Formal group laws for multiple sine functions and applications

In memory of Professor Tsuneo Kanno

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Abstract. We investigate addition relations for multiple sine functions from the view point of formal group laws. We find that the functions which appear in the coefficients are related to classical Eisenstein serires. As application we obtain a limit formula for automorphic forms.

Key words: Multiple sine function, Eisenstein series, modular function

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1 Introduction

The addition theorem for the multiple sine function is a quite important problem. Indeed a suitable addition relation would imply the algebraicity of the division value of the multiple sine function, which would lead to Kronecker's Jugendtraum as studied by Shintani [S1] in 1977.

In this paper we investigate addition relations for multiple sine functions from the view point of formal group laws. The central problem is to determine characters of coefficients. We see that important functions are appearing in these coefficients and that they are intimately related to classical Eisenstein serires. Consequently we obtain some limit formulas for automorphic forms.

Let

$$\begin{aligned} S_r(x,(\omega_1,\cdots,\omega_r)) &= \prod_{\mathbf{n}\geq\mathbf{0}}(\mathbf{n}\cdot\boldsymbol{\omega}+x)\left(\prod_{\mathbf{m}\geq\mathbf{1}}(\mathbf{m}\cdot\boldsymbol{\omega}-x)\right)^{(-1)^{r-1}} \\ &= \Gamma_r(x,\boldsymbol{\omega})^{-1}\Gamma_r(|\boldsymbol{\omega}|-x,\boldsymbol{\omega})^{(-1)^r} \end{aligned}$$

be the multiple sine function, where

$$\Gamma_r(x, \boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s} \zeta_r(s, x, \boldsymbol{\omega}) \bigg|_{s=0} \right)$$

is the normalized multiple gamma function obtained from the multiple Hurwitz zeta function

$$\zeta_r(s, x, \boldsymbol{\omega}) = \sum_{\mathbf{n} \geq \mathbf{0}} (\mathbf{n} \cdot \boldsymbol{\omega} + x)^{-s}.$$

Here we use the notation $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_r)$ and $|\boldsymbol{\omega}| = \omega_1 + \cdots + \omega_r$. It is obvious that $S_r(x, (\omega_1, \cdots, \omega_r))$ is symmetric with respect to $\omega_1, \cdots, \omega_r$.

As was shown in our previous papers, the values of the multiple sine functions at rational points are important for the study of special values of the Riemann zeta and the Dirichlet L-functions. For example, we would obtain the trancendency of $\zeta(3)/\pi^2$, if we prove the rationality of their division values. Thus the addition formula is crutial.

We put the addition formula for $S_r(x, \boldsymbol{\omega})$ as

$$S_r(x + y, \boldsymbol{\omega}) = \Phi(S_r(x, \boldsymbol{\omega}), S_r(y, \boldsymbol{\omega}))$$

with

$$\Phi(X, Y) = X + Y + \sum_{i,j \geq 1} c_{i,j} X^i Y^j. \tag{1.1}$$

Our chief concern lies in the coefficients $c_{i,j}$. In this paper we calculate for $r = 2$ the coefficients $c_{1,1}$, $c_{1,2}$, $c_{2,1}$ and find their mutual relations. We also establish their relation to the Eisenstein series.

We first observe more general facts. Let $F(x)$ be a differentiable function in x with $F(0) = 0$ and $F'(0) \neq 0$. Then there exists a two variable function $\Phi(X, Y)$ given by (1.1) such that

$$F(x + y) = \Phi(F(x), F(y)).$$

In this case it is easily shown that

$$c_{1,1} = \frac{F''(0)}{F'(0)^2} \tag{1.2}$$

and

$$c_{1,2} = c_{2,1} = \frac{F'''(0)F'(0) - F''(0)^2}{2F'(0)^4}. \tag{1.3}$$

Example 1 When $r = 1$, we have $S_1(x) = 2\sin(\pi x)$. The addition formula is

$$S_1(x+y) = S_1(x)\sqrt{1-\frac{S_1(y)^2}{4}} + S_1(y)\sqrt{1-\frac{S_1(x)^2}{4}} = \Phi(S_1(x), S_1(y)).$$

Thus

$$\Phi(X,Y) = X\sqrt{1-\frac{Y^2}{4}} + Y\sqrt{1-\frac{X^2}{4}} = X + Y - \frac{1}{8}X^2Y - \frac{1}{8}XY^2 + \cdots.$$

Then

$$c_{1,1} = 0, \qquad c_{1,2} = c_{2,1} = -\frac{1}{8}.$$

It is easy to verify they satisfy the identities (1.2) and (1.3).

In what follows we fix $F(x) = S_r(x, \boldsymbol{\omega})$ and write $c_{i,j} = c_{i,j}(\boldsymbol{\omega})$. It is easy to see the following facts.

Theorem 0 (1) $c_{1,1}(\boldsymbol{\omega}) = \frac{S_r''(0,\boldsymbol{\omega})}{S_r'(0,\boldsymbol{\omega})^2}$.

(2) $c_{1,1}(\lambda\boldsymbol{\omega}) = c_{1,1}(\boldsymbol{\omega})$ *for any* $\lambda \in \mathbf{R}$.

(3) $c_{1,1}(\boldsymbol{\omega})$ *is symmetric.*

For each non-zero complex number k we put

$$E_k(\tau) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

where τ belongs to the upper half plane and $q = e^{2\pi i\tau}$. As is well-known, for an even integer $k \geq 4$, $E_k(\tau)$ is the Eisenstein series of weight k with respect to the modular group $SL_2(\mathbf{Z})$. Especially, it satisfies the automorphy

$$E_k\left(-\frac{1}{\tau}\right) = \tau^k E_k(\tau)$$

and the corresponding zeta function is given by

$$L(s, E_k) = \zeta(s)\zeta(s-k+1).$$

We consider $E_k(\tau)$ as the principal Eisenstein series of weight k for $k \in \mathbf{C} \setminus \{0\}$ in general, where "principal" is indicating the "principal character." It seems to be an interesting problem to see the automorphy of $E_k(\tau)$. This means to calculate

$$R_k(\tau) = E_k\left(-\frac{1}{\tau}\right) - \tau^k E_k(\tau)$$

concretely. We know the result in the following cases:

(1) $R_k(\tau) = 0$ for each even integer $k \geq 4$,

(2) $R_2(\tau) = -\dfrac{\tau}{4\pi i}$.

From these results we obtain

$$E_6(i) = 0$$

and

$$E_2(i) = -\frac{1}{8\pi}.$$

In other words,

$$\sum_{n=1}^{\infty} \frac{n^5}{e^{2\pi n} - 1} = \frac{1}{504}$$

and

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} = \frac{1}{24} - \frac{1}{8\pi}$$

as noted by Ramanujan. We remark that the case (2) is shown from the transformation formula

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)$$

for the Dedekind η-function. It would be a non-trivial problem to investigate $R_k(\tau)$ for $k \in \mathbf{C} \setminus \{0\}$ in general.

In papers [K2] [KK4] we proved

$$\lim_{\tau \to 1} R_k(\tau) = \frac{(-1)^k B_{k-1}}{2\pi i}$$

and its higher dimensional analogue for each positive integer k, where B_k is the Bernoulli number.

In the previous paper we examined the case $k = 1$, where

$$E_1(\tau) = -\frac{1}{4} + \sum_{n=1}^{\infty} d(n) q^n = -\frac{1}{4} + \sum_{m,n \geq 1} q^{mn}$$

and

$$R_1(\tau) = E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau).$$

We notice that this k is the only one case when $L(s, E_k)$ has the double pole; in fact $L(s, E_1) = \zeta(s)^2$ is having the double pole at $s = 1$.

In this paper we calculate boundary values at all cusps M/N as follows:

Theorem 1 *Let M, $N \geq 1$ be integers which are coprime. It holds that*

$$\lim_{\substack{\tau \to M/N \\ \mathrm{Im}(\tau)>0}} \left(E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau) \right) = \frac{1}{2iN} \sum_{\substack{m=0,\ldots,M-1 \\ n=0,\ldots,N-1 \\ (m,n)\neq(0,0)}} \left|\frac{m}{M} - \frac{n}{N}\right| \cot\left(\pi \left|\frac{m}{M} - \frac{n}{N}\right|\right) - \frac{1}{2\pi i N}.$$

This is a generalization of the previous result obtained in [K1] recalled in Section 2.

2 Preliminary Propositions

In this section we prepare some facts which will be used in the proof of our theorems.

We first recall the properties of the multiple sine function. In the previous paper [K1] we proved the following four facts (i)-(vi).

Proposition 1 (i) *Let N be a positive integer. Then $R_1(\tau)$ has the following transcendental numbers as boundary values at cusps N and $\frac{1}{N}$.*

$$(1)\quad \lim_{\tau \to N} R_1(\tau) = \frac{1}{2i}\left(\frac{1}{N}\sum_{k=1}^{\left[\frac{N}{2}\right]} (N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi} \right).$$

$$(2)\quad \lim_{\tau \to \frac{1}{N}} R_1(\tau) = \frac{1}{2Ni}\left(\frac{1}{N}\sum_{k=1}^{\left[\frac{N}{2}\right]} (N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi} \right).$$

(ii) *For* $\mathrm{Im}(\tau) > 0$

$$R_1(\tau) = \frac{\tau^{\frac{3}{2}}}{8\pi^2 i} S_2''(0, (1, \tau)).$$

(iii) *For each positive real number α*

$$\lim_{\tau \to \alpha} R_1(\tau) = \frac{\alpha^{\frac{3}{2}}}{8\pi^2 i} S_2''(0, (1, \alpha)).$$

(iv) *For each integer* $N \geq 1$

$$S_2''(0,(1,N)) = \frac{4\pi^2}{N^{\frac{3}{2}}}\left(\frac{1}{N}\sum_{k=1}^{\left[\frac{N}{2}\right]}(N-2k)\cot\left(\frac{\pi k}{N}\right) - \frac{1}{\pi}\right).$$

Proposition 2

(1) *(periodicity)*

$$S_r(x+\omega_i, \boldsymbol{\omega}) = S_r(x,\boldsymbol{\omega})S_{r-1}(x,\boldsymbol{\omega}(i))^{-1},$$

where $\boldsymbol{\omega}(i) = (\omega_1, \cdots, \omega_{i-1}, \omega_{i+1}, \cdots, \omega_r)$.

(2) *(homogeneity)*

$$S_r(cx, c\boldsymbol{\omega}) = S_r(x,\boldsymbol{\omega})$$

for $c > 0$.

(3) *(differential equation)*

$$\frac{S_r'}{S_r}(x,(1,\cdots,1)) = (-1)^{r-1}\pi\binom{x-1}{r-1}\cot(\pi x).$$

(4)

$$S_1(x,\omega) = 2\sin\left(\frac{\pi x}{\omega}\right).$$

(5)

$$S_2(\omega_1,(\omega_1,\omega_2)) = \sqrt{\frac{\omega_2}{\omega_1}},$$

$$S_2(\omega_2,(\omega_1,\omega_2)) = \sqrt{\frac{\omega_1}{\omega_2}}.$$

We define the *multiple cotangent function* as

$$\mathrm{Cot}_r(\tau,\boldsymbol{\omega}) = \frac{S_r'}{S_r}(\tau,\boldsymbol{\omega}).$$

We recall the following fact which was proved in [KK1].

Lemma 1

$$\mathrm{Cot}_2(x) = \pi(1-x)\cot(\pi x).$$

Lemma 2

$$\mathrm{Cot}_2(x,(M,N)) = \frac{1}{MN} \sum_{\substack{k=0,\dots,N-1 \\ l=0,\dots,M-1}} \mathrm{Cot}_2\left(\frac{x}{MN}+\frac{k}{N}+\frac{l}{M}\right).$$

Proof. By [KK2] we have

$$S_2(x,(M,N)) = \sum_{\substack{k=0,\dots,N-1 \\ l=0,\dots,M-1}} S_2\left(\frac{x}{MN}+\frac{k}{N}+\frac{l}{M}\right).$$

Its logarithmic derivative gives the result. ∎

By Proposition 2(2), it is easy to see that

Lemma 3

$$\mathrm{Cot}_2(cx,(c\omega_1,c\omega_2)) = \frac{1}{c}\mathrm{Cot}_2(x,(\omega_1,\omega_2)).$$

3 Results

Theorem 2 *For* $\mathrm{Im}(\tau)>0$*, it holds that*

$$c_{1,1}(1,\tau) = \frac{2i}{\sqrt{\tau}}\left(E_1\left(-\frac{1}{\tau}\right)-\tau E_1(\tau)\right).$$

Proof. By Proposition 1(ii), we have

$$S_2''(0,(1,\tau)) = \frac{8\pi^2 i}{\tau^{3/2}}\left(E_1\left(-\frac{1}{\tau}\right)-\tau E_1(\tau)\right).$$

On the other hand, we already know the fact in our previous paper that

$$S_2'(0,(1,\tau)) = \frac{2\pi}{\sqrt{\tau}}.$$

The result follows by (1.2). ∎

Theorem 3 ($r=2$)

$$\begin{aligned} c_{1,1}(\omega_1,\omega_2) &= \frac{\sqrt{\omega_1\omega_2}}{\pi}\mathrm{Cot}_2(\omega_1,(\omega_1,\omega_2)) \\ &= \frac{\sqrt{\omega_1\omega_2}}{\pi}\mathrm{Cot}_2(\omega_2,(\omega_1,\omega_2)). \end{aligned}$$

Proof. In the previous paper [K1, p.399], we proved

$$S_2''(0,(1,\tau)) = \frac{4\pi}{\sqrt{\tau}} \cdot \frac{S_2'}{S_2}(\tau,(1,\tau)). \tag{3.1}$$

Hence

$$\begin{aligned}\mathrm{Cot}_2(\tau,(1,\tau)) &= \frac{S_2'}{S_2}(\tau,(1,\tau)) \\ &= \frac{\sqrt{\tau}}{4\pi} S_2''(0,(1,\tau)) \\ &= \frac{\sqrt{\tau}}{4\pi} S_2'(0,(1,\tau))^2 c_{1,1}(1,\tau)\end{aligned}$$

by (1.2). The fact that

$$S_2'(0,(1,\tau)) = \frac{2\pi}{\sqrt{\tau}}$$

leads to the result. ∎

Combining Theorems 2 and 3 gives the following fact.

Lemma 4

$$E_1\left(-\frac{1}{\tau}\right) - \tau E_1(\tau) = \frac{\tau}{2\pi i}\mathrm{Cot}_2(\tau,(1,\tau))$$

for $\mathrm{Im}(\tau) > 0$.

Theorem 4 ($r = 2$) *Let $M, N \geq 1$ be coprime integers. It holds that*

$$c_{1,1}(M,N) = \frac{1}{\sqrt{MN}} \sum_{\substack{m=0,\ldots,M-1 \\ n=0,\ldots,N-1 \\ (m,n)\neq(0,0)}} \left|\frac{m}{M} - \frac{n}{N}\right| \cot\left(\pi\left|\frac{m}{M} - \frac{n}{N}\right|\right) - \frac{1}{\pi\sqrt{MN}}.$$

Proof. Putting $x = M$ in Lemma 2, it follows that

$$\mathrm{Cot}_2(M,(M,N)) = \frac{1}{MN} \sum_{\substack{k=0,\ldots,N-1 \\ l=0,\ldots,M-1}} \mathrm{Cot}_2\left(\frac{1+k}{N} + \frac{l}{M}\right)$$

under the convention that when $(k,l) = (N-1,0)$, we interpret

$$\mathrm{Cot}_2(1) = \lim_{x\to 1} \mathrm{Cot}_2(x) = \lim_{x\to 1} \pi(1-x)\cot(\pi x) = \lim_{x\to 1} \frac{\pi(1-x)\cos(\pi x)}{\sin(\pi x)} = -1$$

with

$$\mathrm{Cot}_2(x) = \mathrm{Cot}_2(x,(1,1)) = \frac{S_2'}{S_2}(x).$$

Taking Theorem 2 into account, we have

$$
\begin{aligned}
&c_{1,1}(M,N)\\
&= \frac{\sqrt{MN}}{\pi}\mathrm{Cot}_2(M,(M,N))\\
&= \frac{1}{\pi\sqrt{MN}} \sum_{\substack{k=0,\dots,N-1\\ l=0,\dots,M-1}} \mathrm{Cot}_2\left(\frac{1+k}{N}+\frac{l}{M}\right)\\
&= \frac{1}{\pi\sqrt{MN}} \sum_{\substack{k=0,\dots,N-1\\ l=0,\dots,M-1\\ (k,l)\neq(N-1,0)}} \pi\frac{MN-(k+1)M-lN}{MN}\cot\left(\pi\frac{(k+1)M+lN}{MN}\right) - \frac{1}{\pi\sqrt{MN}}\\
&= \frac{1}{\sqrt{MN}} \sum_{\substack{k=0,\dots,N-1\\ l=0,\dots,M-1\\ (k,l)\neq(N-1,0)}} \left(1-\frac{k+1}{N}-\frac{l}{M}\right)\cot\left(\pi\left(\frac{k+1}{N}+\frac{l}{M}\right)\right) - \frac{1}{\pi\sqrt{MN}}\\
&= \frac{1}{\sqrt{MN}} \sum_{\substack{m=0,\dots,M-1\\ n=0,\dots,N-1\\ (m,n)\neq(0,0)}} \left|\frac{m}{M}-\frac{n}{N}\right|\cot\left(\pi\left|\frac{m}{M}-\frac{n}{N}\right|\right) - \frac{1}{\pi\sqrt{MN}}
\end{aligned}
$$

by putting $k+1=N-n$. ∎

4 Proof of Theorem 1

From the results in the preceding section, we extend Proposition 1(i) to generalize rational numbers M/N as follows.

Proof of Theorem 1. We compute

$$
\begin{aligned}
\lim_{\substack{\tau\to M/N\\ \mathrm{Im}(\tau)>0}} \left(E_1\left(-\frac{1}{\tau}\right)-\tau E_1(\tau)\right) &= \lim_{\substack{\tau\to M/N\\ \mathrm{Im}(\tau)>0}} \frac{\sqrt{\tau}}{2i}c_{1,1}(1,\tau) \quad (\because \text{Theorem 2})\\
&= \frac{\sqrt{M}}{2i\sqrt{N}}c_{1,1}(N,M) \quad (\because \text{Theorem 0(2)}).
\end{aligned}
$$

Theorem 4 leads to the result. ∎

5 Relation between Coefficients

Theorem 5 *It holds that*

$$
c_{1,2}(1,\tau)=c_{2,1}(1,\tau)=-\frac{1}{8}c_{1,1}(1,\tau)^2-\frac{1}{16}\left(\tau+\frac{1}{\tau}\right)
$$

for $\tau\in\mathbf{C}\setminus(-\infty,0]$.

Proof. Put
$$\mathcal{S}_k(u_1,\cdots,u_r)=S_{r+1}^{(k)}(0,(u_1,\cdots,u_r,1))$$
for $k\geq 0$. By [KK5, Theorem 4], we have
$$\mathcal{S}_3(\tau)=\frac{3}{4}\mathcal{S}_2(\tau)^2\mathcal{S}_1(\tau)^{-1}-\frac{1}{8}\left(\tau+\frac{1}{\tau}\right)\mathcal{S}_1(\tau)^3.$$
Since
$$\mathcal{S}_1(\tau)=\frac{2\pi}{\sqrt{\tau}},$$
it holds that
$$\mathcal{S}_3(\tau)=\frac{3\sqrt{\tau}}{8\pi}\mathcal{S}_2(\tau)^2-\frac{1}{8}\left(\tau+\frac{1}{\tau}\right)\frac{8\pi^3}{\tau^{\frac{3}{2}}}. \tag{5.1}$$
From the fact that
$$\mathcal{S}_k(\tau)=S_2^{(k)}(0,(1,\tau)), \tag{5.2}$$
the basic identity (1.3) and (1.2) leads to
$$\begin{aligned}c_{1,2}(1,\tau)=c_{2,1}(1,\tau) &= \frac{\mathcal{S}_3(\tau)\mathcal{S}_1(\tau)-\mathcal{S}_2(\tau)^2}{2\mathcal{S}_1(\tau)^4}\\ &= \frac{\mathcal{S}_3(\tau)}{2\mathcal{S}_1(\tau)^3}-\frac{\mathcal{S}_2(\tau)^2}{2\mathcal{S}_1(\tau)^4}\end{aligned}$$
and that
$$c_{1,1}(1,\tau)=\frac{S_2(\tau)}{S_1(\tau)^2}.$$
By combining these identities we have
$$\begin{aligned}c_{1,2}(1,\tau)=c_{2,1}(1,\tau) &= \frac{\mathcal{S}_3(\tau)}{2\mathcal{S}_1(\tau)^3}-\frac{1}{2}c_{1,1}(1,\tau)^2\\ &= \frac{1}{2}\left(\frac{3}{4}\frac{\mathcal{S}_2(\tau)^2}{\mathcal{S}_1(\tau)^4}-\frac{1}{8}\left(\tau+\frac{1}{\tau}\right)\right)-\frac{1}{2}c_{1,1}(1,\tau)^2.\\ &= \frac{3}{8}c_{1,1}(1,\tau)^2-\frac{1}{16}\left(\tau+\frac{1}{\tau}\right)-\frac{1}{2}c_{1,1}(1,\tau)^2\\ &= -\frac{1}{8}c_{1,1}(1,\tau)^2-\frac{1}{16}\left(\tau+\frac{1}{\tau}\right).\end{aligned}$$

∎

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