal{O}'}$ and

$$\text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(M_{\mathcal{O}'}/\mathbf{x}_a M_{\mathcal{O}'}) \subseteq \text{char}_{R_{\mathcal{O}'}/\mathbf{x}_a R_{\mathcal{O}'}}(N_{\mathcal{O}'}/\mathbf{x}_a N_{\mathcal{O}'}).$$

(3) For all but finitely many height-one primes:

$$\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}} \in \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}),$$

we have

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_a N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Note that $\dim R \geq 3$ holds automatically, due to the hypothesis $\text{depth } R \geq 3$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are obvious. So it remains to prove (3) \Rightarrow (1). By Lemma 8.6, it suffices to show that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

Take fundamental pseudo-isomorphisms for M and N :

$$M \rightarrow \bigoplus_i R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} \text{ (resp. } N \rightarrow \bigoplus_j R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_j^{f_j})$$

for a (not necessarily redundant) finite set of height-one primes $\{P_i\}$ (resp. $\{Q_j\}$) of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$. Put $I_M := (\prod_i P_i^{e_i})^{**}$ and $I_N := (\prod_j Q_j^{f_j})^{**}$ and the condition **(B)** allows one to assume that

$$M = \bigoplus_i R_{\widehat{\mathcal{O}}_{\text{ur}}}/P_i^{e_i} \text{ (resp. } N = \bigoplus_j R_{\widehat{\mathcal{O}}_{\text{ur}}}/Q_j^{f_j}).$$

To simplify the notation, assume that $\{P_i\}$ (resp. $\{Q_j\}$) is a redundant set of prime ideals and all relevant modules are defined over $R_{\widehat{\mathcal{O}}_{\text{ur}}}$. We make a step-by-step study to complete the proof. Let

$$\{\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}\}_{i \in \mathbb{N}} \subseteq \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(M_{\widehat{\mathcal{O}}_{\text{ur}}}) \cap \mathcal{L}_{\widehat{\mathcal{O}}_{\text{ur}}}(N_{\widehat{\mathcal{O}}_{\text{ur}}})$$

be any infinite sequence of mutually distinct primes of $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ satisfying the condition (3). In particular, we have $\bigcap_{i \in \mathbb{N}} \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}} = 0$.

Step1: Let us establish $\text{Supp}_{\text{ht}=1} N \subseteq \text{Supp}_{\text{ht}=1} M$, where $\text{Supp}_{\text{ht}=1}(-)$ is the set of height-one primes contained in the support of a module. By assumption, we find that

$$(I_M(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}))^{**} \subseteq (I_N(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}))^{**}$$

for all $i \in \mathbb{N}$. Fix a divisor Q_k . Then for every fixed $i \in \mathbb{N}$, in particular, we have

$$I_M \subseteq \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a},$$

where Λ_i is the set of symbolic power ideals associated to all minimal prime ideals of

$$\text{Ass}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/(Q_k + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}).$$

The image of the natural map

$$\bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a} \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}$$

is the reflexive closure of $Q_k(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}})$. Moreover, we have an injection:

$$R_{\widehat{\mathcal{O}}_{\text{ur}}}/\bigcap_{i \in \mathbb{N}} \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a} \hookrightarrow \prod_{i \in \mathbb{N}} (R_{\widehat{\mathcal{O}}_{\text{ur}}}/\bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a}).$$

Since Q_k is a prime ideal and since we could have chosen the set $\{\mathbf{x}_{a_i}\}_{i \in \mathbb{N}}$ such that

$$\bigcup_{i \in \mathbb{N}} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(Q_k + \mathbf{x}_{a_i} R_{\widehat{\mathcal{O}}_{\text{ur}}}) = \bigcup_{i \in \mathbb{N}} \bigcup_{\mathfrak{a} \in \Lambda_i} \text{Min}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\mathfrak{a})$$

is infinite in view of Lemma 8.5, it follows that

$$Q_k = \bigcap_{i \in \mathbb{N}} \bigcap_{\mathfrak{a} \in \Lambda_i} \mathfrak{a},$$

and thus, we have $I_M \subseteq Q_k$. Since Q_k is arbitrary, $I_M \subseteq (\prod_j Q_j)^{**}$, or equivalently, $\text{Supp}_{\text{ht}=1} N \subseteq \text{Supp}_{\text{ht}=1} M$.

Step2: In this step, we deal with multiplicities of divisors in the characteristic ideal and we complete this step by induction on the number of divisors appearing in I_M .

First, assume $I_M = (P^e)^{**}$ for $e \geq 1$. Then we have $\text{Supp}_{\text{ht}=1} N = \emptyset$ or $\{P\}$ because of **Step1**. If $\text{Supp}_{\text{ht}=1} N = \emptyset$, there is nothing to prove. So assume $\text{Supp}_{\text{ht}=1} N = \{P\}$. Both M and N are assumed to be fundamental torsion $R_{\widehat{\mathcal{O}}_{\text{ur}}}$ -modules, thus $M[\mathbf{x}_{a_i}]$ and $N[\mathbf{x}_{a_i}]$ are trivial modules and Proposition 7.5 yields that $\ell_{(R_{\widehat{\mathcal{O}}_{\text{ur}}})_P}(M_P) \geq \ell_{(R_{\widehat{\mathcal{O}}_{\text{ur}}})_P}(N_P)$.

In the general case, we prove by contradiction and thus, assume that $I_M \not\subseteq I_N$. Then this implies that we have $e_k < f_k$ for some k . Put

$$\tilde{I}_M := (P_k^{-e_k} \cdot I_M)^{**} \text{ (resp. } \tilde{I}_N := (P_k^{-e_k} \cdot I_N)^{**}),$$

which are both integral reflexive ideals. There is a short exact sequence:

$$0 \rightarrow \tilde{I}_M/I_M \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/I_M \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/\tilde{I}_M \rightarrow 0 \text{ (resp. } 0 \rightarrow \tilde{I}_N/I_N \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/I_N \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}}/\tilde{I}_N \rightarrow 0)$$

and it is clear that

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\tilde{I}_M/I_M) = \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(\tilde{I}_N/I_N) = (P_k^{e_k})^{**},$$

which induces the following short exact sequences by the snake lemma:

$$0 \rightarrow \tilde{I}_M / (I_M, \mathbf{x}_{a_i} \tilde{I}_M) \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}} / (I_M, \mathbf{x}_{a_i}) \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}} / (\tilde{I}_M, \mathbf{x}_{a_i}) \rightarrow 0$$

and

$$0 \rightarrow \tilde{I}_N / (I_N, \mathbf{x}_{a_i} \tilde{I}_N) \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}} / (I_N, \mathbf{x}_{a_i}) \rightarrow R_{\widehat{\mathcal{O}}_{\text{ur}}} / (\tilde{I}_N, \mathbf{x}_{a_i}) \rightarrow 0.$$

Taking characteristic ideals, we get by the assumption (3)

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i}R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\tilde{I}_M, \mathbf{x}_{a_i})) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}/\mathbf{x}_{a_i}R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/(\tilde{I}_N, \mathbf{x}_{a_i})).$$

Since the number of primes divisors in $\text{Ass}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\tilde{I}_M)$ is just one less than that of components of prime divisors in I_M , the induction hypothesis on \tilde{I}_M yields

$$\text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\tilde{I}_M) \subseteq \text{char}_{R_{\widehat{\mathcal{O}}_{\text{ur}}}}(R_{\widehat{\mathcal{O}}_{\text{ur}}}/\tilde{I}_N).$$

However, we deduce from these observations that $I_M \subseteq I_N$, which is a contradiction to our assumption $I_M \not\subseteq I_N$. Hence, we obtain $I_M \subseteq I_N$, as desired. \square

Remark 8.8. It is worth pointing out that Theorem 8.7 holds for complete normal local rings of mixed characteristic with arbitrary infinite perfect residue field as well. More precisely, it can be proven that $\text{char}_R(M) \subseteq \text{char}_R(N) \iff \text{char}_{R/xR}(M/xM) \subseteq \text{char}_{R/xR}(N/xN)$ for sufficiently many $x \in R$.

In this article, we presented an application of Bertini theorem to characteristic ideals. However, we believe that the main theorem has more interesting applications such as the study of the restriction map on divisor class (Chow) groups.

9. APPENDIX

In this appendix, we study the relationship between Fitting ideals and characteristic ideals. For Fitting ideals, we refer the reader to Northcott's book [9], but we review the basic part of the theory. Throughout, we assume that R is a Noetherian ring and M is a finitely generated R -module.

Definition 9.1 (Fitting ideal). Let the notation be as above and assume that

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is a finite free resolution of the R -module M , where the mapping $F_1 \rightarrow F_0$ is defined via a $m \times n$ -matrix X with $\text{rank}(F_1) = n$ and $\text{rank}(F_0) = m$. Then $\text{Fitt}_R(M)$ is defined as an ideal of R generated by all m -minors of X .

The Fitting ideal does not depend on the choice of a free resolution and it enjoys the following properties.

Proposition 9.2. *Let M be a finitely generated module over a Noetherian ring R . Then we have the following properties.*

- (1) *Let $I \subseteq R$ be an ideal. Then $\text{Fitt}_R(R/I) = I$.*
- (2) *Let S be any Noetherian R -algebra. Then $\text{Fitt}_S(M \otimes_R S) = \text{Fitt}_R(M)S$.*
- (3) *Let $\text{Ann}_R(M)$ be the annihilator of the R -module M . Then*

$$\text{Fitt}_R(M) \subseteq \text{Ann}_R(M).$$

- (4) *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of R -modules, then*

$$\text{Fitt}_R(L) \cdot \text{Fitt}_R(N) \subseteq \text{Fitt}_R(M).$$

- (5) *Assume that R is a discrete valuation ring with its uniformizing parameter b and M is a torsion R -module. Then $\text{Fitt}_R(M) = (b)^{\ell_R(M)}$.*

Proof. These facts are all well known. For (5), it simply follows from the elementary divisors of modules over a principal ideal domain. \square

For a Noetherian domain R and an R -module M , let $M^* := \text{Hom}_R(M, R)$, the dual of M . We say that M^{**} is the *reflexive closure* of M . Then we have the following lemma.

Lemma 9.3. *Let R be a Noetherian domain and let I be a fractional ideal of R . Then the reflexive closure I^{**} is naturally regarded as a fractional ideal of R .*

Proof. By assumption, there exists $\alpha \in R$ such that $I \simeq \alpha \cdot I \subseteq R$. Let $J := \alpha \cdot I$, an ideal of R . The short exact sequence $0 \rightarrow J \rightarrow R \rightarrow R/J \rightarrow 0$ induces a short exact sequence

$$0 = \text{Hom}_R(R/J, R) \rightarrow R \rightarrow \text{Hom}_R(J, R) \rightarrow N \rightarrow 0,$$

with $N \subseteq \text{Ext}_R^1(R/J, R)$ cokernel of $R \rightarrow \text{Hom}_R(J, R)$. Then applying $\text{Hom}_R(-, R)$, we get an exact sequence

$$0 = \text{Hom}_R(N, R) \rightarrow J^{**} \rightarrow R,$$

because $J \cdot N = 0$. This implies that J^{**} is an ideal of R . Then $I^{**} = \alpha^{-1} \cdot J^{**}$ is a fractional ideal. \square

Note that even when I and J are reflexive, the product $I \cdot J$ need not be reflexive. Any principal ideal is reflexive. Let I be an ideal of a normal domain R . Then we recall the following fact:

$$I^{**} = \bigcap_P I_P,$$

where P ranges over all height-one primes of R . The natural inclusion $I \rightarrow I^{**}$ is a pseudo-isomorphism, since $I_P \rightarrow (I^{**})_P = (I_P)^{**}$ for every height-one prime $P \subseteq R$. The following lemma explains the naturality of reflexive ideals and gives a way to investigate the inclusion relation between characteristic ideals.

Lemma 9.4. *Let R be a Noetherian normal domain and let I and J be reflexive ideals. Then $I \subseteq J$ if and only if $v_P(I) \geq v_P(J)$ for a valuation v attached to every height-one prime P of R . In particular, the only reflexive integral ideal containing a prime ideal of R properly is R itself.*

If R is only assumed to be Cohen-Macaulay, a similar result holds for invertible modules ([1], Lemma 5.3). We defined characteristic ideals as reflexive ideals and this is natural from the viewpoint of Iwasawa theory, because the most interesting arithmetic information may be captured at height-one primes. For finitely generated torsion R -modules M, N , it follows from the above lemma that $\text{char}_R(M) \subseteq \text{char}_R(N)$ if and only if $\text{char}_R(M)_P \subseteq \text{char}_R(N)_P$ for every height-one prime $P \in \text{Supp}(M) \cup \text{Supp}(N)$.

Example 9.5. Suppose I is reflexive and let $\mathfrak{a} \subseteq R$ be such that R/\mathfrak{a} is a normal domain. Then $I(R/\mathfrak{a})$ need not be reflexive. For a general ideal $I \subseteq R$, it can happen that

$$(I(R/\mathfrak{a}))^{**} \neq (I^{**}(R/\mathfrak{a}))^{**}.$$

Here, I^{**} is the reflexive closure with respect to R and $(I(R/\mathfrak{a}))^{**}$ is the reflexive closure with respect to R/\mathfrak{a} . Let us take a look at the following simple example. Take $R = \mathbb{Z}_p[[x, y]]$, $I = (x, y)$, and $\mathfrak{a} = (x)$. Then $(I^{**}(R/\mathfrak{a}))^{**} = R/xR$, since there is no height-one prime of R containing I . But $(I(R/\mathfrak{a}))^{**} = y(R/xR)$.

Proposition 9.6. *Let R be a Noetherian normal domain. Then the following hold.*

- (1) *Let M be a finitely generated torsion R -module. Then we have*

$$\text{char}_R(M) = \left(\prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} \right)^{**} = \text{Fitt}_R(M)^{**}.$$

In particular, $\text{Fitt}_R(M) \subseteq \text{char}_R(M)$ and if R is a UFD, then

$$\text{Fitt}_R(M) \subseteq \prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} = \text{char}_R(M).$$

- (2) *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of finitely generated torsion R -modules. Then*

$$\text{char}_R(M) = (\text{char}_R(L) \cdot \text{char}_R(N))^{**}.$$

Proof. (1): Since the characteristic ideal is reflexive, the first equality follows by taking localization at all height-one primes of R . The second equality follows from the fact

$$\text{Fitt}_{R_P}(M_P) = (PR_P)^{\ell_{R_P}(M_P)}$$

for any height-one prime ideal $P \subseteq R$. The second assertion is due to the fact that a height-one prime in a UFD is principal.

- (2): This is clear, since the length is additive with respect to short exact sequences. \square

Example 9.7. The ordinary power of a height-one prime in a normal domain is not necessarily reflexive. Here is an example. Let $R = \mathbb{Z}_p[[x^2, xy, y^2]]$ and let $\mathfrak{p} = (x^2, xy)$. Then R is a normal domain and $\text{ht}(\mathfrak{p}) = 1$. Then $\mathfrak{p}^2 = (x^4, x^3y, x^2y^2)$ and $\mathfrak{p}^2 \neq \mathfrak{p}^{(2)}$. In fact, $x^2 \notin \mathfrak{p}^2$, but $\mathfrak{p}^{(2)} = (x^2)$.

Now let $\mathfrak{q} = (xy, y^2)$ and $M = R/(\mathfrak{p} \cap \mathfrak{q})$, which is torsion over R . Then one verifies that

$$\text{Fitt}_R(M) = \mathfrak{p} \cap \mathfrak{q} \not\subseteq \prod_{\text{ht } \mathfrak{p}=1} \mathfrak{p}^{\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})} = \mathfrak{p}\mathfrak{q},$$

which tells us that Proposition 9.6 (1) is the most optimal.

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E-mail address: ochiai@math.sci.osaka-u.ac.jp, shimomotokazuma@gmail.com