# ENUMERATIVE PROPERTIES OF BRUHAT INTERVALS OF ABSOLUTE LENGTH TWO

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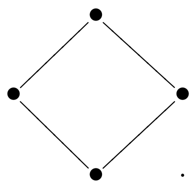
Abstract. It is a well-known fact that every Bruhat interval of length two has precisely two (co)atoms. As a partial generalization of this, we prove that every Bruhat interval of absolute length 2 necessarily has even number of parabolic (co)atoms and its number is bounded by length of the interval.

## 1. Introduction

Bruhat order of Coxeter groups has been of great importance in modern mathematics. It shows up in many areas such as representation theory of Hecke algebras, symmetry of root systems, geometry of Schubert varieties and so on (see books [1, 3]). One particular importance is that this order is locally Eulerian so that every interval of length 2 is isomorphic to



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Observe that this poset has precisely two (co)atoms which is equal to length of the interval. Our main result (Theorem in Section 6) is a partial generalization this; we show that every Bruhat interval of *absolute* length 2 necessarily has even number of parabolic (co)atoms. Its number is bounded by length of the interval. This is a sort of the philosophy "from Hasse diagram to Bruhat graph". A key idea is to make use of derivatives of $R$-polynomials. These are (a family of) polynomials over integer coefficients which play an important role in combinatorics of Bruhat order (see [1, Chapter 5]). Here we (unusually) regard them as real polynomials so that we can differentiate them. We will see that the number of parabolic (co)atoms in an interval of absolute length 2 can be computed in terms of the value $q = 1$ of the second derivative of $R$-polynomials of the interval (Section 5).

Sections 2, 3 provides basic terminology and definitions. Section 4, 5 are the main discussions for a proof of Theorem in Section 6.

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*Date*: November 15, 2011.

2000 *Mathematics Subject Classification.* Primary:20F55; Secondary:51F15.

*Key words and phrases.* Bruhat order, Coxeter groups, $R$-polynomials.

## 2. Notation

We follow notation commonly used in the context of Coxeter groups (such as [1, 3]). By $(W, S)$ (or simply $W$) we mean a Coxeter system with length function $\ell$. Let $T = \cup_{w \in W} w^{-1}Sw$ denote the set of reflections. Write $u \to w$ if $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$. Define *Bruhat order* $u \le w$ if there exist $v_0, v_1, \dots, v_k \in W$ such that $u = v_0 \to v_1 \to \cdots \to v_k = w$. For $u \le w$, let $[u, w] \overset{\text{def}}{=} \{v \in W \mid u \le v \le w\}$ denote a *Bruhat interval.* Often $\ell(u, w) \overset{\text{def}}{=} \ell(w) - \ell(u)$ abbreviates a difference of lengths of $u$ and $w$. For $f \in \mathbb{R}[q]$, $[q^n](f)$ denotes the coefficient of $q^n$ in $f$. By $f'$ we mean the derivative.

**Fact 2.1.** $(W, S, \le, \ell)$ is a locally Eulerian poset (locally means that every interval is Eulerian).

## 3. $R$- and $\widetilde{R}$- polynomials

Following [1, Section 5.1], we begin with a definition of $R$-polynomials.

**Fact 3.1** (*$R$-polynomials*)**.** There exists a unique family of polynomials $\{R_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ such that

(1) $R_{uw}(q) = 0$ if $u \not\le w$,
(2) $R_{uw}(q) = 1$ if $u = w$,
(3) If $s \in S$ and $ws < w$, then

$$R_{uw}(q) = \begin{cases} R_{us,ws}(q) & \text{if } us < u, \\ qR_{us,ws}(q) + (q-1)R_{u,ws}(q) & \text{if } u < us. \end{cases}$$

Next we collect key facts of $R$-polynomials (which we will need later).

**Fact 3.2.** Let $u \le w$.

(1) $R_{uw}(q)$ is a monic polynomial of degree $\ell(u, w)$.
(2) We have

$$R_{uw}(1) = \begin{cases} 1 & \text{if } u = w \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad R'_{uw}(1) = \begin{cases} 1 & \text{if } u \to w \\ 0 & \text{otherwise.} \end{cases}$$

(3) We have

$$\sum_{u \le v \le w} (-1)^{\ell(u,v)} R_{uv}(q) R_{vw}(q) = \delta_{uw} \text{ (Kronecker delta).}$$

There is another family of polynomials indexed by a pair of elements of Coxeter groups:

**Fact 3.3** (*$\widetilde{R}$-polynomials*)**.** There exists a unique family of polynomials $\{\widetilde{R}_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{N}[q]$ such that

(1) $\widetilde{R}_{uw}(q) = 0$ if $u \not\le w$,

(2) $\widetilde{R}_{uw}(q) = 1$ if $u = w$,
(3) If $s \in S$ and $ws < w$, then

$$\widetilde{R}_{uw}(q) = \begin{cases} \widetilde{R}_{us,ws}(q) & \text{if } us < u, \\ \widetilde{R}_{us,ws}(q) + q\widetilde{R}_{u,ws}(q) & \text{if } u < us. \end{cases}$$

## 4. ABSOLUTE LENGTH

**Definition 4.1.** Let $u \leq w$. Define the *absolute length* between $u$ and $w$ to be $a(u,w) = \min\{\alpha \mid w = ut_1t_2\cdots t_\alpha, t_i \in T\}$.

**Remark 4.2.** Note that we have $a(u,w) \leq \ell(u,w)$ as an easy consequence of Chain Property of Bruhat order [1, Theorem 2.2.6] and furthermore $(-1)^{a(u,w)} = (-1)^{\ell(u,w)}$ since the parity of length necessarily changes at each relation $u \to ut$.

**Proposition 4.3.** *Let $u < w$, $\ell = \ell(u,w)$ and $a = a(u,w)$. Then there exist positive integers $c_\ell, c_{\ell-2}, \ldots, c_a$ such that*

$$\widetilde{R}_{uw}(q) = c_\ell q^\ell + c_{\ell-2}q^{\ell-2} + \cdots + c_a q^a.$$

*Consequently, we have*

$$R_{uw}(q) = \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k}\, q^{\frac{\ell-a-2k}{2}} (q-1)^{a+2k}.$$

*Proof.* For the first statement, refer to [4, Theorem 2.5]. Then

$$\begin{aligned} R_{uw}(q) &= q^{\frac{\ell}{2}}\widetilde{R}_{uw}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \\ &= q^{\frac{\ell}{2}} \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{a+2k} \\ &= q^{\frac{\ell}{2}} \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k}(q^{-\frac{1}{2}}(q-1))^{a+2k} \\ &= \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k}\, q^{\frac{\ell-a-2k}{2}} (q-1)^{a+2k}. \end{aligned}$$

□

**Proposition 4.4.** *Let $u \leq w$. For each nonnegative integer $n$, we have $R^{(n)}_{uw}(1) \geq 0$. Moreover, $R^{(n)}_{uw}(1) > 0$ if and only if $a(u,w) \leq n \leq \ell(u,w)$. In particular, $a(u,w)$ is the largest power of $q-1$ that divides $R_{uw}(q)$.*

*Proof.* Let $a = a(u,w)$ and $\ell = \ell(u,w)$. Differentiate the second equation in Proposition 4.3 $n$ times with Leibnitz rule, and let $q = 1$:

$$R^{(n)}_{uw}(1) = \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k} \sum_{i=0}^{n} \binom{n}{i} (q^{\frac{\ell-a-2k}{2}})^{(i)}((q-1)^{a+2k})^{(n-i)}\Big|_{q=1}.$$

Observe that each term is nonnegative since all $c_{a+2k}$ and $\binom{n}{i}$ are positive and the resulting number $(q^{\frac{\ell-a-2k}{2}})^{(i)}((q-1)^{a+2k})^{(n-i)}\Big|_{q=1}$ would be nonnegative in any case. Hence $R^{(n)}_{uw}(1) \geq 0$. To show the second part, suppose $n < a$. Then $R^{(n)}_{uw}(1) = 0$ since $((q-1)^{a+2k})^{(n-i)}\Big|_{q=1} = 0$ for all $i, k$. If $n > \ell$, then $R^{(n)}_{uw}(1) = 0$ since $\deg R_{uw}(q) = \ell$. Suppose now $a \leq n \leq \ell$. (Case 1) If $n \equiv \ell \pmod 2$, then the term $i = 0, k = (n-a)/2$ is positive as

$$c_n q^{\frac{\ell-n}{2}} ((q-1)^n)^{(n)}\Big|_{q=1} = c_n n! > 0.$$

(Case 2) If $n \not\equiv \ell \pmod 2$, then the term $i = 1, k = (n-a-1)/2$ is positive as

$$c_{n-1}\binom{n}{1}\left(q^{\frac{\ell-n+1}{2}}\right)'\left((q-1)^{n-1}\right)^{(n-1)}\Big|_{q=1} = c_{n-1}\,\frac{\ell-n+1}{2}\,n! > 0.$$

Hence we verified $R^{(n)}_{uw}(1) > 0$ whenever $a \leq n \leq \ell$.
Conversely, if $R^{(n)}_{uw}(1) > 0$, then it is easily seen that we must have $a \leq n \leq \ell$. □

## 5. Parabolic (co)atoms

This section first describes the equality between the number of parabolic coatoms and the value $q = 1$ of second derivative of $R$-polynomials. After that we mention an upper bound of $\widetilde{R}$-polynomials related to Fibonacci polynomials. We begin with a recall of a well-known fact in general locally Eulerian posets:

**Fact 5.1.** [1, Lemma 2.7.3] Let $u < w$. If $\ell(u,w) = 2$, then $[u,w]$ is isomorphic to the boolean poset of rank 2 as a graded poset. In other words, $[u,w]$ has precisely two (co)atoms.

**Definition 5.2.** Let $a(u,w) = 2$. Say $v$ is a *parabolic (co)atom* of $[u,w]$ if $u \to v \to w$. Define $m(u,w) = |\{v \in [u,w] \mid u \to v \to w\}|$.

**Proposition 5.3.** *If $a(u,w) = 2$, then $m(u,w) = R''_{uw}(1)$.*

*Proof.* Differentiate

$$\sum_{u \leq v \leq w} (-1)^{\ell(u,v)} R_{uv}(q) R_{vw}(q) = \delta_{uw} = 0$$

twice and then let $q = 1$:

$$\sum_{u \le v \le w} (-1)^{\ell(u,v)} (R''_{uv}(1) R_{vw}(1) + 2R'_{uv}(1) R'_{vw}(1) + R_{uv}(1) R''_{vw}(1)) = 0.$$

By Proposition 4.4, we have $R''_{uv}(1)R_{vw}(1)$ is nonzero if and only if $a(u, v) \ge 2$ and $v = w$. Similarly $R'_{uv}(1)R'_{vw}(1)$ is nonzero (and must be 1) if and only if $u \to v \to w$. Also $R_{uv}(1)R''_{vw}(1)$ is nonzero if and only if $v = u$ and $a(v, w) \ge 2$. Computing signs, we have $R''_{uw}(1) - 2m(u, w) + R''_{uw}(1) = 0$.

□

For a proof of Theorem, we need one more result on an upper bound of coefficients of $\widetilde{R}$-polynomials:

**Definition 5.4.** Let $F_n(q)$ be Fibonacci polynomials defined by $F_0(q) = 1$, $F_1(q) = q$, $F_2(q) = q^2$ and $F_n(q) = F_{n-2}(q) + qF_{n-1}(q)$ $(n \ge 3)$.

**Proposition 5.5.** *For each $n \ge 1$, we have $[q^2](F_{2n}) = n$ and $[q](F_{2n-1}) = 1$.*

*Proof.* Induction on $n$. For $n = 1$, $[q^2](F_2) = 1$ and $[q](F_1) = 1$. As an inductive step, suppose that the assertions are true for $n-1$. Then $[q^2](F_{2n}) = [q^2](F_{2n-2}(q) + qF_{2n-1}(q)) = (n - 1) + 1 = n$ and $[q](F_{2n-1}) = [q](F_{2n-3} + qF_{2n-2}) = [q](F_{2n-3}) = 1$. □

**Fact 5.6.** [2, Proposition 5.3] For all $u, w$, we have $\widetilde{R}_{uw}(q) \le F_{\ell(u,w)}(q)$ coefficientwise.

## 6. Proof of Theorem

We now come to the main result:

**Theorem.** *Let $u < w$. If $a(u, w) = 2$, then $m(u, w)$ is even. Moreover, $m(u, w) \le \ell(u, w)$.*

*Proof.* Write

$$R_{uw}(q) = \sum_{k=0}^{\frac{\ell-a}{2}} c_{a+2k}\, q^{\frac{\ell-a-2k}{2}} (q-1)^{a+2k}.$$

as in Proposition 4.3. Note that $c_2 = [q^2](\widetilde{R}_{uw})$. Since $\widetilde{R}_{uw}(q) \le F_{\ell(u,w)}(q)$ coefficientwise (Fact 5.6), we have $c_2 = [q^2](\widetilde{R}_{uw}) \le [q^2](F_{\ell(u,w)}) \le \ell(u, w)/2$ thanks to Proposition 5.5. Thus $m(u, w) = R''_{uw}(1) = 2c_2$ (even) $\le 2 \cdot \ell(u, w)/2 = \ell(u, w)$. □

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