A proof of the Riemann hypothesis for the Weng zeta function of rank 3 for the rationals

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1   Introduction

Let F be an algebraic number field. Recently, L. Weng introduced zeta functions of rank n associated to F as a generalization of the Iwasawa-Tate zeta integral from the Arakelov geometric point of view. In the article, we call them the Weng zeta function of rank n. In the case of rank one, the Weng zeta function coincide with the Dedekind zeta function of F. The background and precise definition of Weng zeta functions were published in [4, sec.B.4]. One remarkable fact for Weng zeta functions is that we can prove the Riemann hypothesis in the case of rank 2. It had been done by the author and J.C. Lagarias in [2] for the rational number field and was extended to the case of general number field F by Weng in [4, sec.C.4]. The proof of the Riemann hypothesis for a zeta function of rank 2 depends on the explicit expression for it.

In this spring, Weng obtained an explicit expression for the zeta function of rank 3 in the case of $F = \mathbb{Q}$. It is given in [5] this volume. By using his explicit formula, the author proved the Riemann hypothesis for the zeta function of rank 3 over the rational number field. In the article we give the proof of the Riemann hypohesis for it and the idea of the proof in self-contained fashion, as far as possible.

The zeta function $\widehat{\zeta}_{F,3}(s)$ of rank 3 is obtained by an integral of the completed Epstein zeta function of rank 3 over a moduli space of semi-stable lattices of rank 3 over F. In detail, see [5, sec.5.3, chap.9] (in [5] our $\widehat{\zeta}_{F,3}(s)$ is denoted by $\xi_{F,3}(s)$). In the case $F = \mathbb{Q}$, Theorem 4 in [5] assert that the explicit expression for $\widehat{\zeta}_{\mathbb{Q},3}(s)$ is given by

$$
\begin{aligned}
\widehat{\zeta}_{\mathbb{Q},3}(s) = {}& \Big(\frac{\widehat{\zeta}(2)}{3(s-1)} - \frac{1}{2(3s-2)} \Big) \widehat{\zeta}(3s) - \frac{1}{9} \Big(\frac{1}{s-1} - \frac{1}{s} \Big) \widehat{\zeta}(3s-1) \\
& - \Big(\frac{\widehat{\zeta}(2)}{3s} - \frac{1}{2(3s-1)} \Big) \widehat{\zeta}(3s-2),
\end{aligned}
\tag{1.1}
$$

where $\widehat{\zeta}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ and $\zeta(s)$ is the Riemann zeta-function. (Note that there were sign mistakes in [5, Theorem 4]. See [6, §A.1.2] for corrected formula.) From the general theory of zeta functions of rank n, the function $\widehat{\zeta}_{F,3}(s)$ satisfies the functional

equation $\widehat{\zeta}_{F,3}(s) = \widehat{\zeta}_{F,3}(1-s)$. Hence the Riemann hypothesis for $\widehat{\zeta}_{F,3}(s)$ is that all zeros of $\widehat{\zeta}_{F,3}(s)$ lie on the line $\Re(s) = 1/2$. Using (1.1) we can prove the Riemann hypothesis for $\widehat{\zeta}_{\mathbb{Q},3}(s)$.

Theorem 1 *The function $\widehat{\zeta}_{\mathbb{Q},3}(s)$ satisfies the Riemann hypothesis. That is, all zeros of $\widehat{\zeta}_{\mathbb{Q},3}(s)$ lie on the line $\Re(s) = 1/2$ which is the central line of the functional equation $\widehat{\zeta}_{\mathbb{Q},3}(s) = \widehat{\zeta}_{\mathbb{Q},3}(1-s)$.*

The article is organized as follows. In section 2, we explain the idea of the proof. In section 3, we state and prove Theorem 2 which is a easily version of Theorem 1. We describe the frame of the proof of Theorem 1 by proving Theorem 2 by using Lemma 1 and Lemma 2. In section 4, we give the proof of Theorem 1. It is proved by using Lemma 3 and Lemma 4 whose are modifications of Lemma 1 and Lemma 2. In section 5 we give the proof of Lemma 1 which is also used in the proof of Lemma 3. In section 6, we give the proof of Lemma 5 which is used in the proof of Lemma 4. In section 7, we comment on a possibility of applications of Theorem 1 and Theorem 2 to the original Riemann zeta function.

Acknowledgment. The author thanks professor Lin Weng for his helpful information on zeta functions of rank 3 and encouragement for the proof of the Riemann hypothesis.

2   The idea of the proof

For simplicity, we denote by $Z_3(s)$ the function $\widehat{\zeta}_{\mathbb{Q},3}(s)$:

$$
\begin{aligned}
Z_3(s) = {} & \widehat{\zeta}(2) \cdot \frac{1}{3s-3} \cdot \widehat{\zeta}(3s) - \widehat{\zeta}(2) \cdot \frac{1}{3s} \cdot \widehat{\zeta}(3s-2) \\
& - \frac{1}{3} \cdot \frac{1}{3s-3} \cdot \widehat{\zeta}(3s-1) + \frac{1}{3} \cdot \frac{1}{3s} \cdot \widehat{\zeta}(3s-1) \\
& + \frac{1}{2} \cdot \frac{1}{3s-1} \cdot \widehat{\zeta}(3s-2) - \frac{1}{2} \cdot \frac{1}{3s-2} \cdot \widehat{\zeta}(3s).
\end{aligned}
\tag{2.1}
$$

By using the well-known functional equation $\widehat{\zeta}(s) = \widehat{\zeta}(1-s)$, we can directly check that $Z_3(s)$ has the functional equation $Z_3(s) = Z_3(1-s)$. Since $\widehat{\zeta}(s)$ is holomorphic except for two simple poles at $s = 0, 1$ with residues -1 and 1 respectively, the possible poles of $Z_3(s)$ are $s = 0, 1/3, 2/3, 1$. We see that genuine poles are $s = 0, 1$ only. To investigate the zeros of $Z_3(s)$, we consider the entire function

$$
\begin{aligned}
\xi_3(s) = {} & 3s(3s-1)(3s-2)(3s-3)Z_3(s) \\
= {} & \frac{\xi(2)}{2} \cdot (3s-2) \cdot \xi(3s) + \frac{\xi(2)}{2} \cdot (1-3s) \cdot \xi(3s-2) \\
& - \xi(3s-1) + \frac{3}{2} \cdot s \cdot \xi(3s-2) + \frac{3}{2} \cdot (1-s) \cdot \xi(3s),
\end{aligned}
\tag{2.2}
$$

where $\xi(s)$ is Riemann's xi-function $\xi(s) = s(s-1)\widehat{\zeta}(s)$.

To describe the idea of the proof of Theorem 1, we recall the proof of the Riemann hypothesis for the zeta function $\widehat{\zeta}_{\mathbb{Q},2}(s)$ of rank 2. The key of the proof was in the explicit expression

$$\xi_2(s) = 2s(2s-1)(2s-2)\widehat{\zeta}_{\mathbb{Q},2}(s) = \xi(2s) - \xi(2s-1).$$

The right-hand side has the properties

(A) $\xi(2s)$ and $\xi(2s-1)$ are related by the functional equation $\xi(2(1-s)) = \xi(2s-1)$,

(B) all zeros of $\xi(2s)$ lie in the strip $0 < \Re(s) < 1/2$.

By using these properties we derived $|\xi(2s-1)/\xi(2s)| < 1$ for $\Re(s) > 1/2$ and $|\xi(2s)/\xi(2s-1)| < 1$ for $\Re(s) < 1/2$. These inequality and (B) gave the Riemann hypothesis for $\widehat{\zeta}_{\mathbb{Q},2}(s)$. Here we note one more important property,

(C) $\xi(2s)$ has the functional equation $\xi(2(\frac{1}{2} - s)) = \xi(2s)$.

The role of (C) was hidden in the back of (B) in the proof of the Riemann hypothesis for $\widehat{\zeta}_{\mathbb{Q},2}(s)$. However the property (C) is the key of the generalization to the proof of rank 3 case.

The idea of our proof of Theorem 1 is to recast the right-hand side of (2.2) into the form $\xi_3(s) = Y(s) + Y(1-s)$ so that all zeros of $Y(s)$ are in some vertical strip on the left of the line $\Re(s) = 1/2$. To obtain such expression, we notice properties (A) and (C) in rank 2 case. If we ignore polynomial factors, the right-hand side of (2.2) consists of $\xi(3s)$, $\xi(3s-1)$ and $\xi(3s-2)$. Using functional equations

$$\xi(3(1-s)) = \xi(3s-2) \tag{2.3}$$

and

$$\xi(3(1-s) - 1) = \xi(3s-1), \tag{2.4}$$

we can easily obtain an analogue of (A) for $\xi_3(s)$. However such expression has many possibilities. Here we notice the functional equations

$$\xi\left(3\left(\frac{2}{3} - s\right)\right) = \xi(3s-1) \tag{2.5}$$

and

$$\xi\left(3\left(\frac{4}{3} - s\right) - 2\right) = \xi(3s-1). \tag{2.6}$$

These extra functional equations allow us to recast the right-hand side of (2.2) as $\xi_3(s) = Y(s) + Y(1-s)$ so that $Y(s)$ satisfies an analogue of (B). In this process, the extra functional equations (2.5) and (2.6) play a role of (C). At the time, we obtain Theorem 1 by a way similar to the proof of rank 2 case.

Now we put

$$h(s) = \left\{ \frac{3}{2}\big(\xi(2) - 1\big)s - \big(\xi(2) - \frac{3}{2}\big) \right\} \xi(3s). \tag{2.7}$$

This $h(s)$ consists of all terms in the right-hand side of (2.2) containing $\xi(3s)$. We have

$$h(1 - s) = \left\{ \frac{3}{2}\big(\xi(2) - 1\big)(1 - s) - \big(\xi(2) - \frac{3}{2}\big) \right\} \xi(3s - 2). \tag{2.8}$$

This coincides to all terms in the right-hand side of (2.2) containing $\xi(3s - 2)$. Further

$$h\big(\frac{2}{3} - s\big) + h\big(s - \frac{1}{3}\big) = \big(1 - \frac{3}{2}(\xi(2) - 1)\big)\,\xi(3s - 1). \tag{2.9}$$

Hence if we take

$$X(s) = h(s) + h((2/3) - s), \tag{2.10}$$

then we have

$$\xi_3(s) = X(s) + X(1 - s) + \frac{\pi - 7}{2}\xi(3s - 1). \tag{2.11}$$

Here we used $\xi(2) = \pi/3$. Since $\xi(3s - 1) = \xi(3(1 - s) - 1)$, this equality also shows the functional equation $\xi_3(s) = \xi_3(1 - s)$. By taking

$$Y(s) = X(s) + \frac{\pi - 7}{4}\xi(3s - 1), \tag{2.12}$$

we obtain

$$\xi_3(s) = Y(s) + Y(1 - s). \tag{2.13}$$

As shown in below this expression satisfies properties analogous to (A) and (B).

3   The frame of the proof

To describe the frame of our proof of Theorem 1, we give the proof of the following result which is similar to Theorem 1.

Theorem 2 *Let $X(s)$ be the function defined by (2.10) with (2.7). If we take $\xi_3^{\natural}(s) = X(s) + X(1 - s)$, then all zeros of $\xi_3^{\natural}(s)$ lie on the line $\Re(s) = 1/2$.*

To prove Theorem 2, we prepare following two lemmas.

Lemma 1 *Let $F(s)$ be an entire function of genus zero or one, that has the following properties.*

(i) $F(s)$ is real on the real axis and satisfies a functional equation

$$F(s) = \pm F(1 - s), \tag{3.1}$$

for some choice of sign.

(ii) There exists $a > 0$ such that all zeros of $F(s)$ lie in the vertical strip

$$|\Re(s) - 1/2| < a. \qquad (3.2)$$

Then for any real $c \geq a$,

$$\left| \frac{F(s+c)}{F(s-c)} \right| > 1 \quad \text{for} \quad \Re(s) > 1/2, \qquad (3.3)$$

and

$$\left| \frac{F(s+c)}{F(s-c)} \right| < 1 \quad \text{for} \quad \Re(s) < 1/2. \qquad (3.4)$$

In particular, all zeros of $F(s+c) \pm F(s-c)$ lie on the line $\Re(s) = 1/2$.

Proof. We prove the lemma in section 5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma 2 *All zeros of $X(s)$ lie on the line $\Re(s) = 1/3$.*

Remark. From definition (2.10), $X(s)$ has the functional equation $X(s) = X(2/3 - s)$. The line $\Re(s) = 1/3$ is the center of the functional equation.

Proof. We put

$$X^\natural(s) = X(s - 1/6).$$

Then Lemma 2 is equivalent to the assertion that all zeros of $X^\natural(s)$ lie on the line $\Re(s) = 1/2$. Since

$$X(s) = \left\{ \left(\frac{\pi - 3}{2} \right)s - \left(\frac{\pi}{3} - \frac{3}{2} \right) \right\} \xi(3s)$$
$$+ \left\{ \left(\frac{\pi - 3}{2} \right)\left(\frac{2}{3} - s \right) - \left(\frac{\pi}{3} - \frac{3}{2} \right) \right\} \xi(3s - 1), \qquad (3.5)$$

we have

$$X^\natural(s) = \left\{ \left(\frac{\pi - 3}{2} \right)\left(s - \frac{1}{6} \right) - \left(\frac{\pi}{3} - \frac{3}{2} \right) \right\} \cdot \xi(3(s - 1/6))$$
$$\times \left\{ 1 + \frac{\left(\frac{\pi-3}{2} \right)\left(\frac{2}{3} - \left(s - \frac{1}{6} \right) \right) - \left(\frac{\pi}{3} - \frac{3}{2} \right)}{\left(\frac{\pi-3}{2} \right)\left(s - \frac{1}{6} \right) - \left(\frac{\pi}{3} - \frac{3}{2} \right)} \frac{\xi(3(s - 1/6) - 1)}{\xi(3(s - 1/6))} \right\}. \qquad (3.6)$$

If we take $F(s) = \xi(3s - 1)$, then $F(s) = F(1 - s)$ and all zeros of $F(s)$ lie in the strip $|\Re(s) - 1/2| < 1/6$. Applying Lemma 1 to $F(s)$ with $c = 1/6$, we obtain

$$1 > \left| \frac{F(s - 1/6)}{F(s + 1/6)} \right| = \left| \frac{\xi(3(s - 1/6) - 1)}{\xi(3(s - 1/6))} \right| \quad \text{for} \quad \Re(s) > 1/2. \qquad (3.7)$$

On the other hand, we find that

$$\left| \frac{\left(\frac{\pi-3}{2}\right)\left(\frac{2}{3} - \left(s - \frac{1}{6}\right)\right) - \left(\frac{\pi}{3} - \frac{3}{2}\right)}{\left(\frac{\pi-3}{2}\right)\left(s - \frac{1}{6}\right) - \left(\frac{\pi}{3} - \frac{3}{2}\right)} \right| < 1 \quad \text{for} \quad \Re(s) > 1/2 \qquad (3.8)$$

by an elementary calculation. Since

$$\left\{ \left(\frac{\pi - 3}{2}\right)\left(s - \frac{1}{6}\right) - \left(\frac{\pi}{3} - \frac{3}{2}\right) \right\} \xi(3(s - 1/6))$$

has no zeros in the right-half plane $\Re(s) > 1/2$, equality (3.6) and (3.7), (3.8) implies $X^{\natural}(s)$ has no zeros in the right-half plane $\Re(s) > 1/2$. The functional equation $X^{\natural}(s) = X^{\natural}(1-s)$ shows that $X^{\natural}(s)$ also has no zeros in the left-half plane $\Re\Re(s) < 1/2$. Thus we obtain the assertion of Lemma 2. $\qquad\square$

3.1 Proof of Theorem 2

We have

$$\xi_3^{\natural}(s) = X(s)\left(1 + \frac{X(1-s)}{X(s)}\right).$$

We show that

$$\left| \frac{X(1-s)}{X(s)} \right| < 1 \quad \text{for} \quad \Re(s) > 1/2. \qquad (3.9)$$

This yields the non-existence of the zeros of $\xi_3^{\natural}(s)$ in the right-half plane $\Re(s) > 1/2$, since $X(s)$ has no zeros in the right-half plane $\Re(s) > 1/3$ by Lemma 2. We put

$$F(s) = X(s - 1/6).$$

The functional equation $X(s) = X((2/3) - s)$ yields $F(s) = F(1-s)$. By Lemma 2, all zeros of $F(s)$ lie on the line $\Re(s) = 1/2$. These means that $F(s)$ satisfies all conditions of Lemma 1 for any $a > 0$. Applying Lemma 1 to $F(s)$ with $c = 1/6$, we obtain

$$\left| \frac{F(s - 1/6)}{F(s + 1/6)} \right| < 1 \quad \text{for} \quad \Re(s) > 1/2.$$

Using $X(s) = X((2/3) - s)$ we have

$$\frac{F(s - 1/6)}{F(s + 1/6)} = \frac{X(s - 1/3)}{X(s)} = \frac{X(1-s)}{X(s)}.$$

Thus we obtain (3.9). Because of the functional equation $\xi_3^{\natural}(s) = \xi_3^{\natural}(1-s)$, we also obtain the non-existence of the zeros of $\xi_3^{\natural}(s)$ in the left-half plane $\Re(s) < 1/2$. Now we complete the proof of Theorem 2. $\qquad\square$

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As the above, the proof of Theorem 2 consists of two steps. These two steps correspond to different two symmetries. Lemma 1 is used for each symmetry. In the first step corresponding to $X(s)$, we used Lemma 1 for $h(s) + h((2/3) - s)$ which yields the functional equation $X(s) = X((2/3) - s)$. In the second step corresponding to $\xi_3^\natural(s)$, we used Lemma 1 for $X(s) + X(1-s)$ which yields the functional equation $\xi_3^\natural(s) = \xi_3^\natural(1-s)$. This is the rough frame of the proof of Theorem 1.

4   Proof of Theorem 1

Let $Y(s)$ be the function defined by (2.12). Recall equation (2.13);

$$\xi_3(s) = Y(s) + Y(1-s).$$

Since $Y(s)$ does not have a functional equation, to prove Theorem 1, we need the following modification of Lemma 1.

Lemma 3 *Let $F(s)$ be an entire function of genus zero or one. Suppose that*

(i) *$F(s)$ is real on the real axis,*

(ii) *there exists $\sigma_0 < 1/2$ such that all zeros of $F(s)$ lie in the vertical strip*

$$\sigma_0 < \Re(s) < 1/2, \tag{4.1}$$

(iii) *there exists $C > 0$ such that*

$$N(T) \le CT \log T \quad as \quad T \to \infty, \tag{4.2}$$

where $N(T)$ is the number of zeros ρ of $F(s)$ satisfying $\sigma_0 < \Re(\rho) < 1/2$ and $0 \le \Im(\rho) < T$. Further $F(1-\sigma)/F(\sigma) > 0$ for large $\sigma > 1/2$ and

$$\frac{F(1-\sigma)}{F(\sigma)} \to 0 \quad as \quad 1/2 < \sigma \to \infty. \tag{4.3}$$

Then we have

$$\left| \frac{F(1-s)}{F(s)} \right| < 1 \quad for \quad \Re(s) > 1/2, \tag{4.4}$$

and

$$\left| \frac{F(1-s)}{F(s)} \right| > 1 \quad for \quad \Re(s) < 1/2. \tag{4.5}$$

In particular, all zeros of $F(s) \pm F(1-s)$ lie on the line $\Re(s) = 1/2$.

To apply Lemma 3 to $Y(s)$, we prepare the following Lemma 4 which is a modification of Lemma 2.

Lemma 4 *All zeros of $Y(s)$ lie in the strip $1/3 \le \Re(s) < 1/2$.*

4.1   Proof of Theorem 1 under Lemma 3 and 4

Now we give the proof of Theorem 1 by using Lemma 3 and Lemma 4. We prove these lemmas after the proof of the theorem. By definition of $Y(s)$, it is an entire function which is real on the real axis. By Lemma 4, $Y(s)$ satisfies (ii) of Lemma 3. Hence if (4.2) and (4.3) are shown for $Y(s)$, then we obtain Theorem 1 by Lemma 3.

At first we prove (4.2). It is well known that $|\xi(s)| < \exp(M|s|\log|s|)$ as $|s| \to \infty$ for some constant $M > 0$. Since $Y(s)$ is a linear combination of $\xi(s)$ up to degree one polynomials, we have $|Y(s)| < \exp(M'|s|\log|s|)$ as $|s| \to \infty$ for some constant $M' > 0$. Recall Jensen's formula for entire function $f(s)$ with $f(0) \neq 0$:

$$\int_0^R r^{-1} n(r) dr = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|,$$

where $n(r)$ is the number of zeros in $|s| < r$. We apply Jensen's formula to $Y(s)$. Then the right-hand side is estimated as $\ll R\log R$ for sufficiently large R. Since $\int_R^{2R} r^{-1} n(r) dr \geq n(R)\log 2$, it follows that

$$n(R) = O(R\log R).$$

Hence we obtain (4.2). From the definition of $Y(s)$ we have

$$\frac{Y(1-\sigma)}{Y(\sigma)} = \frac{\xi(3\sigma-1)}{\xi(3\sigma)} \frac{\left(\frac{\pi-3}{2} - \frac{\pi-1}{4}\sigma^{-1}\right) - \left(\frac{\pi-3}{2} - \frac{\pi}{6}\sigma^{-1}\right)\frac{\xi(3\sigma-2)}{\xi(3\sigma-1)}}{\left(\frac{\pi-3}{2} - \left(\frac{\pi}{3} - \frac{3}{2}\right)\sigma^{-1}\right) - \left(\frac{\pi-3}{2} - \frac{\pi-5}{4}\sigma^{-1}\right)\frac{\xi(3\sigma-1)}{\xi(3\sigma)}}.$$

Since $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, we have

$$\frac{\xi(3\sigma-1)}{\xi(3\sigma)} = \sqrt{\pi}\left(1 - \frac{2}{3\sigma}\right)\frac{\Gamma((3\sigma-1)/2)}{\Gamma(3\sigma/2)}\frac{\zeta(3\sigma-1)}{\zeta(3\sigma)}.$$

Using the Stirling formula and the Dirichlet series expansion of $\zeta(s)$, we have

$$\frac{\Gamma((3\sigma-1)/2)}{\Gamma(3\sigma/2)} = \frac{e^{-\frac{1-\log 2}{2}+O(\sigma^{-1})}}{\sqrt{3\sigma}}\frac{1+O(\sigma^{-1})}{1+O(\sigma^{-1})} \quad \text{and} \quad \frac{\zeta(3\sigma-1)}{\zeta(3\sigma)} = 1 + O(2^{1-\sigma})$$

for large $\sigma \geq 1$. Hence we obtain $Y(1-\sigma)/Y(\sigma) > 0$ for large $\sigma \geq 1$ and

$$\frac{Y(1-\sigma)}{Y(\sigma)} = O(\sigma^{-1/2}) \quad \text{as} \quad \sigma \to +\infty.$$

Thus (4.3) holds for $Y(s)$. Now we complete the proof of Theorem 1 under Lemma 3 and Lemma 4. $\qquad\square$

4.2 Proof of Lemma 3

We prove the lemma only if $F(s)$ has genus one, since if $F(s)$ has genus zero it is easily proved by a way similar to the case of genus one. The genus one assumption is equivalent to the assertion that the Hadamard product factorization

$$F(s) = e^{A+Bs} s^m \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \quad (m \in \mathbb{Z}_{\geq 0}) \tag{4.6}$$

converges absolutely and uniformly on any compact subsets of $\mathbb{C}$. This assumption is also equivalent to the bound $\sum_\rho |\rho|^{-2} < \infty$. Assumption (i) implies symmetriy of the zeros under $\rho \mapsto \bar{\rho}$. It follows that the set of zeros $\rho = \beta + i\gamma$, counted with multiplicity, can be partitioned into blocks $B(\rho)$ comprising $\{\rho, \bar{\rho}\}$ if $\gamma > 0$ and $\{\rho\}$ if $\beta \neq 0$ and $\gamma = 0$. Each block is labeled with the unique zero in it having $\beta < 1/2$ and $\gamma \geq 0$. Using assumption (ii), we show

$$F(s) = s^m e^{A+B's} \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left(1 - \frac{s}{\rho}\right) \right) \tag{4.7}$$

where the outer product on the right-hand side converges absolutely and uniformly on any compact subsets of $\mathbb{C}$. This assertion holds because the block convergence factors $\exp(c(B(\rho))s)$ are given by $c(B(\rho)) = 2\beta|\rho|^{-2}$ for $\gamma > 0$. Assumption (ii) gives $\sigma_0 < \beta < 1/2$. Hence

$$\sum_{B(\rho)} |c(B(\rho))| \leq \sum_{0 \neq \rho \in (\sigma_0, 1/2)} |\rho|^{-1} + \max\{1, 2|\sigma_0|\} \left(\sum_\rho |\rho|^{-2} \right) < \infty.$$

Thus the convergence factors can be pulled out of the product. Hence we have (4.7) with

$$B' = B + \sum_{B(\rho)} c(B(\rho)). \tag{4.8}$$

To establish (4.4) we proceed block by block in (4.7), using the factorization

$$\left| \frac{F(1-s)}{F(s)} \right| = e^{B'(1-2\Re(s))} \left| \frac{1-s}{s} \right|^m \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left| \frac{1 - \frac{1-s}{\rho}}{1 - \frac{s}{\rho}} \right| \right). \tag{4.9}$$

Using assumption (iii), we show

$$B' \geq 0. \tag{4.10}$$

We will prove this later. If (4.10) is shown, we have

$$e^{B'(1-2\Re(s))} \left| \frac{1-s}{s} \right|^m \leq 1 \quad \text{for} \quad \Re(s) > \frac{1}{2}. \tag{4.11}$$

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In a single block we can clear denominators to obtain

$$\prod_{\rho \in B(\rho)} \left| \frac{1 - \frac{1-s}{\rho}}{1 - \frac{s}{\rho}} \right| = \prod_{\rho \in B(\rho)} \left| \frac{s - 1 + \rho}{s - \rho} \right|.$$

We compare the term in the numerator with ρ against the term in the denominator with $\rho' := 1 - \bar{\rho}$. We find that

$$\left| \frac{s - 1 + \bar{\rho}}{s - \rho} \right|^2 < 1 \quad \text{for} \quad \Re(s) > \frac{1}{2}. \tag{4.12}$$

In fact, writing $s = \sigma + it$, we have

$$\left| \frac{s - 1 + \bar{\rho}}{s - \rho} \right|^2 = \frac{(\sigma - 1 + \beta)^2 + (t - \gamma)^2}{(\sigma - \beta)^2 + (t - \gamma)^2} = 1 - \frac{(1 - 2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + (t - \gamma)^2}.$$

This implies (4.12) since $\beta < 1/2$ by assumption (ii). Thus we conclude for $\Re(s) > 1/2$ that the absolute value of the product over terms in each block on the right in (4.9) is less than 1. Hence (4.4) holds by (4.11) and (4.12).

Therefore it remains to show (4.10). By (4.3), we have

$$\mathbb{R} \ni \log\left(\frac{F(1 - \sigma)}{F(\sigma)} \right) \to -\infty \quad \text{as} \quad 1/2 < \sigma \to +\infty. \tag{4.13}$$

Using (4.3) and (4.7), we have

$$\frac{F(1 - \sigma)}{F(\sigma)} = e^{B'(1 - 2\sigma)} \left(\frac{\sigma - 1}{\sigma} \right)^m \prod_{\rho = \beta \in \mathbb{R}} \frac{\sigma - 1 + \beta}{\sigma - \beta} \prod_{\substack{\rho = \beta + i\gamma \\ \gamma > 0}} \frac{(\sigma - 1 + \beta)^2 + \gamma^2}{(\sigma - \beta)^2 + \gamma^2}.$$

Thus

$$\log\left(\frac{F(1 - \sigma)}{F(\sigma)} \right) = B'(1 - 2\sigma) + m \log\left(1 - \frac{1}{\sigma} \right) + \sum_{\rho = \beta \in \mathbb{R}} \log\left(1 - \frac{1 - 2\beta}{\sigma - \beta} \right)$$
$$+ \sum_{\substack{\rho = \beta + i\gamma \\ \gamma > 0}} \log\left(1 - \frac{(1 - 2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2} \right). \tag{4.14}$$

Here we note that

$$\log\left(1 - \frac{(1 - 2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2} \right) < 0 \quad \text{for} \quad \sigma > 1/2$$

by assumption (ii). Suppose that $B' < 0$. Then (4.13) and (4.14) claim that

$$\left| \sum_{\substack{\rho = \beta + i\gamma \\ \gamma > 0}} \log\left(1 - \frac{(1 - 2\beta)(2\sigma - 1)}{(\sigma - \beta)^2 + \gamma^2} \right) \right| \geq 2|B'|\sigma \tag{4.15}$$

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for sufficiently large $\sigma > 1/2$, because the number of real zeros of $F(s)$ is finite by assumption (ii). On the other hand, for large $\sigma > 1/2$, we have

$$\left| \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2 + \gamma^2}\right) \right| \leq \left| \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\sigma_0)(2\sigma-1)}{(\sigma-1/2)^2 + \gamma^2}\right) \right|$$

$$\ll (2\sigma - 1) \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \frac{1}{(\sigma-1/2)^2 + \gamma^2}.$$

The sum in the right-hand side can be written as the Stieltjes integral

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2 + t^2}.$$

Using (4.2) we have

$$\int_{\gamma_0}^{\infty} \frac{dN(t)}{(\sigma-1/2)^2 + t^2} \ll \int_{\gamma_0}^{\infty} \frac{(\log t)\, dt}{(\sigma-1/2)^2 + t^2} \ll \frac{\log(\sigma+\gamma_0)}{\sigma-1/2}.$$

Hence we obtain

$$\left| \sum_{\substack{\rho=\beta+i\gamma \\ \gamma>0}} \log\left(1 - \frac{(1-2\beta)(2\sigma-1)}{(\sigma-\beta)^2 + \gamma^2}\right) \right| \ll \log(\sigma+\gamma_0) \tag{4.16}$$

for sufficiently large $\sigma > 1/2$. This contradict (4.15). Thus (4.10) holds.

Inequality (4.5) is proved by a way similar to the proof of (4.4). We complete the proof of Lemma 3. $\qquad\square$

4.3   Proof of Lemma 4

Now we complete the proof of Theorem 1 by proving Lemma 4. We put

$$p(s) = \left(\frac{\pi-3}{2}\right)s - \left(\frac{\pi}{3} - \frac{3}{2}\right). \tag{4.17}$$

Then $X(s)$ and $Y(s)$ are written as

$$X(s) = p(s)\,\xi(3s) + p((2/3) - s)\,\xi(3s - 1) \tag{4.18}$$

and

$$Y(s) = p(s)\,\xi(3s) + \left\{ p\left(\frac{2}{3} - s\right) + \frac{\pi-7}{4} \right\}\xi(3s - 1). \tag{4.19}$$

We find that

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(i) the only one zero of $p(s)$ is $\dfrac{2\pi - 9}{3\pi - 9} \simeq -6.40$,

(ii) the only one zero of $p\left(\dfrac{2}{3} - s\right)$ is $\dfrac{1}{\pi - 3} \simeq 7.06$,

(iii) the only one zero of $p\left(\dfrac{2}{3} - s\right) + \dfrac{\pi - 7}{4}$ is $\dfrac{\pi - 5}{2\pi - 6} \simeq -6.56$.

At the first, we show that $Y(s)$ has no zeros in the left-half plane $\Re(s) < 1/3$. To prove the assertion, we divide $Y(s)$ as

$$Y(s) = \left\{ p\left(\frac{2}{3} - s\right) + \frac{\pi - 7}{4} \right\} \xi(3s - 1) \cdot \left(1 + \frac{p(s)}{p((2/3) - s) + (\pi - 7)/4} \cdot \frac{\xi(3s)}{\xi(3s - 1)} \right).$$

Here we note that

$$\left\{ p\left(\frac{2}{3} - s\right) + \frac{\pi - 7}{4} \right\} \xi(3s - 1)$$

has no zeros in the left-half plane $\Re(s) < 1/3$. Therefore if we show that

$$\left| \frac{p(s)}{p((2/3) - s) + (\pi - 7)/4} \cdot \frac{\xi(3s)}{\xi(3s - 1)} \right| < 1 \quad \text{for} \quad \Re(s) < \frac{1}{3}, \tag{4.20}$$

then $Y(s)$ has no zeros in $\Re(s) < 1/3$. By a way similar to the proof of Lemma 2,

$$\left| \frac{\xi(3s)}{\xi(3s - 1)} \right| < 1 \quad \text{for} \quad \Re(s) < \frac{1}{3}.$$

On the other hand, from (i) and (iii), we have

$$\frac{p(s)}{p((2/3) - s) + (\pi - 7)/4} = -\frac{2}{3} \cdot \frac{s - (2\pi - 9)/(3\pi - 9)}{s - (\pi - 5)/(2\pi - 6)}.$$

Thus

$$\left| \frac{p(s)}{p((2/3) - s) + (\pi - 7)/4} \right| < 1 \quad \text{for} \quad \Re(s) > \frac{7\pi - 33}{12(\pi - 3)} \simeq -6.48.$$

Hence we obtain (4.20).

Next, we prove that $Y(s)$ has no zeros in the right-half plane $\Re(s) \geq 1/2$. To prove the assertion, we divide $Y(s)$ as

$$Y(s) = p(s)\,\xi(3s) \cdot \left(1 + \frac{p((2/3) - s) + (\pi - 7)/4}{p(s)} \cdot \frac{\xi(3s - 1)}{\xi(3s)} \right).$$

Here we note that $p(s)\xi(3s)$ has no zeros in the right-half plane $\Re(s) > 1/3$. Therefore we show that

$$V(s) = 1 + \frac{p((2/3) - s) + (\pi - 7)/4}{p(s)} \cdot \frac{\xi(3s - 1)}{\xi(3s)} \tag{4.21}$$

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has no zeros in the right-half plane $\Re(s) \geq 1/2$. By a way similar to the proof of Lemma 2, we obtain

$$\left|\frac{\xi(3s-1)}{\xi(3s)}\right| < 1 \quad \text{for} \quad \Re(s) > \frac{1}{3}.$$

On the other hand, we obtain

$$\left|\frac{p((2/3)-s)+(\pi-7)/4}{p(s)}\right| \leq 1 \quad \text{for} \quad \Re(s) \leq \frac{7\pi-33}{12(\pi-3)} \simeq -6.48$$

by an elementary calculation. Thus $V(s) \neq 0$ for $\Re(s) \geq \dfrac{7\pi-33}{12(\pi-3)} \simeq -6.48$. Hence to complete the proof of Lemma 4 we should prove that $V(s)$ has no zeros in the strip

$$\frac{1}{2} \leq \Re(s) < \frac{7\pi-21}{12(\pi-3)}. \tag{4.22}$$

Let R be the rectangle $[1/2, (7\pi-21)/(12(\pi-3))] \times [-15, 15]$. Clearly $V(s)$ is holomorphic in R. From a numerical computation (for example by using MATHEMATICA), we obtain

$$\frac{1}{2\pi i} \int_{\partial R} \frac{V'}{V}(s)ds = 0$$

where ∂R is the boundary of R. This means $V(s) \neq 0$ in R by the argument principle.

Now we prove $V(\sigma + it) \neq 0$ for σ in (4.22) and $|t| \geq 15$ by showing that

$$\left|\frac{\xi(3s-1)}{\xi(3s)}\right| \left|\frac{p((2/3)-s)+(\pi-3)/4}{p(s)}\right| < 1 \tag{4.23}$$

holds in this region. In the proof of Lemma 2, we obtained

$$\left|\frac{\xi(3s-1)}{\xi(3s)}\right| < 1 \quad \text{for} \quad \Re(s) > \frac{1}{3} \tag{4.24}$$

by using the method established in [2], which is also explained in section 5. In detail, (4.24) is obtained by showing that this inequality holds term-by-term in the (modified) Hadamard product on a zero-by-zero basis, where the zero $\rho = \beta + i\gamma$ of $\xi(s)$ in the numerator is paired against the zero $\rho' = 1 - \beta + i\gamma$ in the denominator. That is, we established (4.24) using the fact that

$$\left|\frac{3s-1-\rho'}{3s-\rho}\right| < 1 \quad \text{for} \quad \Re(s) > \frac{1}{3}. \tag{4.25}$$

Hence, to establish (4.23), it suffices to show that for each s in (4.22) with $|t| \geq 15$ there exists a zero ρ of $\xi(s)$ such that

$$\left|\frac{3s-1-\rho'}{3s-\rho}\right| \left|\frac{p((2/3)-s)+(\pi-3)/4}{p(s)}\right| < 1. \tag{4.26}$$

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Then (4.25) and (4.26) give (4.23). To prove the existence of a zero ρ of $\zeta(s)$ satisfying (4.26), we need the following lemma which will be proved in section 6.

Lemma 5 (Lagarias) *For any real $|t| \geq 14$ there exists a zero $\rho = \beta + i\gamma$ of $\xi(s)$ such that $0 < \beta \leq 1/2$ and $|t - \gamma| \leq 5$.*

For each $s = \sigma + it$ satisfying $|t| \geq 15$, Lemma 5 gives a zero $\rho = \beta + i\gamma$ with $0 < \beta \leq 1/2$ and $|3t - \gamma| \leq 5$. Therefore, to prove (4.26), it is sufficient that

$$\left| \frac{3\sigma - 2 + \beta + it_0}{3\sigma - \beta + it_0} \right| < \left| \frac{p(s)}{p((2/3) - s) + (\pi - 3)/4} \right| \tag{4.27}$$

for any $1/2 \leq \sigma \leq (7\pi - 21)/(12\pi - 36)$, $|t| \geq 15$, $0 < \beta \leq 1/2$ and $|t_0| \leq 5$. Denote by μ and ν the zeros of $p(s)$ and $p((2/3) - s) + (\pi - 3)/4$, respectively. By squaring the inequality, (4.27) is equivalent to

$$\frac{(3\sigma - 2 + \beta)^2 + t_0^2}{(3\sigma - \beta)^2 + t_0^2} < \frac{(\sigma - \mu)^2 + t^2}{(\sigma - \nu)^2 + t^2}. \tag{4.28}$$

For fixed s and β it suffices to prove (4.27) with $t_0 = 5$, for it would then hold for $|t_0| \leq 5$, using the fact that $(3\sigma - \beta)^2 \geq (3\sigma - 2 + \beta)^2$ for $\Re(s) > 1/3$. Next, when $t_0 = 5$, it suffices to verify the inequality for $\beta = 1/2$ since if it holds there then the left-hand side of (4.28) with $t_0 = 5$ increases as $\beta > 0$ increases, while the right side is fixed. To establish (4.27) it thus suffices to verify the inequality

$$\frac{(3\sigma - \frac{3}{2})^2 + 25}{(3\sigma - \frac{1}{2})^2 + 25} < \frac{(\sigma - \mu)^2 + t^2}{(\sigma - \nu)^2 + t^2}. \tag{4.29}$$

The left-hand side of (4.29) decreases as σ increases in $1/2 \leq \sigma \leq (7\pi - 21)/(12\pi - 36)$. On the other hand, for fixed $|t| \geq 15$, the right-hand side of (4.29) increases as σ increases in $1/2 \leq \sigma \leq (7\pi - 21)/(12\pi - 36)$. Hence to establish (4.29) it suffices to verify the inequality

$$\frac{25}{26} < \frac{((1/2) - \mu)^2 + t^2}{((1/2) - \nu)^2 + t^2}. \tag{4.30}$$

By an elementary calculation, inequality (4.30) is valid for $|t| \geq 3$. Now we obtain (4.26) and complete the proof of Lemma 3. $\square$

5 Proof of Lemma 1

In this section, we give a proof of Lemma 1 according to Lagarias-Suzuki [2].

The genus one assumption is equivalent to the assertion that the Hadamard product factorization

$$F(s) = e^{A + Bs} s^m \prod_\rho \left(1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}} \tag{5.1}$$

converges absolutely and uniformly on any compact subsets of $\mathbb{C}$. This assumption is also equivalent to the bound $\sum_\rho |\rho|^{-2} < \infty$. Assumption (i) implies symmetries of the zeros under $\rho \mapsto \bar\rho$ and $\rho \mapsto 1 - \rho$. It follows that the set of zeros $\rho = \beta + i\gamma$, counted with multiplicity, can be partitioned into blocks $B(\rho)$ comprising $\{\rho, 1 - \rho, \bar\rho, 1 - \bar\rho\}$ if $\beta \neq 1/2$, $\{\rho, 1 - \rho\}$ if $\beta = 1/2$ and $\gamma \neq 0$, and $\{\rho\}$ if $\rho = 1/2$. Each block is labeled with the unique zero in it having $\beta \leq 1/2$ and $\gamma \geq 0$. Using assumption (ii), we show

$$F(s) = e^{A+B's} \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left(1 - \frac{s}{\rho} \right) \right), \tag{5.2}$$

where the outer product on the right-hand side converges absolutely and uniformly on any compact subsets of $\mathbb{C}$. This assertion holds because the block convergence factors $\exp(c(B(\rho))s)$ are given by

$$c(B(\rho)) = \begin{cases} \beta|\rho|^{-2} + (1-\beta)|1-\rho|^{-2} & \text{if } \beta \neq 1/2, \\ |\rho|^{-2} & \text{if } \beta = 1/2 \text{ and } \gamma \neq 0, \\ 2 & \text{if } \rho = 1/2. \end{cases}$$

Assumption (ii) gives $-a < \beta - 1/2 < a$. Hence

$$\sum_{B(\rho)} |c(B(\rho))| \leq (1 + 2a)\left(\sum_\rho |\rho|^{-2} \right) < \infty.$$

Thus the convergence factors can be pulled out of the product. Hence we have (5.2) with $B' = B + \sum_{B(\rho)} c(B(\rho))$.

Using the functional equation (3.1) we infer that $B' = 0$ in (5.2), so that

$$F(s) = e^{A} \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left(1 - \frac{s}{\rho} \right) \right). \tag{5.3}$$

Indeed the change of variable $s \mapsto 1 - s$ permutes the factors in each block $B(\rho)$, with a possible sign change for $\rho = 1/2$, so it must be that $e^{A+B's} = \pm e^{A+B'(1-s)}$, which forces $B' = 0$.

To establish (3.3) and (3.4) we proceed block by block in (5.3), using the factorization

$$\left| \frac{F(s+c)}{F(s-c)} \right| = \prod_{B(\rho)} \left(\prod_{\rho \in B(\rho)} \left| \frac{1 - \frac{s+c}{\rho}}{1 - \frac{s-c}{\rho}} \right| \right). \tag{5.4}$$

In a single block we can clear denominators to obtain

$$\prod_{\rho \in B(\rho)} \left| \frac{1 - \frac{s+c}{\rho}}{1 - \frac{s-c}{\rho}} \right| = \prod_{\rho \in B(\rho)} \left| \frac{s + c - \rho}{s - c - \rho} \right|.$$

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The main point is to compare the term in the numerator with ρ against the term in the denominator with $\rho' := \overline{1 - \rho} = 1 - \bar{\rho}$. We show that

$$\left| \frac{s + c - \rho}{s - c - (1 - \bar{\rho})} \right|^2 > 1 \quad \text{for} \quad \Re(s) > \frac{1}{2}, \tag{5.5}$$

and

$$\left| \frac{s + c - \rho}{s - c - (1 - \bar{\rho})} \right|^2 < 1 \quad \text{for} \quad \Re(s) < \frac{1}{2}. \tag{5.6}$$

If (5.5) is shown, then we may conclude for $\Re(s) > 1/2$ that the absolute value of the product over terms in each block on the right in (5.4) exceeds 1, and (3.3) follows. Similarly (5.6) implies that for $\Re(s) < 1/2$ the product of terms over each block is smaller than 1, and (3.4) follows.

Therefore it remains to show (5.5) and (5.6). Writing $s = \sigma + it$, we have

$$\left| \frac{s + c - \rho}{s - c - (1 - \bar{\rho})} \right|^2 = \frac{(\sigma + c - \beta)^2 + (t - \gamma)^2}{(\sigma - c - 1 + \beta)^2 + (t - \gamma)^2}$$

Now (5.5) reduces to the assertion that

$$(\sigma + c - \beta)^2 > (\sigma - c - 1 + \beta)^2 \quad \text{for} \quad \Re(s) > \frac{1}{2}. \tag{5.7}$$

To show this we note that $\Re(s) > 1/2$ gives

$$\sigma + c - \beta > \frac{1}{2} + a - \beta > 0,$$

whence (5.7) makes the two assertions

$$\sigma + c - \beta > \sigma - c - 1 + \beta,$$

$$\sigma + c - \beta > -(\sigma - c - 1 + \beta).$$

The second of these asserts that $\sigma > 1/2$. While the first asserts that $2c > 2(\beta - 1/2)$. This holds since $c \geq a > \beta - 1/2$. Thus (5.7) holds, whence (5.5) holds.

A similar argument is used to establish (5.6). It reduces to the assertion that

$$(\sigma + c - \beta)^2 < (\sigma - c - 1 + \beta)^2 \quad \text{for} \quad \Re(s) < \frac{1}{2}. \tag{5.8}$$

We have

$$-(\sigma - c - 1 + \beta) \geq \frac{1}{2} + a - \beta > 0,$$

so that (5.8) is equivalent to the two assertions

$$-\sigma + c + 1 - \beta > -(\sigma + c - \beta),$$

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$$-\sigma + c + 1 - \beta > \sigma + c - \beta.$$

The second of these is equivalent to $\sigma < 1/2$. While the first of these is equivalent to $c + 1/2 - \beta > 0$. This holds by our choice of c.

The conclusion that all zeros of $F(s+c) \pm F(s-c)$ lie on the line $\Re(s) = 1/2$ follows, because two terms $F(s+c)$ and $F(s-c)$ have different absolute values off the line $\Re(s) = 1/2$. □

6   Proof of Lemma 5

In this section, we give a proof of Lemma 5 according to Lagarias [1]. Throughout the section, we denote by γ the imaginary part of the non-trivial zeros of $\zeta(s)$. The spacing between consecutive ordinate γ goes to zero as $\gamma \to \infty$. Therefore the result of Lemma 5 holds for $|t|$ exceeding some bound and gives an explicit bound.

First, we prove that the result of the lemma holds for $|t| \geq 14$. Since the zeros are symmetric around the real axis, it suffices to consider the case $t \geq 14$. We verify the lemma directly for $14 \leq t \leq 168\pi + 5 < 525$ by inspection of a table of zeros of $\zeta(s)$. In fact there is no gap of size 5 between any consecutive zeros of $\zeta(s)$ starting with $\gamma_2 \simeq 21.02$, and the smallest zeros have ordinates $\gamma_1 \simeq 14.13$.

For the remaining range we use numerical estimates of Turing [3]. Let $N(T)$ be the number of zeros ρ with $0 < \Im(\rho) < T$ and let $\pi S(T)$ be the argument of $\zeta(1/2 + iT)$ obtained by analytic continuation along a horizontal line from $\infty + iT$. We have

$$N(T) = 2\,\kappa\left(\frac{T}{2\pi}\right) + 1 + S(T), \tag{6.1}$$

where $\kappa(\tau) = (4\pi i)^{-1} \log\left(\Gamma(1/4 + \pi i\tau)/\Gamma(1/4 - \pi i\tau)\right) - (1/4)\tau \log \pi$. Theorem 1 of Turing [3] gives

$$\kappa(\tau) = \frac{1}{2}\left(\tau \log \tau - \tau - \frac{1}{2}\right) + \varepsilon(\tau) \quad \text{with} \quad |\varepsilon(\tau)| \leq \frac{0.006}{\tau} \quad \text{for} \quad \tau \geq 64. \tag{6.2}$$

For $S_1(t) = \int_0^t S(u)du$, Theorem 4 of Turing [3] assert that if $t_2 > t_1 > 168\pi$ then

$$|S_1(t_2) - S_1(t_1)| \leq 2.30 + 0.128 \log\left(\frac{t_2}{2\pi}\right). \tag{6.3}$$

Now we prove Lemma 5 under the following Lemma 6 which is shown in the end of the section.

Lemma 6   *Suppose that there is no zero ρ of $\zeta(s)$ with $168\pi \leq t_1 < \Im(\rho) < t_2$ and that $S(T)$ has one sign over the interval $[t_1, t_2]$. Then*

$$t_2 - t_1 < 10/3. \tag{6.4}$$

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Suppose that $t \geq 168\pi + 5$ and there is no zero on $[t - 2, t + 2]$, that is, $N(T)$ is constant on $[t - 2, t + 2]$. Because of Lemma 6, inside this interval $S(T)$ must have a zero-crossing in each subinterval of length $10/3$. Hence it must have a zero-crossing at some point $t_1 = t + x$ with $|x| \leq 4/3$. Since $N(T)$ is constant, $S(T)$ varies like $-2\kappa(T/2\pi) - N(t_1) - 1$. Therefore (6.2) implies that all other zero-crossing of $S(T)$ in $[t-2, t+2]$ are localized within a distance $\varepsilon = 0.006/(t_1 \log(t_1/2\pi))$ of this one. If there were no zero on $[t-5, t]$, then $N(T)$ is constant there. Hence $S(T)$ varies approximately linearly on the interval. If $t + x$ falls in $[t - 5, t]$, zero-crossings of $S(T)$ are located within 0.001 of $t + x$, and otherwise it has no zero-crossings. Since $|x| \leq 4/3$, $S(T)$ has single sign on $[t - 5, t - 5 + 10/3]$. This contradicts Lemma 6. Thus there is a zero with the ordinate γ in $[t - 5, t - 2]$. By a similar argument there is a zero with the ordinate γ in $[t + 2, t + 5]$.

If $t \geq 168\pi + 5$ and there is a zero with the ordinate γ in $[t - 2, t + 2]$, this of course implies our assertion. We complete the proof of Lemma 5. $\qquad\square$

Proof of Lemma 6. We take $t_2 = t_1 + c$ with $c \geq 10/3$. From the assumption $S(T)$ has single sign on $[t_1, t_2]$. Hence we have

$$|S_1(t_2) - S_1(t_1)| = \left| \int_0^c S(t_1 + u) du \right| = \int_0^c |S(t_1 + u)| du \geq \int_0^{10/3} |S(t_1 + u)| du.$$

Suppose that $S(T)$ is positive. Since $N(t)$ is constant on $[t_1, t_2]$, we obtain

$$S(t_1 + u) = N(t_1) - 2\kappa\left(\frac{t_1 + u}{2\pi}\right) - 1 = 2\kappa\left(\frac{t_1}{2\pi}\right) - 2\kappa\left(\frac{t_1 + u}{2\pi}\right) + S(t_1) > 0 \qquad (6.5)$$

for any $0 \leq u \leq 10/3$. Thus

$$S(t_1) > 2\kappa\left(\frac{t_2}{2\pi}\right) - 2\kappa\left(\frac{t_1}{2\pi}\right). \qquad (6.6)$$

Combining (6.5) and (6.6) we obtain

$$S(t_1 + u) > 2\kappa\left(\frac{t_2}{2\pi}\right) - 2\kappa\left(\frac{t_1 + u}{2\pi}\right)$$

for any $0 \leq u \leq 10/3$. Hence we have

$$|S_1(t_2) - S_1(t_1)| > \int_0^{10/3} \left| 2\kappa\left(\frac{t_2}{2\pi}\right) - 2\kappa\left(\frac{t_1 + u}{2\pi}\right) \right| du. \qquad (6.7)$$

Thus (6.2) yields

$$
\begin{aligned}
|S_1(t_2) - S_1(t_1)| &\geq \int_0^{10/3} \left(\frac{t_2}{2\pi} - \frac{t_1 + u}{2\pi}\right)\left(\log \frac{t_2}{2\pi} - 1\right) du - 3.5\left(\frac{0.006}{t_1}\right) \\
&\geq 0.884 \log\left(\frac{t_2}{2\pi}\right) - 0.886.
\end{aligned}
\qquad (6.8)
$$

Suppose that $S(T)$ is negative. By (6.1) we have $N(t_1) < 2\kappa(t_1/(2\pi)) + 1$. Since $N(T)$ is constant on $[t_1, t_2]$, we obtain

$$-S(t_1 + u) = 2\kappa\left(\frac{t_1 + u}{2\pi}\right) - N(t_1) + 1 > 2\kappa\left(\frac{t_1 + u}{2\pi}\right) - 2\kappa\left(\frac{t_1}{2\pi}\right)$$

for $0 \le u \le 10/3$. Hence we have

$$|S_1(t_2) - S_1(t_1)| > \int_0^{10/3} \left|2\kappa\left(\frac{t_1 + u}{2\pi}\right) - 2\kappa\left(\frac{t_1}{2\pi}\right)\right| du. \tag{6.9}$$

Thus (6.2) yields

$$|S_1(t_2) - S_1(t_1)| \ge \int_0^{10/3} \frac{u}{2\pi}\left(\log\frac{t_1}{2\pi} - 1\right) du - 3.5\left(\frac{0.006}{t_1}\right)$$
$$\ge 0.884\log\left(\frac{t_1}{2\pi}\right) - 0.886. \tag{6.10}$$

Now $t_1 \ge 168\pi$ gives $\log(t_1/(2\pi)) \ge 4.4$ and $\log(t_1/(2\pi)) \ge \log(t_2/(2\pi)) - 0.01$. Hence (6.8) and (6.10) gives

$$|S_1(t_2) - S_1(t_1)| \ge 0.884\log\left(\frac{t_2}{2\pi}\right) - 0.886 \tag{6.11}$$

under the assumptions of Lemma 6. This contradicts (6.3). We complete the proof. $\square$

7   A question

Equality (2.11) can be written as

$$\xi_3(s) = \xi_3^\natural(s) + \frac{\pi - 3}{2}\xi(3s - 1), \tag{7.1}$$

where $\xi_3^\natural(s) = X(s) + X(1 - s)$. This yields

$$\frac{\pi - 3}{2}\xi(3s - 1) = \xi_3(s) - \xi_3^\natural(s). \tag{7.2}$$

As this, the Riemann xi function $\xi(s)$ can be written as the sum of $\xi_3(s)$ and $\xi_3^\natural(s)$. Equality (7.2) was first pointed out by Weng. We have already known that all zeros of $\xi_3(s)$ and $\xi_3^\natural(s)$ lie on the line $\Re(s) = 1/2$. Therefore, equality (7.2) may be a remarkable relation. We conclude the article by the following natural question:

Can one say some nice things for the zeros of $\xi(s)$ by using the properties of $\xi_3(s)$ and $\xi_3^\natural(s)$ via relation (7.2) ?

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