The final problem on the optimality of the general theory for nonlinear wave equations and recent topics[1]

Hiroyuki Takamura

Department of Complex and Intelligent Systems, Faculty of Systems Information Science, Future University Hakodate,
116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan.
e-mail: takamura@fun.ac.jp

We are interested in the following theorem.

Theorem 1 ([1], [2]) The lifespan $T(\varepsilon)$, the maximal existence time, of a classical solution of

$$\begin{cases} u_{tt} - \Delta u = u^2 \quad \text{in} \quad \mathbf{R}^4 \times [0, \infty), \\ u(x,0) = \varepsilon f(x), u_t(x,0) = \varepsilon g(x) \end{cases} \tag{1}$$

of a small parameter $\varepsilon > 0$, compactly supported smooth functions f and g, has an estimate

$$\exp\left(c\varepsilon^{-2}\right) \le T(\varepsilon) \le \exp\left(C\varepsilon^{-2}\right), \tag{2}$$

where c and C are positive constants depending only on f and g.

This result is due to Li and Zhou [1] for the lower bound and to Takamura and Wakasa [2] for the upper bound. We note that its importance is extremely huge as the problem is related to the final open part of Strauss' conjecture on semilinear wave equations as well as one of the last open optimality of the general theory for nonlinear wave equations.

In this talk, I would like to present the blow-up part above, the upper bound of the lifespan. Moreover, I would like to introduce you the following related theorems.

Theorem 2 ([3]) (2) still holds even if u^2 in (1) is replaced by

$$\begin{aligned} u(x,t)^2 \quad & -\frac{1}{\pi^2}\int_0^t d\tau \int_{|\xi|\le 1} \frac{(u_t u)(x+(t-\tau)\xi, \tau)}{\sqrt{1-|\xi|^2}} d\xi \\ & -\frac{\varepsilon^2}{2\pi^2}\int_{|\xi|\le 1} \frac{f(x+t\xi)^2}{\sqrt{1-|\xi|^2}} d\xi. \end{aligned} \tag{3}$$

Theorem 3 ([4]) $T(\varepsilon) = \infty$ holds if u^2 in (1) is replaced by

$$\begin{aligned} u(x,t)^2 \quad & -\frac{1}{2\pi^2}\int_0^t d\tau \int_{|\omega|=1} (u_t u)(x+(t-\tau)\omega, \tau) dS_\omega \\ & -\frac{\varepsilon}{4\pi^2}\int_{|\omega|=1} (\varepsilon f^2 + \Delta f + 2\omega\cdot\nabla g)(x+t\omega) dS_\omega. \end{aligned} \tag{4}$$

[1]All the results in this talk are based on joint works with Kyouhei Wakasa (DC2, Hokkaido Univ.), which are partially supported by the Grant-in-Aid for Scientific Research (C)(No.24540183), Japan Society for the Promotion of Science.

Both non-local nonlinear terms in (3) and (4) are due to “derivative loss” in high dimensions. We are looking for a criterion to get the global in time existence of a solution, $T(\varepsilon) = \infty$, for more general terms of the critical power in four space dimensions. Except for this situation, there is no possibility to discuss this kind of problems for “classical” solutions in high dimensions.

References

[1] T-T.Li and Y.Zhou, *A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions*, Indiana Univ. Math. J., **44**, 1207-1248 (1995).

[2] H.Takamura and K.Wakasa, *The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions*, J. Differential Equations **251**, 1157-1171 (2011).

[3] H.Takamura and K.Wakasa, *Almost global solutions of semilinear wave equations with the critical exponent in high dimensions*, arXiv:1404.4471v1 [math.AP] 17 Apr.2014, to appear in Nonlinear Analysis, TMA.

[4] H.Takamura and K.Wakasa, *Global existence for semilinear wave equations with the critical blow-up term in high dimensions*, arXiv:1405.0120v1 [math.AP] 1 May 2014.