ANALYTIC APPROACH FOR BRENTI'S CONJECTURE ON R-POLYNOMIALS

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Abstract. We make the conjecture that all complex roots of R-polynomials are on the unit circle. Our main result shows that this conjecture implies that the conjecture by Brenti on a bound of coefficients of R-polynomials [Brenti, Europ. J. of Combin., 1998, **19**, 283-297]. The idea for a proof is simply to consider values of n-th derivative of R-polynomials at $q = 0$.

1. Introduction

R-polynomials play an important role for study in representation theory of Hecke algebras as well as combinatorics of Coxeter groups. For its details and history, see books [1, 3]. In 1998, F. Brenti made a conjecture on a bound of coefficients of R-polynomials:

Conjecture 1 (Brenti)**.** [2, Problem 5.2.] *Let $u < w$ in Bruhat order of Coxeter groups. Then for all nonnegative integers n, we have*

$$|[q^n](R_{uw}(q))| \leq \binom{\deg R_{uw}(q)}{n}$$

where $[q^n](f(q))$ denotes a coefficient of q^n in a polynomial $f(q)$ and $\binom{\deg R_{uw}(q)}{n}$ denotes binomial coefficients.

He proved the cases for $n = 1$ and 2. A proof for general cases remains open at time of writing (November, 2011). Toward a complete solution of his conjecture, this short article suggests another conjecture; regarding R-polynomials as complex polynomials, all of its roots are on the unit circle (Conjecture 2). We further show that if this is the case, then Brenti's conjecture is true (Theorem). The idea is rather "analytical"; To compute coefficients, we differentiate R-polynomials as we often do in a calculus class.

2. Notation

We follow notation commonly used in the context of Coxeter groups (such as [1] and [3]). By (W, S) (or simply W) we mean a Coxeter system with length function ℓ. Let $T = \cup_{w \in W} w^{-1}Sw$ denote the set of reflections. Write $u \to w$ if $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$. Define *Bruhat order* $u \leq w$ if

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there exist $v_0, v_1, \dots, v_k \in W$ such that $u = v_0 \to v_1 \to \cdots \to v_k = w$. Often $\ell(u,w) \overset{\text{def}}{=} \ell(w) - \ell(u)$ abbreviates a difference of lengths of u and w.

3. R- and $\widetilde{R}$- polynomials

Following [1, Section 5.1], we begin with a definition of R-polynomials.

Fact 3.1 (*R-polynomials*)**.** There exists a unique family of polynomials $\{R_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{Z}[q]$ such that

(1) $R_{uw}(q) = 0$ if $u \not\leq w$,
(2) $R_{uw}(q) = 1$ if $u = w$,
(3) If $s \in S$ and $ws < w$, then

$$R_{uw}(q) = \begin{cases} R_{us,ws}(q) & \text{if } us < u, \\ qR_{us,ws}(q) + (q-1)R_{u,ws}(q) & \text{if } u < us. \end{cases}$$

Next we collect key facts of R-polynomials (which we will need later).

Fact 3.2. Let $u \leq w$.

(1) $R_{uw}(q)$ is a monic polynomial of degree $\ell(u,w)$ with contant term $R_{uw}(0) = (-1)^{\ell(u,w)}$.
(2) If $u \neq w$, then $(q-1)$ divides $R_{uw}(q)$, i.e., $R_{uw}(1) = 0$.
(3) $(-q)^{\ell(u,w)} R_{uw}(q^{-1}) = R_{uw}(q)$.

There is another family of polynomials indexed by a pair of elements of Coxeter groups:

Fact 3.3 (*$\widetilde{R}$-polynomials*)**.** There exists a unique family of polynomials $\{\widetilde{R}_{uw}(q) \mid u, w \in W\} \subseteq \mathbb{N}[q]$ such that

(1) $\widetilde{R}_{uw}(q) = 0$ if $u \not\leq w$,
(2) $\widetilde{R}_{uw}(q) = 1$ if $u = w$,
(3) If $s \in S$ and $ws < w$, then

$$\widetilde{R}_{uw}(q) = \begin{cases} \widetilde{R}_{us,ws}(q) & \text{if } us < u, \\ \widetilde{R}_{us,ws}(q) + q\widetilde{R}_{u,ws}(q) & \text{if } u < us. \end{cases}$$

4. Conjecture

Here is our concerning conjecture:

Conjecture 2. *Let (W,S) be a Coxeter system, $u < w$ and $\alpha \in \mathbb{C}$. If $R_{uw}(\alpha) = 0$, then $|\alpha| = 1$.*

The author has verified that his conjecture is valid for $\deg R_{uw} \leq 7$ (type A) and $\deg R_{uw} \leq 5$ (types B, D) according to Incitti's table of R- and $\widetilde{R}$-polynomials [4, 5]. In addition, the following result supports our conjecture.

Proposition 4.1. *Let $u < w$ and $\alpha \in \mathbb{R}$. If $\alpha > 0, \alpha \neq 1$, then $R_{uw}(\alpha) \neq 0$. Consequently, $\alpha = 1$ is the only nonnegative real root of $R_{uw}(q)$.*

Proof. Let $\ell = \ell(u, w)$ for simplicity.
(Case 1) Suppose $\alpha > 1$. Let $\beta := \alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}$ (> 0). By Fact 3.3, we have $R_{uw}(\alpha) = \alpha^{\frac{\ell}{2}} \widetilde{R}_{uw}(\beta)$. Observe that $\widetilde{R}_{uw}(\beta) = \beta^{\ell} + \cdots > 0$ since $\widetilde{R}$-polynomials have all nonnegative coefficients. Hence $R_{uw}(\alpha) = \alpha^{\frac{\ell}{2}} \widetilde{R}_{uw}(\beta) \neq 0$.
(Case 2) Suppose $0 < \alpha < 1$. Due to Fact 3.2 (3), we know $R_{uw}(\alpha) = 0$ if and only if $R_{uw}(\alpha^{-1}) = 0$ (provided $\alpha \neq 0$). We thus deduce this case to (Case 1).
(Case 3) Suppose $\alpha = 0$. Then $R_{uw}(0) = (-1)^{\ell(u,w)} \neq 0$ by Fact 3.2 (1).
(Case 4) Finally, 1 is a root of whenever $u < w$ (Fact 3.2 (2)). □

5. Proof of Theorem

Theorem. *Let (W, S) be a Coxeter system. Suppose Conjecture 2 holds in W. Then Brenti's conjecture follows: Let $u < w$ and $\ell = \ell(u, w)$. Then for all integers n such that $0 \leq n \leq \ell$, we have*

$$|[q^n](R_{uw}(q))| \leq \binom{\ell}{n}.$$

Proof. Regard $R_{uw}(q)$ as a polynomial in $\mathbb{C}[q]$. Then we can factor it into linear polynomials, say

$$R_{uw}(q) = (q - \alpha_1) \cdots (q - \alpha_\ell), \text{ for } \alpha_k \in \mathbb{C}$$

and moreover $|\alpha_k| = 1$ for all k assuming Conjecture 2. Differentiate this n times with multinomial Leibnitz rule:

$$R_{uw}(q) = \sum_m \binom{n}{m_1, \ldots, m_\ell} (q - \alpha_1)^{(m_1)} \cdots (q - \alpha_\ell)^{(m_\ell)}$$

where the sum is taken over $m = (m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$ such that $m_1 + \cdots + m_\ell = n$. Since each factor is just linear, whenever $m_k \geq 2$ for some k, then that term is zero. It turns out that, in the above sum, it is enough to consider $m = (m_1, \ldots, m_\ell)$, $m_1 + \cdots + m_\ell = n$ such that $0 \leq m_k \leq 1$ for all k. There are precisely $\binom{\ell}{n}$ choices of such m (choose n numbers from $m_1, \ldots, m_\ell$ for which $m_k = 1$). Moreover, for all such m, its multinomial coefficient is $\binom{n}{m_1, \ldots, m_\ell} = \frac{n!}{m_1! \cdots m_\ell!} = n!$ since $m_k!$ $(= 0! \text{ or } 1!) = 1$ for all k. Note also that

$$\left| (q - \alpha_k)^{(m_k)} \Big|_{q=0} \right| \left(= \begin{cases} |-\alpha_k| & (m_k = 0) \\ 1 & (m_k = 1) \end{cases} \right) = 1$$

because of Conjecture 2. As a result, we obtain

$$\begin{aligned}|R^{(n)}_{uw}(0)| &= \left| \sum_{0 \le m_k \le 1} \binom{n}{m_1, \ldots, m_\ell} (q-\alpha_1)^{(m_1)} \cdots (q-\alpha_\ell)^{(m_\ell)} \Big|_{q=0} \right| \\ &\le \sum_{0 \le m_k \le 1} \left| \binom{n}{m_1, \ldots, m_\ell} (q-\alpha_1)^{(m_1)} \cdots (q-\alpha_\ell)^{(m_\ell)} \Big|_{q=0} \right| \\ &= \sum_{0 \le m_k \le 1} n! \\ &\le \binom{\ell}{n} n!.\end{aligned}$$

Since $R^{(n)}_{uw}(0) = n![q^n](R_{uw}(q))$, we conlude that

$$|[q^n](R_{uw}(q))| \le \binom{\ell}{n}.$$

□

Remark 5.1. As Brenti mentioned [2, p.295], this bound is best possible. If $R_{uw}(q) = (q-1)^{\ell(u,w)}$ (which certainly occurs, for example, when $[u, w]$ is boolean), then obviously equalities hold for all n.

References

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