Morse-Floer theory for superquadratic Dirac equations, I: relative Morse indices and compactness

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Abstract

In this paper and its sequel [37], we study Morse-Floer theory for superquadratic Dirac functionals associated with a class of nonlinear Dirac equations on compact spin manifolds. We are interested in two topics: (i) relative Morse indices and its relation to compactness issues of critical points, (ii) construction and computation of the Morse-Floer homology and its application to the existence problem for solutions to nonlinear Dirac equations. In this part I, we focus on the topic (i). One of our main results is a compactness of critical points under the boundedness assumption of their relative Morse indices which is an analogue of the results of Bahri-Lions [16] and Angenent-van der Vorst [13] for Dirac functionals. To prove this, we give an appropriate definition of relative Morse indices for bounded solutions to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$. We show that for $m \geq 3$ and $1 < p < \frac{m+1}{m-1}$, the relative Morse index of any non-trivial bounded solution to that equation is $+\infty$. We also give some properties of the relative Morse indices of Dirac functionals which will be useful in the study of the topic (ii) above.

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1 Introduction

Let (M, g, σ) be an m-dimensional compact spin manifold, where g is a Riemannian metric on M and σ is a spin structure on M. We denote by $\mathbb{S}(M) \to M$ the spinor bundle and $\mathsf{D}_g : C^\infty(M, \mathbb{S}(M)) \to C^\infty(M, \mathbb{S}(M))$ the (Atiyah-Singer-)Dirac operator. See §2 for notation and definitions of basic concepts which will be used throughout this paper. In this paper and its sequel [37], we are concerned with nonlinear Dirac equations of the following form

$$\mathsf{D}_g\psi = h(x, \psi) \quad \text{on } M, \tag{1.1}$$

where $h : \mathbb{S}(M) \to \mathbb{S}(M)$ is a fiber preserving nonlinear map. We consider the case where h has a potential, that is, there exists a smooth function H defined on $\mathbb{S}(M)$ such that $h(x, \psi) = \nabla_\psi H(x, \psi)$, where $\nabla_\psi H(x, \psi)$ is the vertical gradient of H with respect to the metric on $\mathbb{S}(M)$ and we write a general point on $\mathbb{S}(M)$ as (x, ψ), where ψ is a point on the fiber $\mathbb{S}(M)_x$ over $x \in M$ of the bundle $\mathbb{S}(M) \to M$. In such a case, the equation (1.1) has a variational structure: We define a functional $\mathcal{L}_H$ by

$$\mathcal{L}_H(\psi) = \frac{1}{2}\int_M \langle \psi, \mathsf{D}_g\psi\rangle \, d\mathrm{vol}_g - \int_M H(x, \psi)\, d\mathrm{vol}_g \tag{1.2}$$

for $\psi \in C^1(M, \mathbb{S}(M))$. Then, at least formally, $\psi \in C^1(M, \mathbb{S}(M))$ is a solution to (1.1) if and only if ψ is a critical point of $\mathcal{L}_H$, i.e., $d\mathcal{L}_H(\psi) = 0$.

The equations of the form (1.1) arise naturally in some problems in geometry and physics. In the following, we give two examples which motivate our study of the equation of the form (1.1). Although we will not mention the details here, there are many other important problems related to the equation (1.1): for example, in [23], [24] interesting problems which come from quantum physics are studied from a variational point of view. See [23], [24] and references therein for details about these problems.

Our first example is the supersymmetric Dirac-harmonic maps introduced in [19], [18]. We consider a Riemann surface (Σ, g, σ) with a metric g and a spin structure σ and a Riemannian manifold (N, h) with a metric h. In this model, we consider two basic fields, $\phi \in C^\infty(\Sigma, N)$ and $\psi \in C^\infty(\Sigma, \mathbb{S}(\Sigma) \otimes \phi^*TN)$. In physics, the pair (ϕ, ψ) describes a supersymmetric string moving on N. We consider the following action functional defined on the configuration manifold $\mathcal{F} := \{(\phi, \psi) : \phi \in C^\infty(\Sigma, N), \psi \in C^\infty(\Sigma, \mathbb{S}(\Sigma) \otimes \phi^*TN)\}$:

$$\begin{aligned}\mathcal{L}(\phi, \psi) = \frac{1}{2}\int_\Sigma |d\phi|^2 \, d\mathrm{vol}_g + \frac{1}{2}\int_\Sigma \langle \psi, \mathsf{D}_\phi \psi\rangle \, d\mathrm{vol}_g \\ - \frac{1}{12}\int_\Sigma R_{ikjl}(\phi)\langle \psi^i, \psi^j\rangle\langle \psi^k, \psi^l\rangle \, d\mathrm{vol}_g,\end{aligned} \tag{1.3}$$

where D_ϕ is the Dirac operator associated with a natural connection on $\mathbb{S}(\Sigma) \otimes \phi^*TN$, R is the curvature tensor of (N, h) and $\psi = \psi^k \otimes \frac{\partial}{\partial y^k}(\phi)$ is a local expression of ψ with respect to a local coordinate $y = (y^k)$ on N. The Euler-Lagrange equations of the action $\mathcal{L}$ take the following form

$$\tau^m(\phi) - \frac{1}{2}R^m_{lij}(\phi)\langle \psi^i, \nabla\phi^l \cdot \psi^j\rangle + \frac{1}{12}g^{mp}R_{ikjl;p}(\phi)\langle \psi^i, \psi^j\rangle\langle \psi^k, \psi^l\rangle = 0, \tag{1.4}$$

$$\mathsf{D}_\phi \psi^m = \frac{1}{3}R^m_{jkl}(\phi)\langle \psi^j, \psi^l\rangle \psi^k, \tag{1.5}$$

where $\tau(\phi) = \mathrm{tr}\nabla d\phi$ is the tension field of ϕ. The second equation (1.5) is a type of the equation (1.1). For the derivation of (1.4), (1.5), please refer to [19], [18].

The second example is the so called spinorial Yamabe equations, see [8], [9], [35], [36]. Consider a spin manifold (M, g, σ) as before. Let $H \in C^\infty(M)$ be a given function. We consider the following equation on M:

$$\mathsf{D}_g \psi = H(x)|\psi|^{\frac{2}{m-1}}\psi. \tag{1.6}$$

At least for the cases $m = 2, 3$, solutions to the equation (1.6) have interesting geometric meanings. For the case $m = 2$, the equation (1.6) arises from the spinorial Weierstrass representation of a conformally immersed surface $M \to \mathbb{R}^3$: a solution to (1.6) gives a conformal immersion $M \to \mathbb{R}^3$ with mean curvature H (see [8], [9] and reference therein for more details). For the case $m = 3$, the equation (1.6) is related to the Cauchy problem for Einstein metrics: a solution to (1.6) gives a initial hypersurface $\{0\} \times M$ with mean curvature H on an Einstein manifold $\mathbb{R} \times M$. See [12], for more details. As in the previous example, (1.6) has a variational structure: ψ is a solution to (1.6) if and only if ψ is a critical point of the following functional

$$\mathcal{L}(\psi) = \frac{1}{2}\int_M \langle \psi, \mathsf{D}_g \psi\rangle \, d\mathrm{vol}_g - \frac{m-1}{2m}\int_M H(x)|\psi|^{\frac{2m}{m-1}} \, d\mathrm{vol}_g. \tag{1.7}$$

In both examples, the action functionals take the similar from (1.2). For both of these examples, it is a fundamental problem to determine the structure of the set of solutions. For Dirac-harmonic maps, some explicit constructions of solutions are known, see [38], [10], [11]. In [34], some existence results were established for Dirac-geodesics (Dirac-geodesic is the 1-dimensional version of Dirac-harmonic map) via linking argument. For the spinorial Yamabe equation, in [8], [9], [44], [35], [36] some existence results are established. But for both problems, general existence results are still lacking. Since, at least formally, solutions to (1.4), (1.5) and (1.6) are obtained as critical points of the action functionals (1.3) and (1.7), respectively, one of powerful approaches to such problems may be based on Morse theory. But constructing suitable Morse theory for both problems are not so straightforward: due to the presence of the pure spinor action $\int_M \langle \psi, \mathsf{D}_g\psi\rangle \, d\mathrm{vol}_g$, we observe that the action functionals (1.3) and (1.7) are strongly indefinite in the sense that the Morse index and the co-index at any critical point are infinite. Thus the classical Morse theory is not available to these examples. To construct meaningful theory, we need a Floer type construction [25], [26], [27], [46]. In addition to the strong indefinite character of the functionals, there are additional difficulties in both problems. Namely, both of the functionals are invariant under conformal changes of the metrics on the domain manifolds. Due to this invariance, the associated variational problems are not compact in the sense that both functionals never satisfy the Palais-Smale like compactness condition.

Motivated by the above examples, in this paper and its sequel [37], we are concerned with the Morse theoretic properties of the model functional (1.2) and we will study the global structure of solutions to the equation (1.1) from a Morse theoretic point of view. We focus our attention mainly on the strong indefinite character of the variational problem, so we will assume that $\mathcal{L}_H$ is subcritical in the sense that the term $\int_M H(x, \psi) \, d\mathrm{vol}_g$ in (1.2) is a compact perturbation of the pure spinor action $\frac{1}{2}\int_M \langle \psi, \mathsf{D}_g\rangle \, d\mathrm{vol}_g$, see §3 for the precise meaning of this. Though we do not address the problem of non-compact perturbations in this paper, in [33], [35] and [36] some compactness and existence issues were treated from a different perspective for some class of non-compact perturbations.

Morse theory for strongly indefinite functionals were previously studied by some authors, see [1], [2], [3], [5], [6], [13], [14], [25], [26], [27], [28], [39], [48] and references therein. Except for 1-dimensional variational problems, however, there are few satisfactory results which can be applied to problems arising from geometry and physics, including, in particular, our model example. Motivated by the works of Bahri-Lions [16], Angenent-van der Vorst [13], [14] and Abbondandolo-Majer [3], [5], [6], in this paper and its sequel [37], we study some aspects of the Morse-Floer theory for the Dirac functionals of the from (1.2). More precisely, we will focus on the following topics:

(i) Relative Morse indices and its relation with compactness issues of solutions to (1.1).

(ii) Construction and computation of the Morse-Floer homology of $\mathcal{L}_H$ on the set of spinors and its implications for the global structure of the set of solutions to (1.1).

Note that both of the above topics are closely related to each other. Recall that the Morse-Floer homology is the homology of a chain complex generated by critical points of the functional. Thus in order to construct the Morse-Floer homology with coefficient Λ, where Λ is an arbitrary abelian group, we first need to define a graded Λ-module $\{C_p(\mathcal{L}_H; \Lambda)\}_{p\in\mathbb{Z}}$ (the grading is given by a "relative Morse index") by $C_p(\mathcal{L}_H; \Lambda) = C_p(\mathcal{L}_H) \otimes \Lambda$, where $C_p(\mathcal{L}_H)$ is the free abelian group generated by critical points of $\mathcal{L}_H$ with relative Morse index p:

$$C_p(\mathcal{L}_H) = \bigoplus_{\psi\in \mathrm{crit}_p(\mathcal{L}_H)} \mathbb{Z}\langle\psi\rangle, \tag{1.8}$$

where $\mathrm{crit}_p(\mathcal{L}_H) := \{\psi : d\mathcal{L}_H(\psi) = 0,\, m(\psi) = p\}$ and $m(\psi)$ is the relative Morse index of ψ. Since $\mathcal{L}_H$ is defined on an infinite dimensional space of spinors, even in the case where $\mathcal{L}_H$ is a Morse function, it is not necessary that $C_p(\mathcal{L}_H;\Lambda)$ is finitely generated. Thus, even in the case where the boundary operator $\partial_p : C_p(\mathcal{L}_H;\Lambda) \to C_{p-1}(\mathcal{L}_H;\Lambda)$ is defined properly such that $\partial_p \circ \partial_{p-1} = 0$, it is not necessary that its homology $HF_p(\mathcal{L}_H;\Lambda) = \ker \partial_p/\mathrm{Im}\, \partial_{p+1}$ is finitely generated. However, if we have a suitable compactness property of critical points under the boundedness assumption of their relative Morse indices, we will obtain such a finiteness result. Preceding results in this direction [16], [14] suggest that such a compactness property will be expected if the functional satisfies a suitable condition.

In this paper, we study the Morse-Floer theory for the functional $\mathcal{L}_H$, mainly focusing on the topic related to (i). Our main result proved in this paper is the compactness of critical points of $\mathcal{L}_H$ under the boundedness assumption of their relative Morse indices for a class of superquadratic functionals $\mathcal{L}_H$. Thus a finiteness result as mentioned above holds for such a class of functionals $\mathcal{L}_H$. For clarity of our exposition, we first consider a simple model case $h(x,\psi) = H(x)|\psi|^{p-1}\psi$, where H is a smooth positive function on M. Later, we will show that, with a suitable modification, essentially the same argument applies for a more general class of nonlinearities, see §7.3 for details. Thus we shall first consider the following functional L_H defined on $H^{1/2}(M,\mathbb{S}(M))$ ($H^{1/2}(M,\mathbb{S}(M))$ is the $H^{1/2}$-spinor space which will be defined in §2):

$$L_H(\psi) = \frac{1}{2}\int_M \langle \psi, \mathsf{D}_g\psi\rangle\, d\mathrm{vol}_g - \frac{1}{p+1}\int_M H(x)|\psi|^{p+1}\, d\mathrm{vol}_g. \tag{1.9}$$

Our main result concerning the topic (i) above is the following:

Theorem 1.1 *Let us assume that $m \geq 3$ and $1 < p < \frac{m+1}{m-1}$. For $H \in C^0(M)$ with $H > 0$ on M, $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, there exists a constant $C(\lambda,k,H) > 0$ such that the following holds: For any solution $\psi \in H^{1/2}(M,\mathbb{S}(M))$ to the equation $\mathsf{D}_g\psi = H(x)|\psi|^{p-1}\psi$ on M with its relative Morse index satisfying $\mu_\lambda(\psi) \leq k$ (for the definition of $\mu_\lambda(\psi)$, see Definition 3.2 below), there holds $\|\psi\|_{L^\infty(M)} \leq C(\lambda,k,H)$.*

As a corollary, we have

Corollary 1.1 *Let us assume that $m \geq 3$, $1 < p < \frac{m+1}{m-1}$ and $\lambda \in \mathbb{R}$. Assume also that $H \in C^0(M)$ and $H > 0$ on M. Then the following assertions (1) and (2) are equivalent for a sequence $\{\psi_n\}_{n=1}^\infty \subset H^{1/2}(M,\mathbb{S}(M))$ of solutions to (1.1).*

(1) $\{L_H(\psi_n)\}_{n=1}^\infty \subset \mathbb{R}$ is unbounded from above: $\sup_{n\geq 1} L_H(\psi_n) = +\infty$.

(2) $\{\mu_\lambda(\psi_n)\}_{n=1}^\infty$ is unbounded from above: $\sup_{n\geq 1} \mu_\lambda(\psi_n) = +\infty$.

A sequence of solutions to (1.1) which satisfies the condition (1) of the above corollary was constructed in [32] via a linking argument. Thus, we have by the above corollary an additional information about the solutions constructed in [32]: For any given integer k, there exists a solution ψ to (1.1) such that its relative Morse index satisfies $\mu_\lambda(\psi) \geq k$.

Theorem 1.1 implies the following finiteness result for the chain group $\{(C_p(L_H;\Lambda)\}_{p\in\mathbb{Z}}$, where the grading is given by the relative Morse index μ_λ. Note that L_H is S^1-invariant (S^1 action on spinors is given by $S^1 \times H^{1/2}(M,\mathbb{S}(M)) \ni (z,\psi) \mapsto z\psi \in H^{1/2}(M,\mathbb{S}(M))$), so any non-trivial critical point of L_H is degenerate. For this reason, the next result is stated for a more general class of nonlinearities.

Corollary 1.2 *Let us assume that* $m \geq 3$, $1 < p < \frac{m+1}{m-1}$ *and* $H = H(x,\psi) \in C^2(\mathbb{S}(M))$ *is a lower order perturbation of* $\frac{1}{p+1}H(x)|\psi|^{p+1}$ *in the sense of* §7.3. *Let us also assume that* $\mathfrak{L}_H$ *is a Morse function on* $H^{1/2}(M,\mathbb{S}(M))$. *Assume that the chain complex* $\{(C_p(\mathfrak{L}_H;\Lambda)\}_{p\in\mathbb{Z}}$ *is graded by the relative Morse index* μ_λ. *Then for any* $p \in \mathbb{Z}$, $C_p(\mathfrak{L}_H;\Lambda)$ *is finitely generated.*

In particular, if the homology of the chain complex $\{C_*(L_H;\Lambda), \partial_*(L_H)\}$ is defined properly, then it is finitely generated at any degree. (In fact, we will prove in [37] that its homology is well-defined. Moreover, it will be shown that we can explicitly calculate it for more general class of H.)

The proof of Theorem 1.1 relies on the existence of a suitable relative Morse index $m_{\mathbb{R}^m}$ defined for $\psi \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ satisfying $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$. Formally, solutions to the equation $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ is obtained as a critical point of the functional $L_{\mathbb{R}^m}(\psi) = \frac{1}{2}\int_{\mathbb{R}^m}\langle\psi, \mathsf{D}_{g_{\mathbb{R}^m}}\psi\rangle\, d\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{1}{p+1}\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_{g_{\mathbb{R}^m}}$. Notice, however, that it is not obvious at all how to define relative Morse index for a critical point ψ of $L_{\mathbb{R}^m}$. The reason is that natural notion of relative Morse indices defined for solutions to (1.1) on compact manifolds (which we will review in §3) depends crucially on the compactness of M, thus does not apply for critical points of $L_{\mathbb{R}^m}$. Therefore, we first need to investigate how to extend the notion of the relative Morse indices to such a non-compact setting. As for this problem, we will prove the existence of a "relative Morse index" $m_{\mathbb{R}^m}$ which has the following properties (I-1), (I-2):
(I-1) $m_{\mathbb{R}^m}(\psi)$ is defined for $\psi \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$, takes values in $\mathbb{Z}\cup\{+\infty\}$ and lower-semicontinuous with respect to $L^\infty_{\mathrm{loc}}(\mathbb{R}^m)$-convergence.
(I-2) $m_{\mathbb{R}^m}$ is a natural continuation of μ_λ along a conformal blow up of a compact spin manifold (M, g, σ), where μ_λ is the relative Morse index defined for critical points of $\mathfrak{L}_H$ on $H^{1/2}(M,\mathbb{S}(M))$.

In addition, it has the following crucial property:

Theorem 1.2 *Assume* $m \geq 3$ *and* $1 < p < \frac{m+1}{m-1}$. *Then for any non-trivial solution* $\psi \in L^\infty(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ *to* $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ *on* $\mathbb{R}^m$, *we have* $m_{\mathbb{R}^m}(\psi) = +\infty$.

Note that the above theorem gives a positive answer to the conjecture proposed by Maalaoui in [42]. Note, however, in [42] the author does not give any meaning of the "relative Morse index" for solutions to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$. In fact, as we have already mentioned above, due to the non-compactness of $\mathbb{R}^m$, giving a suitable meaning of the relative Morse index for such a solution is not trivial and, in fact, it is an important part of the problem. The definition of $m_{\mathbb{R}^m}$ is given in §6, Definition 6.1.

Remark 1.1 *(1) As we mentioned before, Theorem 1.1 is motivated by the works of Bahri-Lions [16] and Angenent-van der Vorst [14]. In [16], Bahri-Lions obtained a result similar to Theorem 1.1 for equation of the form* $-\Delta u = a(x)|u|^{p-1}u$ *on* $\Omega \subset \mathbb{R}^m$ *under the Dirichlet boundary condition* $u = 0$ *on* $\partial\Omega$, *where* a *is a positive function on* Ω *and* u *is a scalar function on* Ω. *In their work, the Morse index is the classical Morse index. Later, Angenent-van der Vorst [14] extended the result of [16] to a system of equations of the type* $-\Delta u = a(x)|v|^{p-1}v$, $-\Delta v = b(x)|u|^{q-1}u$ *on* $\Omega \subset \mathbb{R}^m$ *under the Dirichlet boundary conditions* $u = v = 0$ *on* $\partial\Omega$, *where* a, b *are positive functions on* Ω *and* u, v *are scalar functions on* Ω. *In this case, the equations have a strongly indefinite variational structure and the Morse index is a relative Morse index, thus closely related to ours. The idea of the paper [14] is based on the following observation: due to the special structure of the equations, one of the functions, for example* v *can be eliminated from the*

second equation, and the system of equations becomes a single equation for u. In a similar way, it was shown in [14] paper that the index problem can be reduced to the classical index problem for the reduced equation for u. In fact, this reduction process was done for the linearized equations and, therefore, applied for more general class of equations of the form $-\Delta u = H_v(x,u,v)$, $-\Delta v = H_u(x,u,v)$. Anyway, it relies on the special structure of the equations and it does not seem to be applicable to our case and more general context such as a functional defined on a general Hilbert manifold. Thus, we must take another route to approach our index problem. Note that our approach applied to the above mentioned Angenent-van der Vorst's elliptic systems will give a different index formula. However, it can be shown that it coincides with the one of [14] up to the addition of a constant.
(2) It is natural to expect that the conclusions of Theorem 1.1 and Theorem 1.2 also hold for $m = 2$. Some parts of our arguments apply to the case $m = 2$. But, they are not sufficient to conclude the above results for this case.

As for the topic (ii), in the sequel of this paper [37], we will give a construction and computation of the Morse-Floer homology for a class of superquadratic functionals (including our model example L_H as a special case). As a consequence of the invariance of the homology within a class of superquadratic Dirac functionals (which is one of the main results proved in [37]), the *superquadratic Dirac-Morse-Floer homology* will be defined. As an application of the computation of this homology, we will prove the existence results for a class of superquadratic Dirac equations which extends the results proved before in [32].

This paper is organized as follows: in the next section, we introduce basic definitions and notation which are used throughout this paper and its sequel [37]. In §3, we give three different definitions of the relative Morse indices for critical points of $\mathcal{L}_H$. Some fundamental properties of these indices and relations between these different definitions are also discussed in that section. In §4, we give a reformulation of the relative Morse index for some class of functionals, thus giving one more another formula of the relative Morse index. This reformulation is necessary to "continue" the relative Morse indices defined on compact spin manifolds to $\mathbb{R}^m$ via a blow-up procedure. It is essentially accomplished by comparing the relative Morse index of $\mathcal{L}_H$ with the classical Morse index of the Legendre-Fenchel dual of the second order approximation of $\mathcal{L}_H$. In §5, we prove a Liouville theorem which asserts that a solution $\psi \in L^{p+1}(\mathbb{R}^m)$ to $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$ is identically 0. This result is essential for the proof of Theorem 1.2. In §6, we prove Theorem 1.2. Then, in the final section §7, we prove Theorem 1.1 and its generalization.

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2 Preliminaries

In this section, we collect some basic definitions and notation which will be used throughout this paper. We will use the same notation used in [32], [33]. Please consult these papers, if necessary. See also [29], [30], [40] for more detailed exposition about fundamental concepts from spin geometry.

Let (M, g, σ) be an m-dimensional spin manifold. This means that M is an oriented Riemannian manifold with a metric g and the oriented frame bundle $P_{SO}(M) \to M$ has a non-trivial double covering $\sigma : P_{Spin}(M) \to P_{SO}(M)$ whose restriction to each fiber is the universal double covering $Spin(m) \to SO(m)$. We denote by Cl_m the Clifford algebra generated by $\mathbb{R}^m$. For m even, it is a fundamental fact that there exists an irreducible complex Cl_m-module $\mathbb{S}_m$ such that $\mathbb{Cl}_m := \mathsf{Cl}_m \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}_m)$ (as $\mathbb{C}$-algebras). $\mathbb{S}_m$ is determined uniquely up to isomorphism and is usually called the spinor module. The isomorphism $\mathbb{Cl}_m \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}_m)$ induces a representation (unique up to isomorphism) $\tau : Spin(m) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_m)$ which is called the spin representation. On the other hand, the canonical orientation on $\mathbb{R}^m$ induces a $\mathbb{Z}_2$-grading on $\mathbb{S}_m$; $\mathbb{S}_m = \mathbb{S}_m^+ \oplus \mathbb{S}_m^-$, see [29], [40] for details. Since $Spin(m) \subset \mathsf{Cl}_m^+$, where the subalgebra of Cl_m generated by elements which can be written as the product of even (odd) number of vectors in $\mathbb{R}^m$ is denoted by Cl_m^+ (Cl_m^-), $\mathbb{S}_m^\pm$ are preserved by τ, i.e., $\mathbb{S}_m^\pm$ are representations of $Spin(m)$. It is a fundamental fact that they are irreducible and non-isomorphic as complex $Spin(m)$-modules. $\mathbb{S}_m^\pm$ are called positive/negative complex spin representations. The spinor bundle $\mathbb{S}(M) \to M$ is defined as the associated vector bundle of $P_{Spin}(M) \to M$ via the representation $\tau : Spin(m) \to \mathrm{End}(\mathbb{S}_m)$, $\mathbb{S}(M) = P_{Spin}(M) \times_\tau \mathbb{S}_m$.

For m odd, we have a natural isomorphism $\mathsf{Cl}_m \cong \mathsf{Cl}_{m+1}^+$ given by $x^+ + x^- \mapsto x^+ + e_0 \cdot x^-$ ($\cdot$ denotes the Clifford product), where $x^\pm \in \mathsf{Cl}_m^\pm$, $\{e_0, e_1, \ldots, e_m\}$ and $\{e_1, \ldots, e_m\}$ are orthonormal bases of $\mathbb{R}^{m+1}$ and $\mathbb{R}^m$, respectively, and we consider $\mathbb{R}^m$ as a subspace of $\mathbb{R}^{m+1}$ via the canonical inclusion $\mathbb{R}^m \subset \mathbb{R}^{m+1}$. Under these, we have the following $\mathbb{C}$-algebra isomorphism:

$$\mathbb{Cl}_m \cong \mathbb{Cl}_{m+1}^+ \cong \mathrm{End}_{\mathbb{C}}^+(\mathbb{S}_{m+1}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{S}_{m+1}^+) \oplus \mathrm{End}_{\mathbb{C}}(\mathbb{S}_{m+1}^-).$$

It is also a fundamental fact that $\mathbb{S}_{m+1}^\pm$ are irreducible $\mathbb{Cl}_m$-modules and, as representation spaces of $Spin(m)$, they are irreducible but isomorphic. Thus, we define $\mathbb{S}_m = \mathbb{S}_{m+1}^+ (\cong \mathbb{S}_{m+1}^-)$ and call the spinor representation for odd m. Denoting the spinor representation so obtained as $\tau : Spin(m) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_m)$, we define the spinor bundle $\mathbb{S}(M) \to M$ as in the even dimensional case.

Sections of $\mathbb{S}(M) \to M$ are simply called spinors. The space of C^∞-spinors is denoted by $\mathcal{C}^\infty(M) := C^\infty(M, \mathbb{S}(M))$.

The spinor bundle is a Dirac bundle in the sense that the following conditions (i)–(iv) are satisfied: (i) $X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X,Y)\psi$ for $X, Y \in C^\infty(M, TM)$ and $\psi \in \mathcal{C}^\infty(M)$. (ii) $(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi)$ for $X \in C^\infty(M, TM)$ and $\psi, \varphi \in \mathcal{C}^\infty(M)$ ($(\cdot,\cdot)$ is the hermitian metric on $\mathbb{S}(M)$ induced from a natural hermitian metric on $\mathbb{S}_m$). (iii) ∇ is metric in the sense that $X(\psi, \varphi) = (\nabla_X \psi, \varphi) + (\psi, \nabla_X \varphi)$ for $X \in C^\infty(M, TM)$ and $\psi, \varphi \in \mathcal{C}^\infty(M)$, where ∇ is a canonical connection on $P_{Spin}(M)$ obtained by lifting the Levi-Civita connection on $P_{SO}(M)$. (iv) The Clifford product is constant in the sense that $\nabla_X(Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi$ for $X, Y \in C^\infty(M, TM)$ and $\psi \in \mathcal{C}^\infty(M)$.

The (Atiyah-Singer-)Dirac operator D_g is defined by the following:

$$\mathsf{D}_g = c \circ \nabla : \mathcal{C}^\infty(M) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes \mathbb{S}(M)) \cong C^\infty(M, TM \otimes \mathbb{S}(M)) \xrightarrow{c} \mathcal{C}^\infty(M),$$

where c denotes the Clifford multiplication $c : TM \otimes \mathbb{S}(M) \ni X \otimes \psi \mapsto X \cdot \psi \in \mathbb{S}(M)$ and the identification $T^*M \cong TM$ by the metric g on M is used.

The nonlinear Dirac equation of the from (1.1) is obtained formally as a solution to the Euler-Lagrange equation of the action functional $\mathcal{L}_H$ defined on spinors:

$$\mathcal{L}_H(\psi) = \frac{1}{2} \int_M \langle \psi, \mathsf{D}_g \psi \rangle \, d\mathrm{vol}_g - \int_M H(x, \psi(x)) \, d\mathrm{vol}_g,$$

where $\langle\cdot,\cdot\rangle$ is the metric on $\mathbb{S}(M)$, i.e., $\langle\cdot,\cdot\rangle = \mathrm{Re}\,(\cdot,\cdot)$.

In view of the presence of the pure spinor action $\int_M \langle\psi, \mathsf{D}_g\psi\rangle\, d\mathrm{vol}_g$ term, from a functional analytical point of view, a natural functional space to work with $\mathcal{L}_H$ is the Sobolev space of $H^{1/2}$-spinors, see [32]. It is defined as follows. Let $\{\psi_k\}_{k=1}^\infty$ be a complete orthonormal basis of $L^2(M,\mathbb{S}(M))$, the space of L^2-spinors on M, consisting of eigenspinors of the Dirac operator D_g with corresponding eigenvalues $\{\lambda_k\}_{k=1}^\infty$. Recall that $\{\lambda_k\}_{k=1}^\infty$ is unbounded from above and below. Then for $s>0$, the unbounded operator $|\mathsf{D}_g|^s : L^2(M,\mathbb{S}(M)) \to L^2(M,\mathbb{S}(M))$ is defined by

$$|\mathsf{D}_g|^s\psi = \sum_{k=1}^\infty |\lambda_k|^s a_k\psi_k,$$

where $\psi = \sum_{k=1}^\infty a_k\psi_k \in L^2(M,\mathbb{S}(M))$.

The domain of $|\mathsf{D}_g|^s$ is, by definition, the space of H^s-spinors and is denoted by $H^s(M,\mathbb{S}(M))$. For simplicity, throughout this paper, we will use the shorthand notation $\mathcal{H}^s(M) := H^s(M,\mathbb{S}(M))$ and $\mathcal{L}^q(M) := L^q(M,\mathbb{S}(M))$ $(1 \le q \le \infty)$, the space of L^q-spinors on M. Thus $\psi = \sum_{k=1}^\infty a_k\psi_k$ is in $\mathcal{H}^s(M)$ if and only if $\sum_{k=1}^\infty (1+|\lambda_k|^{2s})|a_k|^2 < +\infty$. $\mathcal{H}^s(M)$ is a Hilbert space with the inner product $(\psi,\varphi)_{\mathcal{H}^s(M)} := (\psi,\varphi)_{\mathcal{L}^2(M)} + (|\mathsf{D}_g|^s\psi, |\mathsf{D}_g|^s\varphi)_{\mathcal{L}^2(M)}$, where $(\cdot,\cdot)_{\mathcal{L}^2(M)}$ is the L^2-inner product on $\mathcal{L}^2(M)$. We are mainly concerned with the case $s = 1/2$. By the Sobolev embedding theorem [7], we have a continuous inclusion $\mathcal{H}^{1/2}(M) \subset \mathcal{L}^{p+1}(M)$ for $1 \le p \le \frac{m+1}{m-1}$. It is compact for $1 \le p < \frac{m+1}{m-1}$. Thus the functional $\mathcal{L}_H$ is well-defined on $\mathcal{H}^{1/2}(M)$ if H satisfies a growth condition like $|H(x,\psi)| \le C(1+|\psi|^{p+1})$ for $1 \le p \le \frac{m+1}{m-1}$. Furthermore, if H is differentiable in the fiber direction and $h(x,\psi) = \nabla_\psi H(x,\psi)$ satisfies a growth condition like $|h(x,\psi)| \le C(1+|\psi|^p)$ for $1 \le p \le \frac{m+1}{m-1}$, then $\mathcal{L}_H$ is differentiable on $\mathcal{H}^{1/2}(M)$ and a critical point $\psi \in \mathcal{H}^{1/2}(M)$ of $\mathcal{L}_H$ is a weak solution to the equation (1.1). Further regularity property of H (for example, if h is Hölder continuous) implies that any $H^{1/2}$-weak solution is C^1 and is a classical solution of (1.1). For the proof of the regularity of weak solutions, see [8], [33, Appendix].

A typical example which we should have in mind is $H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1}$, where H is a positive smooth function on M and $1 \le p \le \frac{m+1}{m-1}$. In this special case, we will denote the associated functional $\mathcal{L}_H$ by L_H. It is easy to verify that L_H is C^2 on $\mathcal{H}^{1/2}(M)$.

3 Relative Morse indices

For the moment, let us assume that $H(x,\psi)$ is twice differentiable in the fiber direction ψ and there exists a constant $C_5 > 0$ such that the following folds for all $(x,\psi) \in \mathbb{S}(M)$:

$$|H(x,\psi)| \le C_5(1+|\psi|^{p+1}), \tag{3.1}$$

$$|d_\psi H(x,\psi)| \le C_5(1+|\psi|^p), \tag{3.2}$$

$$|d_\psi^2 H(x,\psi)| \le C_5(1+|\psi|^{p-1}), \tag{3.3}$$

where $1 < p < \frac{m+1}{m-1}$ and $d_\psi H$ and $d_\psi^2 H$ denote the first and the second order derivatives of H with respect to the fiber variable ψ.

Under the condition (3.1)–(3.3), by the Sobolev embedding theorem $\mathcal{H}^{1/2}(M) \subset \mathcal{L}^{p+1}(M)$, it is easy to see that $\mathcal{L}_H$ is C^2 on $\mathcal{H}^{1/2}(M)$ and the first and the second order derivatives $d\mathcal{L}_H$

and $d^2\mathcal{L}_H$, respectively, are given by

$$d\mathcal{L}_H(\psi)[\varphi] = \int_M \langle \psi, \mathsf{D}_g\varphi\rangle \,d\mathrm{vol}_g - \int_M d_\psi H(x,\psi)[\varphi]\,d\mathrm{vol}_g,$$

$$d^2\mathcal{L}_H(\psi)[\varphi,\phi] = \int_M \langle \varphi, \mathsf{D}_g\phi\rangle \,d\mathrm{vol}_g - \int_M d^2_\psi H(x,\psi)[\varphi,\phi]\,d\mathrm{vol}_g$$

for $\psi,\varphi,\phi \in \mathcal{H}^{1/2}(M)$. From the first equation, the equation $d\mathcal{L}_H(\psi) = 0$ is the weak form of the equation (1.1) with $h(x,\psi) = \nabla_\psi H(x,\psi)$. From the second equation, the self-adjoint realization of $d^2\mathcal{L}_H(\psi)$ on $\mathcal{H}^{1/2}(M)$ is given by

$$d^2\mathcal{L}_H(\psi) = (1+|\mathsf{D}_g|)^{-1}\mathsf{D}_g - (1+|\mathsf{D}_g|)^{-1} d_\psi \nabla_\psi H(x,\psi). \tag{3.4}$$

Recall that the Morse index of $\mathcal{L}_H$ at $\psi \in H^{1/2}(M,\mathbb{S}(M))$ is the dimension of the maximal subspace of $\mathcal{H}^{1/2}(M)$ on which $d^2\mathcal{L}_H(\psi)$ is negative definite. Under the condition (3.1)–(3.3), by the Sobolev embedding theorem, $(1+|\mathsf{D}_g|)^{-1}d_\psi\nabla_\psi H(x,\psi)$ is a compact operator on $\mathcal{H}^{1/2}(M)$ and, therefore, $d^2\mathcal{L}_H(\psi)$ is a compact perturbation of $(1+|\mathsf{D}_g|)^{-1}\mathsf{D}_g$. Since ± 1 are essential spectrum of $(1+|\mathsf{D}_g|)^{-1}\mathsf{D}_g$, the same is true for $d^2\mathcal{L}_H(\psi)$ and $d^2\mathcal{L}_H(\psi)$ is negative and positive definite on some infinite dimensional subspaces of $\mathcal{H}^{1/2}(M)$. Thus the Morse index and co-index at any critical point of $\mathcal{L}_H$ are infinite. Therefore, to obtain a meaningful quantity, we need a relative (or renormalized) version of the Morse index. It is known at present that there are some definitions of such relative Morse indices based on different concepts. In the following, we give three of them. These are based on (i) the relative dimension of commensurable subspaces, (ii) the spectral flow of a path of Dirac type operators, (iii) the Fredholm index of the linearization of the negative gradient flow equation connecting two critical points. Though all of these are well-known, we will shortly review them in the following subsections because we will use all of these versions throughout this paper and its sequel [37].

3.1 Relative Morse index as a relative dimension

The idea of this definition is to compare the negative subspace of $d^2\mathcal{L}_H(\psi)$ with some fixed subspace of $\mathcal{H}^{1/2}(M)$. Thus, let us first define a notion of relative dimension for two infinite dimensional commensurable subspaces. Let us consider two closed subspaces $V, W \subset \mathcal{H}^{1/2}(M)$. Following [2], [3], we say that V and W are *commensurable* if the operator $P_V - P_W$ is compact, where P_V is the orthogonal projection onto V. For commensurable subspaces V, W, we define their *relative dimension* as $\dim(W,V) := \dim(W\cap V^\perp) - \dim(W^\perp \cap V)$, where $V^\perp$ is the orthogonal complement of V in $\mathcal{H}^{1/2}(M)$. For commensurable subspaces V, W, $\dim(W,V)$ is well-defined, i.e., $\dim(W,V)$ is an integer (see [2], [3] for the proof).

For a given bounded self-adjoint operator T, we denote by $E^+(T)$ and $E^-(T)$ the maximal T-invariant subspaces on which T is positive and negative definite, respectively. We will simply call $E^+(T)$ and $E^-(T)$ the positive and the negative subspaces of T, respectively.

For $\lambda \in \mathbb{R}$, we set $\mathsf{D}_\lambda = \mathsf{D}_g - \lambda$. We define a relative Morse index of $\mathcal{L}_H$ at ψ by comparing $E^-(d^2\mathcal{L}_H(\psi))$ with the negative subspace of D_λ.

Definition 3.1 *We write $E^\pm_H(\psi) := E^\pm(d^2_\psi\mathcal{L}_H(\psi))$, where $d^2\mathcal{L}_H(\psi)$ is given by (3.4). We also define $E^\pm_\lambda := E^\pm((1+|\mathsf{D}_g|)^{-1}\mathsf{D}_\lambda)$ for $\lambda\in\mathbb{R}$. We define the λ-relative Morse index $m_\lambda(\psi)$ of $\mathcal{L}_H$ at $\psi \in \mathcal{H}^{1/2}(M)$ by*

$$m_\lambda(\psi) := \dim(E^-_H(\psi), E^-_\lambda).$$

To see that the above definition is well-defined (i.e., $m_\lambda(\psi)$ is an integer), we observe that, by the Sobolev embedding and the elliptic regularity theory, the difference of the operators $d^2\mathcal{L}_H(\psi)$ and $(1+|\mathsf{D}_g|)^{-1}\mathsf{D}_\lambda$ is compact for any $\lambda \in \mathbb{R}$:

$$d^2\mathcal{L}_H(\psi) - (1+|\mathsf{D}|)^{-1}\mathsf{D}_\lambda \in K(\mathcal{H}^{1/2}(M)),$$

where $K(\mathcal{H}^{1/2}(M)) \subset L(\mathcal{H}^{1/2}(M))$ and $L(\mathcal{H}^{1/2}(M))$ are spaces of compact and bounded operators on $\mathcal{H}^{1/2}(M)$, respectively. Thus, as is easily seen from functional calculus, the difference of the spectral projections of $d^2_\psi\mathcal{L}_H(\psi)$ and $(1+|\mathsf{D}_g|)^{-1}\mathsf{D}_\lambda$ on the spectral set $(-\infty, 0)$ is compact, see [3, Proposition 2.2] for details. Therefore, $E^-_H(\psi)$ and E^-_λ are commensurable and $m_\lambda(\psi) \in \mathbb{Z}$ as asserted.

3.2 Relative Morse index as a spectral flow

There is also a useful formula of the relative Morse index in terms of the *spectral flow* of a path of Dirac operators. We first observe that the L^2-realization of $d^2\mathcal{L}_H(\psi)$ (i.e., the representation of $d^2\mathcal{L}_H(\psi)$ with respect to the L^2-inner product on $\mathcal{H}^{1/2}(M)$) is given by the unbounded self-adjoint operator $\mathsf{D}_g - d_\psi\nabla_\psi H(x,\psi)$ on $\mathcal{L}^2(M)$. For $\psi \in \mathcal{H}^{1/2}(M)$, we set

$$A_\psi = d_\psi\nabla_\psi H(x,\psi). \tag{3.5}$$

Then we have $\mathsf{D}_g - d_\psi\nabla_\psi H(x,\psi) = \mathsf{D}_g - A_\psi$. Note that $A_\psi(x) : \mathbb{S}(M)_x \to \mathbb{S}(M)_x$ is symmetric for a.e. $x \in M$, i.e., A_ψ is a measurable section of $\mathrm{Sym}(\mathbb{S}(M)) \to M$, where $\mathrm{Sym}(\mathbb{S}(M)) = \bigsqcup_{x\in M}\mathrm{Sym}(\mathbb{S}(M)_x)$ and $\mathrm{Sym}(\mathbb{S}(M)_x)$ is the set of symmetric endomorphisms on $\mathbb{S}(M)_x$. We henceforth assume that $\psi \in \mathcal{L}^\infty(M)$. We then define

$$\begin{aligned}\mathcal{A} :&= L^\infty(M, \mathrm{Sym}(\mathbb{S}(M))) \\ &= \{A : A \text{ is a } L^\infty\text{-section of } \mathrm{Sym}(\mathbb{S}(M)) \to M\}\end{aligned}$$

and $\mathsf{D}_A = \mathsf{D} - A$ for $A \in \mathcal{A}$.

We denote the set of bounded symmetric operators from $\mathcal{H}^1(M)$ to $\mathcal{L}^2(M)$ by $L_{\mathrm{sym}}(\mathcal{H}^1(M), \mathcal{L}^2(M))$. We denote by $\mathcal{S}(\mathcal{H}^1(M), \mathcal{L}^2(M)) \subset L_{\mathrm{sym}}(\mathcal{H}^1(M), \mathcal{L}^2(M))$ the open subset consisting of those operators with non-empty resolvent set. For $k \geq 0$, we define

$$\mathcal{S}_k(\mathcal{H}^1(M), \mathcal{L}^2(M)) := \{L \in \mathcal{S}(\mathcal{H}^1(M), \mathcal{L}^2(M)) : \dim\ker L = k\}.$$

For brevity, we write $\mathcal{S} = \mathcal{S}(\mathcal{H}^1(M), \mathcal{L}^2(M))$ and $\mathcal{S}_k = \mathcal{S}_k(\mathcal{H}^1(M), \mathcal{L}^2(M))$. By the compactness of the embedding $\mathcal{H}^1(M) \to \mathcal{L}^2(M)$, for $L \in \mathcal{S}(\mathcal{H}^1(M), \mathcal{L}^2(M))$ and $\lambda \notin \mathsf{Spec}(L)$, the resolvent $(L-\lambda)^{-1} : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$ is compact and $\dim\ker L < +\infty$ for any $L \in \mathcal{S}(\mathcal{H}^1(M), \mathcal{L}^2(M))$. We thus have $\mathcal{S} = \bigcup_{k=0}^\infty \mathcal{S}_k$. By Duistermaat [22] (see also [45]), $\mathcal{S}_k \subset \mathcal{S}$ is a smooth submanifold of codim $\mathcal{S}_k = \frac{k(k+1)}{2}$. In particular, codim $\mathcal{S}_k \geq 3$ for $k \geq 2$. From this, a generic path in $\mathcal{S}$ does not intersect $\mathcal{S}_k$ for $k \geq 2$. Also, a generic path in $\mathcal{S}$ intersect transversally with $\mathcal{S}_1$. The spectral flow of a path of operators $\{L_t\}_{t\in I} \subset \mathcal{S}$ ($I \subset \mathbb{R}$ is an interval) is defined as the intersection number of the path $\{L_t\}_{t\in I}$ with $\mathcal{S}_1$. More precisely, it is defined as follows. Let $L_0 \in \mathcal{S}_1$. By definition, we have $\dim\ker L_0 = 1$ and L_0 has 0 as a simple eigenvalue. Furthermore, for any $L \in \mathcal{S}$ close to L_0, L has a simple eigenvalue $\lambda(L)$ with $\lambda(L_0) = 0$ and $\lambda(L)$ depends smoothly on L near L_0. There exists a neighborhood $U(L_0)$ of L_0 such that $\mathcal{S}_1 \cap U(L_0)$ is represented as $\mathcal{S}_1 \cap U(L_0) = \{L \in \mathcal{S} : \lambda(L) = 0\}$. Thus the 1-form $d\lambda$ defines an orientation on $T_{L_0}\mathcal{S}_1^\perp = T_{L_0}\mathcal{S}/T_{L_0}\mathcal{S}_1$. With this orientation, the spectral flow $\mathsf{sf}\{L_t\}_{t\in I}$ of a generic path $\{L_t\}_{t\in I}$ is defined as the intersection number of $\{L_t\}_{t\in I}$ with $\mathcal{S}_1$:

$$\begin{aligned}\mathsf{sf}\{L_t\}_{t\in I} &= \text{the intersection number of } \{L_t\}_{t\in I} \text{ with } \mathcal{S}_1\\ &= \#\{\text{eigenvalues of } L_t \text{ flowing from negative to positive}\\ &\qquad \text{as } t \text{ varies from the left end of } I \text{ to the right end of } I\}\\ &\quad - \#\{\text{eigenvalues of } L_t \text{ flowing from positive to negative}\\ &\qquad \text{as } t \text{ varies from the left end of } I \text{ to the right end of } I\},\end{aligned}$$

where in the above, for simplicity, we have assumed that L_t is invertible at the both ends of I. For $k \geq 0$, we define $\mathcal{A}_k = \{A \in \mathcal{A} : \mathsf{D}_A \in \mathcal{S}_k\}$. Let us assume that $A_0, A_1 \in \mathcal{A}_0$. We choose a smooth path $\{A_t\}_{t\in[0,1]}$ in $\mathcal{A}$ which connects A_0 to A_1. We may assume that, after slightly perturbing the path if necessary, $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ intersects with $\mathcal{S}_1$ transversally. We then define:

Definition 3.2 *Under the above condition, we define the index $\mu(\mathsf{D}_{A_1}, \mathsf{D}_{A_0})$ by*

$$\mu(\mathsf{D}_{A_1}, \mathsf{D}_{A_0}) = -\mathsf{sf}\{D_{A_t}\}_{t\in[0,1]}.$$

For $A \in \mathcal{A}_0$ and $\lambda \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$, we define

$$\mu_\lambda(\mathsf{D}_A) := \mu(\mathsf{D}_A, \mathsf{D}_\lambda).$$

Finally, for $\psi \in \mathcal{L}^\infty(M)$ with $A_\psi \in \mathcal{A}_0$ and $\lambda \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$, we define $\mu_\lambda(\psi)$ by

$$\mu_\lambda(\psi) = \mu(\mathsf{D}_{A_\psi}, \mathsf{D}_\lambda),$$

where A_ψ is defined in (3.5).

Note that, since $\operatorname{codim} \mathcal{S}_k \geq 3$ for $k \geq 2$, a generic homotopy of generic paths does not intersect with $\mathcal{S}_k$ for $k \geq 2$. Thus the above definition does not depend on the particular choice of the generic path and is well-defined.

A critical point $\psi \in \operatorname{crit}(\mathcal{L}_H)$ is called non-degenerate if $d^2\mathcal{L}_H(\psi)$ is invertible. This is the case if and only if $A_\psi \in \mathcal{A}_0$. For degenerate case $A_\psi \notin \mathcal{A}_0$, we define:

Definition 3.3 *Let $\psi \in \mathcal{L}^\infty(M)$ and $\lambda \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$. We define the lower λ-index $\mu_{\lambda,-}(\psi)$ and the upper λ-index $\mu_{\lambda,+}(\psi)$ as follows:*

$$\mu_{\lambda,-}(\psi) = \lim_{\epsilon\downarrow 0} \inf\{\mu_\lambda(\mathsf{D}_A) : \|A_\psi - A\|_{L^\infty} < \epsilon,\ A \in \mathcal{A}_0\},$$

$$\mu_{\lambda,+}(\psi) = \lim_{\epsilon\downarrow 0} \sup\{\mu_\lambda(D_A) : \|A_\psi - A\|_{L^\infty} < \epsilon,\ A \in \mathcal{A}_0\}.$$

By definition, μ_λ and $\mu_{\lambda,\pm}$ depend on λ. Its dependence is easily seen by the following lemma:

Lemma 3.1 *Let $\lambda, \lambda' \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$ with $\lambda' < \lambda$. We then have*

$$\mu_{\lambda'}(\psi) = \mu_\lambda(\psi) + \#\{\lambda_k \in \mathsf{Spec}(\mathsf{D}_g) : \lambda' < \lambda_k < \lambda\}.$$

Proof. We have

$$\begin{aligned}\mu_{\lambda'}(\psi) &= \mu(\mathsf{D}_{A_\psi}, \mathsf{D}_{\lambda'}) \\ &= \mu(\mathsf{D}_{A_\psi}, \mathsf{D}_{\lambda}) + \mu(\mathsf{D}_{\lambda}, \mathsf{D}_{\lambda'}) \end{aligned} \tag{3.6}$$

by the additivity property of the spectral flow. Consider the path $A_t = (t\lambda + (1-t)\lambda')\mathbf{1} \in \mathcal{A}$. Since $\mathsf{Spec}(\mathsf{D}_{A_t}) = \mathsf{Spec}(\mathsf{D}_g) - t\lambda - (1-t)\lambda'$, an eigenvalue of D_{A_t} which changes its sign as t changes from 0 to 1 is of the form $\lambda_k(t) = \lambda_k - t\lambda - (1-t)\lambda'$, where $\lambda_k \in \mathsf{Spec}(\mathsf{D}_g)$ lies between λ and λ'. Under the assumption $\lambda' < \lambda$, all of these change from negative to positive. Thus the assertion is proved if all $\lambda_k \in \mathsf{Spec}(\mathsf{D}_g)$ lying between λ' and λ are simple. Otherwise, we slightly perturb the path $\{A_t\}_{t\in[0,1]}$ in such a way that the resulting path $\{\tilde{A}_t\}_{t\in[0,1]}$ does not intersect with $\mathcal{A}_k$ for all $k \geq 2$ and intersects with $\mathcal{A}_1$ transversally. By the assumption $\lambda, \lambda' \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$, we have $A_0, A_1 \notin \bigcup_{k=1}^\infty \mathcal{A}_k$ and, as we have seen above, $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ already intersects transversally with $\mathcal{S}_1$. Therefore, the perturbation $\{\tilde{A}_t\}_{t\in[0,1]}$ can be chosen in such a way that $\{\tilde{A}_t\}_{t\in[0,1]}$ coincides with $\{A_t\}_{t\in[0,1]}$ except on neighborhoods of points at which $\{A_t\}_{t\in[0,1]}$ intersect with $\bigcup_{k=2}^\infty \mathcal{A}_k$. In particular, we may assume that $\tilde{A}_0 = A_0$ and $\tilde{A}_1 = A_1$. Then taking the perturbation small enough, we see that for each multiple eigenvalue λ_k of D_g, $\lambda_k(t)$ splits into $\dim\ker(\mathsf{D}_g - \lambda_k)$ simple eigenvalues of $\mathsf{D}_{\tilde{A}_t}$ near $t = \frac{\lambda_k - \lambda'}{\lambda - \lambda'}$. These are only eigenvalues of $\mathsf{D}_{\tilde{A}_t}$ which change their signs near $t = \frac{\lambda_k - \lambda'}{\lambda - \lambda'}$. They all change from negative to positive as t varies under the assumption $\lambda' < \lambda$. Thus each multiple eigenvalue λ_k contributes $\dim\ker(\mathsf{D}_g - \lambda_k)$ to the spectral flow. Therefore, the assertion is proved also in the case where there are multiple eigenvalues. This completes the proof. □

In the next lemma, we prove monotonicity of $\mu_\lambda(\mathsf{D}_A)$ with respect to variations of A. This result will be also useful in part II of this paper, see [37]. For $A, A' \in \mathcal{A}$, we write $A' \leq A$ if $A - A'$ is non-negative pointwise a.e..

Lemma 3.2 *Assume $A_0, A_1 \in \mathcal{A}_0$ and $A_0 \leq A_1$. For any $\lambda \in \mathbb{R}\setminus\mathsf{Spec}(\mathsf{D}_g)$, we have $\mu_\lambda(\mathsf{D}_{A_0}) \leq \mu_\lambda(\mathsf{D}_{A_1})$.*

Proof. Since $\mathcal{A}_0 \subset \mathcal{A}$ is open, there exists a small connected neighborhood $U(A_0)$ of A_0 such that $U(A_0) \subset \mathcal{A}_0$. For $A \in U(A_0)$, consider a generic path connecting A_0 to A in $U(A_0)$. It is easy to see that $\mu_\lambda(\mathsf{D}_A) = \mu_\lambda(\mathsf{D}_{A_0})$ for all $A \in U(A_0)$ if $U(A_0)$ is small enough, i.e., $\mu_\lambda(\mathsf{D}_A)$ is locally constant on $\mathcal{A}_0$. Thus considering a neighboring A_0' of A_0 if necessary, we may assume at the beginning that A_1, A_0 satisfy $A_1 \geq A_0 + \delta\mathbf{1}$ for some $\delta > 0$. We then consider the linear path $A_t = tA_1 + (1-t)A_0$ for $0 \leq t \leq 1$. We have $\frac{d}{dt}A_t = A_1 - A_0 \geq \delta\mathbf{1}$ and a generic perturbation (in the sense of C^1) $\{\tilde{A}_t\}_{t\in[0,1]}$ of $\{A_t\}_{t\in[0,1]}$ satisfies

$$\frac{d}{dt}\tilde{A}_t \geq \frac{\delta}{2}\mathbf{1}. \tag{3.7}$$

We need to calculate the intersection number of $\{\mathsf{D}_{\tilde{A}_t}\}_{t\in[0,1]}$ with $\mathcal{S}_1$. To this end, let us assume that $B \in \mathcal{A}_1$ and $\lambda = \lambda(A)$ is a smooth function defined in a neighborhood $U(B)$ of $B \in \mathcal{A}$ such that $\lambda(B) = 0$ and $U(B) \cap \mathcal{A}_1 = \{A \in \mathcal{A} : \lambda(A) = 0\}$. That is to say, $\lambda(A)$ is an eigenvalue of D_A defined for A near B and $\lambda(B) = 0$ is a simple eigenvalue. Let $\psi(A)$ be a corresponding L^2-normalized eigenspinor which depends smoothly on A near B. Such a choice is possible by the implicit function theorem: by the assumption $B \in \mathcal{A}_1$ and the implicit function theorem, for A near B there exists an eigenspinor $\psi(A)$ which depends smoothly on A. To obtain a

L^2-normalized eigenspinor, we simply replace $\psi(A)$ by $\psi(A)/\|\psi(A)\|_{\mathcal{L}^2(M)}$ if necessary. Taking the derivative of $\mathsf{D}_g\psi(A) - A\psi(A) = \lambda(A)\psi(A)$ with respect to A at $A = B$ in the direction $\alpha \in T_B\mathcal{A} = \mathcal{A}$ and taking the L^2-inner product with $\psi(B)$, we obtain

$$\langle d\lambda(B), \alpha\rangle_{T^*_B\mathcal{A}\times T_B\mathcal{A}} = -(\alpha\psi(B), \psi(B))_{\mathcal{L}^2(M)}, \tag{3.8}$$

where the left hand side is the duality pairing between $T^*_B\mathcal{A}$ and $T_B\mathcal{A}$. Thus at an intersection point $t_0 \in (0,1)$ of $\{\mathsf{D}_{\tilde{A}_t}\}_{t\in[0,1]}$ with $\mathcal{S}_1$, by (3.7) and (3.8) we have

$$\begin{aligned}\Big\langle d\lambda(\tilde{A}_{t_0}), \frac{d}{dt}\tilde{A}_{t_0}\Big\rangle &= -\Big(\Big(\frac{d}{dt}\tilde{A}_{t_0}\Big)\psi(A_{t_0}), \psi(A_{t_0})\Big)_{\mathcal{L}^2(M)} \\ &\le -\Big(\frac{\delta}{2}\psi(A_{t_0}), \psi(A_{t_0})\Big)_{L^2} = -\frac{\delta}{2} < 0.\end{aligned}$$

This proves that at any intersection point, the intersection number is -1. Thus $\mu(\mathsf{D}_{A_1}, \mathsf{D}_{A_0}) = -\mathsf{sf}\{\mathsf{D}_{\tilde{A}_t}\}_{t\in[0,1]} \ge 0$ ($= 0$ occurs for $\{\mathsf{D}_{\tilde{A}_t}\}_{t\in[0,1]}$ which does not intersect with $\mathcal{S}_1$) and the assertion follows from the additivity property of the spectral flow:

$$\begin{aligned}\mu_\lambda(\mathsf{D}_{A_1}) &= \mu(\mathsf{D}_{A_1}, \mathsf{D}_\lambda) \\ &= \mu(\mathsf{D}_{A_1}, \mathsf{D}_{A_0}) + \mu(\mathsf{D}_{A_0}, \mathsf{D}_\lambda) \\ &\ge \mu(D_{A_0}, \mathsf{D}_\lambda) = \mu_\lambda(\mathsf{D}_{A_0}).\end{aligned}$$

□

3.3 Relative Morse index as the Fredholm index of the linearized gradient flow equation

There is also another useful formula which characterizes the relative Morse index. Let us consider the negative $H^{1/2}$-gradient flow of $\mathcal{L}_H$ connecting two critical points $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$:

$$\frac{\partial\psi}{\partial t} = -\nabla_{1/2}\mathcal{L}_H(\psi), \quad \psi(-\infty) = \mathsf{x},\ \psi(+\infty) = \mathsf{y}, \tag{3.9}$$

where $\nabla_{1/2}\mathcal{L}_H(\psi) = (|\mathsf{D}_g| + 1)^{-1}\mathsf{D}_g - (|\mathsf{D}_g| + 1)^{-1}\nabla_\psi H(x, \psi)$ is the $H^{1/2}$-gradient of $\mathcal{L}_H$ at ψ.

The equation (3.9) plays a fundamental role for the construction of the Morse-Floer homology of $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$ which is the topic we will study in [37].

By definition, the boundary value problem (3.9) is Fredholm if its linearization at any solution ψ to (3.9)

$$\frac{\partial u}{\partial t} = -d\nabla_{1/2}\mathcal{L}_H(\psi)u, \quad u(-\infty) = 0,\ u(+\infty) = 0, \tag{3.10}$$

is a Fredholm boundary value problem, where $d\nabla_{1/2}\mathcal{L}_H(\psi) = (|\mathsf{D}_g| + 1)^{-1}\mathsf{D}_g - d_\psi\nabla_\psi H(x, \psi)$. This means that the operator $F_\psi : u \mapsto \dot{u} - d\nabla_{1/2}\mathcal{L}_H(\psi)u$ considered as $F_\psi : C^1_0(\mathbb{R}, \mathcal{H}^{1/2}(M)) \to C^0_0(\mathbb{R}, \mathcal{H}^{1/2}(M))$ (or $F_\psi : W^{1,2}_0(\mathbb{R}, \mathcal{H}^{1/2}(M)) \to L^2(\mathbb{R}, \mathcal{H}^{1/2}(M))$, or $F_\psi : W^{1,2}_0(\mathbb{R}, X) \to L^2(\mathbb{R}, X)$ for some appropriate subspace $X \subset \mathcal{H}^{1/2}(M)$) is a Fredholm operator.

Notice that, for any $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{H}^{1/2}(M)$, we can write $d\nabla_{1/2}\mathcal{L}_H(\psi) = (|\mathsf{D}_g|+1)^{-1}\mathsf{D}_\lambda - (|\mathsf{D}_g|+1)^{-1}(d_\psi\nabla_\psi H(x, \psi) - \lambda)$. Note that when $\lambda \in \mathbb{R}\setminus\mathsf{Spec}(\mathsf{D}_g)$, $(|\mathsf{D}_g|+1)^{-1}\mathsf{D}_\lambda$ is a hyperbolic operator (in the sense that its spectrum does not intersect with the imaginary axis) and $(|\mathsf{D}_g| + 1)^{-1}(d_\psi\nabla_\psi H(x, \psi) - \lambda) : \mathcal{H}^{1/2}(M) \to \mathcal{H}^{1/2}(M)$ is compact under the assumption (3.3). Thus, under the assumption $\lambda \in \mathbb{R}\setminus\mathsf{Spec}(\mathsf{D}_g)$ and $\mathsf{x}, \mathsf{y} \in \mathrm{crit}(\mathcal{L}_H)$ are non-degenerate, we have, by [13, Lemmas 15, 16], [3, Theorem 3.4], [4, Theorem E] and [21, Theorem 3.3] the following:

Theorem 3.1 *Under the above assumption, the boundary value problem (3.10) is Fredholm. Moreover, its Fredholm index (i.e., the Fredholm index of F_ψ) depends only on $d\nabla_{1/2}\mathfrak{L}_H(\mathsf{x})$ and $d\nabla_{1/2}\mathfrak{L}_H(\mathsf{y})$. We thus denote the Fredholm index* $\mathrm{index}(F_\psi)$ *by* $\mu(\mathsf{x},\mathsf{y})$.

3.4 Relations between the relative Morse indices

We have defined three indices $m_\lambda(\psi)$, $\mu_\lambda(\psi)$ (or $\mu_{\lambda,\pm}(\psi)$ for the degenerate case) and $\mu(\mathsf{x},\mathsf{y})$. We have the following relations between these indices:

Lemma 3.3 *Let us assume that* $\lambda \in \mathbb{R} \setminus \mathsf{Spec}(\mathsf{D}_g)$. *Then we have the following:*

(1) If $\psi \in crit(\mathfrak{L}_H)$ *is non-degenerate, we have* $m_\lambda(\psi) = \mu_\lambda(\psi)$.

(2) For any non-degenerate $\mathsf{x},\mathsf{y} \in crit(\mathfrak{L}_H)$, *we have* $\mu(\mathsf{x},\mathsf{y}) = m_\lambda(\mathsf{x}) - m_\lambda(\mathsf{y})$.

(3) If $\psi \in crit(\mathfrak{L}_H)$ *is degenerate, we have* $m_\lambda(\psi) = \mu_{\lambda,-}(\psi) = \mu_{\lambda,+}(\psi) - \dim\ker \mathsf{D}_{A_\psi}$.

Proof. The assertion (1) follows from [41, Theorem 3.6]. In fact, all the assumptions of that theorem are satisfied for the path $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ of Dirac type operators with $\{A_t\}_{t\in[0,1]} \subset \mathcal{A}$ which connects $\mathsf{D}_{A_0} = \mathsf{D}_\lambda$ and $\mathsf{D}_{A_1} = \mathsf{D}_{A_\psi}$: the Riesz continuity of $\{\mathsf{D}_{A_t}\}_{t\in[0,1]}$ is a consequence of the d_W-continuity (see [41, Definition 2.1]) of that path and the latter property follows from the L^∞-continuity of $\{A_t\}_{t\in[0,1]}$. D_λ-compactness of $\mathsf{D}_{A_t} - \mathsf{D}_\lambda$ (i.e., the compactness of $(\mathsf{D}_{A_t} - \mathsf{D}_\lambda)(1 + |\mathsf{D}_\lambda|^2)^{-1/2}$) is also clear from the elliptic regularity and the Sobolev embedding theorem. Thus by [41, Theorem 3.6], we have

$$\begin{aligned} \mathsf{sf}\{\mathsf{D}_{A_t}\}_{t\in[0,1]} &= \dim(E^{[0,\infty)}(\mathsf{D}_{A_1}), E^{[0,\infty)}(\mathsf{D}_{A_0})) \\ &= \dim(E^{[0,\infty)}(\mathsf{D}_{A_\psi}), E^{[0,\infty)}(\mathsf{D}_\lambda)) \\ &= -\dim(E^-(\mathsf{D}_{A_\psi}), E^-(\mathsf{D}_\lambda)), \end{aligned} \tag{3.11}$$

where $E^{[0,\infty)}(\mathsf{D}_{A_t})$ denotes the range of the spectral projection of the operator D_{A_t} with the spectral set $[0,\infty)$, i.e., $\mathrm{ran}(\mathbf{1}_{[0,\infty)}(\mathsf{D}_{A_t}))$. Although we have not defined a spectral flow for a path of Dirac type operators where at least one of the ends $\mathsf{D}_{A_0}, \mathsf{D}_{A_1}$ is degenerate, there is a suitable definition of the spectral flow for such a case and the above equality (3.11) still holds. See [41] for more details. (3.11) is exactly the assertion $m_\lambda(\psi) = \mu_\lambda(\psi)$.

The assertion (2) is a consequence of [13, Lemmas 15, 16], [3, Theorem 3.4], [4, Theorem E] and [21, Theorem 3.3].

To prove (3), for $\epsilon > 0$, we define $\mathsf{D}_{A_\psi,\pm\epsilon} := \mathsf{D}_{A_\psi} \mp \epsilon$. We take $\epsilon_0 > 0$ such that $((-\epsilon_0,\epsilon_0) \setminus \{0\}) \cap \mathrm{Spec}(\mathsf{D}_{A_\psi}) = \emptyset$. For $0 < \epsilon < \epsilon_0$, we consider $\mu_\lambda(\mathsf{D}_{A_\psi,\epsilon})$. Since $A_\psi - \epsilon\mathbf{1} \in \mathcal{A}_0$, by (1) we have

$$\mu_\lambda(\mathsf{D}_{A_\psi,\epsilon}) = \dim(E^-(\mathsf{D}_{A_\psi,\epsilon}), E^-(D_\lambda)). \tag{3.12}$$

The left hand side of (3.12) becomes, by the additivity property of the relative dimension

$$\begin{aligned} &\dim(E^-(\mathsf{D}_{A_\psi,\epsilon}), E^-(\mathsf{D}_\lambda)) \\ &= \dim(E^-(\mathsf{D}_{A_\psi,\epsilon}), E^-(\mathsf{D}_{A_\psi})) + \dim(E^-(\mathsf{D}_{A_\psi}), E^-(\mathsf{D}_\lambda)) \\ &= \dim(E^-(\mathsf{D}_{A_\psi,\epsilon}), E^-(\mathsf{D}_{A_\psi})) + m_\lambda(\psi) \\ &= \dim\ker \mathsf{D}_{A_\psi} + m_\lambda(\psi), \end{aligned} \tag{3.13}$$

where the last equality follows from $E^-(\mathsf{D}_{A_\psi,\epsilon}) = E^-(\mathsf{D}_{A_\psi}) \oplus \ker \mathsf{D}_{A_\psi}$ by our choice of ϵ. Combining (3.12) and (3.13), we have

$$\mu_\lambda(\mathsf{D}_{A_\psi,\epsilon}) = m_\lambda(\psi) + \dim\ker \mathsf{D}_{A_\psi}. \tag{3.14}$$

Similarly, we have

$$\begin{aligned}\mu_\lambda(\mathsf{D}_{A_\psi,-\epsilon}) &= \dim(E^-(\mathsf{D}_{A_\psi,-\epsilon}), E^-(\mathsf{D}_\lambda)) \\ &= \dim(E^-(\mathsf{D}_{A_\psi,-\epsilon}), E^-(\mathsf{D}_{A_\psi})) + \dim(E^-(\mathsf{D}_{A_\psi}), E^-(\mathsf{D}_\lambda)) \\ &= m_\lambda(\psi), \end{aligned} \tag{3.15}$$

since this time, we have $E^-(\mathsf{D}_{A_\psi,-\epsilon}) = E^-(\mathsf{D}_{A_\psi})$. Since $0 < \epsilon < \epsilon_0$ was arbitrary, (3.14) and (3.15) imply

$$\mu_{\lambda,+}(\psi) \geq m_\lambda(\psi) + \dim\ker \mathsf{D}_{A_\psi} \tag{3.16}$$

and

$$\mu_{\lambda,-}(\psi) \leq m_\lambda(\psi). \tag{3.17}$$

On the other hand, for any $A \in \mathcal{A}_0$ with $\|A_\psi - A\|_{L^\infty} < \epsilon$ ($\epsilon > 0$ is taken as before: $0 < \epsilon < \epsilon_0$), we have $A_\psi - \epsilon\mathbf{1} \leq A \leq A_\psi + \epsilon\mathbf{1}$ and by the monotonicity of μ_λ (Lemma 3.2), we obtain

$$\mu_\lambda(\mathsf{D}_{A_\psi,-\epsilon}) \leq \mu_\lambda(\mathsf{D}_A) \leq \mu_\lambda(\mathsf{D}_{A_\psi,\epsilon}).$$

Thus for any $A, A' \in \mathcal{A}_0$ with $\|A_\psi - A\|_{L^\infty}$, $\|A_\psi - A'\|_{L^\infty} < \epsilon$, we have by (3.14) and (3.15)

$$|\mu_\lambda(\mathsf{D}_A) - \mu_\lambda(\mathsf{D}_{A'})| \leq \dim\ker \mathsf{D}_{A_\psi}.$$

This implies that

$$0 \leq \mu_{\lambda,+}(\psi) - \mu_{\lambda,-}(\psi) \leq \dim\ker \mathsf{D}_{A_\psi}. \tag{3.18}$$

Combining (3.16)–(3.18), we see that all the inequalities (3.16)–(3.18) are equalities and we have the assertion (3). This completes the proof. □

Remark 3.1 *By Lemma 3.3 (3), $\mu_{\lambda,-}(\psi)$ plays the role of the relative Morse index $m_\lambda(\psi)$. On the other hand, $\mu_{\lambda,+}(\psi)$ is suited to be called the large relative Morse index.*

4 A reformulation of the relative Morse index

We first observe that all of the definitions of the relative Morse indices of critical points of $\mathcal{L}_H$ on $\mathcal{H}^{1/2}(M)$ given in the previous section depend crucially on the compactness of M. Thus, these definitions do not apply to solutions of (1.1) defined on non-compact manifolds such as $\mathbb{R}^m$. On the other hand, in the proof of Theorem 1.1, we will need a suitable "relative Morse index" defined for solutions to (1.1) on $\mathbb{R}^m$ which satisfies some natural properties. In order to define such a relative Morse index for solutions to (1.1) defined on non-compact manifolds, in this section, we first give one more another formulation of the relative Morse index for solutions to (1.1) defined on compact manifolds. We then show that, at least formally, our new formulation can be naturally extended to non-compact manifolds as well. Here, we recall that a similar index problem arose in the work of Angenent-van der Vorst [13] in their study of indefinite elliptic systems. As already mentioned in Remark 1.1 (1), however, their approach crucially depends on the special structure of the equations and can not be applied to our case.

Our reformulation of the relative Morse index is inspired from a classical work of Duistermaat [22] about an index problem arising from classical mechanics: in his work [22], Duistermaat considered the Morse index of the Lagrangian action functional $\mathcal{L}(q) = \int_{S^1} L(t, q(t), \dot{q}(t))\, dt$ defined on the free loop space $\Lambda(M) = C^\infty(S^1, M)$ of a manifold M, where the Lagrangian $L : \mathbb{R} \times TM \to \mathbb{R}$ is a smooth function (we note that $(q, \dot{q}) \in TM$). When the Lagrangian L satisfies some convexity condition with respect to $v = \dot{q}$, there corresponds to a dual Hamiltonian system on the cotangent bundle T^*M. The corresponding Hamiltonian $H : \mathbb{R} \times T^*M \to \mathbb{R}$ is given by the Legendre dual of L which is defined by

$$H(t, q, p) = \max_{v \in T_qM} (\langle p, v\rangle - L(t, q, v)),$$

where $p \in T_q^*M$. The corresponding Hamiltonian action is given by $\mathcal{A}(x) = \int_{S^1} x^*\lambda - H(t, x(t))\, dt$, where $\lambda = p \cdot dq$ is the Liouville form and $x(t) = (q(t), p(t)) \in C^\infty(S^1, T^*M)$ is a loop on T^*M. There is a one to one correspondence (so called the Legendre transformation) between critical points of $\mathcal{L}$ on $\Lambda(M)$ and critical points of $\mathcal{A}$ on $C^\infty(S^1, T^*M)$ given by $\mathrm{crit}(\mathcal{L}) \ni q \mapsto x = (q, p) \in \mathrm{crit}(\mathcal{A})$, where $p = \partial_v L(t, q, \dot{q})$. The result of Duistermaat asserts that the Morse index of $\mathcal{L}$ at $q \in \mathrm{crit}(\mathcal{L})$ is equal to the Maslov index (sometimes it is called the Conley-Zehnder index (see [20], [15])) of the corresponding critical point $x = (q, p) \in \mathrm{crit}(\mathcal{A})$, $p = \partial_v L(t, q, \dot{q})$, of $\mathcal{A}$. Recall that the Maslov index plays the same role as the relative Morse index for the action $\mathcal{A}$ (see [2]). In fact, the Maslov index at $x = (p, q) \in \mathrm{crit}(\mathcal{A})$ is equal to the relative Morse index of $\mathcal{A}$ at x up to an irrelevant constant (if the latter is defined), see [2].

This example suggests that the relative Morse indices $m_\lambda(\psi)$, $\mu_\lambda(\psi)$, $\mu(\mathsf{x}, \mathsf{y})$ of our action functional $\mathcal{L}_H$ should also be given by the classical Morse index of a certain "dual" action functional if a suitable condition is satisfied. One candidate for such a dual action is the Legendre-Fenchel dual $\mathcal{L}_H^*$ of $\mathcal{L}_H$ (see [43] for details about the Legendre-Fenchel duality in a general framework). In fact, for a certain class of H, the dual action $\mathcal{L}_H^*$ is well-defined. However, due to the superquadratic growth assumption of $\mathcal{L}_H$, $\mathcal{L}_H^*$ is subquadratic growth and it is only C^1 in general. Thus the classical Morse index may not be defined for $\mathcal{L}_H^*$. However, recall that the (relative) Morse index at a critical point is defined only through the information about the quadratic approximation of the action at that point, so it is not necessary to consider $\mathcal{L}_H^*$ itself at all: it is substituted by the Legendre-Fenchel dual of the quadratic approximation of $\mathcal{L}_H$, $\mathcal{A}_H(\varphi) = \frac{1}{2} d^2\mathcal{L}_H(\psi)[\varphi, \varphi]$. In fact, we shall show that for some class of H, including the model example $H(x, \psi) = \frac{1}{p+1} H(x)|\psi|^{p+1}$, the relative Morse index of $\mathcal{L}_H$ coincides with the classical Morse index of the Legendre-Fenchel dual of the quadratic approximation $\mathcal{A}_H$. But, as we shall see, this reformulation is not satisfactory for our purpose: it does not naturally continue to give a "relative Morse index" for solutions to (1.1) on $\mathbb{R}^m$. Thus we will make a slight modification of this dual construction, i.e., we will make a further regularization of it. After such a regularization, we will obtain a correct reformulation of the relative Morse index: it naturally continues from m_λ (or μ_λ) defined for solutions to (1.1) on a compact manifold to a "relative Morse index" defined for solutions to (1.1) on $\mathbb{R}^m$ along a conformal blow-up of the manifold.

In the following, for clarity and simplicity of the presentation, we first consider the model example $H(x, \psi) = \frac{1}{p+1} H(x)|\psi|^{p+1}$ and give an explicit computation for this special case. Later in §4.3, we shall show that essentially the same idea applies for a more general class of H and we give a general result.

4.1 The Legendre-Fenchel dual of the quadratic approximation of L_H

Let us fix $\lambda > 0$ with $-\lambda \notin \mathsf{Spec}(\mathsf{D})$. We assume that $H \in C^0(M)$ is such that $H > 0$ on M. The action functional L_H (see (1.9)) can be written as

$$L_H(\psi) = \frac{1}{2}\int_M \langle \psi, \mathsf{D}_{-\lambda}\psi\rangle \, d\mathrm{vol}_g - \frac{\lambda}{2}\int_M |\psi|^2 \, d\mathrm{vol}_g - \frac{1}{p+1}\int_M H(x)|\psi|^{p+1} \, d\mathrm{vol}_g,$$

where we recall $\mathsf{D}_{-\lambda} = \mathsf{D}_g + \lambda$.

The Hessian $d^2L_H(\psi)$ at $\psi \in \mathcal{L}^\infty(M)$ is given by

$$\begin{aligned} d^2L_H(\psi)[\varphi,\phi] &= \int_M \langle \varphi, \mathsf{D}_{-\lambda}\phi\rangle \, d\mathrm{vol}_g - \lambda\int_M \langle\varphi,\phi\rangle \, d\mathrm{vol}_g \\ &\quad - \int_M H(x)|\psi|^{p-1}\langle\varphi,\phi\rangle \, d\mathrm{vol}_g - (p-1)\int_M H(x)|\psi|^{p-3}\langle\psi,\varphi\rangle\langle\psi,\phi\rangle \, d\mathrm{vol}_g \end{aligned}$$

for $\varphi, \phi \in \mathcal{H}^{1/2}(M)$. We denote by $\mathcal{A}_{\psi,H}$ the quadratic approximation of L_H at ψ, i.e.,

$$\begin{aligned} \mathcal{A}_{\psi,H}(\varphi) &:= \frac{1}{2}d^2L_H(\psi)[\varphi,\varphi] \\ &= \frac{1}{2}\int_M \langle\varphi, \mathsf{D}_{-\lambda}\varphi\rangle \, d\mathrm{vol}_g - \frac{\lambda}{2}\int_M |\varphi|^2 \, d\mathrm{vol}_g \\ &\quad - \frac{1}{2}\int_M H(x)|\psi|^{p-1}|\varphi|^2 \, d\mathrm{vol}_g - \frac{p-1}{2}\int_M H(x)|\psi|^{p-3}|\langle\psi,\varphi\rangle|^2 \, d\mathrm{vol}_g. \end{aligned}$$

Let us recall the shorthand notation introduced in §2: $\mathcal{H}^{1/2}(M) := H^{1/2}(M, \mathbb{S}(M))$, $\mathcal{H}^1(M) := H^1(M, \mathbb{S}(M))$ and $\mathcal{L}^q(M) := L^q(M, \mathbb{S}(M))$ for $1 \le q \le \infty$. We shall also use a shorthand notation $\mathcal{H}^{-1/2}(M) = H^{-1/2}(M, \mathbb{S}(M)^*) \cong H^{-1/2}(M, \mathbb{S}(M))$, the dual of $\mathcal{H}^{1/2}(M, \mathbb{S}(M))$, where the identification $\mathbb{S}(M)^* \cong \mathbb{S}(M)$ is given via the hermitian metric on $\mathbb{S}(M)$. Since $-\lambda \notin \mathsf{Spec}(\mathsf{D})$,

$$\mathsf{D}_{-\lambda} = \mathsf{D}_g + \lambda : \mathcal{H}^{1/2}(M) \to \mathcal{H}^{-1/2}(M)$$

is an isomorphism. We denote its inverse by A_λ:

$$A_\lambda = \mathsf{D}_{-\lambda}^{-1} : \mathcal{H}^{-1/2}(M) \to \mathcal{H}^{1/2}(M).$$

Also

$$\mathsf{D}_{-\lambda} : \mathcal{H}^1(M) \to \mathcal{L}^2(M)$$

is an isomorphism and we denote its inverse by B_λ:

$$B_\lambda = \mathsf{D}_{-\lambda}^{-1} : \mathcal{L}^2(M) \to \mathcal{H}^1(M).$$

Let $\iota : \mathcal{H}^{1/2}(M) \ni \varphi \mapsto \varphi \in \mathcal{L}^2(M)$ be the canonical inclusion. We consider

$$K_\lambda : \mathcal{L}^2(M) \xrightarrow{\iota^*} \mathcal{H}^{-1/2}(M) \xrightarrow{A_\lambda} \mathcal{H}^{1/2}(M) \xrightarrow{\iota} \mathcal{L}^2(M)$$

and

$$\mathcal{L}^2(M) \xrightarrow{B_\lambda} \mathcal{H}^1(M) \xrightarrow{j} \mathcal{H}^{1/2}(M),$$

where ι^* is the dual of ι and j is the canonical inclusion $j : \mathcal{H}^1(M) \ni \varphi \mapsto \varphi \in \mathcal{H}^{1/2}(M)$. We have $A_\lambda \circ \iota^* = j \circ B_\lambda$ and K_λ is self-adjoint $K_\lambda^* = K_\lambda$ and compact by the Sobolev embedding theorem.

Using these notation, the action $\mathcal{A}_{\psi,H}$ is written as the following from:

$$\mathcal{A}_{\psi,H}(\varphi) = \frac{1}{2}\langle \mathsf{D}_{-\lambda}\varphi, \varphi\rangle_{\mathcal{H}^{-1/2}(M)\times\mathcal{H}^{1/2}(M)} - G_{H,\lambda}(\iota(\varphi)),$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}^{-1/2}(M)\times\mathcal{H}^{1/2}(M)}$ is the duality pairing between $\mathcal{H}^{-1/2}(M)$ and $\mathcal{H}^{1/2}(M)$ and $G_{H,\lambda} : \mathcal{L}^2(M) \to \mathbb{R}$ is defined by

$$G_{H,\lambda}(\varphi) = \frac{\lambda}{2}\int_M |\varphi|^2 \, d\mathrm{vol}_g + \frac{1}{2}\int_M H(x)|\psi|^{p-1}|\varphi|^2 \, d\mathrm{vol}_g + \frac{p-1}{2}\int_M H(x)|\psi|^{p-3}|\langle\psi,\varphi\rangle|^2 \, d\mathrm{vol}_g.$$

With the above notation, the differential of $\mathcal{A}_{\psi,H}$ is given by

$$d\mathcal{A}_{\psi,H}(\varphi) = \mathsf{D}_{-\lambda}\varphi - \iota^* dG_{H,\lambda}(\iota(\varphi)) \in \mathcal{H}^{-1/2}(M)$$

for $\varphi \in \mathcal{H}^{1/2}(M)$.

We next define the dual action $\mathcal{A}^*_{\psi,H,\lambda}$ of $\mathcal{A}_{\psi,H}$. For this, we first define the Legendre-Fenchel dual $G^*_{H,\lambda}$ of $G_{H,\lambda}$ on $\mathcal{L}^2(M)$ by (see [43] for more details about the Legendre-Fenchel duality)

$$G^*_{H,\lambda}(\varphi) := \max\{\langle\phi,\varphi\rangle_{\mathcal{L}^2(M)\times\mathcal{L}^2(M)} - G_{H,\lambda}(\phi) : \phi \in \mathcal{L}^2(M)\},$$

where $\varphi \in \mathcal{L}^2(M)$ and $\langle\cdot,\cdot\rangle_{\mathcal{L}^2\times\mathcal{L}^2}$ is the (real) duality pairing between $\mathcal{L}^2(M)$ and $\mathcal{L}^2(M)$, i.e., the real part of the L^2-inner product on $\mathcal{L}^2(M)$. The above maximum is attained at ϕ which satisfies $\varphi = dG_{H,\lambda}(\phi)$, i.e.,

$$\varphi = \lambda\phi + H(x)|\psi|^{p-1}\phi + (p-1)H(x)|\psi|^{p-3}\langle\psi,\phi\rangle\psi. \tag{4.1}$$

To solve ϕ in terms of φ, it is convenient to decompose spinors into two components $\mathrm{span}\{\psi\}$ and $\mathrm{span}\{\psi\}^\perp$. For this purpose, we take a measurable section e of $\mathbb{S}(M) \to M$ such that $|e(x)| = 1$ for a.e. $x \in M$. Such a section is easily constructed by locally trivializing the bundle $\mathbb{S}(M) \to M$. We then define $e_\psi \in \mathcal{L}^\infty(M)$ by setting $e_\psi(x) = \psi(x)/|\psi(x)|$ for a.e. $x \in M^*_\psi := \{x \in M : \psi(x) \neq 0\}$ and $e_\psi(x) = e(x)$ for a.e. $x \in M \setminus M^*_\psi$. Notice that such an e_ψ is not uniquely determined by ψ (since it depends on the choice of e), but we fix one such e_ψ throughout the following argument. In fact, it is easy to see that the resulting formula (4.5) below does not depend on the choice of e and the conclusion is independent of such a choice. We define $P_\psi, P^\perp_\psi \in L^\infty(M, \mathrm{Sym}(\mathbb{S}(M))$ by $P_\psi(\phi) = \langle e_\psi, \phi\rangle e_\psi$ and $P^\perp_\psi(\phi) = \phi - P_\psi(\phi)$. Then (4.1) can be written as

$$\varphi = (\lambda + pH(x)|\psi|^{p-1})P_\psi(\phi) + (\lambda + H(x)|\psi|^{p-1})P^\perp_\psi(\phi). \tag{4.2}$$

Comparing $\mathrm{span}\{e_\psi\}$ and $\mathrm{span}\{e_\psi\}^\perp$ components of the both sides of (4.2), we obtain $P_\psi(\varphi) = (\lambda + pH(x)|\psi|^{p-1})P_\psi(\phi)$ and $P^\perp_\psi(\varphi) = (\lambda + H(x)|\psi|^{p-1})P^\perp_\psi(\phi)$ and

$$\begin{aligned}\phi &= P_\psi(\phi) + P^\perp_\psi(\phi)\\ &= \frac{1}{\lambda + pH(x)|\psi|^{p-1}}P_\psi(\varphi) + \frac{1}{\lambda + H(x)|\psi|^{p-1}}P^\perp_\psi(\varphi).\end{aligned} \tag{4.3}$$

We therefore have the following formula for $G^*_{H,\lambda}(\varphi)$ after a simple calculation:

$$G^*_{H,\lambda}(\varphi) = \frac{1}{2}\int_M \frac{1}{\lambda + pH(x)|\psi|^{p-1}}|P_\psi(\varphi)|^2 \, d\mathrm{vol}_g + \frac{1}{2}\int_M \frac{1}{\lambda + H(x)|\psi|^{p-1}}|P^\perp_\psi(\varphi)|^2 \, d\mathrm{vol}_g. \tag{4.4}$$

The dual action $\mathcal{A}^*_{\psi,H,\lambda}$ of $\mathcal{A}_{\psi,H}$ is then defined by

$$\begin{aligned}\mathcal{A}^*_{\psi,H,\lambda}(\varphi) &= G^*_{H,\lambda}(\varphi) - \frac{1}{2}\langle K_\lambda\varphi,\varphi\rangle_{\mathcal{L}^2(M)\times\mathcal{L}^2(M)}\\ &= \frac{1}{2}\int_M \frac{1}{\lambda+pH(x)|\psi|^{p-1}}|P_\psi(\varphi)|^2\,\mathrm{dvol}_g + \frac{1}{2}\int_M \frac{1}{\lambda+H(x)|\psi|^{p-1}}|P_\psi^\perp(\varphi)|^2\,\mathrm{dvol}_g\\ &\quad - \frac{1}{2}\int_M \langle K_\lambda\varphi,\varphi\rangle\,\mathrm{dvol}_g. \end{aligned}\tag{4.5}$$

We then have the following index formula:

Proposition 4.1 *Let us assume that $\psi \in \mathcal{L}^\infty(M)$ and $\lambda > 0$ with $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. We then have the following formula of the relative Morse index $m_{-\lambda}(\psi)$:*

$$m_{-\lambda}(\psi) = m\text{-}ind(\mathcal{A}^*_{\psi,H,\lambda}),\tag{4.6}$$

where $m\text{-}ind(\mathcal{A}^*_{\psi,H,\lambda})$ is the Morse index of $\mathcal{A}^*_{\psi,H,\lambda}$.

We will not give a proof of this proposition since it easily follows from the index theorem given in the next subsection.

As for the action $\mathcal{A}^*_{\psi,H,\lambda}$, note that the defining formula (4.5) is only meaningful for $\lambda > 0$: in fact, for $\lambda \le 0$, $G_{H,\lambda}$ is not strictly convex and its Legendre-Fenchel dual $G^*_{H,\lambda}$ is not defined. This is also seen from the formula (4.4): for $\lambda \le 0$, it does not have a well-defined meaning in general since one of $\lambda + pH(x)|\psi|^{p-1}$ and $\lambda + H(x)|\psi|^{p-1}$ may vanish on a large subset of M. This will cause a problem if we want to "continue" the relative Morse indices $m_{-\lambda}$ defined on compact spin manifolds to the one defined on $\mathbb{R}^m$ by blowing up the metric based on the index formula (4.6). In fact, to retain appropriate continuity property of the relative Morse index along such a blow-up process, we need continuity at $\lambda = 0$ for $\mathbb{R}^m$. But the action $\mathcal{A}^*_{\psi,H,\lambda}$ becomes degenerate (or singular) at the limit. For this reason, in the next subsection, we give a further regularization which will remove the possible degeneracy after the blow-up of the manifold.

4.2 Index formulas

In this subsection, we give a modification of the index formula given in the previous subsection, see Proposition 4.1. To motivate our construction, we first observe that the dual action $\mathcal{A}^*_{\psi,H,\lambda}$ is rewritten as follows:

$$\mathcal{A}^*_{\psi,H,\lambda}(\varphi) = \frac{1}{2}((H(x)|\psi|^{p-1} + \lambda L_\psi)^{-1}L_\psi(\varphi),\varphi)_{L^2(M)} - \frac{1}{2}(K_\lambda(\varphi),\varphi)_{L^2(M)},\tag{4.7}$$

where $L_\psi : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$ is an isomorphism defined by $L_\psi(\varphi) = \frac{1}{p}P_\psi + P_\psi^\perp$.

After conjugation with the operator $\sqrt{2}(H(x)|\psi|^{p-1}+\lambda L_\psi)^{1/2}$, we have a self-adjoint operator $L_\psi - T_{\psi,H,\lambda}$ on $\mathcal{L}^2(M)$ which is conjugate to $\mathcal{A}^*_{\psi,H,\lambda}$:

$$\mathcal{A}^*_{\psi,H,\lambda} \overset{\text{conj.}}{\cong} L_\psi - T_{\psi,H,\lambda},\tag{4.8}$$

where $T_{\psi,H,\lambda} = (H(x)|\psi|^{p-1} + \lambda L_\psi)^{1/2} \circ K_\lambda \circ (H(x)|\psi|^{p-1} + \lambda L_\psi)^{1/2}$.

In view of the discussion given in the previous subsection, it is natural to expect that the relative Morse index $m_{-\lambda}(\psi)$ is expressed in terms of a spectral information of $L_\psi - T_{\psi,H,\lambda}$.

Notice also that the operator $L_\psi - T_{\psi,H,\lambda}$ has a well-defined meaning for $\lambda = 0$. Thus, as we remarked at the end of the last subsection, we also expect that the spectral information of $L_\psi - T_{\psi,H,\lambda}$ naturally continues along blow-up process to the corresponding one for solutions to (1.1) defined on the blow-up manifold $\mathbb{R}^m$. In the following, we show that these expectations are realized and we give a formula of $m_{-\lambda}(\psi)$ in terms of the spectral property of $L_\psi - T_{\psi,H,\lambda}$.

Let us assume that $\lambda > 0$ and $-\lambda \notin \mathsf{Spec}(\mathsf{D})$. We also assume that $\psi \in \mathcal{H}^{1/2}(M)$ is a non-trivial critical point of L_H so that it satisfies

$$\mathsf{D}\psi = H(x)|\psi|^{p-1}\psi \tag{4.9}$$

weakly on M. We henceforth assume that $H \in C^{0,\alpha}(M)$ for some $0 < \alpha < 1$. Then by the elliptic regularity theory (see [8], [9], or [33, Appendix]), we have $\psi \in C^{1,\alpha}(M)$ and ψ is a classical solution to (4.9). Since we have assumed that ψ is not trivial $\psi \not\equiv 0$, by the weak unique continuation principle for Dirac operators which can be applied to a solution φ of the equation of the form $\mathsf{D}_g\varphi + V\varphi = 0$, where $V \in L^\infty(M)$, ψ does not vanish identically on any non-empty open subset, see [17]. Thus the set $M^* = \{x \in M : \psi(x) \neq 0\}$ is an open and dense subset of M. We define e_ψ, P_ψ and $P_\psi^\perp$ as in the previous subsection. We first give the following characterization of $\ker \mathcal{A}_{\psi,H}$, where we recall $\mathcal{A}_{\psi,H} = \frac{1}{2}d^2L_H(\psi)$.

Lemma 4.1 *The following map is an isomorphism:*

$$\mathcal{H}^{1/2}(M) \supset \ker \mathcal{A}_{\psi,H} \ni \varphi \mapsto \mathsf{D}_{-\lambda}\varphi \in \ker(L_\psi - (H(x)|\psi|^{p-1} + \lambda L_\psi)K_\lambda) \subset \mathcal{L}^2(M).$$

Remark 4.1 *Observe that, by definition, we have*

$$\ker \mathcal{A}_{\psi,H} = \{\varphi \in \mathcal{H}^{1/2}(M) : d^2\mathcal{L}_H(\psi)[\varphi,\phi] = 0 \text{ for any } \phi \in \mathcal{H}^{1/2}(M)\}$$

and $\ker \mathcal{A}_{\psi,H} \subset C^{1,\beta}(M)$ *for some* $0 < \beta < 1$ *by the elliptic regularity theory, since* $\varphi \in \ker \mathcal{A}_{\psi,H}$ *satisfies (4.10) below in the weak sense. Recall that we have assumed* $H \in C^{0,\alpha}(M)$ *and thus* $\psi \in C^{1,\alpha}(M)$. *From this, we have* $\mathsf{D}_{-\lambda}\varphi \in \mathcal{L}^2(M)$ *for any* $\varphi \in \ker \mathcal{A}_{\psi,H}$.

Proof of Lemma 4.1. Let $\varphi \in \ker \mathcal{A}_{\psi,H}$. By Remark 4.1 above, we have $\varphi \in C^{1,\beta}(M)$ for some $0 < \beta < 1$ and φ satisfies the following equation

$$\mathsf{D}_{-\lambda}\varphi - (H(x)|\psi|^{p-1} + \lambda)\varphi - (p-1)H(x)|\psi|^{p-3}\langle\psi,\varphi\rangle\psi = 0. \tag{4.10}$$

We set $\phi := \mathsf{D}_{-\lambda}\varphi \in \mathcal{L}^2(M)$. By the notation of the previous subsection, we have $\varphi = A_\lambda(\iota^*(\phi))$. From (4.10), $\phi \in \mathcal{L}^2(M)$ satisfies the following equation:

$$\phi - (H(x)|\psi|^{p-1} + \lambda)K_\lambda(\phi) - (p-1)H(x)|\psi|^{p-3}\langle\psi, K_\lambda(\phi)\rangle\psi = 0. \tag{4.11}$$

Taking $\mathrm{span}\{e_\psi\}$ and $\mathrm{span}\{e_\psi\}^\perp$ components of (4.11), respectively, we obtain: denoting $\phi^\intercal := P_\psi(\phi)$ and $\phi^\perp := P_\psi^\perp(\phi)$,

$$\begin{aligned}\phi^\intercal &= (H(x)|\psi|^{p-1} + \lambda)(K_\lambda(\phi))^\intercal + (p-1)H(x)|\psi|^{p-3}\langle\psi, K_\lambda(\phi)\rangle\psi \\ &= (pH(x)|\psi|^{p-1} + \lambda)(K_\lambda(\phi))^\intercal,\end{aligned}$$

$$\phi^\perp = (H(x)|\psi|^{p-1} + \lambda)(K_\lambda(\phi))^\perp.$$

We therefore have:

$$
\begin{aligned}
L_\psi(\phi) &= \frac{1}{p}\phi^{\intercal} + \phi^{\perp} \\
&= H(x)|\psi|^{p-1}K_\lambda(\phi) + \lambda\Big(\frac{1}{p}(K_\lambda(\phi))^{\intercal} + (K_\lambda(\phi))^{\perp}\Big) \\
&= (H(x)|\psi|^{p-1} + \lambda L_\psi)K_\lambda(\phi).
\end{aligned}
$$

This proves $\phi := \mathsf{D}_{-\lambda}\varphi \in \ker(L_\psi - (H(x)|\psi|^{p-1} + \lambda L_\psi)K_\lambda)$ if $\varphi \in \ker \mathcal{A}_{\psi,H}$. The injectivity is clear since $-\lambda \notin \mathsf{Spec}(\mathsf{D})$.

To prove the surjectivity, assume $\phi \in \ker(L_\psi - (H(x)|\psi|^{p-1} + \lambda L_\psi)K_\lambda)$. We define $\varphi := \mathsf{D}_{-\lambda}^{-1}\phi = K_\lambda(\phi)$. Here, we understand this equality as elements of $\mathcal{L}^2(M)$, i.e., after embedding $\mathsf{D}_{-\lambda}^{-1}\phi \in \mathcal{H}^1(M)$ into $\mathcal{L}^2(M)$ by the canonical embedding. Comparing span$\{e_\psi\}$ and span$\{e_\psi\}^\perp$-components of $L_\psi(\phi) = (H(x)|\psi|^{p-1} + \lambda L_\psi)K_\lambda(\phi)$, respectively, we have

$$
\begin{aligned}
\phi^{\intercal} &= pH(x)|\psi|^{p-1}(K_\lambda(\phi))^{\intercal} + \lambda(K_\lambda(\phi))^{\intercal}, \\
\phi^{\perp} &= H(x)|\psi|^{p-1}(K_\lambda(\phi))^{\perp} + \lambda(K_\lambda(\phi))^{\perp}.
\end{aligned}
$$

From these, we have

$$
\begin{aligned}
\phi &= \phi^{\intercal} + \phi^{\perp} \\
&= \lambda K_\lambda(\phi) + H(x)|\psi|^{p-1}K_\lambda(\phi) + (p-1)H(x)|\psi|^{p-1}(K_\lambda(\phi))^{\intercal} \\
&= \lambda K_\lambda(\phi) + H(x)|\psi|^{p-1}K_\lambda(\phi) + (p-1)H(x)|\psi|^{p-3}\langle\psi, K_\lambda(\phi)\rangle\psi \\
&= \lambda\varphi + H(x)|\psi|^{p-1}\varphi + (p-1)H(x)|\psi|^{p-3}\langle\psi,\varphi\rangle\psi. \qquad (4.12)
\end{aligned}
$$

Since $\phi = \mathsf{D}_{-\lambda}\varphi$, by (4.12), we have

$$
\mathsf{D}_{-\lambda}\varphi = \lambda\varphi + H(x)|\psi|^{p-1}\varphi + (p-1)H(x)|\psi|^{p-3}\langle\psi,\varphi\rangle\psi,
$$

i.e., $\varphi \in \ker \mathcal{A}_{\psi,H}$. This completes the proof. $\square$

In the next lemma, we rewrite the result of Lemma 4.1 in terms of the kernel of the operator $L_\psi - T_{\psi,H,\lambda}$ defined above. For simplicity, we set $a_{H,\psi}(x) = H(x)|\psi(x)|^{p-1}$. For $\lambda \geq 0$, we observe that $a_{H,\psi} + \lambda L_\psi : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$ is a non-negative self-adjoint operator. We denote by $(a_{H,\psi} + \lambda L_\psi)^{1/2}$ its square root. It is given by $(a_{H,\psi} + \lambda L_\psi)^{1/2} = (a_{H,\psi} + \lambda/p)^{1/2}P_\psi + (a_{H,\psi} + \lambda)^{1/2}P_\psi^\perp$. Recall that we have defined $T_{\psi,H,\lambda}$ (see (4.8)) by conjugating K_λ with $(a_{H,\psi} + \lambda L_\psi)^{1/2}$:

$$
T_{\psi,H,\lambda} = (a_{H,\psi} + \lambda L_\psi)^{1/2} \circ K_\lambda \circ (a_{H,\psi} + \lambda L_\psi)^{1/2}.
$$

We thus have a compact self-adjoint operator $T_{\psi,H,\lambda} : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$.

Lemma 4.2 *The following map is an isomorphism*

$$
\mathcal{L}^2(M) \supset \ker(L_\psi - (a_{H,\psi} + \lambda L_\psi)K_\lambda) \ni \phi \mapsto (a_{H,\psi} + \lambda L_\psi)^{-1/2}\phi \in \ker(L_\psi - T_{\psi,H,\lambda}) \subset \mathcal{L}^2(M).
$$

Remark 4.2 *Note that for $\phi \in \ker(L_\psi - (a_{H,\psi} + \lambda L_\psi)K_\lambda)$, we have $(a_{H,\psi} + \lambda L_\psi)^{-1/2}\phi = (a_{H,\psi} + \lambda L_\psi)^{1/2}L_\psi^{-1}K_\lambda\phi \in \mathcal{L}^2(M)$ and the above map is well-defined.*

Proof of Lemma 4.2. Assume $\phi \in \ker(L_\psi - (a_{H,\psi} + \lambda L_\psi)K_\lambda)$. We set $\varphi = (a_{H,\psi} + \lambda L_\psi)^{1/2} L_\psi^{-1} K_\lambda \phi$. Note that any two of the operators L_ψ, $a_{H,\psi} + \lambda L_\psi$ and $(a_{H,\psi} + \lambda L_\psi)^{1/2}$ commute with each other and we have

$$\phi = (a_{H,\psi} + \lambda L_\psi) L_\psi^{-1} K_\lambda \phi = (a_{H,\psi} + \lambda L_\psi)^{1/2} \varphi. \tag{4.13}$$

(4.13) implies that

$$L_\psi \varphi = (a_{H,\psi} + \lambda L_\psi)^{1/2} K_\lambda \phi = (a_{H,\psi} + \lambda L_\psi)^{1/2} K_\lambda (a_{H,\psi} + \lambda L_\psi)^{1/2} \varphi = T_{\psi,H,\lambda} \varphi,$$

i.e., $\varphi \in \ker(L_\psi - T_{\psi,H,\lambda})$. Thus, by Remark 4.2, the map in the statement of the lemma takes values in $\ker(L_\psi - T_{\psi,H,\lambda})$ and it is injective since $a_{H,\psi} \geq 0$ and $\lambda > 0$.

To prove surjectivity, assume $\varphi \in \ker(L_\psi - T_{\psi,H,\lambda})$. We set $\phi := (a_{H,\psi} + \lambda L_\psi)^{1/2} \varphi$. Operating $(a_{H,\psi} + \lambda L_\psi)^{1/2}$ on both sides of the equation $L_\psi \varphi = (a_{H,\psi} + \lambda L_\psi)^{1/2} K_\lambda (a_{H,\psi} + \lambda L_\psi)^{1/2} \varphi$, we have

$$L_\psi \phi = (a_{H,\psi} + \lambda L_\psi) K_\lambda \phi.$$

This implies that $\phi \in \ker(L_\psi - (a_{H,\psi} + \lambda L_\psi)K_\lambda)$ and the surjectivity is proved. □

Combining Lemma 4.1 and Lemma 4.2, we obtain:

Corollary 4.1 *We have the following isomorphism*

$$\mathcal{L}^2(M) \supset \ker \mathcal{A}_{\psi,H} \ni \varphi \mapsto (a_{H,\psi} + \lambda L_\psi)^{-1/2} \mathsf{D}_{-\lambda} \varphi \in \ker(L_\psi - T_{\psi,H,\lambda}) \subset \mathcal{L}^2(M).$$

Corollary 4.1 gives a characterization of the nullity of $d^2\mathcal{L}_H(\psi)$, $\mathrm{null}(\psi) := \dim\ker d^2\mathcal{L}_H(\psi)$, in terms of the dimension of the kernel of $L_\psi - T_{\psi,H,\lambda}$. Notice that $\ker(L_\psi - T_{\psi,H,\lambda}) = \ker(\mathbf{1} - L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2})$, where $\mathbf{1} : \mathcal{L}^2(M) \to \mathcal{L}^2(M)$ is the identity. We also observe that the operator $L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}$ does not depend on the choice of e_ψ in the definition of L_ψ. This is easily seen by writing

$$L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2} = (a_{H,\psi} L_\psi^{-1} + \lambda)^{1/2} \circ K_\lambda \circ (a_{H,\psi} L_\psi^{-1} + \lambda)^{1/2}$$

and noting that $a_{H,\psi} L_\psi^{-1}$ does not depend on the choice of e_ψ.

The following index formula is the main result of this section:

Proposition 4.2 *Assume $\lambda > 0$, $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. Let $\psi \in \mathcal{H}^{1/2}(M)$ be a critical point of L_H. We have the following:*

(1) $\mathrm{null}(\psi) = \dim\ker(\mathbf{1} - L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2})$.

(2) $\mu_{-\lambda,-}(\psi) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu > 1\}$.

(3) $\mu_{-\lambda,+}(\psi) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu \geq 1\}$.

Proof. (1) follows from Corollary 4.1 since $\ker(L_\psi - T_{\psi,H,\lambda}) = \ker(\mathbf{1} - L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2})$ via the conjugation with $L_\psi^{-1/2}$. Recall that the Hessian $d^2 L_H(\psi) = 2\mathcal{A}_{\psi,H}$ is given by

$$d^2\mathcal{L}_H(\psi)[\varphi,\varphi] = 2\mathcal{A}_{\psi,H}(\varphi) = \int_M \langle \varphi, (\mathsf{D}_g - A_\psi)\varphi \rangle \, d\mathrm{vol}_g,$$

where A_ψ is defined in (3.5). Notice that A_ψ is written as

$$A_\psi = a_{H,\psi}(x)(\mathbf{1} + (p-1)P_\psi),$$

where $a_{H,\psi}(x) = H(x)|\psi|^{p-1}$ is as before. We then have

$$\begin{aligned}\mathsf{D}_{A_\psi} &= \mathsf{D}_g - A_\psi \\ &= \mathsf{D}_{-\lambda} - [(a_{H,\psi} + \lambda)\mathbf{1} + (p-1)a_{H,\psi}(x)P_\psi].\end{aligned}$$

We take $0 \le \epsilon \le \lambda$ such that $A_\psi - \epsilon \in \mathcal{A}_0$. For $0 \le \theta \le 1$, define $A_{\lambda,\psi,\epsilon,\theta} := \theta[(a_{H,\psi} - \epsilon + \lambda)\mathbf{1} + (p-1)a_{H,\psi}(x)P_\psi]$. Since $a_{H,\psi} \ge 0$ and $\lambda - \epsilon \ge 0$, $A_{\lambda,\psi,\epsilon,\theta}$ is monotonically non-decreasing with respect to θ: for $0 \le \theta \le \theta' \le 1$, we have $A_{\lambda,\psi,\epsilon,\theta} \le A_{\lambda,\psi,\epsilon,\theta'}$. Furthermore, since $m_{-\lambda}(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,0}) = m_{-\lambda}(\mathsf{D}_{-\lambda}) = 0$, we see by Lemma 3.2 and its proof that $m_{-\lambda}(\mathsf{D}_{A_\psi - \epsilon}) = m_{-\lambda}(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,1})$ is the sum of $\dim\ker(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,\theta})$ for θ running from 0 to 1:

$$\mu_{-\lambda}(\mathsf{D}_{A_\psi - \epsilon}) = \sum_{0<\theta<1} \dim\ker(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,\theta}). \tag{4.14}$$

Examining the proof of Lemma 4.1 and Lemma 4.2, we have the following isomorphisms

$$\ker(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,\theta}) \ni \varphi \mapsto \mathsf{D}_{-\lambda}\varphi \in \ker(L_\psi - \theta(a_{H,\psi} + (\lambda-\epsilon)L_\psi)K_\lambda), \tag{4.15}$$

$$\ker(L_\psi - \theta(a_{H,\psi} + (\lambda-\epsilon)L_\psi)K_\lambda) \ni \phi \mapsto \theta^{-1}(a_{H,\psi} + (\lambda-\epsilon)L_\psi)^{-1/2}\phi \in \ker(L_\psi - \theta T_{\psi,H,\epsilon,\lambda}), \tag{4.16}$$

where $T_{\psi,H,\epsilon,\lambda} = (a_{H,\psi} + (\lambda-\epsilon)L_\psi)^{1/2} \circ K_\lambda \circ (a_{H,\psi} + (\lambda-\epsilon)L_\psi)^{1/2}$. Combining (4.15) and (4.16), as in Corollary 4.1, we obtain an isomorphism

$$\begin{aligned}\ker(\mathsf{D}_{-\lambda} - A_{\lambda,\psi,\epsilon,\theta}) \ni \varphi \mapsto \theta^{-1}(a_{H,\psi} + (\lambda-\epsilon)L_\psi)^{-1/2}\mathsf{D}_{-\lambda}\varphi &\in \ker(L_\psi - \theta T_{\psi,H,\epsilon,\lambda}) \\ &= \ker(\mathbf{1} - \theta L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}).\end{aligned} \tag{4.17}$$

Recall that $L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}$ is a compact self-adjoint operator. Thus its non-zero spectrum is an eigenvalue with finite multiplicity and 0 is the only accumulation point of its spectrum. Let $\mu_1, \ldots, \mu_k$ be eigenvalues of $L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}$ larger than 1/2 counted with multiplicities. For any eigenvalue μ of $L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}$ with $\mu \le 1/2$, we have $\theta\mu < 1/2$ for $0 < \theta < 1$ and the corresponding eigenvalue $\theta\mu - 1$ of $\theta L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2} - \mathbf{1}$ is smaller than $-1/2$ for any $0 < \theta < 1$. Thus in order to count eigenvalues of $\theta L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2} - \mathbf{1}$ which change their signs when θ running from 0 to 1, it is sufficient to consider the family of eigenvalues $\{\theta\mu_i - 1\}_{0<\theta<1}$ $(1 \le i \le k)$ listed above. Among them, only eigenvalues $\{\theta\mu_i - 1\}_{0<\theta<1}$ with $\mu_i > 1$ change their signs when θ running from 0 to 1. Thus we have

$$\begin{aligned}&\#\{\mu_\theta \in \mathsf{Spec}(\theta L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2} - \mathbf{1}) : \mu_\theta \text{ changes the sign when } \theta \text{ running from 0 to 1}\} \\ &= \sum_{0<\theta<1} \dim\ker(\mathbf{1} - \theta L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}) \\ &= \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}) : \mu > 1\}\end{aligned} \tag{4.18}$$

By (4.14), (4.17) and (4.18), we obtain

$$\mu_{-\lambda}(\mathsf{D}_{A_\psi - \epsilon}) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}) : \mu > 1\}. \tag{4.19}$$

By the monotonicity Lemma 3.2, we have $\mu_{-\lambda}(\mathsf{D}_{A_\psi-\epsilon}) \nearrow \mu_{-\lambda,-}(\psi)$ as $\epsilon \searrow 0$. Similarly, we have

$$\mu_{-\lambda}(\mathsf{D}_{A_\psi+\epsilon}) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,-\epsilon,\lambda} L_\psi^{-1/2}) : \mu > 1\} \tag{4.20}$$

for $\epsilon > 0$ and $\mu_{-\lambda}(\mathsf{D}_{A_\psi+\epsilon}) \searrow \mu_{-\lambda,+}(\psi)$ as $\epsilon \searrow 0$.

By Lemma 3.3 (3), (4.19), (4.20) and the continuity of the spectrum of $L_\psi^{-1/2} T_{\psi,H,\pm\epsilon,\lambda} L_\psi^{-1/2}$ with respect to ϵ, we have

$$\begin{aligned} \dim\ker \mathsf{D}_{A_\psi} &= \mu_{-\lambda,+}(\psi) - \mu_{-\lambda,-}(\psi) \\ &= \lim_{\epsilon\searrow 0}(\mu_{-\lambda}(\mathsf{D}_{A_\psi+\epsilon}) - \mu_{-\lambda}(\mathsf{D}_{A_\psi-\epsilon})) \\ &= \lim_{\epsilon\searrow 0}(\#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,-\epsilon,\lambda} L_\psi^{-1/2}) : \mu > 1\} \\ &\quad - \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\epsilon,\lambda} L_\psi^{-1/2}) : \mu > 1\}) \\ &\le \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu \ge 1\} \\ &\quad - \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu > 1\} \\ &= \dim\ker(\mathbf{1} - L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) = \dim\ker \mathsf{D}_{A_\psi}, \end{aligned} \tag{4.21}$$

where the the number of the spectrum is counted with multiplicities. Therefore, all the inequalities in (4.21) are equalities and we have $\mu_{-\lambda,-}(\psi) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu > 1\}$ and $\mu_{-\lambda,+}(\psi) = \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}) : \mu \ge 1\}$. This completes the proof. □

4.3 A generalization

The results of §4.1 and §4.2 continue to hold for more general class of H. We assume that $H \in C^2(\mathbb{S}(M))$ and it is convex in the vertical direction in the sense that

$$d_\psi \nabla_\psi H(x,\psi) \ge 0 \quad \text{on } \mathbb{S}_x(M). \tag{4.22}$$

Assume that $\psi \in \mathcal{L}^\infty(M) \cap \mathcal{H}^{1/2}(M)$ and $\lambda > 0$ with $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. The second order approximation of $\mathcal{L}_H$ at ψ is given by

$$\mathcal{A}_{\psi,H}(\varphi) = \frac{1}{2}\langle \mathsf{D}_{-\lambda}\varphi, \varphi\rangle_{\mathcal{H}^{-1/2}(M)\times\mathcal{H}^{1/2}(M)} - G_{H,\psi,\lambda}(\iota(\varphi)), \tag{4.23}$$

where in this general case

$$G_{H,\psi,\lambda}(\varphi) = \frac{1}{2}\int_M (\lambda|\varphi|^2 + \langle d_\psi \nabla_\psi H(x,\psi)\varphi, \varphi\rangle)\, d\mathrm{vol}_g \tag{4.24}$$

for $\varphi \in \mathcal{L}^2(M)$.

The Legendre-Fenchel dual $G^*_{\psi,H,\lambda}$ of $G_{\psi,H,\lambda}$ is defined as in §4.1 and we have

$$G^*_{\psi,H,\lambda}(\phi) = \frac{1}{2}\langle (\lambda + d_\psi \nabla_\psi H(x,\psi))^{-1}\phi, \phi\rangle_{\mathcal{L}^2(M)\times\mathcal{L}^2(M)}. \tag{4.25}$$

Thus the dual action of $\mathcal{A}_{\psi,H}$ in this case is defined by

$$\mathcal{A}^*_{\psi,H,\lambda}(\varphi) = \frac{1}{2}\langle (\lambda + d_\psi \nabla_\psi H(x,\psi))^{-1}\phi, \phi\rangle_{\mathcal{L}^2(M)\times\mathcal{L}^2(M)} - \frac{1}{2}\langle K_\lambda \varphi, \varphi\rangle_{\mathcal{L}^2(M)\times\mathcal{L}^2(M)}. \tag{4.26}$$

After conjugation with a non-negative self-adjoint operator $\sqrt{2}(\lambda + d_\psi \nabla_\psi H(x,\psi))^{1/2}$, we have a self-adjoint operator $\mathbf{1} - \mathcal{T}_{\psi,H,\lambda}$ on $\mathcal{L}^2(M)$:

$$\mathcal{A}^*_{\psi,H,\lambda} \overset{\text{conj.}}{\cong} \mathbf{1} - \mathcal{T}_{\psi,H,\lambda}, \tag{4.27}$$

where in this case $\mathcal{T}_{\psi,H,\lambda} = (\lambda + d_\psi \nabla_\psi H(x,\psi))^{1/2} \circ K_\lambda \circ (\lambda + d_\psi \nabla_\psi H(x,\psi))^{1/2}$.

The resulting formula (4.27) is slightly different from (4.8) when $H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1}$, but it is essentially the same: conjugating the right side of (4.27) with $L_\psi^{1/2}$, we have (4.8) for the special case $H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1}$: $L_\psi^{1/2}(\mathbf{1} - \mathcal{T}_{\psi,H,\lambda})L_\psi^{1/2} = L_\psi - T_{\psi,H,\lambda}$. Then repeating the similar arguments as in the previous subsections, we have the following generalization:

Proposition 4.3 *Assume $\lambda > 0$, $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. Assume that $H \in C^2(\mathbb{S}(M))$ satisfies (4.22). Let $\psi \in \mathcal{H}^{1/2}(M) \cap \mathcal{L}^\infty(M)$ be a critical point of $\mathcal{L}_H$. We then have the following:*

(1) $\mathrm{null}(\psi) := \dim\ker d^2\mathcal{L}_H(\psi) = \dim\ker(\mathbf{1} - \mathcal{T}_{\psi,H,\lambda})$.

(2) $\mu_{-\lambda,-}(\psi) = \#\{\mu \in \mathsf{Spec}(\mathcal{T}_{\psi,H,\lambda}) : \mu > 1\}$.

(3) $\mu_{-\lambda,+}(\psi) = \#\{\mu \in \mathsf{Spec}(\mathcal{T}_{\psi,H,\lambda}) : \mu \geq 1\}$.

Remark 4.3 *(i) In Proposition 4.2 and Proposition 4.3, we have given a characterization of $\mu_\lambda(\psi)$ (more generally, $\mu_{\lambda,\pm}(\psi)$) only for $\lambda < 0$ with $\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. This is because $T_{\psi,H,-\lambda}$ and $\mathcal{T}_{\psi,H,-\lambda}$ are defined only for $\lambda < 0$ with $\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$. However, for $\lambda \geq 0$ with $\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$, by using Lemma 3.1, it is also possible to give a similar characterization. Namely, take $\lambda_0 < 0$ with $\lambda_0 \notin \mathsf{Spec}(\mathsf{D}_g)$. By Lemma 3.1, we have*

$$\begin{aligned}\mu_{\lambda,-}(\psi) &= \mu_{\lambda_0,-}(\psi) - \#\{\lambda_k \in \mathsf{Spec}(\mathsf{D}_g) : \lambda_0 < \lambda_k < \lambda\} \\ &= \#\{\mu \in \mathsf{Spec}(L_\psi^{-1/2}T_{\psi,H,-\lambda_0}L_\psi^{-1/2}) : \mu > 1\} - \#\{\lambda_k \in \mathsf{Spec}(\mathsf{D}_g) : \lambda_0 < \lambda_k < \lambda\}\end{aligned}$$

Similar formula also holds for $\mu_{\lambda,+}(\psi)$ and for the case $H \in C^2(\mathbb{S}(M))$ satisfying the condition (4.22).

(ii) The definitions of the operators $L_\psi^{-1/2}T_{\psi,H,\lambda}L_\psi^{-1/2}$ and $\mathcal{T}_{\psi,H,\lambda}$ of Propositions 4.2 and 4.3 can be extended to solutions (1.1) defined on non-compact manifolds such as $\mathbb{R}^m$. Thus, using these operators and the characterization of the relative Morse index given by Propositions 4.2 and 4.3, it is possible to define relative Morse indices for solutions to (1.1) defined on such non-compact manifolds. However, the spectral properties of these operators for such a non-compact case are not easy to see, because, in general, they are unbounded operators acting on $\mathcal{L}^2(M)$. In §6, we will study the spectral properties of these operators for the special case $M = \mathbb{R}^m$ which will be sufficient for our purposes.

5 A Liouville type theorem for $\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi$ on $\mathbb{R}^m$

In this section, we consider the following superquadratic Dirac equation defined on $\mathbb{R}^m$

$$\mathsf{D}_{g_{\mathbb{R}^m}}\psi = |\psi|^{p-1}\psi \qquad \text{on } \mathbb{R}^m, \tag{5.1}$$

where $\mathsf{D}_{g_{\mathbb{R}^m}}$ is the Dirac operator acting spinors on $\mathbb{R}^m$, $\psi \in C^1(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ is a spinor on $\mathbb{R}^m$ and $1 < p < \frac{m+1}{m-1}$. Recall that the canonical spinor bundle is $\mathbb{S}(\mathbb{R}^m) = \mathbb{R}^m \times \mathbb{S}_m \to \mathbb{R}^m$. For simplicity, we write $\mathsf{D}_0 = \mathsf{D}_{g_{\mathbb{R}^m}}$.

As we shall see, (5.1) arises as the conformal blow-up equation of (1.1) (up to an irrelevant constant factor) when $H(x,\psi)$ is a lower order perturbation of $\frac{1}{p+1}H(x)|\psi|^{p+1}$, that is, it arises as the limiting equation of a sequence of equations (1.1) for $g = g_n$, where $g_n = \rho_n^2 g$ with $\rho_n \to \infty$. Our purpose in this and the next section is to define an appropriate relative Morse index for solutions to (5.1). As a natural requirement, we want the relative Morse index m_λ (or, equivalently, μ_λ) defined for solutions to (1.1) on compact manifolds naturally continues to the one of solutions to (5.1) after the blow-up of the metric on M. This continuity (in fact, lower-semicontinuity) requirement will give us a natural definition of the relative Morse index for solutions to (5.1), see Remark 4.3 (ii) in the previous section. But, before defining it, in this subsection we shall prove the following Liouville type theorem for solutions to (5.1). This will give us an important information about the spectral property of the blow-up of the operator $L_\psi^{-1/2} T_{\psi,H,\lambda} L_\psi^{-1/2}$.

Theorem 5.1 *Let us assume that $m \geq 2$ and $1 < p < \frac{m+1}{m-1}$. Let $\psi \in \mathcal{L}^{p+1}(\mathbb{R}^m) := L^{p+1}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ be a solution to (5.1). Then we have $\psi \equiv 0$.*

Proof. By the elliptic regularity theory, any weak solution $\psi \in \mathcal{L}^{p+1}(\mathbb{R}^m)$ to (5.1) is in fact $\psi \in C^{2,\alpha}(\mathbb{R}^m)$ for some $0 < \alpha < 1$, see [8], [33, Appendix]. Since the spinor bundle $\mathbb{S}(\mathbb{R}^m)$ is trivial, we may consider ψ as a function $\psi : \mathbb{R}^m \to \mathbb{S}_m$.

Choose a cut-off function $\rho \in C^\infty(\mathbb{R}^m)$ such that $\rho(x) = 1$ for $|x| \leq 1/2$ and $\rho(x) = 0$ for $|x| \geq 1$. For $R > 0$, we define $\rho_R(x) = \rho(x/R)$. For $\epsilon > 0$, we set $\psi_\epsilon(x) = \epsilon^{-\frac{1}{p-1}}\psi(x/\epsilon)$. For large $R > 0$ and small $\epsilon > 0$ we define

$$\mathcal{L}(R,\epsilon) = \int_{\mathbb{R}^m} \rho_R(x)\Big(\frac{1}{2}\langle\psi_\epsilon, \mathsf{D}_0\psi_\epsilon\rangle - \frac{1}{p+1}|\psi_\epsilon|^{p+1}\Big)\, d\mathrm{vol}_0,$$

where, for simplicity, we write $d\mathrm{vol}_0 = d\mathrm{vol}_{g_{\mathbb{R}^m}}$.

We set $\varphi(x) = \frac{d}{d\epsilon}\big|_{\epsilon=1}\psi_\epsilon(x) = -\frac{1}{p-1}\psi(x) - \nabla\psi(x)\cdot x$, where $\nabla\psi(x)\cdot x = \sum_{j=1}^m \nabla_{\partial_j}\psi(x)x^j$ (note that, here "$\cdot$" is not the Clifford product). Since $\psi \in C^{2,\alpha}(\mathbb{R}^m)$, $\mathcal{L}(R,\epsilon)$ is differentiable with respect to ϵ at $\epsilon = 1$ and we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=1}\mathcal{L}(R,\epsilon) = \int_{\mathbb{R}^m}\rho_R(x)\Big(\frac{1}{2}\langle\varphi, \mathsf{D}_0\psi\rangle + \frac{1}{2}\langle\psi, \mathsf{D}_0\varphi\rangle - |\psi|^{p-1}\langle\psi,\varphi\rangle\Big)\, d\mathrm{vol}_0 \tag{5.2}$$

$$= \int_{\mathbb{R}^m}\rho_R(x)(\langle\varphi, \mathsf{D}_0\psi\rangle - |\psi|^{p-1}\langle\psi,\varphi\rangle)\, d\mathrm{vol}_0 + \frac{1}{2}\int_{\mathbb{R}^m}\langle\nabla\rho_R\cdot\psi, \varphi\rangle\, d\mathrm{vol}_0 \tag{5.3}$$

$$= \frac{1}{2}\int_{\mathbb{R}^m}\langle\nabla\rho_R\cdot\psi, \varphi\rangle\, d\mathrm{vol}_0 \tag{5.4}$$

$$= \frac{1}{2}\int_{\mathbb{R}^m}\Big\langle\nabla\rho_R\cdot\psi, -\frac{1}{p-1}\psi - \nabla\psi\cdot x\Big\rangle\, d\mathrm{vol}_0, \tag{5.5}$$

where (5.3) follows form (5.2) by integrating by parts, $\nabla\rho_R\cdot\psi$ in (5.3) is the Clifford multiplication of ψ by $\nabla\rho_R$ and (5.4) follows from the fact that ψ satisfies (5.1).

We claim that $\frac{d}{d\epsilon}\big|_{\epsilon=1}\mathcal{L}(R,\epsilon)$ tends to 0 as $R \to \infty$. To see this, we need to estimate two terms in (5.5). First, we have by the Hölder's inequality

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\langle\nabla\rho_R\cdot\psi,\psi\rangle\, d\mathrm{vol}_0\right| &\le CR^{-1}\int_{\frac{R}{2}\le|x|\le R}|\psi|^2\, d\mathrm{vol}_0\\ &\le CR^{-1}\Big(\int_{\frac{R}{2}\le|x|\le R}|\psi|^{p+1}\, d\mathrm{vol}_0\Big)^{\frac{2}{p+1}}R^{m\left(1-\frac{2}{p+1}\right)}\\ &\le C\Big(\int_{\frac{R}{2}\le|x|\le R}|\psi|^{p+1}\, d\mathrm{vol}_0\Big)^{\frac{2}{p+1}}R^{\frac{m(p-1)}{p+1}-1}.\end{aligned} \tag{5.6}$$

Since $p<\frac{m+1}{m-1}$, we have $\frac{m(p-1)}{p+1}-1<0$ and (5.6) converges to 0 as $R\to\infty$ under the assumption $\psi\in\mathcal{L}^{p+1}(\mathbb{R}^m)$.

To estimate the second term of (5.5), we observe that the Green kernel of the Dirac operator D_0 is written as $\mathsf{D}_0^{-1}=(-\Delta_{g_{\mathbb{R}^m}})^{-1/2}R\,\cdot$, where R is the Riesz operator whose Fourier symbol is $\mathcal{F}(R)(\xi)=\frac{i\xi}{|\xi|}$, $(-\Delta_{g_{\mathbb{R}^m}})^{-1/2}$ is the so-called Riesz potential operator (in the notation of [47], it is denoted by I_1) whose Fourier symbol is $\mathcal{F}(-\Delta_{g_{\mathbb{R}^m}})^{-1/2})(\xi)=|\xi|^{-1}$ and $\cdot$ is the Clifford multiplication. For basic properties of the Riesz operators, see [47] for details. Thus, for ψ satisfying the equation (5.1), we have

$$\nabla\psi=\nabla(-\Delta_{g_{\mathbb{R}^m}})^{-1/2}R\cdot(|\psi|^{p-1}\psi). \tag{5.7}$$

Observe that $\nabla(-\Delta_{g_{\mathbb{R}^m}})^{-1/2}$ is the Riesz operator R and we thus have $\nabla\psi=R\circ R\cdot(|\psi|^{p-1}\psi)$ by (5.7). Since $|\psi|^{p-1}\psi\in\mathcal{L}^{\frac{p+1}{p}}(\mathbb{R}^m)$ and the Riesz operator $R:L^{\frac{p+1}{p}}(\mathbb{R}^m)\to L^{\frac{p+1}{p}}(\mathbb{R}^m)$ is bounded ([47]), we conclude that $\nabla\psi\in\mathcal{L}^{\frac{p+1}{p}}(\mathbb{R}^m)$. Thus, by the Hölder's inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\langle\nabla\rho_R\cdot\psi,\nabla\psi\cdot x\rangle\, d\mathrm{vol}_0\right| &\le C\int_{\frac{R}{2}\le|x|\le R}|\psi||\nabla\psi|\, d\mathrm{vol}_0\\ &\le C\Big(\int_{\frac{R}{2}\le|x|\le R}|\psi|^{p+1}\, d\mathrm{vol}_0\Big)^{\frac{1}{p+1}}\Big(\int_{\frac{R}{2}\le|x|\le R}|\nabla\psi|^{\frac{p+1}{p}}\, d\mathrm{vol}_0\Big)^{\frac{p}{p+1}}\end{aligned} \tag{5.8}$$

and (5.8) converges to 0 by the fact that $\psi\in\mathcal{L}^{p+1}(\mathbb{R}^m)$ and $\nabla\psi\in\mathcal{L}^{\frac{p+1}{p}}(\mathbb{R}^m)$.

Combining (5.5), (5.6) and (5.8), we have $\frac{d}{d\epsilon}\mathcal{L}(R,\epsilon)\big|_{\epsilon=1}\to 0$ as $R\to\infty$. Since ψ_ϵ is also a solution to (5.1) with $\psi_\epsilon\in\mathcal{L}^{p+1}(\mathbb{R}^m)$ for any $\epsilon>0$, replacing ψ with ψ_ϵ in the above argument, we have

$$\frac{d}{d\epsilon}\mathcal{L}(R,\epsilon)\to 0 \tag{5.9}$$

as $R\to\infty$ for any $\epsilon>0$.

On the other hand, by the dominated convergence theorem, we have

$$\mathcal{L}(R,\epsilon)\to\mathcal{L}(\epsilon):=\frac{1}{2}\int_{\mathbb{R}^m}\langle\psi_\epsilon,\mathsf{D}_0\psi_\epsilon\rangle\, d\mathrm{vol}_0-\frac{1}{p+1}\int_{\mathbb{R}^m}|\psi_\epsilon|^{p+1}\, d\mathrm{vol}_0 \tag{5.10}$$

as $R\to\infty$, where the convergence is locally uniformly on $\epsilon\in(0,\infty)$. By (5.9) and (5.10), we have $\frac{d}{d\epsilon}\mathcal{L}(\epsilon)=0$ in the sense of distributions. Since $\mathcal{L}(\epsilon)=\epsilon^{m-\frac{p+1}{p-1}}\mathcal{L}(\psi)$, where $\mathcal{L}(\psi)=\frac{1}{2}\int_{R^m}\langle\psi,\mathsf{D}_0\psi\rangle\, d\mathrm{vol}_0-\frac{1}{p+1}\int_{R^m}|\psi|^{p+1}\, d\mathrm{vol}_0$, we therefore obtain

$$0=\mathcal{L}(\psi)=\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_0.$$

This implies that $\psi\equiv 0$ and completes the proof of Theorem 5.1. $\square$

6 Relative Morse index $m_{\mathbb{R}^m}$ and the proof of Theorem 1.2

In this section, we give the definition of the relative Morse index $m_{\mathbb{R}^m}$ and prove one of our main results Theorem 1.2. Throughout this section, we assume $\psi \in \mathcal{L}^\infty(\mathbb{R}^m)$.

6.1 Relative Morse index $m_{\mathbb{R}^m}$

Motivated by Proposition 4.2, we define an (unbounded, in general,) operator T_ψ on $\mathcal{L}^2(\mathbb{R}^m)$ by

$$(T_\psi \varphi)(x) = -\frac{1}{\omega_{m-1}} |\psi|^{\frac{p-1}{2}}(x) \int_{\mathbb{R}^m} \frac{(x-y)}{|x-y|^m} \cdot (|\psi|^{\frac{p-1}{2}}(y)\varphi(y))\, d\mathrm{vol}_0(y), \tag{6.1}$$

where $(x-y)\cdot$ denotes the Clifford multiplication by the vector $x - y \in \mathbb{R}^m$.

Observe that T_ψ is written as $T_\psi = |\psi|^{\frac{p-1}{2}} \circ \mathsf{D}_0^{-1} \circ |\psi|^{\frac{p-1}{2}}$, where $|\psi|^{\frac{p-1}{2}}$ is the multiplication operator $\mathcal{L}^2(\mathbb{R}^m) \ni \varphi \mapsto |\psi|^{\frac{p-1}{2}}\varphi \in \mathcal{L}^2(\mathbb{R}^m)$ and D_0^{-1} is the Green kernel of D_0 which is given by $G_{\mathsf{D}_0}(x, y) = -\omega_{m-1}^{-1} \frac{(x-y)}{|x-y|^m} \cdot d\mathrm{vol}_0$ (see, for example, [8]). T_ψ is obtained formally by setting $(M, g, \rho) = (\mathbb{R}^m, g_{\mathbb{R}^m}, \rho_{\mathrm{can}})$ (ρ_{can} is the canonical spin structure on $(\mathbb{R}^m, g_{\mathbb{R}^m})$), $H \equiv 1$ and $\lambda = 0$ in the definition of $T_{\psi,H,\lambda}$ in §4.

As is guessed from Propositions 4.2 and 4.3, the operator T_ψ (more precisely, the spectrum of $L_\psi^{-1/2} T_\psi L_\psi^{-1/2}$) will play a fundamental in the definition of an appropriate notion of "relative Morse index" for a solution ψ to (5.1). Unfortunately, at present, we do not know much about T_ψ. For example, we do not know whether it is bounded or not as an operator $\mathcal{L}^2(\mathbb{R}^m) \to \mathcal{L}^2(\mathbb{R}^m)$. This is due to the fact that we do not know much about solutions ψ to (5.1). Since T_ψ explicitly depends on ψ, it is necessary to know some properties of ψ, for example the size of the nodal set $\{x \in \mathbb{R}^m : \psi(x) = 0\}$ of ψ (see [8, conjecture] for a conjecture about the size of the nodal set of the solutions to (5.1)) and the behavior of ψ near its zeros, in order to deduce precise properties of T_ψ, in particular, the spectral property of $L_\psi^{-1/2} T_\psi L_\psi^{-1/2}$. We will see, however, for $m \geq 3$ results of §5 and Lemma 6.1 below are sufficient in order to prove the assertion of our main results.

We begin with the proof of the following simple properties of T_ψ.

Lemma 6.1 *Assume $m \geq 3$. We have the following:*

(1) The domain of T_ψ, $\mathcal{D}(T_\psi) := \{\varphi \in \mathcal{L}^2(\mathbb{R}^m) : T_\psi(\varphi) \in \mathcal{L}^2(\mathbb{R}^m)\}$, contains compactly supported L^∞-spinors, $\mathcal{L}_c^\infty(\mathbb{R}^m) := \{\varphi \in \mathcal{L}^\infty(\mathbb{R}^m) : \mathrm{supp}\,\varphi \subset \mathbb{R}^m;$ compact$\}$. In particular, it is densely defined.

(2) T_ψ has the following symmetry property: for any $\phi \in \mathcal{L}_c^\infty(\mathbb{R}^m)$ and $\varphi \in \mathcal{L}^2(\mathbb{R}^m)$, there holds

$$(T_\psi(\phi), \varphi)_{\mathcal{L}^2(\mathbb{R}^m)} = (\phi, T_\psi(\varphi))_{\mathcal{L}^2(\mathbb{R}^m)}.$$

Proof. We first observe that for any $\varphi \in \mathcal{L}^2(\mathbb{R}^m)$, we have $T_\psi(\varphi) \in \mathcal{L}^2(\mathbb{R}^m) + \mathcal{L}^\infty(\mathbb{R}^m)$. In fact, since $\psi \in \mathcal{L}^\infty(\mathbb{R}^m)$, there exists $M > 0$ such that $|\psi(x)| \leq M$ for a.e. $x \in \mathbb{R}^m$. We define $K(x) = \frac{1}{|x|^{m-1}}$. Since $|T_\psi(\varphi)(x)| \leq \omega_{m-1}^{-1} M^{p-1} (K * |\varphi|)(x)$, it suffices to prove $(K * |\varphi|)(x) = \int_{\mathbb{R}^m} \frac{|\varphi(y)|}{|x-y|^{m-1}}\, d\mathrm{vol}_0(y) \in L^2(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$. We write $K(x) = K_1(x) + K_2(x)$, where $K_1(x) = \mathbf{1}_{\{|x|\leq 1\}}(x) K(x)$ and $K_2(x) = \mathbf{1}_{\{|x|>1\}}(x) K(x)$, where $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ for $x \notin A$. For $m \geq 3$, we have $K_1 \in L^1(\mathbb{R}^m)$ and $K_2 \in L^2(\mathbb{R}^m)$ and

$$(K * |\varphi|)(x) = \int_{\mathbb{R}^m} K_1(x-y)|\varphi(y)|\, d\mathrm{vol}_0(y) + \int_{\mathbb{R}^m} K_2(x-y)|\varphi(y)|\, d\mathrm{vol}_0(y). \tag{6.2}$$

By the Hausdorff-Young's inequality, the first term of (6.2) is in $L^2(\mathbb{R}^m)$. The second term of (6.2) is in $L^\infty(\mathbb{R}^m)$ by the Hölder's inequality:

$$\|K * |\varphi|\|_{L^\infty(\mathbb{R}^m)} \le \|K_2\|_{L^2(\mathbb{R}^m)}\|\varphi\|_{\mathcal{L}^2(\mathbb{R}^m)}.$$

Thus the claim is proved. In particular, it implies that for any $\varphi \in \mathcal{L}^2(\mathbb{R}^m)$, $T_\psi(\varphi)(x)$ defined by (6.1) is meaningful for a.e. $x \in \mathbb{R}^m$ and the right hand side of the equation in the statement of the Lemma (2) is well-defined.

(1) Now, we prove $\mathcal{L}_c^\infty(\mathbb{R}^m) \subset \mathcal{D}(T_\psi)$. To see this, for any $\varphi \in \mathcal{L}_c^\infty(\mathbb{R}^m)$, as we have observed above $T_\psi(\varphi)(x)$ is finite for a.e. $x \in \mathbb{R}^m$. Assuming that the support of φ is contained in $B_R = \{x \in \mathbb{R}^m : |x| \le R\}$, we have the following estimate:

$$\begin{aligned}|T_\psi(\varphi)(x)| &\le \omega_{m-1}^{-1}M^{p-1}\int_{\mathbb{R}^m}\frac{|\varphi(y)|}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y)\\ &\le \omega_{m-1}^{-1}M^{p-1}\|\varphi\|_{\mathcal{L}^\infty(\mathbb{R}^m)}\int_{B_R}\frac{1}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y).\end{aligned} \tag{6.3}$$

Here, we have ($B_R(x)$ will denote the euclidean ball of radius R with center at x)

$$\begin{aligned}\int_{B_R}\frac{1}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y) &= \int_{B_R\cap B_R(x)}\frac{1}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y) + \int_{B_R\setminus B_R(x)}\frac{1}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y)\\ &\le \int_{B_R\cap B_R(x)}\frac{1}{|x-y|^{m-1}}\,d\mathrm{vol}_0(y) + \int_{B_R}\frac{1}{|y|^{m-1}}\,d\mathrm{vol}_0(y)\\ &\le 2\int_{B_R}\frac{1}{|y|^{m-1}}\,d\mathrm{vol}_0(y) = 2\omega_{m-1}R,\end{aligned} \tag{6.4}$$

where we have used $|x-y| \ge R \ge |y|$ for $y \in B_R \setminus B_R(x)$.

By (6.3) and (6.4), we have

$$T_\psi(\varphi) \in \mathcal{L}^\infty(\mathbb{R}^m). \tag{6.5}$$

On the other hand, by (6.3), we have $|T_\psi(\varphi)(x)| \le CM^{p-1}\|\varphi\|_{\mathcal{L}^\infty(\mathbb{R}^m)}|x|^{1-m}$ for some $C > 0$ independent of $x \in \mathbb{R}^m$. This combined with (6.5) implies that $T_\psi(\varphi) \in \mathcal{L}^2(\mathbb{R}^m)$ when $m \ge 3$. This proves that $\mathcal{L}_c^\infty(\mathbb{R}^m) \subset \mathcal{D}(T_\psi)$ and T_ψ is densely defined.

(2) By the observation we have made at the beginning of the proof, we have $T_\psi(\varphi) \in \mathcal{L}^2(\mathbb{R}^m) + \mathcal{L}^\infty(\mathbb{R}^m)$ for $\varphi \in \mathcal{L}^2(\mathbb{R}^m)$. Thus, we can apply the Fubini-Tonelli theorem and obtain the following for any $\phi \in \mathcal{L}_0^\infty(\mathbb{R}^m)$ and $\varphi \in \mathcal{L}^2(\mathbb{R}^m)$:

$$\begin{aligned}(T_\psi(\phi),\varphi)_{\mathcal{L}^2(\mathbb{R}^m)} &= \int_{\mathbb{R}^m}(T_\psi(\phi),\varphi)\,d\mathrm{vol}_0\\ &= -\omega_{m-1}^{-1}\int_{\mathbb{R}^m}|\psi|^{\frac{p-1}{2}}(x)\Big(\int_{\mathbb{R}^m}\frac{(x-y)}{|x-y|^{m-1}}\cdot(|\psi|^{\frac{p-1}{2}}(y)\phi(y))\,d\mathrm{vol}_0(y),\varphi(x)\Big)\,d\mathrm{vol}_0(x)\\ &= -\omega_{m-1}^{-1}\int_{\mathbb{R}^m}\Big(\phi(y),|\psi(y)|^{\frac{p-1}{2}}\int_{\mathbb{R}^m}\frac{(y-x)}{|x-y|^{m-1}}\cdot(|\psi(x)|^{\frac{p-1}{2}}\varphi(x))\,d\mathrm{vol}_0(x)\Big)\,d\mathrm{vol}_0(y)\\ &= (\phi, T_\psi(\varphi))_{\mathcal{L}^2(\mathbb{R}^m)},\end{aligned} \tag{6.6}$$

where we have used the skew-adjointness of the Clifford multiplication by vectors in the third line. This completes the proof. □

We are now in a position to give a definition of the relative Morse index for solutions to (5.1). Once again, we recall that the difficulty of defining relative Morse index for solutions to

(5.1) comes from the fact that $\mathbb{R}^m$ is not compact, so all of the definitions of the relative Morse indices given in §3 are not applicable to the present case. Moreover, by the result of Theorem 5.1, any non-trivial solution to (5.1) has infinite action

$$\frac{1}{2}\int_{\mathbb{R}^m}\langle\psi,\mathsf{D}_0\psi\rangle\, d\mathrm{vol}_0-\frac{1}{p+1}\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_0=\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^m}|\psi|^{p+1}\, d\mathrm{vol}_0=+\infty.$$

Motivated by the formula of the relative Morse index given in Proposition 4.2, we define the relative Morse index for $\mathcal{L}_H$ at $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$ as follows:

Definition 6.1 *Let us assume that $m\geq 3$. For $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$, the relative Morse index $m_{\mathbb{R}^m}(\psi)$ is defined as the dimension of the maximal subspace of $\mathcal{L}_c^\infty(\mathbb{R}^m)\subset\mathcal{D}(T_\psi)$ on which the following inequality holds*

$$\frac{(L_\psi^{-1/2}T_\psi L_\psi^{-1/2}(\varphi),\varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(\varphi,\varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}>1$$

for all non-zero φ, where $L_\psi=\frac{1}{p}P_\psi+P_\psi^\perp$ ($\perp$ is taken with respect to the metric on $\mathbb{S}(\mathbb{R}^m)$).

Remark 6.1 *(i) As we have remarked after Corollary 4.1, also in the present case $L_\psi^{-1/2}T_\psi L_\psi^{-1/2}$ is well-defined for any $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$, i.e., it does not depend on the choice of the section e_ψ used to define L_ψ since $|\psi|^{\frac{p-1}{2}}L_\psi^{-1/2}=L_\psi^{-1/2}|\psi|^{\frac{p-1}{2}}$ does not depend on such a choice. From this observation, by conjugation with $L_\psi^{1/2}$, $m_{\mathbb{R}^m}(\psi)$ is defined alternatively as the dimension of the maximal subspace of $\mathcal{L}_c^\infty(\mathbb{R}^m)\subset\mathcal{D}(T_\psi)$ on which*

$$\frac{(T_\psi(\varphi),\varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\varphi),\varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}>1$$

holds for all non-zero φ.

(ii) Notice that it may be possible that ψ has infinite index, $m_{\mathbb{R}^m}(\psi)=+\infty$, in general.

In the following, we give some basic properties of $m_{\mathbb{R}^m}$ from which Theorem 1.2 will follow.

Theorem 6.1 *$m_{\mathbb{R}^m}$ (for $m\geq 3$) has the following properties:*

(1) $m_{\mathbb{R}^m}(\psi)$ is defined for $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$, takes values in $\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ and lower semi-continuous with respect to the $L^\infty_{loc}(\mathbb{R}^m)$-convergence of ψ.

(2) Let $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$ and $x_0\in M$ be arbitrary. Assume also that $H\in C^0(M)$. For any sequence $\{(M,g_n,\sigma)\}$ (where $g_n=\rho_n^2 g$ with $\rho_n\to\infty$) of conformal blowing-up of the manifold (M,g,σ) at x_0 and any sequence $\psi_n\in\mathcal{L}^\infty(M,g_n,\sigma)$ with $\psi_n\to\psi$ in $L^\infty_{loc}(\mathbb{R}^m)$, there holds

$$m_{\mathbb{R}^m}(\psi)\leq\liminf_{n\to\infty}\mu_{-\rho_n^{-1}\lambda}(\psi_n).$$

(3) For any non-trivial solution $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$ to (5.1), we have $m_{\mathbb{R}^m}(\psi)=+\infty$.

Remark 6.2 *(i) In (2) above, on any compact subset K of $\mathbb{R}^m$ and large n, we have identified $\psi_n\in\mathcal{L}^\infty(M,g_n)$ with a spinor on $K\subset\mathbb{R}^m$ obtained as the pull-back of ψ_n by the conformal blow-up maps at x_0, $\exp_{x_0}(\rho_n^{-1}\cdot):(B_{\rho_n r_0},h_n)\to(M,g_n)$, where $h_n=\exp_{x_0}(\rho_n^{-1}\cdot)^*g_n$ and $0<r_0$*

is smaller than the injectivity radius of M. Note that $(M, g_n, \sigma) \to (\mathbb{R}^m, g_{\mathbb{R}^m}, \sigma_{can})$ in $C^\infty_{loc}(\mathbb{R}^m)$ and $H \circ \exp_{x_0}(\rho_n^{-1}\cdot) \to H(x_0)$ in $C^0_{loc}(\mathbb{R}^m)$ as $n \to \infty$.

(ii) It is not difficult to see that the relative Morse index m_λ (or μ_λ) is lower semi-continuous with respect to a continuous deformation of the metric on M. Theorem 6.1 (2) asserts that, with our definition of $m_{\mathbb{R}^m}$, it also holds along a conformal blow-up of M.

The assertion (1) of Theorem 6.1 is easily seen from the definition and we omit its proof, see also Remark 6.1 (i). (2) will be proved in the course of the proof of Theorem 1.1 given in the next section. The assertion (3) is exactly the assertion of Theorem 1.2 which we will prove in this subsection. Its proof requires some more preparations.

To estimate the relative Morse index of a solution ψ to (5.1), we need a suitable family of test spinors. Let $\eta \in C^\infty_0(\mathbb{R}^m)$ be such that $\eta(x) = 1$ for $|x| \le 1$, $\eta(x) = 0$ for $|x| \ge 2$ and $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}^m$. For $\ell \ge 1$ and $R > 0$, we define $\eta_R(x) = \eta(x/R)$, $\eta_{\ell,R} = \eta_R^\ell$ and $\varphi_{\ell,R} = \eta_{\ell,R}|\psi|^{\frac{p-1}{2}}\psi$. We then have:

Lemma 6.2 *Assume that $m \ge 3$ and $\psi \in \mathcal{L}^\infty(\mathbb{R}^m)$ is a non-trivial solution to (5.1). For $\ell \ge \frac{p+1}{p-1}$, we have $T_\psi(\varphi_{\ell,R}) \in \mathcal{L}^2(\mathbb{R}^m)$ and*

$$\varliminf_{R\to\infty} \frac{(T_\psi(\varphi_{\ell,R}), \varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\varphi_{\ell,R}), \varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}} \ge p.$$

Proof. We first note that it is not necessary that the above limit is finite. The assertion $T_\psi(\varphi_{\ell,R}) \in \mathcal{L}^2(\mathbb{R}^m)$ follows from (1) of Lemma 6.1.

By definition, we have $T_\psi(\varphi_{\ell,R}) = |\psi|^{\frac{p-1}{2}}\mathsf{D}_0^{-1}(\eta_{\ell,R}|\psi|^{p-1}\psi)$. We define $\phi_{\ell,R}$ by $\mathsf{D}_0^{-1}(\eta_{\ell,R}|\psi|^{p-1}\psi) = \eta_{\ell,R}\psi + \phi_{\ell,R}$. We thus have

$$\begin{aligned}\mathsf{D}_0\phi_{\ell,R} &= \eta_{\ell,R}|\psi|^{p-1}\psi - \mathsf{D}_0(\eta_{\ell,R}\psi)\\ &= \eta_{\ell,R}|\psi|^{p-1}\psi - \eta_{\ell,R}\mathsf{D}_0\psi - \nabla\eta_{\ell,R}\cdot\psi\\ &= -\nabla\eta_{\ell,R}\cdot\psi, \end{aligned} \tag{6.7}$$

where in the last line, we have used equation (5.1) which is satisfied by ψ. We therefore have

$$\phi_{\ell,R} = -\mathsf{D}_0^{-1}(\nabla\eta_{\ell,R}\cdot\psi). \tag{6.8}$$

By definition, we have $T_\psi(\varphi_{\ell,R}) = |\psi|^{\frac{p-1}{2}}(\eta_{\ell,R}\psi + \phi_{\ell,R})$ and

$$(T_\psi(\varphi_{\ell,R}), \varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)} = \int_{\mathbb{R}^m} \eta_{\ell,R}^2|\psi|^{p+1}\,d\mathrm{vol}_0 + \int_{\mathbb{R}^m} \eta_{\ell,R}|\psi|^{p-1}(\psi, \phi_{\ell,R})\,d\mathrm{vol}_0. \tag{6.9}$$

We estimate the second term of (6.9) as follows: By (5.1), we have

$$\begin{aligned}\int_{\mathbb{R}^m} \eta_{\ell,R}|\psi|^{p-1}(\psi, \phi_{\ell,R})\,d\mathrm{vol}_0 &= \int_{\mathbb{R}^m} \eta_{\ell,R}(\mathsf{D}_0\psi, \phi_{\ell,R})\,d\mathrm{vol}_0\\ &= \int_{\mathbb{R}^m} (\psi, \mathsf{D}_0(\eta_{\ell,R}\phi_{\ell,R}))\,d\mathrm{vol}_0\\ &= \int_{\mathbb{R}^m} (\psi, \nabla\eta_{\ell,R}\cdot\phi_{\ell,R})\,d\mathrm{vol}_0 + \int_{\mathbb{R}^m} (\psi, \eta_{\ell,R}\mathsf{D}_0\phi_{\ell,R})\,d\mathrm{vol}_0\\ &= \int_{\mathbb{R}^m} (\psi, \nabla\eta_{\ell,R}\cdot\phi_{\ell,R})\,d\mathrm{vol}_0 - \int_{\mathbb{R}^m} \eta_{\ell,R}(\psi, \nabla\eta_{\ell,R}\cdot\psi)\,d\mathrm{vol}_0, \end{aligned} \tag{6.10}$$

where we have used (6.8) in the last line.

Since $\nabla\eta_{\ell,R} = \ell\eta_{\ell,R}^{1-\frac{1}{\ell}}\nabla\eta_R$, the second term of (6.10) is estimated as

$$\begin{aligned}\left|\int_{\mathbb{R}^m}\eta_{\ell,R}(\psi,\nabla\eta_{\ell,R}\cdot\psi)\,d\mathrm{vol}_0\right| &= \left|\ell\int_{\mathbb{R}^m}\eta_{\ell,R}^{2-\frac{1}{\ell}}(\psi,\nabla\eta_R\cdot\psi)\,d\mathrm{vol}_0\right| \\ &\le C\ell R^{-1}\int_{\mathbb{R}^m}\eta_{\ell,R}^{2-\frac{1}{\ell}}|\psi|^2\,d\mathrm{vol}_0 \\ &\le C\ell R^{m\frac{p-1}{p+1}-1}\left(\int_{\mathbb{R}^m}\eta_{\ell,R}^{(2-\frac{1}{\ell})\frac{p+1}{2}}|\psi|^{p+1}\,d\mathrm{vol}_0\right)^{\frac{2}{p+1}}, \end{aligned} \tag{6.11}$$

where here and in the following $C>0$ denotes various constants independent of R and ψ and we have used in the third line the Hölder's inequality.

By our choice of ℓ, we have $\left(2-\frac{1}{\ell}\right)\frac{p+1}{2}\ge 2$. (In fact, this only requires $\ell\ge\frac{p+1}{2(p-1)}$. For the latter purpose which will become clear soon, we take ℓ even larger such that $\ell\ge\frac{p+1}{p-1}$.) Thus the inequality (6.11) becomes

$$\left|\int_{\mathbb{R}^m}\eta_{\ell,R}(\psi,\nabla\eta_{\ell,R}\cdot\psi)\,d\mathrm{vol}_0\right| \le C\ell R^{m\frac{p-1}{p+1}-1}\left(\int_{\mathbb{R}^m}\eta_{\ell,R}^2|\psi|^{p+1}\,d\mathrm{vol}_0\right)^{\frac{2}{p+1}}. \tag{6.12}$$

We set $I_R := \int_{\mathbb{R}^m}\eta_{\ell,R}^2|\psi|^{p+1}\,d\mathrm{vol}_0$. Then by Theorem 5.1, we have $I_R\to\infty$ as $R\to\infty$; otherwise we have $\psi\in\mathcal{L}^{p+1}(\mathbb{R}^m)$ and $\psi\equiv 0$ by Theorem 5.1. But this contradicts our assumption $\psi\not\equiv 0$. With this notation, (6.12) becomes

$$\left|\int_{\mathbb{R}^m}\eta_{\ell,R}(\psi,\nabla\eta_{\ell,R}\cdot\psi)\,d\mathrm{vol}_0\right| \le CR^{m\frac{p-1}{p+1}-1}I_R^{\frac{2}{p+1}}. \tag{6.13}$$

We next estimate the first term of (6.10). We have

$$\int_{\mathbb{R}^m}(\psi,\nabla\eta_{\ell,R}\cdot\phi_{\ell,R})\,d\mathrm{vol}_0 = \ell\int_{\mathbb{R}^m}(\psi,\eta_{\ell,R}^{1-\frac{1}{\ell}}\nabla\eta_R\cdot\phi_{\ell,R})\,d\mathrm{vol}_0. \tag{6.14}$$

Here, we recall that $\mathsf{D}_0^{-1} = (-\Delta)^{-1/2}R\,\cdot$, where R is the Riesz operator and $(-\Delta)^{-1/2}$ is the Riesz potential operator I_1 whose integral kernel is $\frac{1}{\gamma(1)}\frac{1}{|x-y|^{n-1}}$, where $\gamma(1) = \frac{2\pi^{n/2}\Gamma(1/2)}{\Gamma\left(\frac{n-1}{2}\right)}$ (see [47]). By the L^q-boundedness of R ($1<q<\infty$) and the Hardy-Littlewood-Sobolev inequality for $(-\Delta)^{-1/2}$ (see [47, p119, Theorem 1]), we have $\phi_{\ell,R}\in\mathcal{L}^{\frac{m(p+1)}{m-p-1}}(\mathbb{R}^m)$ since $\nabla\eta_{\ell,R}\cdot\psi\in\mathcal{L}^{p+1}(\mathbb{R}^m)$ (and note that for $m\ge 3$, we have $p+1<\frac{2m}{m-1}\le m$). Thus we obtain

$$\|\phi_{\ell,R}\|_{\mathcal{L}^{\frac{m(p+1)}{m-p-1}}(\mathbb{R}^m)} \le C\|\nabla\eta_{\ell,R}\cdot\psi\|_{\mathcal{L}^{p+1}(\mathbb{R}^m)}. \tag{6.15}$$

Using again the relation $\nabla\eta_{\ell,R} = \ell\eta_{\ell,R}^{1-\frac{1}{\ell}}\nabla\eta_R$, we have

$$\begin{aligned}\|\nabla\eta_{\ell,R}\cdot\psi\|_{\mathcal{L}^{p+1}(\mathbb{R}^m)}^{p+1} &\le \int_{\mathbb{R}^m}|\nabla\eta_{\ell,R}|^{p+1}|\psi|^{p+1}\,d\mathrm{vol}_0 \\ &\le C\ell^{p+1}R^{-p-1}\int_{\mathbb{R}^m}\eta_{\ell,R}^{(1-\frac{1}{\ell})(p+1)}|\psi|^{p+1}\,d\mathrm{vol}_0 \\ &\le CR^{-p-1}\int_{\mathbb{R}^m}\eta_{\ell,R}^2|\psi|^{p+1}\,d\mathrm{vol}_0 = CR^{-p-1}I_R, \end{aligned} \tag{6.16}$$

where we have used $\left(1-\frac{1}{\ell}\right)(p+1)\ge 2$ by our choice of ℓ, $\ell\ge\frac{p+1}{p-1}$.

Combining (6.15) and (6.16), we have

$$\|\phi_{\ell,R}\|_{\mathcal{L}^{\frac{m(p+1)}{m-p-1}}(\mathbb{R}^m)} \le CR^{-1} I_R^{\frac{1}{p+1}}. \tag{6.17}$$

From (6.14) and (6.17), we obtain by the Hölder's inequality

$$\begin{aligned}\left|\int_{\mathbb{R}^m} (\psi, \nabla\eta_{\ell,R}\cdot\phi_{\ell,R})\,\mathrm{dvol}_0\right| &\le C\int_{\mathbb{R}^m} |\psi|\eta_{\ell,R}^{1-\frac{1}{\ell}}|\nabla\eta_R||\phi_{\ell,R}|\,\mathrm{dvol}_0 \\ &\le CR^{-1}\int_{\mathbb{R}^m} |\psi|\eta_{\ell,R}^{\frac{2}{p+1}}\eta_{\ell,R}^{1-\frac{1}{\ell}-\frac{2}{p+1}}|\phi_{\ell,R}|\,\mathrm{dvol}_0 \\ &\le CR^{-1}\left(\int_{\mathbb{R}^m}\eta_{\ell,R}^2|\psi|^{p+1}\,\mathrm{dvol}_0\right)^{\frac{1}{p+1}}\|\phi_{\ell,R}\|_{\mathcal{L}^{\frac{m(p+1)}{m-p-1}}(\mathbb{R}^m)} R^{\frac{m}{r}}, \end{aligned}\tag{6.18}$$

where we have used $1-\frac{1}{\ell}-\frac{2}{p+1}\ge 0$ for $\ell \ge \frac{p+1}{p-1}$ and r is defined by the relation $\frac{1}{p+1}+\frac{m-p-1}{m(p+1)}+\frac{1}{r}=1$. Note that such a $r>1$ exists if and only if $p<2m-1$. Since we have $p<p+1<\frac{2m}{m-1}\le 2m-1$ for $m\ge 3$, such a $r>1$ indeed exists for $m\ge 3$.

Combining (6.17) and (6.18), we obtain

$$\left|\int_{\mathbb{R}^m} (\psi, \nabla\eta_{\ell,R}\cdot\phi_{\ell,R})\,\mathrm{dvol}_0\right| \le CR^{-2+\frac{m}{r}} I_R^{\frac{2}{p+1}}. \tag{6.19}$$

We notice here that, under the assumption $p<\frac{m+1}{m-1}$, we have

$$-2+\frac{m}{r} = m-1-\frac{2m}{p+1} = m\frac{p-1}{p+1}-1<0. \tag{6.20}$$

Combining (6.9), (6.10), (6.13), (6.19) and (6.20), we obtain

$$(T_\psi(\varphi_R),\varphi_R)_{\mathcal{L}^2(\mathbb{R}^m)} \ge I_R - CR^{m-1-\frac{2m}{p+1}} I_R^{\frac{2}{p+1}}. \tag{6.21}$$

We next estimate the denominator $(L_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}$. Recall that $\varphi_{\ell,R}=\eta_{\ell,R}|\psi|^{\frac{p-1}{2}}\psi$ and we have $P_\psi^\perp(\varphi_{\ell,R})=0$ and

$$(L_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)} = \frac{1}{p}\int_{\mathbb{R}^m}\eta_{\ell,R}^2|\psi|^{p+1}\,\mathrm{dvol}_0 = \frac{1}{p}I_R. \tag{6.22}$$

By (6.21) and (6.22), we finally obtain

$$\frac{(T_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\varphi_{\ell,R}),\varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}} \ge p - CR^{m-1-\frac{2m}{p+1}} I_R^{-\frac{p-1}{p+1}}. \tag{6.23}$$

By (6.20) and $I_R\to\infty$ as $R\to\infty$, the assertion of the lemma follows from (6.23). □

In view of Remark 6.1 (i), Theorem 6.1 (3) is a direct consequence of the following:

Lemma 6.3 *Assume that $m\ge 3$ and $\psi\in\mathcal{L}^\infty(\mathbb{R}^m)$ is a non-trivial solution to (5.1). For any $\epsilon>0$ and $k\in\mathbb{N}$, there exist $0<R_1<R_2<\cdots<R_k$ such that the following holds: For $\varphi_j := \frac{\varphi_{\ell,R_j}}{\|\varphi_{\ell,R_j}\|_{\mathcal{L}^2(\mathbb{R}^m)}}$, where $\varphi_{\ell,R}=\eta_{\ell,R}|\psi|^{\frac{p-1}{2}}\psi$ is as in Lemma 6.2 with $\ell\ge\frac{p+1}{p-1}$, we have*

$$|(\varphi_i,\varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}|<\epsilon, \quad |(T_\psi(\varphi_i),\varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}|<\epsilon$$

for $1\le i\ne j\le k$ and

$$\frac{(T_\psi(\varphi_j),\varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\varphi_j),\varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}} \ge p-\epsilon$$

for $1\le j\le k$.

Proof. Let $\epsilon > 0$ be arbitrary. By Lemma 6.2, there exists $R_0 > 0$ such that

$$\frac{(T_\psi(\varphi_{\ell,R}), \varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\varphi_{\ell,R}), \varphi_{\ell,R})_{\mathcal{L}^2(\mathbb{R}^m)}} \geq p - \epsilon \tag{6.24}$$

for $R \geq R_0$. We take $R_1 = R_0$ and set $\varphi_1 = \frac{\varphi_{\ell,R_1}}{\|\varphi_{\ell,R_1}\|_{\mathcal{L}^2(\mathbb{R}^m)}}$.

For $R_2 > R_1$ (R_2 will be determined soon), we set $\varphi_2 = \frac{\varphi_{\ell,R_2}}{\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)}}$. We estimate the interaction $(T_\psi(\varphi_2), \varphi_1)_{\mathcal{L}^2(\mathbb{R}^m)}$. For any $R > 0$, we have by Lemma 6.1 (2)

$$\begin{aligned}(T_\psi(\varphi_2), \varphi_1)_{\mathcal{L}^2(\mathbb{R}^m)} &= \frac{(\varphi_{\ell,R_2}, T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}}{\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)}} \\ &= \frac{1}{\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)}}\{(\varphi_{\ell,R_2}, \eta_{\ell,R} T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)} + (\varphi_{\ell,R_2}, (1-\eta_{\ell,R}) T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}\}.\end{aligned} \tag{6.25}$$

Since the term $(\varphi_{\ell,R_2}, \eta_{\ell,R} T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}$ is independent of R_2 for $R_2 > 2R$ and $\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)} = \big(\int_{\mathbb{R}^m} \eta_{\ell,R_2}^2 |\psi|^{p+1}\, d\mathrm{vol}_0\big)^{1/2} \to +\infty$ as $R_2 \to \infty$, we have

$$\frac{(\varphi_{\ell,R_2}, \eta_{\ell,R} T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}}{\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)}} \to 0 \tag{6.26}$$

as $R_2 \to \infty$. On the other hand, we have

$$\Big|\frac{1}{\|\varphi_{\ell,R_2}\|_{\mathcal{L}^2(\mathbb{R}^m)}}(\varphi_{\ell,R_2}, (1-\eta_{\ell,R}) T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}\Big| \leq \|(1-\eta_{\ell,R}) T_\psi(\varphi_1)\|_{\mathcal{L}^2(\mathbb{R}^m)} \to 0 \tag{6.27}$$

as $R \to \infty$ (recall that, by Lemma 6.1 (1), $T_\psi(\varphi_1) \in \mathcal{L}^2(\mathbb{R}^m)$). From (6.25)–(6.27), we can choose $R_2 > R_1$ such that

$$|(T_\psi(\varphi_2), \varphi_1)_{\mathcal{L}^2(\mathbb{R}^m)}| = |(\varphi_2, T_\psi(\varphi_1))_{\mathcal{L}^2(\mathbb{R}^m)}| < \epsilon. \tag{6.28}$$

Arguing similarly, taking $R_2 > R_1$ even larger if necessary, we also have

$$|(\varphi_2, \varphi_1)_{\mathcal{L}^2(\mathbb{R}^m)}| < \epsilon. \tag{6.29}$$

Now, we assume that we have chosen $R_0 = R_1 < R_2 < \cdots < R_h$ such that the assertion of the lemma holds for $k = h$. By the same argument as in the proof of (6.28) and (6.29), we can find $R_{h+1} > R_h$ such that both of the following inequalities

$$|(T_\psi(\varphi_{h+1}), \varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}| < \epsilon, \quad |(\varphi_{h+1}, \varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}| < \epsilon \tag{6.30}$$

hold for all $1 \leq j \leq h$, where $\varphi_{h+1} = \frac{\varphi_{\ell,R_{h+1}}}{\|\varphi_{\ell,R_{h+1}}\|_{\mathcal{L}^2(\mathbb{R}^m)}}$. Therefore, by induction, for any given $k \in \mathbb{N}$, we have $0 < R_1 < \cdots < R_k$ such that the assertion of the lemma holds. □

Completion of the proof of Theorem 6.1 (3). For any $k \in \mathbb{N}$ and $\epsilon > 0$, let $\varphi_1, \ldots, \varphi_k \in \mathcal{L}^2(\mathbb{R}^m)$ be as in the statement of Lemma 6.3. Observe that $\varphi_i \in \mathcal{L}_c^\infty(\mathbb{R}^m) \subset \mathcal{D}(T_\psi)$ for $1 \leq i \leq k$ and $\varphi_1, \ldots, \varphi_k$ are linearly independent if $\epsilon > 0$ is small enough. Let $\phi \in \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \setminus \{0\}$ be arbitrary. Thus $\phi = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_k\varphi_k$ for some $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{C}$. Since $L_\psi(\varphi_i) = \frac{1}{p}\varphi_i$, we have

$$(L_\psi(\varphi_i), \varphi_i)_{\mathcal{L}^2(\mathbb{R}^m)} = \frac{1}{p} \tag{6.31}$$

for $1 \leq i \leq k$ and

$$|(L_\psi(\varphi_i), \varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}| \leq \frac{\epsilon}{p} \tag{6.32}$$

for $1 \leq i \neq j \leq k$ by Lemma 6.3.

By (6.31) and (6.32), we obtain

$$\begin{aligned}(L_\psi(\phi), \phi)_{\mathcal{L}^2(\mathbb{R}^m)} &\leq \frac{1}{p}\sum_{i=1}^{k}|\alpha_i|^2 + \sum_{1\leq i\neq j\leq k}\frac{\epsilon}{p}|\alpha_i||\alpha_j| \\ &\leq \frac{1}{p}\sum_{i=1}^{k}|\alpha_i|^2 + \frac{k\epsilon}{p}\sum_{i=1}^{k}|\alpha_i|^2 \\ &\leq \frac{1}{p}(1+k\epsilon)\sum_{i=1}^{k}|\alpha_i|^2. \end{aligned}\tag{6.33}$$

On the other hand, we have

$$(T_\psi(\phi), \phi)_{\mathcal{L}^2(\mathbb{R}^m)} = \sum_{i=1}^{k}|\alpha_i|^2(T_\psi(\varphi_i), \varphi_i)_{\mathcal{L}^2(\mathbb{R}^m)} + \sum_{1\leq i\neq j\leq k}\alpha_i\overline{\alpha_j}(T_\psi(\varphi_i), \varphi_j)_{\mathcal{L}^2(\mathbb{R}^m)}. \tag{6.34}$$

Here, we have

$$(T_\psi(\varphi_i), \varphi_i)_{\mathcal{L}^2(\mathbb{R}^m)} \geq (p-\epsilon)(L_\psi(\varphi_i), \varphi_i)_{\mathcal{L}^2(\mathbb{R}^m)} \tag{6.35}$$

for $1 \leq i \leq k$ by Lemma 6.3. Combining (6.31), (6.32), (6.34), (6.35) and Lemma 6.3, we have

$$(T_\psi(\phi), \phi)_{\mathcal{L}^2(\mathbb{R}^m)} \geq \Big(1 - \frac{\epsilon}{p} - k\epsilon\Big)\sum_{i=1}^{k}|\alpha_i|^2. \tag{6.36}$$

By (6.33) and (6.36), we finally obtain

$$\frac{(T_\psi(\phi), \phi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(L_\psi(\phi), \phi)_{\mathcal{L}^2(\mathbb{R}^m)}} \geq \frac{p\big(1-\frac{\epsilon}{p}-k\epsilon\big)}{1+k\epsilon}. \tag{6.37}$$

Since $\epsilon > 0$ was arbitrary, the assertion of Theorem 6.1 (3) follows from (6.37) and Remark 6.1 (i). □

7 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We thus consider the equation

$$\mathsf{D}_g\varphi = H(x)|\varphi|^{p-1}\varphi \qquad \text{on } M. \tag{7.1}$$

Notice that, if the assertion of Theorem 1.1 holds under the assumption $\mu_{-\lambda}(\psi) \leq k$ for some $\lambda > 0$, $-\lambda \notin \mathsf{Spec}(\mathsf{D})$, the same assertion under the assumption $\mu_\lambda(\psi) \leq k$ also holds for a general $\lambda \in \mathbb{R}$ in view of Lemma 3.1. Thus, throughout this section, we fix $\lambda > 0$ with $-\lambda \notin \mathsf{Spec}(\mathsf{D}_g)$ and prove Theorem 1.1 under the assumption $\mu_{-\lambda}(\psi) \leq k$.

7.1 Proof of Theorem 1.1, I: Blow-up argument

We shall prove Theorem 1.1 by contradiction. Thus, we assume that there exist a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of solutions to (7.1) and $k \in \mathbb{N}$ such that

$$\mu_{-\lambda}(\varphi_n) \leq k \tag{7.2}$$

for all $n \geq 1$ and

$$\|\varphi_n\|_{\mathcal{L}^\infty(M)} \to +\infty \tag{7.3}$$

as $n \to \infty$.

We set $\rho_n = \|\varphi_n\|_{\mathcal{L}^\infty(M)}^{p-1}$. Thus, we have $\rho_n \to \infty$ as $n \to \infty$. We conformally blow-up (M, g, ρ) by defining new metrics $g_n = \rho_n^2 g$ on M. Under such a change of the metric, the Dirac operators D_{g_n} are related to D_g as follows: denoting by $\mathbb{S}(M, g)$ and $\mathbb{S}(M, g_n)$ the spinor bundles of (M, g, σ) and (M, g_n, σ), respectively, there exists a bundle isomorphism $F_n : \mathbb{S}(M, g) \to \mathbb{S}(M, g_n)$ such that F_n is a fiberwise isometry and

$$\mathsf{D}_{g_n}(F_n(\varphi)) = F_n(\rho_n^{-\frac{m+1}{2}}(\mathsf{D}_g(\rho_n^{\frac{m-1}{2}}\varphi))), \tag{7.4}$$

see [31], [30]. We set $\psi_n := \rho_n^{-\frac{1}{p-1}} F_n(\varphi_n)$. This is a spinor on (M, g_n) and by (7.4), we easily see that ψ_n satisfies the following equation:

$$\mathsf{D}_{g_n}\psi_n = H(x)|\psi_n|^{p-1}\psi_n \quad \text{on } M, \tag{7.5}$$

where $|\psi_n|$ is the norm of ψ_n with respect to the metric g_n. For clarity, we henceforth denote it by $|\psi_n|_{g_n}$ to indicate its dependence on g_n.

Moreover, by the definition of ρ_n, we have

$$|\psi_n(x)|_{g_n} = \rho_n^{-\frac{1}{p-1}}|\varphi_n(x)|_g \leq 1 \tag{7.6}$$

for any $x \in M$ and

$$\|\psi_n\|_{\mathcal{L}^\infty(M,g_n)} = 1, \tag{7.7}$$

where $\mathcal{L}^\infty(M, g_n)$ is the set of L^∞-spinors on (M, g_n) whose norm is defined by the metric g_n. By the compactness of (M, g_n), there exists $x_n \in M$ such that

$$|\psi(x_n)|_{g_n} = \|\psi_n\|_{\mathcal{L}^\infty(M,g_n)} = 1. \tag{7.8}$$

We may assume that, after taking a subsequence if necessary, $x_n \to x_\infty$ for some $x_\infty \in M$.

Let $f_n : B_{r_0} \to (M, g)$ and $f_\infty : B_{r_0} \to (M, g)$ be normal charts at x_n and x_∞, respectively, where B_{r_0} is the Euclidean ball with center at $0 \in \mathbb{R}^m$ and radius $r_0 > 0$ and r_0 is smaller than the injectivity radius of (M, g). We denote by $h_n := f_n^* g$ and $h_\infty := f_\infty^* g$ the pull-back metrics on B_{r_0}. Then $f_n : (B_{r_0}, \rho_n^2 h_n) \to (M, \rho_n^2 g)$ is an isometric embedding. Define $s_n : (B_{\rho_n r_0}, \tilde{h}_n) \to (B_{r_0}, \rho_n^2 h_n)$ by $s_n(x) = \rho_n^{-1}x$, where $\tilde{h}_n = \rho_n^2 s_n^* h_n$. Then the map $\tilde{f}_n$ defined by $\tilde{f}_n = f_n \circ s_n : (B_{\rho_n r_0}, \tilde{h}_n) \to (M, \rho_n^2 g)$ is an isometric embedding and

$$(\tilde{h}_n)_{ij} = \tilde{h}_n(\partial_i, \partial_j) = (h_n)_{ij}(\rho_n^{-1}\,\cdot\,) \to (h_\infty)_{ij}(0) = \delta_{ij} \tag{7.9}$$

in $C^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ as $n \to \infty$.

Let us consider the chart defined by $\tilde{f}_n$, $\tilde{f}_n : (B_{\rho_n r_0}, \tilde{h}_n) \to (M, \rho_n^2 g = g_n)$. We call $\{(M, g_n, \sigma)\}$ conformal blow-up sequence of (M, g, σ) with respect to centers $\{x_n\}$. In this chart, the equation (7.5) is transformed into the following equation on $(B_{\rho_n r_0}, \tilde{h}_n)$

$$\mathsf{D}_{\tilde{h}_n} \tilde{\psi}_n = \tilde{H}_n(x) |\tilde{\psi}_n|_{\tilde{h}_n}^{p-1} \tilde{\psi}_n, \tag{7.10}$$

where $\tilde{\psi}_n$ is the pull-back spinor of ψ_n on $(B_{\rho_n r_0}, \tilde{h}_n)$ by the isometry $\tilde{f}_n$, $\tilde{H}_n = \tilde{f}_n^* H$ is the pull-back of H on $(B_{\rho_n r_0}, \tilde{h}_n)$ and the spin structure on $(B_{\rho_n r_0}, \tilde{h}_n)$ is the pull-back one from (M, g_n, σ). Note that, as $n \to \infty$,

$$\tilde{H}_n \to H(x_\infty) \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^m). \tag{7.11}$$

Since $\|\tilde{\psi}_n\|_{\mathcal{L}^\infty(B_{\rho_n r_0})} = 1$, by (7.9)–(7.11) and the elliptic regularity theory, there exists $\psi_\infty \in C^0(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ such that

$$\tilde{\psi}_n \to \psi_\infty \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^m) \tag{7.12}$$

and

$$\mathsf{D}_0 \psi_\infty = H(x_\infty) |\psi_\infty|^{p-1} \psi_\infty \quad \text{in } \mathbb{R}^m. \tag{7.13}$$

We also have

$$|\psi_\infty(0)| = 1 \tag{7.14}$$

since $|\tilde{\psi}_n(0)|_{\tilde{h}_n} = |\psi_n(x_n)|_{g_n} = 1$ by (7.8). By (7.14), we have, in particular, $\psi_\infty \not\equiv 0$ and $|\psi_\infty(x)| \le 1$ for any $x \in \mathbb{R}^m$. Thus, ψ_∞ is a bounded non-trivial solution to (7.13). Since $H(x_\infty) > 0$, after multiplying a suitable positive constant to the spinor ψ_∞ if necessary, we may assume that $H(x_\infty) = 1$.

We next want to express the operator $L_{\varphi_n}^{-1/2} T_{\varphi_n, H, \lambda} L_{\varphi_n}^{-1/2}$ associated to the spinor φ_n on (M, g, σ) in terms of the spinor ψ_n and the Dirac operator D_{g_n} on (M, g_n, σ). We recall that $T_{\varphi_n, H, \lambda}$ is defined by

$$T_{\varphi_n, H, \lambda} = (H(x)|\varphi_n|_g^{p-1} + \lambda L_{\varphi_n})^{1/2} \circ K_\lambda \circ (H(x)|\varphi_n|_g^{p-1} + \lambda L_{\varphi_n})^{1/2}, \tag{7.15}$$

where $K_\lambda = (\mathsf{D}_g + \lambda)^{-1}$.

Since $|\varphi_n|_g^{p-1} = \rho_n |\psi_n|_{g_n}^{p-1}$, $F_n \circ L_{\varphi_n} \circ F_n^{-1} = L_{\psi_n}$ and

$$F_n((\mathsf{D}_g + \lambda)\varphi) = \rho_n (\mathsf{D}_{g_n} + \rho_n^{-1}\lambda) F_n(\varphi)$$

by (7.4), we have

$$\begin{aligned} \mathsf{D}_g + \lambda &= \rho_n F_n^{-1} \circ (\mathsf{D}_{g_n} + \rho_n^{-1}\lambda) \circ F_n, \\ K_\lambda &= \rho_n^{-1} F_n^{-1} \circ (\mathsf{D}_{g_n} + \rho_n^{-1}\lambda)^{-1} \circ F_n \end{aligned} \tag{7.16}$$

and

$$T_{\varphi_n, H, \lambda} = F_n^{-1} \circ (H(x)|\psi_n|_{g_n}^{p-1} + \rho_n^{-1}\lambda L_{\psi_n})^{1/2} \circ K_{\lambda, n} \circ (H(x)|\psi_n|_{g_n}^{p-1} + \rho_n^{-1}\lambda L_{\psi_n})^{1/2} \circ F_n, \tag{7.17}$$

where $K_{\lambda,n} = (\mathsf{D}_{g_n} + \rho_n^{-1}\lambda)^{-1}$. Thus, by (7.17), we have $L_{\varphi_n}^{-1/2} T_{\varphi_n, H, \lambda} L_{\varphi_n}^{-1/2} = F_n^{-1} L_{\psi_n}^{-1/2} T_{\psi_n, H, \rho_n^{-1}\lambda} L_{\psi_n}^{-1/2} F_n$ and $\mu_{-\lambda}(\varphi_n) = \mu_{-\rho_n^{-1}\lambda}(\psi_n)$ by Proposition 4.2. But, by Theorem 6.1 (2) and (3) (note that we have not proved Theorem 6.1 (2) yet. We assume it here for the moment. It will be proved in the next subsection), we have a desired contradiction:

$$+\infty = m_{\mathbb{R}^m}(\psi_\infty) \le \liminf_{n \to \infty} \mu_{-\rho_n^{-1}\lambda}(\psi_n) \le k.$$

This completes the proof of Theorem 1.1 under assuming Theorem 6.1 (2). □

Thus, it remains to prove the assertion of Theorem 6.1 (2), namely, under the above notation and the assumption, we shall prove

$$m_{\mathbb{R}^m}(\psi_\infty) \leq \liminf_{n\to\infty} \mu_{-\rho_n^{-1}\lambda}(\psi_n). \tag{7.18}$$

For the proof of (7.18), we need to show that $L_{\psi_n}^{-1/2} T_{\psi_n,H,\rho_n^{-1}\lambda} L_{\psi_n}^{-1/2}$ converges to $L_\psi^{-1/2} T_\psi L_\psi^{-1/2}$ in a certain sense. This will be proved in the next subsection.

7.2 Proof of Theorem 1.1, II: the lower semi-continuity of the relative Morse index

To study the convergence property of $L_{\psi_n}^{-1/2} T_{\psi_n,H,\rho^{-1}\lambda} L_{\psi_n}^{-1/2}$, we need to estimate the Schwartz kernel of $(\mathsf{D}_{g_n} + \rho_n^{-1}\lambda)^{-1}$. Let us denote by $k(x,y)$ and $k_n(x,y)$ the Schwartz kernels of $(\mathsf{D}_g+\lambda)^{-1}$ and $(\mathsf{D}_{g_n} + \rho_n^{-1}\lambda)^{-1}$, respectively. More precisely, since the kernels of $(\mathsf{D}_g + \lambda)^{-1}$ and $(\mathsf{D}_{g_n} + \rho_n^{-1}\lambda)^{-1}$ are considered as densities on (M, g) and (M, g_n), respectively, $k(x, y)$ and $k_n(x, y)$ are defined as

$$(\mathsf{D}_g + \lambda)^{-1} = k(x,y)\, d\mathrm{vol}_g, \quad (D_{g_n} + \rho_n^{-1}\lambda)^{-1} = k_n(x,y)\, d\mathrm{vol}_{g_n}. \tag{7.19}$$

Then by (7.16), (7.19) and $d\mathrm{vol}_{g_n} = \rho_n^m\, d\mathrm{vol}_g$ we have

$$k_n(x,y) = \rho_n^{1-m} F_n \circ k(x,y) \circ F_n^{-1}. \tag{7.20}$$

With respect to the local coordinate $\tilde{f}_n : (B_{\rho_n r}, \tilde{h}_n) \to (M, g_n)$ defined in the previous subsection, we have

$$\begin{aligned} \tilde{k}_n(x,y) :&= \tilde{f}_n^* k_n(x,y) \\ &= k_n(\rho_n^{-1}x, \rho_n^{-1}y) \\ &= \rho_n^{1-m} F_n \circ k(\rho_n^{-1}x, \rho_n^{-1}y) \circ F_n^{-1} \end{aligned} \tag{7.21}$$

for $x, y \in B_{\rho_n r_0}$.

As for the Schwartz kernel $k(x,y)$ of $(\mathsf{D}_g + \lambda)^{-1}$, we have the following estimate in the local coordinate $f_n : (B_{r_0}, h_n) \to (M, g)$ introduced in the previous subsection:

$$|k(x,y)|_{h_n} \leq \alpha(x,y)|x-y|^{1-m}, \tag{7.22}$$

where $\alpha(x, y) > 0$ is determined by the metric g and it remains bounded when x, y vary within B_{r_0}. This is a well-known fact and is proved with the help of the theory of pseudo-differential operators. Here we only give a sketch of the argument. For details, see [49, I&II]. We write $(\mathsf{D}_g+\lambda)^{-1} = (\mathsf{D}_g+\lambda)(\mathsf{D}_g+\lambda)^{-2}$. The principal part of the operator $(\mathsf{D}_g+\lambda)^2$ is $-\Delta_g \otimes \mathbf{1}_{\mathbb{S}(M,g)}$ and it is the scalar Laplace operator. So in order to construct a parametrix of $(\mathsf{D}_g+\lambda)^2$, we can argue as in [49, Chapter 3, §9]. Since the singular part of the parametrix coincides with that of the Schwartz kernel K of $(D_g + \lambda)^{-2}$ (in fact, they coincides up to a smoothing factor) and the singular part of the parametrix of $(\mathsf{D}_g + \lambda)^2$ behaves like $|x - y|^{2-m}$ (see [49, I, Chapter 3, §9; II, Chapter 7]), (7.22) follows from $k = (\mathsf{D}_g + \lambda)K$.

By (7.21) and (7.22), we have

$$\begin{aligned} |\tilde{k}_n(x,y)|_{\tilde{h}_n} &\leq \rho_n^{1-m}\alpha(\rho_n^{-1}x, \rho_n^{-1}y)|\rho_n^{-1}x - \rho_n^{-1}y|^{1-m} \\ &\leq \alpha(\rho_n^{-1}x, \rho_n^{-1}y)|x-y|^{1-m}. \end{aligned} \tag{7.23}$$

From (7.23), we have the following uniform estimate of $\tilde{k}_n(x,y)$ for $x,y \in \rho_n B_{r_0} = B_{\rho_n r_0}$:

$$|\tilde{k}_n(x,y)|_{\tilde{h}_n} \le C|x-y|^{1-m}, \tag{7.24}$$

where $C > 0$ is independent of $x, y \in B_{\rho_n r_0}$ for all large n.

Since $B_{\rho_n r} \to \mathbb{R}^m$ as $n \to \infty$, we have: for any compact subset $K \subset \mathbb{R}^m$, there exist $n(K) \in \mathbb{N}$ such that

$$|\tilde{k}_n(x,y)|_{\tilde{h}_n} \le C|x-y|^{1-m} \tag{7.25}$$

for any $x, y \in K$ and $n \ge n(K)$.

From the uniform estimate (7.25), arguing as in (6.3) and (6.4), we have the following: for any $\phi \in \mathcal{L}_c^\infty(\mathbb{R}^m)$ and any compact subset $K \subset \mathbb{R}^m$, there exist $C = C(K, \mathrm{supp}(\phi)) > 0$ and $n(K, \mathrm{supp}(\phi)) \in \mathbb{N}$ depending only on K and the diameter of the support of ϕ such that

$$\begin{aligned}|(\mathsf{D}_{\tilde{h}_n} + \rho_n^{-1}\lambda)^{-1}\phi(x)| &\le \int_{\mathbb{R}^m} |\tilde{k}_n(x,y)||\phi(y)|\, d\mathrm{vol}_{\tilde{h}_n}(y) \\ &\le C\int_{\mathbb{R}^m} \frac{|\phi(y)|}{|x-y|^{m-1}}\, d\mathrm{vol}_0(y) \le C\|\phi\|_{\mathcal{L}^\infty(\mathbb{R}^m)} \end{aligned}\tag{7.26}$$

for any $x \in K$ and $n \ge n(K, \mathrm{supp}(\phi))$, where we have used $\tilde{h}_n \to g_{\mathbb{R}^m}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ as $n \to \infty$.

Set $(\mathsf{D}_{\tilde{h}_n} + \rho_n^{-1}\lambda)^{-1}\phi =: u_n$. Then u_n satisfies $\mathsf{D}_{\tilde{h}_n} u_n + \rho_n^{-1}\lambda u_n = \phi$. By (7.26) and the elliptic estimate for the Dirac operator $\mathsf{D}_{\tilde{h}_n}$, we have that $\{u_n\}$ is bounded in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^m)$ for any $0 < \alpha < 1$, where we have used $\tilde{h}_n \to g_{\mathbb{R}^m}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^m)$ again. Thus by the Ascoli-Arzelà theorem, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n \to u_\infty$ locally uniformly on $\mathbb{R}^m$ for some $u_\infty \in C^0(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$. Then u_∞ satisfies $\mathsf{D}_{g_{\mathbb{R}^m}} u_\infty = \phi$ and we have $u_\infty \in C^{0,\alpha}(\mathbb{R}^m)$ for any $0 < \alpha < 1$ by the elliptic regularity. Combining this with (7.12), for any $\phi \in \mathcal{L}_c^\infty(\mathbb{R}^m)$ we have

$$\tilde{f}_n^*(L_{\psi_n}^{-1/2} T_{\psi_n, H, \rho_n^{-1}\lambda} L_{\psi_n}^{-1/2})\phi = L_{\tilde{\psi}_n}^{-1/2} T_{\tilde{\psi}_n, \tilde{H}_n, \rho_n^{-1}\lambda} L_{\tilde{\psi}_n}^{-1/2}\phi \to L_{\psi_\infty}^{-1/2} T_{\psi_\infty} L_{\psi_\infty}^{-1/2}\phi \tag{7.27}$$

in $C^0_{\mathrm{loc}}(\mathbb{R}^m)$, where $L_{\tilde{\psi}_n} := \frac{1}{p}P_{\tilde{\psi}_n} + P^\perp_{\tilde{\psi}_n}$ (the orthogonal projections are taken with respect to the metric $\tilde{h}_n$), $T_{\tilde{\psi}_n, \tilde{H}_n, \rho_n^{-1}\lambda} := (\tilde{H}_n|\tilde{\psi}_n|^{p-1} + \rho_n^{-1}\lambda L_{\tilde{\psi}_n})^{1/2} \circ \tilde{K}_{n,\lambda} \circ (\tilde{H}_n|\tilde{\psi}_n|^{p-1} + \rho_n^{-1}\lambda L_{\tilde{\psi}_n})^{1/2}$, $\tilde{K}_{n,\lambda} = (\mathsf{D}_{\tilde{h}_n} + \rho_n^{-1}\lambda)^{-1}$ $(= \tilde{k}_n(x,y)\, d\mathrm{vol}_{\tilde{h}_n})$ and L_{ψ_∞} and T_{ψ_∞} are defined in the previous subsection.

Under the convergence (7.27), we easily obtain (7.18). Namely, by the definition of the index $m_{\mathbb{R}^m}(\psi_\infty)$, for any $l \in \mathbb{N}$ with $l \le m_{\mathbb{R}^m}(\psi_\infty)$, there exist $\epsilon > 0$ and linearly independent spinors $\varphi_1, \varphi_2, \ldots, \varphi_l \in \mathcal{L}_c^\infty(\mathbb{R}^m)$ such that

$$\frac{(L_{\psi_\infty}^{-1/2} T_{\psi_\infty} L_{\psi_\infty}^{-1/2}(\varphi), \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(\varphi, \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}} \ge 1 + \epsilon \tag{7.28}$$

for any $\varphi \in \mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_l\}$.

On the other hand, there exists a compact subset $K \subset \mathbb{R}^m$ such that $\mathrm{supp}\,\varphi_j \subset K \subset B_{\rho_n r_0}$ for all $1 \le j \le l$ and large n. Then by (7.27) and (7.28), we have

$$\frac{(L_{\tilde{\psi}_n}^{-1/2} T_{\tilde{\psi}_n, \tilde{H}_n, \rho_n^{-1}\lambda} L_{\tilde{\psi}_n}^{-1/2}(\varphi), \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(\varphi, \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}} \to \frac{(L_{\psi_\infty}^{-1/2} T_{\psi_\infty} L_{\psi_\infty}^{-1/2}(\varphi), \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}}{(\varphi, \varphi)_{\mathcal{L}^2(\mathbb{R}^m)}} \ge 1 + \epsilon \tag{7.29}$$

as $n \to \infty$ uniformly for $\varphi \in \mathrm{span}\{\varphi_1, \varphi_2, \ldots, \varphi_l\}$. By Proposition 4.2 (2), (7.29) implies that $\mu_{-\rho_n^{-1}\lambda}(\psi_n) \ge l$ for all large n. Since $l \le m_{\mathbb{R}^m}(\psi_\infty)$ was arbitrary, this implies (7.18) and completes the proof of Theorem 1.1. □

7.3 A generalization

The above argument applies for a more general class of nonlinearities. As in §4.3, we assume that $H \in C^2(\mathbb{S}(M))$ satisfies (4.22). Moreover, we assume that the following condition is satisfied for H: for any $\{x_n\}_{n=1}^\infty \subset M$ and $\{\lambda_n\}_{n=1}^\infty$ with $x_n \to x_\infty$ and $\lambda_n \to \infty$, there exists a positive constant H_∞ such that

$$H_n(x,\varphi) := \lambda_n^{-p-1} H(\exp_{x_n}(\lambda_n^{1-p}\cdot), \lambda_n F_n(\varphi)) \to H_\infty |\varphi|^{p+1} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{S}(\mathbb{R}^m)) \tag{7.30}$$

as $n \to \infty$, where $F_n : \mathbb{S}(\mathbb{R}^m)|_{B_{\lambda_n^{p-1} r_0}} \to \mathbb{S}(M)$ is the fiberwise isometry which is defined as in §7.1.

For example, the condition (7.30) is satisfied for H which is a lower order perturbation of $\frac{1}{p+1}H(x)|\psi|^{p+1}$. More precisely, for H of the following form

$$H(x,\psi) = \frac{1}{p+1}H(x)|\psi|^{p+1} + h(x,\psi),$$

where $h(x,\psi)$ satisfies

$$\lim_{|\psi|\to\infty} \frac{|h(x,\psi)|}{|\psi|^{p+1}} = 0.$$

In this case, H satisfies (7.30) with $H_\infty = \frac{1}{p+1}H(x_\infty)$.

By repeating the same argument as in the proof of Theorem 1.1, we have the following generalization of Theorem 1.1:

Theorem 7.1 *Assume $H \in C^2(\mathbb{S}(M))$ satisfies (4.22) and (7.30). For any $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, there exists a constant $C(\lambda,k,H) > 0$ such that the following holds: for any solution $\psi \in \mathcal{H}^{1/2}(M)\cap\mathcal{L}^\infty(M)$ to the equation $\mathsf{D}_g\psi = H_\psi(x,\psi)$ on M with the relative Morse index $\mu_\lambda(\psi) \le k$, there holds $\|\psi\|_{\mathcal{L}^\infty(M)} \le C(\lambda,k,H)$.*

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