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Stable maps of surfaces into the plane

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Abstract

In this paper we investigate $\Sigma^{1,0}$-maps of closed surfaces into the plane, specifically, the singular sets of such maps. This set is the disjoint union of finitely many embedded circles in the surface; we will determine all possible numbers of components for each surface. During this survey we will construct singular maps of all closed surfaces into the plane which are simplest in the sense that they have the least possible number of cusps (0 or 1) and under this condition their singular sets have the least possible number of components (1 or 2). Additionally, we will provide a simplified and shortened proof of the dimension 2 case of the theorem concerning the elimination of cusps (due to Millett, and Levine for the higher-dimensional cases). © 2000 Elsevier Science B.V. All rights reserved.

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1. Obstructions

Any stable map f of an n-manifold M ($n \geqslant 2$) into a 2-manifold can have fold singularities (constituting embedded arcs in M) and isolated cusp singularities (see [10]). Their sets are denoted by $\Sigma^{1,0}(f)$ and $\Sigma^{1,1}(f)$, respectively and the union of these by $S(f)$. Fold maps or $\Sigma^{1,0}$-maps are those without cusps. The next theorem of Thom [8] is well known.

Theorem 1.1. *Let M be a closed n-dimensional manifold ($n \geqslant 2$), N an orientable surface and $f : M \to N$ a stable map. Then*

$$\left|\Sigma^{1,1}(f)\right| \equiv \chi(M) \pmod 2.$$

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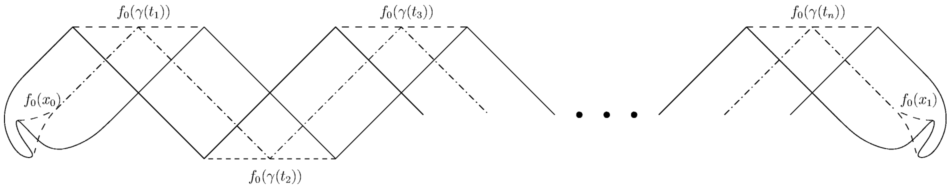

Fig. 1. $f_0(N(\gamma))$ intersects $f_0(S(f_0))$ in $n+2$ disjoint segments.

Corollary 1.2. *No closed surface with odd Euler characteristic can be mapped into the plane without cusps.*

As we will soon see, fold maps of surfaces with even Euler characteristic into the plane do exist for all such surfaces (see also Propositions 2.1 and 2.4). The next theorem shows that there is no further restriction on the number of cusps than the one given in Theorem 1.1.

Theorem 1.3. *Any continuous map $f: M^n \to N^2$, with M a connected closed manifold of dimension $n \geqslant 2$ and N an orientable surface, is homotopic to a stable map with at most* 1 *cusp* (1, *if $\chi(M)$ is odd and* 0 *otherwise*).

This is a result of Levine [4] for $n \geqslant 3$ and was shown by Èliašberg [1] and Millett [7] for $n = 2$. This latter case is obviously a consequence of the lemma below, stating that a pair of cusps can always be eliminated by a homotopy.

Lemma 1.4. *Let F and S be arbitrary surfaces, $f_0: F \to S$ a stable map, $x_0, x_1 \in \Sigma^{1,1}(f_0)$ different points and $\gamma: [0,1] \to F$ an embedded curve transversal to $S(f_0)$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ but $\gamma((0,1)) \cap \Sigma^{1,1}(f_0) = \emptyset$. Assume that $T_{x_0} f_0(\gamma'(0))$ and $-T_{x_1} f_0(\gamma'(1))$ point to the same region as the respective cusps themselves. Let $N(\gamma)$ be a tubular neighborhood of the image of γ in F. Then there exists a homotopy $H: F \times [0,1] \to S$ such that $H|_{(F \setminus N(\gamma)) \times [0,1]} = f_0 \circ \mathrm{Pr}_1$ (i.e., we only change f_0 in $N(\gamma)$) and the map $f_1 = H|_{F \times \{1\}}$ has no cusps in $N(\gamma) \times \{1\}$.*

Proof. Of course if F is connected and x_0 and x_1 are given then there is always an arc γ with the above properties; we will call such an arc suitable. Let $0 < t_1 < t_2 < \cdots < t_n < 1$ be the γ^{-1}-images of the points in $\gamma((0,1)) \cap S(f_0)$. The image of $N(\gamma)$ can be seen on Fig. 1.

Apply the operation visualized on Fig. 2 to this n times bent band. Thus we obtain a map $f_{1/3}$ homotopic to f_0 which no longer has the "original" cusps and which maps a neighborhood $U_i \subset N(\gamma)$ of $\gamma(t_i)$ as it can be seen on Fig. 3 (imagine the right hand part of Fig. 2 fold in two). This map $U_i \to S$ can be approximated by a stable map (see Fig. 3) in such a way that the two coincide in a neighborhood of ∂U_i.

The latter has two cusps, but these two can be joined by a suitable arc in U_i consisting only of regular points, so they can be eliminated with no new cusps emerging. □

Recall the following theorem of Hopf [2,3]:

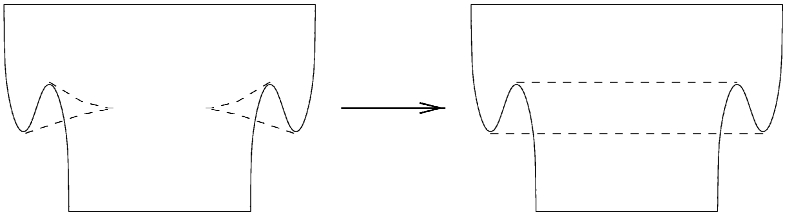

Fig. 2. Elimination of two cusps.

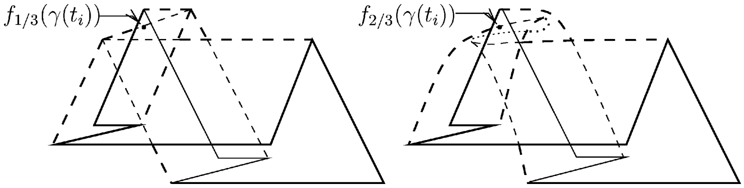

Fig. 3. The change of the map in a neighborhood of an intersection point of $\mathrm{im}(\gamma)$ and the singular set.

Lemma 1.5. *Let M be a $2k$-dimensional compact manifold with boundary, $g : M \to \mathbb{R}^{2k}$ an immersion and $\nu : \partial M \to S^{2k-1}$ the normal map of the immersion $g|_{\partial M}$ (for $x \in \partial M$ let $\nu(x)$ denote the outward pointing unit normal vector to $T_x g(T_x \partial M)$). Orient M by pulling back the orientation of $\mathbb{R}^{2k}$ and set an orientation for ∂M by the outward normal first convention. Then* $\deg(\nu) = \chi(M)$.

Remark 1.6. We have $\deg(\nu) = \deg(-\nu)$ since the antipodal map $S^{2k-1} \to S^{2k-1}$ has degree 1. We assume throughout that there are fixed orientations for the spaces $\mathbb{R}^n$ and thus for $D^n \subset \mathbb{R}^n$ and $S^{n-1} = \partial D^n$.

Proposition 1.7. *Let F be an orientable closed surface and $f : F \to \mathbb{R}^2$ a fold map. Then the number of components of $S(f)$ is of the same parity as $\frac{1}{2}\chi(F)$.*

Proof. Set an orientation for F arbitrarily. Let $F_+(f)$ and $F_-(f)$ denote the subsets of $F \setminus S(f)$ in the points of which f is orientation preserving and reversing, respectively. Let $N(S(f))$ be a closed tubular neighborhood of $S(f)$ and

$$F'_+ := \overline{F_+(f) \setminus N(S(f))}, \quad F'_- := \overline{F_-(f) \setminus N(S(f))}.$$

$S(f)$ is contained in the closure of both $F_+(f)$ and $F_-(f)$, i.e., $\partial\overline{F_+(f)} = \partial\overline{F_-(f)} = S(f)$. This yields that the embedding $S(f) \subset F$ has trivial normal bundle and as a consequence $\partial F'_+$ and $\partial F'_-$ are diffeomorphic by the map ι taking each point of $\partial F'_+$ to the point in the same fibre of $\nu(S(f))$. It is also clear that the immersions $f|_{\partial F'_+}$ and $(f|_{\partial F'_-}) \circ \iota$ are regularly homotopic and that

$$F'_+ \cong \overline{F_+(f)} \quad \text{and} \quad F'_- \cong \overline{F_-(f)}.$$

Applying Lemma 1.5 we have $\chi(F'_+) = \chi(F'_-)$ as both sides coincide with the degree of the normal map of $f|_{\partial F'_+}$. This means that $\chi(\overline{F_+(f)}) = \chi(\overline{F_-(f)})$ and denoting the common value by χ we have $\chi(F) = 2\chi - \chi(S(f)) = 2\chi$. On the other hand, attaching disks to the boundary components of $\overline{F_+(f)}$ we obtain an orientable surface (thus one with even Euler characteristic). This proves that $\chi(\overline{F_+(f)}) = \chi = \frac{1}{2}\chi(F)$ and the number of components of $S(f)$ have the same parity. □

Remark 1.8. Proposition 1.7 fails in the nonorientable case; see Proposition 2.4.

2. Constructions

In the following we show that Proposition 1.7 gives all restrictions to the number of components of the singular set of a fold map of a closed orientable surface into the plane (together with the easy fact that there is no stable map of that type without fold singularities).

Proposition 2.1. *Let F be an orientable closed surface and k a positive integer such that $k \equiv \frac{1}{2}\chi(F) \pmod 2$. Then there exists a fold map $f : F \to \mathbb{R}^2$ with its singular set being the union of k disjoint circles.*

Proof. Let us denote by A_t the orientable closed surface of genus t and by A_t^0 the surface with boundary obtained by deleting the interior of a disk from A_t.

First we show that for $t = 2m$ the surface A_t can be mapped into the plane with only fold singularities and a connected singular set.

Let $\varphi : A_m^0 \to \mathbb{R}^2$ be an immersion (see Fig. 4) and $\psi : A_t \to A_t$ an involution with $A_t/\psi \cong A_m^0$ (identifying A_t by the the surface on Fig. 5, which is embedded in $\mathbb{R}^3$, ψ can be chosen to a reflection through the plane indicated). With $\pi : A_t \to A_t/\psi$ the factorization, $\vartheta_m := \varphi \circ \iota \circ \pi$ is a map satisfying the conditions, where $\iota : A_t/\psi \to A_m^0$ is a diffeomorphism.

For $t = 2m + 1$ we have $A_t \cong A_{2m}\#A_1$ and one can easily construct a fold map $\tilde{\vartheta}_m : A_t \to \mathbb{R}^2$ with a two-component singular set (see Fig. 4).

Finally, it is not difficult to define a homotopy that increases the number of fold components of any stable map $A_t \to \mathbb{R}^2$ by adding a pair of concentric circles around an arbitrary regular point. □

As we have already mentioned, the parity of the number of components is not determined in the nonorientable case. To illustrate this by a pair of examples first we construct two maps of the projective plane $\mathbb{R}P^2$ into $\mathbb{R}^2$.

Proposition 2.2. *There exist stable maps $\varphi, \psi : \mathbb{R}P^2 \to \mathbb{R}^2$ with one cusp each and a one-component and a two-component singular set, respectively.*

Proof. First we construct ψ. Let us embed a Möbius band into $\mathbb{R}^3$ as on Fig. 6 and compose the embedding with the orthogonal projection onto the plane of the figure.

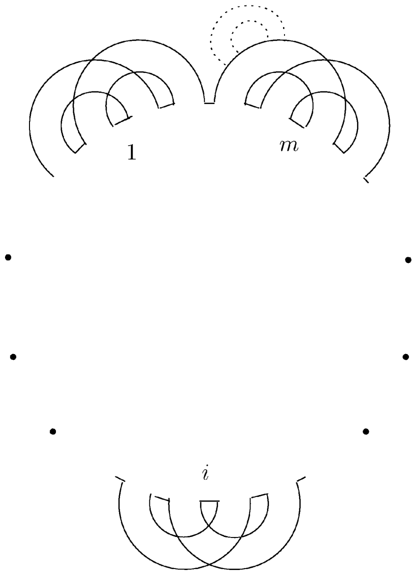

Fig. 4. The set $\varphi(A_m^0) = \vartheta_m(A_{2m})$; adding the strip bounded by the dotted lines we obtain $\tilde{\vartheta}_m(A_{2m+1})$.

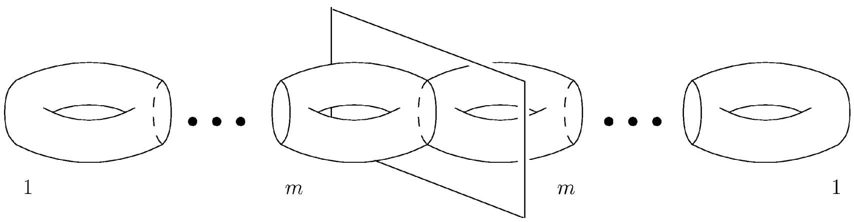

Fig. 5. A_{2m}.

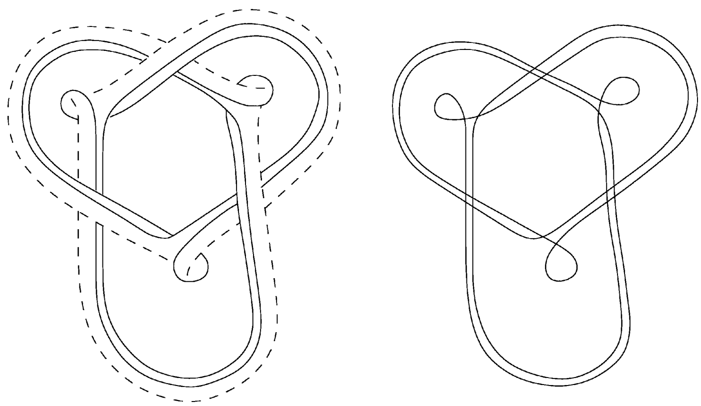

Fig. 6. The embedding of the Möbius band into $\mathbb{R}^3$ and the image of its boundary after the projection.

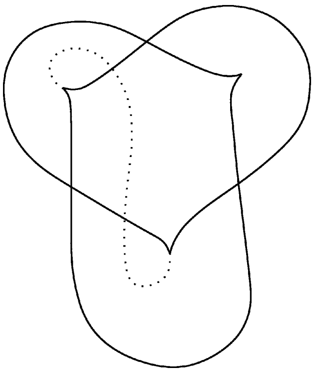

Fig. 7. The set $\psi_0(S(\psi_0))$; eliminating two cusps along the dotted arc we obtain ψ, the image of the singular set of which can be seen on the right-hand side.

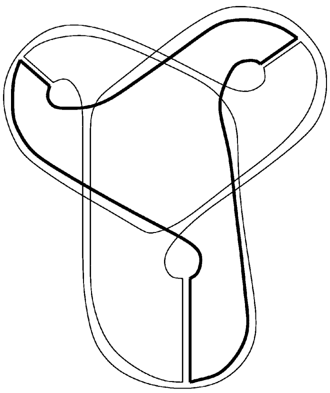

Fig. 8. Two extensions of the curve on Fig. 6 to immersions of D^2.

This maps the boundary of the band to the curve on the right hand side of Fig. 6 and the image of the singular set can be seen on the left-hand side of Fig. 7.

The image of the boundary can be obtained as the boundary of an immersed disc D^2 in two different ways (see Fig. 8); choosing any of these possibilities (say the one on the right) a map $\psi_0 : \mathbb{R}P^2 \to \mathbb{R}^2$ can be obtained with three cusps and a connected singular set. Join now two of the cusps by an arc in D^2 and eliminate them (see Fig. 7), obtaining a map ψ with one cusp and a two-component singular set. The definition of ψ_0 is taken from Levine [6, pp. 155–156].

The construction of φ is similar and can be found entirely at Millett [7] so we only sketch it for completeness. Above we mapped a band into the plane following the projection of the trefoil knot and used three twists to make it Möbius-type; now take the Milnor curve and only one twist (Fig. 9). The singular set is again the center circle of the Möbius band and the image of it is the Milnor curve with a cusp on it. The boundary of the band is mapped

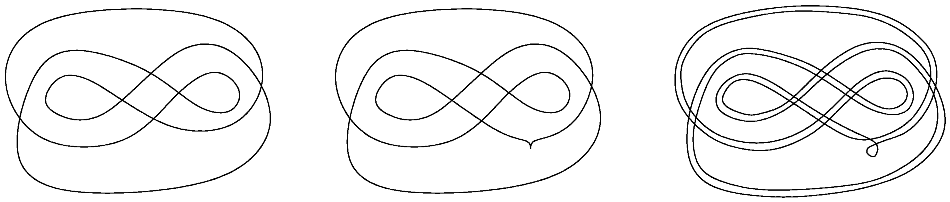

Fig. 9. Milnor curve; $\varphi(S(\varphi))$; the φ-image of the common boundary of the Möbius band and the disk, the union of which is $\mathbb{R}P^2$.

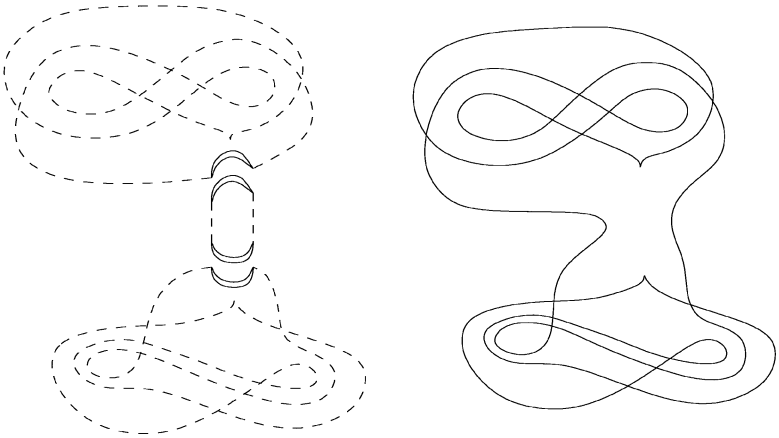

Fig. 10. Stable map of the Klein bottle into the plane with a two-component singular set and one cusp on each component.

to the curve on the right side of Fig. 9 which also turns out to be the the image of ∂D^2 under an immersion $D^2 \to \mathbb{R}^2$. Thus $S(\varphi)$ is connected. □

Let us denote by $\widehat{A}_t$ the nonorientable closed surface with genus t and by $\widehat{A}_t^0$ the surface with boundary obtained by deleting an open disk from it.

Proposition 2.3. *There exist fold maps $f, g : \widehat{A}_2 \to \mathbb{R}^2$ with $S(f)$ connected and $S(g)$ being the union of two circles.*

Proof. It is easy to find a map g as above, see, for example, [6, p. 153]. In the construction of f we will use the maps φ and ψ defined in the proof of Proposition 2.2. Consider $\widehat{A}_2$ as the union of two disjoint samples of $\widehat{A}_1^0$ and a cylinder, attached along their boundaries. Map these into $\mathbb{R}^2$ as on Fig. 10, i.e., the Möbius bands by restrictions of φ and ψ and the cylinder by a projection. Attaching these maps along boundaries we obtain a stable map $\tilde{f} : \widehat{A}_2 \to \mathbb{R}^2$ and one can do it such a way that the two cusps can be joined by an arc through regular points of $\tilde{f}$. One can check it easily that $\tilde{f}$ has a two-component singular set with one cusp on each component. So after eliminating these along the above mentioned suitable arc the resulting map will have no cusps and only one fold component. □

Proposition 2.4. *The nonorientable closed surface $\widehat{A}_{2m}$ admits a fold map into the plane with a k-component singular set for all positive integers k.*

Proof. We prove the cases $k=1$ and $k=2$ by induction on m. For $m=1$ this is just the statement of Proposition 2.3. Assume that we have the proposition for m. Map $\widehat{A}_{2m}$ into the plane with a connected singular set and $\widehat{A}_2$ by f or g of Proposition 2.3; attaching these two maps the same way as in the previous proof we obtain a fold map of $\widehat{A}_{2m}\#\widehat{A}_2=\widehat{A}_{2m+2}$ into $\mathbb{R}^2$ with one or two components in its singular set. We conclude with the same remark as at the end of the proof of Proposition 2.1. □

Summarizing our results we state the next theorem:

Theorem 2.5. *Let F be a closed surface. If $\chi(F)$ is odd then there is no $\Sigma^{1,0}$-map of F into the plane. For $\chi(F)$ even we have exactly the following possibilities.*

(1) *If F is orientable then the singular set of any fold map $F\to\mathbb{R}^2$ has a number of components of the same parity as $\frac{1}{2}\chi(F)$ and all such positive integers occur.*
(2) *If F is nonorientable then all positive integers occur as the number of components of $S(f)$ for a $\Sigma^{1,0}$-map $f:F\to\mathbb{R}^2$.*

Remark 2.6. Using Propositions 2.2 and 2.4 one can prove easily that any closed surface with odd Euler characteristic can be mapped into the plane with a single cusp and an arbitrary number of components in the singular set (e.g., 1).

The map f constructed in the proof of Proposition 2.3 can be modified by a homotopy to obtain the symmetric curve on the right hand side of Fig. 11 as the image of the singular set. On Fig. 12 two other fold maps $\widehat{A}_2\to\mathbb{R}^2$ are shown (with connected singular sets). The left is obtained by composing ψ (see Proposition 2.2) by a reflection before attaching to φ and the right by eliminating two cusps of ψ_0 along a proper arc not in the disc as before, but in its complementer.

If an immersion $S^1\to\mathbb{R}^2$ can be obtained as $f|_{S(f)}$ for some fold map $f:\widehat{A}_2\to\mathbb{R}^2$, then the following theorem of Levine [5] (which we state only for our case) and the Whitney–Graustein Theorem [9] yield that it must be regularly homotopic to the figure eight immersion (and hence it must have an odd number of double points).

Theorem 2.7. *Let F be a closed surface and $f:F\to\mathbb{R}^2$ a stable map with c a component of $S(f)$. Orient $c\setminus\Sigma^{1,1}(f)$ such a way that the f-image of its neighborhood lies always on the left of $f(c)$. Define $k_c:c\to S^1$ as the composition of the normal map of $f|_c$ and the map $(\cos\vartheta,\sin\vartheta)\mapsto(\cos 2\vartheta,\sin 2\vartheta)$ on fold points of c and extend it continuously to c. Then*

$$\chi(M)=\sum_{c:\ c\text{ a component of }S(f)}\deg(k_c).$$

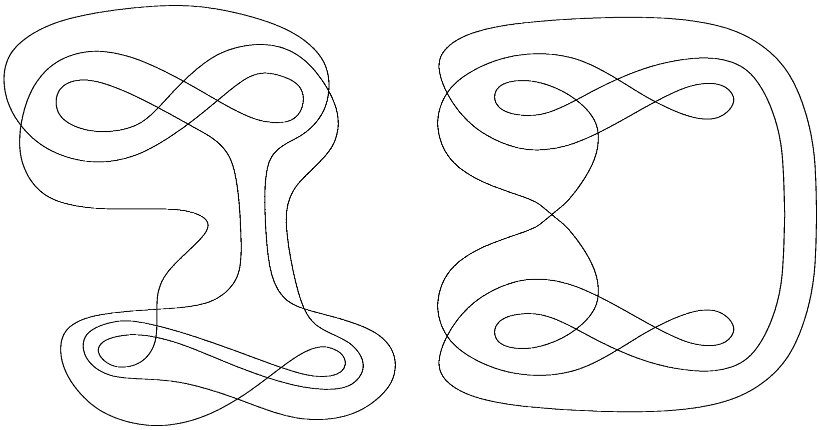

Fig. 11. Fold maps of the Klein bottle into the plane with connected singular sets.

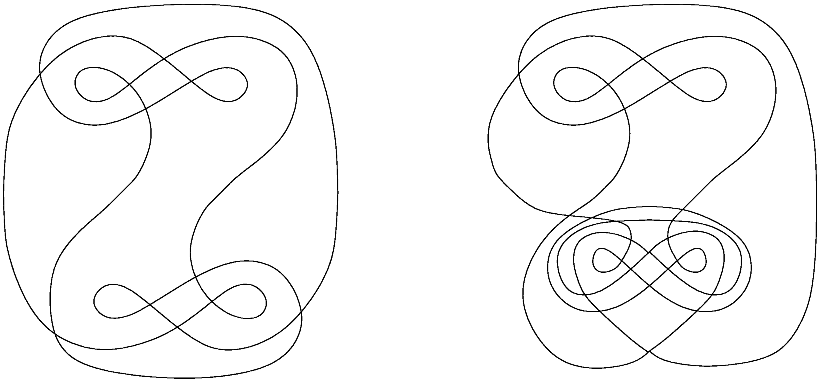

Fig. 12. Two additional $\Sigma^{1,0}$-maps of the Klein bottle into the plane with connected singular sets.

Finally we recall that for any stable map $h:\widehat{A}_2 \to \mathbb{R}^2$ the singular set $S(h)$ represents $w_1(\widehat{A}_2) \in H^1(\widehat{A}_2; \mathbb{Z}_2)$ (of course this is true for all other surfaces as well). In fact, for the map f of Proposition 2.3 $S(f)$ is just the meridian of the Klein bottle.

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