A new type of combinatorics in knot theory

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Outline

Motivating questions:

- What does the Homfly polynomial "measure"?
- What does Floer homology "look like"?

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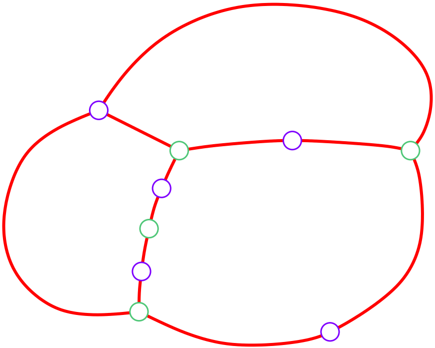
- What does the Homfly polynomial "measure"?
- What does Floer homology "look like"?

We will do a case study on special alternating links and their Seifert surfaces.

We will discuss:

- A combinatorial theory (W. Tutte, A. Postnikov, K)
- Its relation to the Homfly polynomial (joint with H. Murakami)
- Its relation to sutured Floer homology (joint with A. Juhász and J. Rasmussen).

Plane bipartite graphs



They occur in sets of three, i.e., in *trinities*.

G_R : red edges
emerald and violet pts

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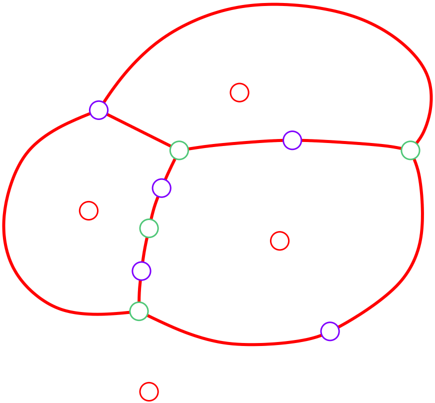

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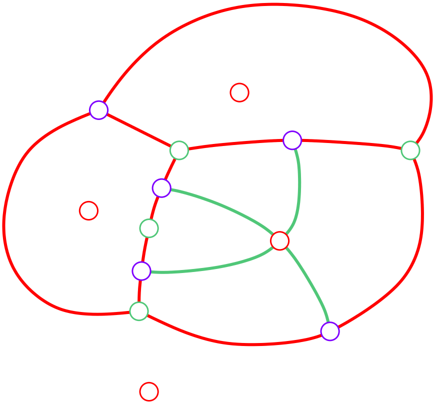

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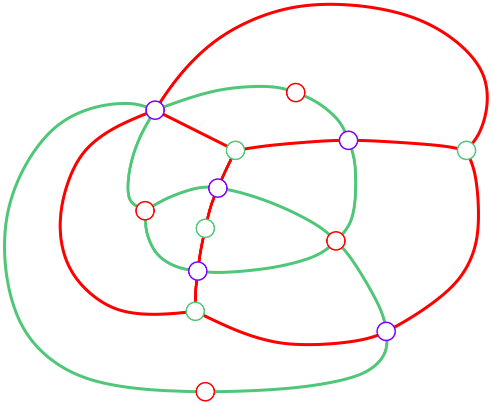

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G_E : emerald edges
violet and red pts

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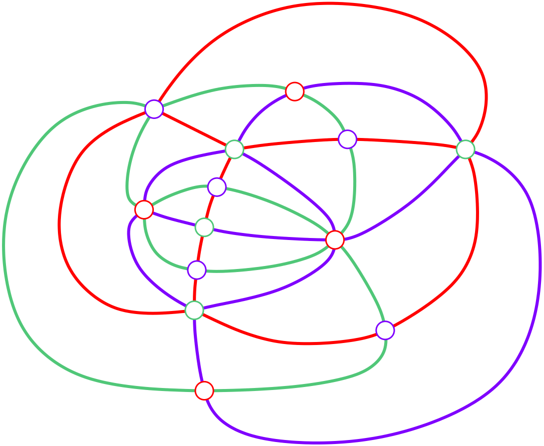

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G_V : violet edges
red and emerald pts

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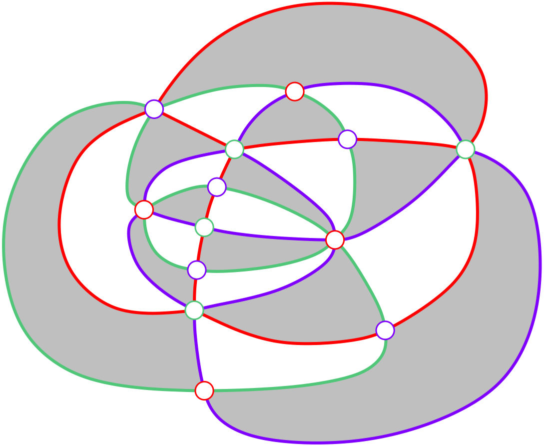

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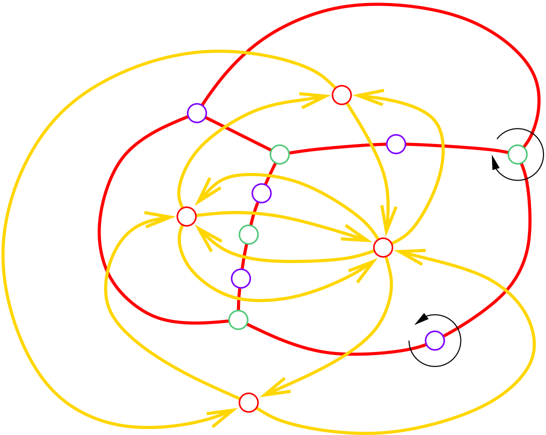
G_R :	red edges
	emerald and violet pts
G_E :	emerald edges
	violet and red pts
G_V :	violet edges
	red and emerald pts

Together they form a triangulation of S^2 with a black/white coloring. Red, emerald, and violet play symmetric roles.

Arborescences



G: plane bipartite graph

G^*: balanced directed graph

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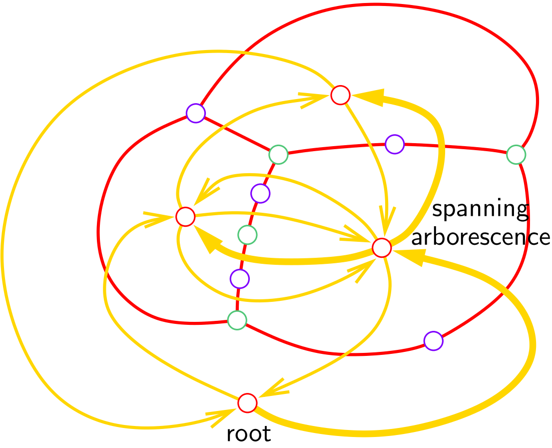


G: plane bipartite graph

G^*: balanced directed graph

Tutte showed:

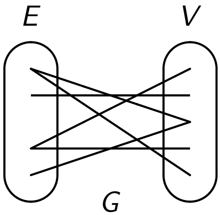
1. The number of spanning arborescences in such a graph is independent of root:

$$G^* \mapsto \text{arborescence number } \rho(G^*).$$

2. In a trinity,

$$\rho(G_R^*) = \rho(G_E^*) = \rho(G_V^*).$$

Root polytope

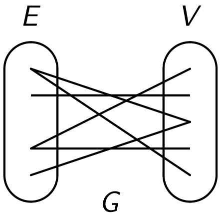


bipartite graph $G \quad \mapsto \quad$ root polytope

$$Q_G = \mathrm{Conv}\{\,\mathbf{e}+\mathbf{v} \mid ev \text{ is an edge in } G\,\} \subset \mathbf{R}^E \oplus \mathbf{R}^V.$$

$$\begin{aligned} \text{edge in } G \quad &= \quad \text{vertex in } Q_G \\ \text{spanning tree in } G \quad &= \quad \text{maximal simplex in } Q_G \end{aligned}$$

Root polytope



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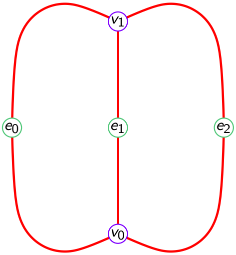
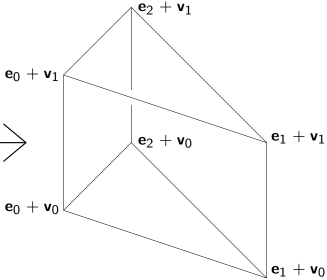
spanning tree in G = maximal simplex in Q_G

Example:



$\longmapsto$



Triangulating Q_G

Proposition

If we fix a root in G^* and consider all spanning arborescences, then the simplices corresponding to their dual trees triangulate Q_G.

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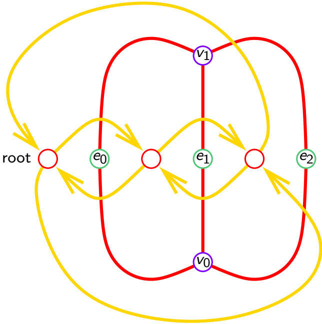
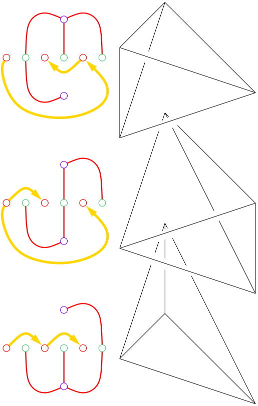
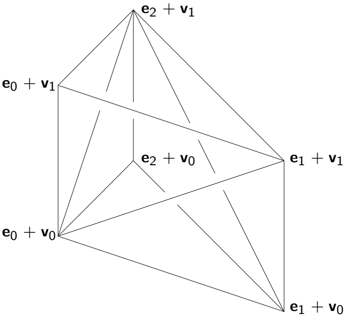
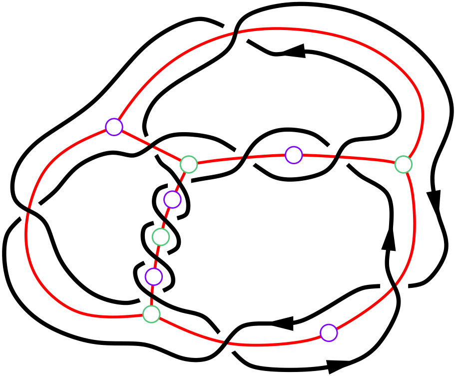






Special alternating links



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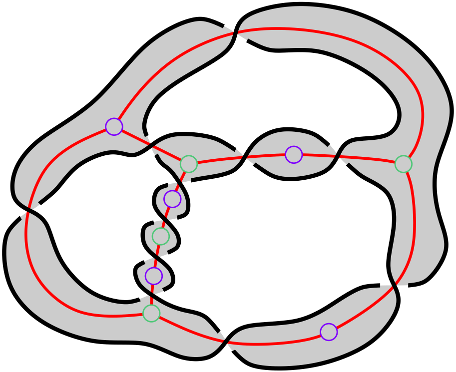

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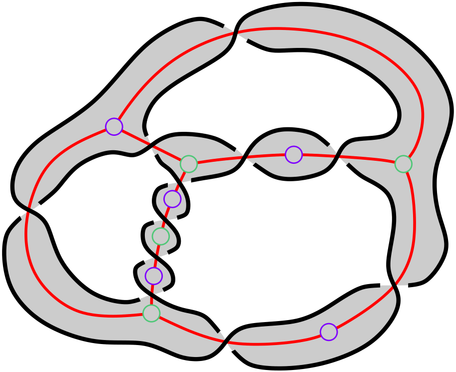

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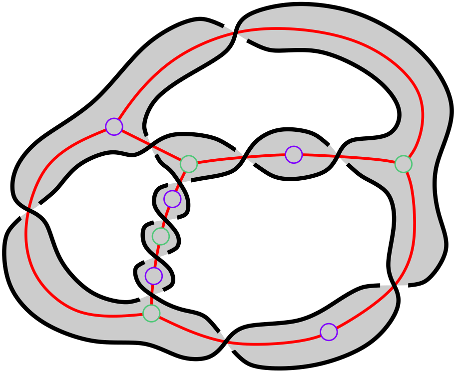

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Thus, $\Delta_{L_{G_R}}$, $\Delta_{L_{G_E}}$, and $\Delta_{L_{G_V}}$ share the same leading coefficient ρ.

Homfly polynomial

$$\text{Defined by: } v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0} \quad (P_{L_+} = v^2 P_{L_-} + vzP_{L_0})$$
$$P_{\bigcirc} = 1.$$

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The part that projects onto the leading term is called the *top* of $P(v,z)$.

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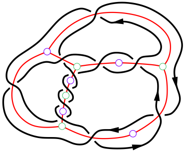

Example: the leading coefficient in ∇ and Δ is $\rho(G^*) = 11$, and the top of the Homfly polynomial is

$$(1 + 3v^2 + 4v^4 + 3v^6)v^3z^3.$$

Computation tree

Question: how is the top of P_{L_G} derived from G?

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Idea: build a computation tree $\mathcal{T}$ for P_{L_G} based on spanning arborescences of G^*.

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We fix a root and keep "poking" G in a simple systematic manner so that a backtrack algorithm finds all arborescences.

At the leaves of $\mathcal{T}$, what remains of G is either

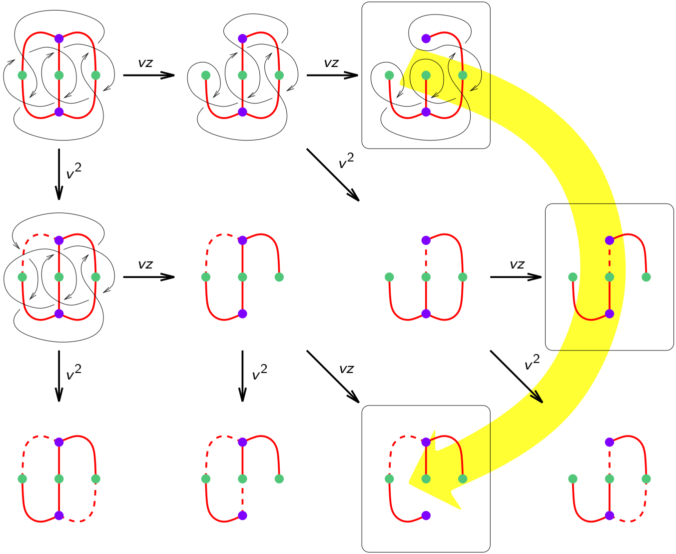
(a) a spanning tree or

(b) a graph with solid and dashed edges alternating along its outside contour.

Lemma

The graphs under (b) *do not contribute to the top of* P_{L_G}.

Computation tree example



Homfly polynomial and root polytope

The trees from $\mathcal{T}$ contribute one monomial each to the top, namely

$$(vz)^{\text{first Betti number of } G} \cdot v^{2(\text{number of dashed edges in the tree})}.$$

In the example, top of $P = (vz)^2(1 + 2v^2)$.

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Translating trees in G to simplices in Q_G, we get

Theorem (K–Murakami)

The computation tree $\mathcal{T}$ triangulates the root polytope Q_G. The trees/simplices appear in such an order that

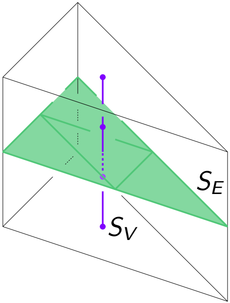
1. *each simplex intersects the union of the previous ones in a collection of codimension one faces (facets)*
2. *the number of such facets is the number of dashed edges in the tree.*

In other words, $\mathcal{T}$ induces a shelling order for the triangulation and the top of P_{L_G} is equivalent to the h-vector of the triangulation.

Floer homology and root polytope

The root polytope Q_G contains certain *SFH* polytopes, as follows.

$Q_G \subset \mathbf{R}^E \oplus \mathbf{R}^V$ projects onto the unit simplices $\Delta_E \subset \mathbf{R}^E$ and $\Delta_V \subset \mathbf{R}^V$. Let the two barycenters have the pre-images S_V and S_E. Then we have



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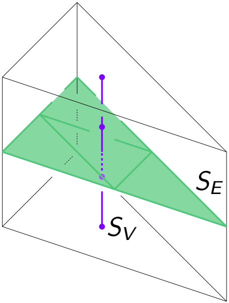


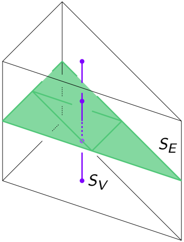
Theorem (Juhász–K–Rasmussen)

$$(|E|S_V - \Delta_V) \cap \mathbf{Z}^V \cong \chi\left(SFH(S^3 \setminus F_{G_V}, L_{G_V})\right) \quad \text{and}$$
$$(|V|S_E - \Delta_E) \cap \mathbf{Z}^E \cong \chi\left(SFH(S^3 \setminus F_{G_E}, L_{G_E})\right),$$

where the lhs's involve Minkowski differences and the rhs's are Euler characteristics, with respect to the Maslov grading, of sutured Floer homology groups.

In this case, Spin^c-structures form a lattice and χ is a 0-1–valued function so it can be identified with a set of lattice points.

Triangulations and cross-sections



The Spin^c-structures supporting Floer homology are identified with certain copies of $\frac{1}{|E|}\Delta_V$ in S_V and of $\frac{1}{|V|}\Delta_E$ in S_E. Call these the small simplices.

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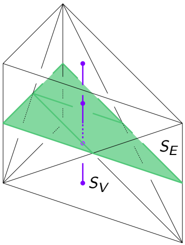

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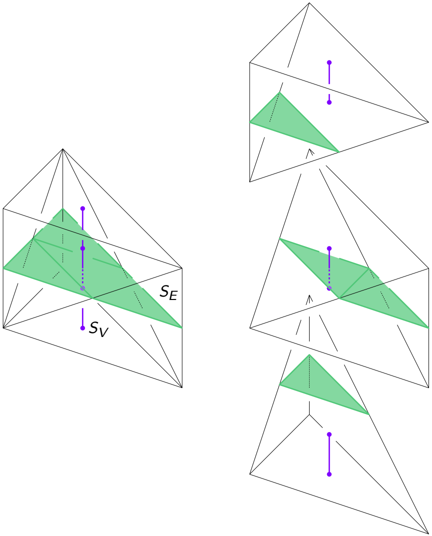

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So the number of small simplices of each color is $\rho(G^*)$. But can we read off the h-vector from the cross-section, i.e., can we get information on the Homfly polynomial from Floer homology?

Conjecture

Yes we can, by a construction called the interior polynomial.