Eigenvalues of elliptic operators and geometric applications

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Abstract

The purpose of this talk is to present a certain method of obtaining upper estimates of eigenvalues of Schrödinger type operators on Riemannian manifolds, which was introduced in the paper

Grigor'yan A., Netrusov Yu., Yau S.-T. *Eigenvalues of elliptic operators and geometric applications*, Surveys in Differential Geometry **IX** (2004), pp.147-218.

The core of the method is the construction that allows to choose a given number of disjoint sets on a manifold such that one can control simultaneously their volumes from below and their capacities from above. The main technical tool for that is the following theorem.

Theorem 1 *Let (X, d) be a metric space satisfying the following covering property: there exists a constant N such that any metric ball of radius r in X can be covered by at most N balls of radii $r/2$. Let all metric balls in X be precompact sets, and let ν be a non-atomic Radon measure on X. Then, for any positive integer k, there exists a sequence $\{A_i\}_{i=1}^k$ of k annuli in X such that the annuli $\{2A_i\}_{i=1}^k$ are disjoint and, for any $i = 1, 2, ..., k$,*

$$\nu(A_i) \geq c\frac{\nu(X)}{k},$$

where $c = c(N) > 0$.

The following results are obtained using this theorem.

Theorem 2 *Let X be a complete Riemannian manifold and μ be its Riemannian measure. Assume that, for some constants N and M, the following is true:*

- (i) *any geodesic ball $B(x, r)$ in X can be covered by at most N balls of radii $r/2$;*
- (ii) *for all $x \in X$ and $r > 0$, we have $\mu(B(x, r)) \leq Mr^2$.*

Then, for any $q \in C(X)$, the number of negative eigenvalues of the operator $-\Delta - q$ admits the estimate

$$Neg(-\Delta - q) \geq c\int_X (\delta q_+ - q_-)\, d\mu, \tag{1}$$

where $\delta = \delta(N) \in (0, 1)$, and $c = c(N, M) > 0$.

For example, this result applies for Schrödinger operators in $\mathbb{R}^2$. However, in $\mathbb{R}^n$, $n > 2$, the estimate (1) fails. It is conjectured that (1) holds with $\delta = 1$ so that $(\delta q_+ - q_-)$ can be replaced by q.

Theorem 3 *Let X be a compact oriented Riemann surface of genus g, equipped with some Riemannian metric. Then, for any $q \in C(X)$,and, for any $k = 1, 2, ...,$*

$$\lambda_k(-\Delta - q) \leq \frac{C(g+1)(k-1) - \int_X (\delta q_+ - q_-)\, d\mu}{\sqrt{\delta}\mu(X)}.$$

where $C > 0$ and $0 < \delta < 1$ are absolute constants.

Theorem 4 *For any connected complete oriented minimal surface X embedded in $\mathbb{R}^3$, we have*

$$\text{ind}(X) \geq c\sqrt{K_{total}(X)},$$

where $\text{ind}(X)$ *is the stability index of X, $K_{total}(X)$ is the absolute value of the total curvature of X, and $c > 0$ is an absolute constant.*