ENUMERATION OF EDGES ON BRUHAT GRAPHS OF THE SYMMETRIC GROUPS

MASATO KOBAYASHI

ABSTRACT. The Bruhat graph is a powerful tool to study combinatorics of Bruhat order in Coxeter groups. This article solves two problems on enumeration of edges on Bruhat graphs on the symmetric groups. First, we show existence of permutations x such that there exists a one-to-one correspondence between descent edges and ascent ones of x. Second, the length of longest descent edges is monotonously increasing in weak order, but not in Bruhat order.

1. INTRODUCTION

Coxeter systems (W, S) are of great importance in modern mathematics. It has a graded partial order structure (*Bruhat order*) with the length function ℓ and Coxeter generators S. There is an important characterization that we define this order to be the transitive closure of the basic relations $w \longrightarrow wt, t \in T$ (meaning $\ell(w) < \ell(wt)$), called *reductions* where T is the conjugate of S. To study partial order structure in general, we often draw the Hasse diagram. However, for Coxeter groups, it is much more useful to draw the Bruhat graphs instead due to these reductions as just mentioned. Its (directed) edges are reductions $w \longrightarrow wt$ which may or may not be a covering relation, that is, Bruhat graphs have many more edges. It is thus a $|T|$-regular graph, i.e., each vertex is incident to exactly $|T|$ edges. Further we can split edges incident to w into two parts: $wt \longrightarrow w$ is a *descent edge* of w and $w \longrightarrow wt$ is an *ascent edge*. It follows from the fact in theory of root systems that $\ell(w)$ is equal to the number of descent edges of w. For example, Figure 1 illustrates that the number of decent edges increases one by one as w gets larger under Bruhat order on S_3.

This article concerns two enumeration problems on edges of Bruhat graphs on the symmetric groups $W = S_n$ (type A):

1. Symmetric Reductions. Suppose $wt \longrightarrow w$. As is well-known, $\ell(w) - \ell(wt)$ is always odd so that we can associate to the edge the grade by natural numbers $(\ell(w) - \ell(wt) + 1)/2$. This grading for edges leads us to the following idea.

If $|T|$ is even, then there exists some $w \in S_n$ such that $\ell(w) = |T|/2$ because $(S_n, \leq, \ell)$ is a graded poset. Then w has $|T|/2$ descent edges and $|T|/2$ ascent ones.

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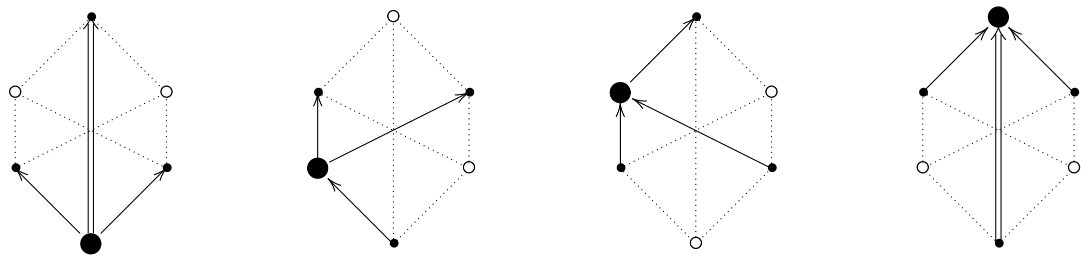
Figure 1. Descent and ascent edges



Among those, does there exist w such that edges incident to w are "symmetric upside down", i.e., for each $t \in T$ such that $wt \longrightarrow w$ there exists a unique $t' \in T$ such that $w \longrightarrow wt'$ and moreover $\ell(w)-\ell(wt) = \ell(wt')-\ell(w)$? Theorem 1 asserts that this guess is true whenever $|T|$ is even.

Theorem 1. *Let n be a positive integer ≥ 4. If $n \equiv 0$ or $1 \pmod 4$ (so that $|T| = n(n-1)/2$ is even), then there exist at least two $x \in S_n$ such that $\ell_{-r}(x) = \ell_{+r}(x)$ for all $1 \leq r \leq n-1$ where $\ell_{-r}(w)(\ell_{+r}(w))$ is the number of edges $wt \longrightarrow w$ ($w \longrightarrow wt$) on the Bruhat graph of S_n such that $\ell(w) - \ell(wt) = 2r - 1$ ($\ell(wt) - \ell(w) = 2r - 1$).*

2. Longest descent edge under weak order. As seen, $\ell(w)$ is equal to the number of descent edges of w. We can write it as a sum of non-negative functions

$$\ell(w) = \ell_{-1}(w) + \ell_{-2}(w) + \cdots ,$$

with respect to our grading. Needless to say, $\ell(w)$ is monotonously increasing in Bruhat order. However, each part $\ell_{-r}(w)$ is not necessarily. Is there any more kinds of monotony about descent edges on Bruhat graphs? If we consider the following suborder of Bruhat order, we can prove hidden and little more monotony of descent edges: The *right weak order* on S_n is the transitive closure of $w \longrightarrow ws$, $s \in S$. This order also plays an important role in theory of Coxeter groups [2, Chapter 3]. Theorem 2 shows one difference between Bruhat and weak orders.

Theorem 2. *Define $\rho(w) = \max\{r \mid \ell_{-r}(w) > 0\}$. If $w \leq x$ in the right weak order, then $\rho(w) \leq \rho(x)$. This is not necessarily true in Bruhat order.*

Section 2 provides preliminaries on Bruhat order and Bruhat graphs. Section 3 prepares a lemma for Theorem 1 and Section 4 gives a proof. Similarly Section 5 prepares a lemma for Theorem 2 and Section 6 gives a proof.

2. Bruhat order and Bruhat graphs

This section provides preliminaries on Bruhat order and Bruhat graphs on the symmetric groups as in [2, 3].

Definition 2.1. Let $w, x, y \in S_n$.

(1) By $w = i_1 \cdots i_n$ we mean $w(k) = i_k$ (*one-line notation*). For example, $w = 312$ means $w(1) = 3, w(2) = 1$ and $w(3) = 2$.
(2) For $i < j$, denote by t_{ij} the transposition switching i and j. Let $T = \{t_{ij}\}$ denote the set of all transpositions.
(3) In particular, set $s_i = (i, i+1)$ and $S = \{s_1, \dots, s_{n-1}\}$ (*simple reflections*). It is well-known that S generates S_n. That is, given $w \in S_n$, there exists $s_{i_1}, \dots, s_{i_k}$ such that $w = s_{i_1} \cdots s_{i_k}$. The *length* $\ell(w)$ is the minimum of such k among all expressions of w. We interpret the length of the identity permutation e as 0 with the empty expression.
(4) We write $w \longrightarrow x$ and say that w is a *reduction* of x if $x = wt$ for some $t \in T$ and $\ell(w) < \ell(x)$.
(5) Define *Bruhat order* on S_n: $w \leq y$ if there exists $x_0, x_1, \dots, x_m$ in S_n such that $w = x_0$, $x_m = y$ and $x_i \longrightarrow x_{i+1}$ for all $0 \leq i \leq m-1$.
(6) The *Bruhat graph* of S_n is the one for nodes $w \in S_n$ and for edges $w \longrightarrow wt$, $t \in T$. For each subset $X \subseteq S_n$ we can naturally define the Bruhat subgraph of X in the same way.
(7) Define the permutation w_0 by $w_0(i) = n - i + 1$ for all i.

Next we recall some basic facts needed later for a proof of main theorems.

Fact 2.2. (See [2, Chapter 2])

(1) S_n is a graded poset of rank $|T| = \ell(w_0) = n(n-1)/2$ with the minimum element e and the maximum w_0.
(2) The group inverse $x \mapsto x^{-1}$ (multiplication $x \mapsto xw_0$) is an order-preserving (-reversing) automorphism on S_n. In particular, $\ell(x) = \ell(x^{-1}) = \ell(w_0) - \ell(xw_0)$.
(3) $\ell(w) = |\{t \in T \mid wt < w\}|$.

Definition 2.3. Let $w \in S_n$. Then for integers $1 \leq r \leq n-1$, define

$$T_{-r}(w) = \{t \in T \mid \ell(w) - \ell(wt) = 2r - 1\},$$
$$T_{+r}(w) = \{t \in T \mid \ell(wt) - \ell(w) = 2r - 1\}.$$

Define $\ell_{-r}(w) = |T_{-r}(w)|, \ell_{+r}(w) = |T_{+r}(w)|$.

3. Lemma for Theorem 1

Before a proof of Theorem 1, we need the following lemma.

Lemma 1. *Let $x \in S_n$. Then for each $t \in T$, the following are equivalent:*

(1) $x \longrightarrow xt$,
(2) $x^{-1} \longrightarrow (xt)^{-1}$,
(3) $(xt)w_0 \longrightarrow xw_0$.

Moreover, we have $\ell(xt) - \ell(x) = \ell((xt)^{-1}) - \ell(x^{-1}) = \ell(xw_0) - \ell((xt)w_0)$*. As a result, the automorphism* $x \mapsto x^{-1}$ *(*$x \mapsto xw_0$*) on* S_n *preserves (reverses) not only Bruhat order but also reductions keeping a grade. Hence* $\ell_{-r}(x) = \ell_{-r}(x^{-1}) = \ell_{+r}(xw_0)$ *for all* r*.*

Proof. Note that $y = xt \iff y^{-1} = tx^{-1} = x^{-1}(xtx^{-1}) \iff yw_0 = xw_0(w_0tw_0)$. Then use Fact 2.2 (2) for the part of keeping a grade. □

4. Theorem 1

Theorem 1. *Let* n *be a positive integer* ≥ 4*. If* $n \equiv 0$ *or* $1 \pmod 4$*, then there exist at least two* $x \in S_n$ *such that* $\ell_{-r}(x) = \ell_{+r}(x)$ *for all* $1 \leq r \leq n-1$*.*

Proof. The idea is to find x_0 such that $x_0w_0x_0 = e$. It then follows that $x_0 = x_0^{-1}w_0$ and hence $\ell_{-r}(x_0) = \ell_{-r}(x_0^{-1}w_0) = \ell_{+r}(x_0^{-1}) = \ell_{+r}(x_0)$ for r all $1 \leq r \leq n-1$ by Lemma 1. Needless to say, we also have $\ell_{-r}(x_0^{-1}) = \ell_{+r}(x_0^{-1})$. Note that $n(n-1)/2$ is even $\iff n \equiv 0$ or $1 \pmod 4$.

(1) First suppose that $n \equiv 0 \pmod 4$. Define

$$x_0(i) = \begin{cases} i + (n/2) & 1 \leq i \leq n/4, \\ -i + (n/2) + 1 & (n/4) + 1 \leq i \leq n/2, \\ -i + (3n/2) + 1 & (n/2) + 1 \leq i \leq 3n/4, \\ i - (n/2) & (3n/4) + 1 \leq i \leq n. \end{cases}$$

If $1 \leq i \leq n/4$, then $(n/4) + 1 \leq (n/2) - i + 1 \leq n/2$. We have

$$\begin{aligned} x_0w_0x_0(i) &= x_0\left(w_0\left(i + \frac{n}{2}\right)\right) \\ &= x_0\left(n - \left(i + \frac{n}{2}\right) + 1\right) \\ &= x_0\left(\frac{n}{2} - i + 1\right) \\ &= -\left(\frac{n}{2} - i + 1\right) + \frac{n}{2} + 1 \\ &= i. \end{aligned}$$

If $(n/4) + 1 \leq i \leq n/2$, then $(3n/4) + 1 \leq (n/2) + i \leq n$. We have

$$\begin{aligned} x_0w_0x_0(i) &= x_0\left(w_0\left(-i + \frac{n}{2} + 1\right)\right) \\ &= x_0\left(n - \left(-i + \frac{n}{2} + 1\right) + 1\right) \\ &= x_0\left(\frac{n}{2} + i\right) \\ &= \left(\frac{n}{2} + i\right) - \frac{n}{2} \\ &= i. \end{aligned}$$

It is quite analogous to check the other cases.

(2) Next suppose that $n \equiv 1 \pmod 4$. Define

$$x_0(i) = \begin{cases} i + (n+1)/2 & 1 \le i \le (n-1)/4, \\ -i + (n+1)/2 + 1 & (n-1)/4 + 1 \le i \le (n-1)/2, \\ (n+1)/2 & i = (n+1)/2, \\ -i + (3n+3)/2 & (n+3)/2 \le i \le (3n+1)/4, \\ i - (n+1)/2 & (3n+5)/4 \le i \le n. \end{cases}$$

Actually construction of this x_0 is quite similar to the previous one except the middle fixed point. It is similar to verify that $x_0 w_0 x_0 = e$. So the same conclusion follows.

Finally, we see that $x_0 \neq x_0^{-1}$ in any case since if $x_0 = x_0^{-1}$, then $x_0 = x_0^{-1} w_0 = x_0 w_0$. But this implies that $w_0 = e$, a contradiction. □

Example 4.1. Let $n = 8$ and $x_0 = 56218734$. Then $x_0^{-1} = 43781265 = x_0 w_0$. For $n = 9$ let $x_0 = 672159834$. Then $x_0^{-1} = 438951276 = x_0 w_0$.

5. Lemma for Theorem 2

Definition 5.1. A reduction of the form $w \longrightarrow ws, s \in S$ is said to be *right weak*. Define the *right weak order* on S_n by the transitive closure of this relation.

This is also a graded partial order on S_n. In particular, $w \le x$ in the right weak order implies $\ell(w) \le \ell(x)$, i.e., $\ell(w)$ is an increasing function. Recall that $\ell(w) = \ell_{-1}(w) + \ell_{-2}(w) + \cdots$. Notice that each part $\ell_{-r}(w)$ is not necessarily increasing. For example, $2431 \longrightarrow 4231$ is right weak and $\ell_{-2}(2431) = 1 > 0 = \ell_{-2}(4231)$. However, as mentioned in the introduction, there is another monotony for descent edges. To be more precise, we need the following definition.

Definition 5.2. For $w \in S_n$, define $\rho(w) = \max\{r \ge 0 \mid T_{-r}(w) \neq \varnothing\}$.

The following lemma will be essential for Theorem 2.

Lemma 2. [1, Observation 1.2 and 5.3] *Let $x \in S_n$ and $t_{ik} \in T$. Suppose $x(i) > x(k)$. Then $t_{ik} \in T_{-r}(x)$ if and only if*

$$r - 1 = \#\{j \mid i < j < k \text{ and } x(i) > x(j) > x(k)\}.$$

Consequently, $ws_i \longrightarrow w \iff w(i) > w(i+1)$. We also have $\ell_{-r}(w) = 0$ for all $r \ge n$.

6. Theorem 2

Theorem 2. *If $w \le x$ in the right weak order on S_n, then $\rho(w) \le \rho(x)$.*

Proof. It is little tedious to write a proof in full detail. Instead we give an idea and a sketch of proof. We may assume that $w \longrightarrow ws_m = x$ and moreover $\rho(w) \geq 1$. For convenience let $r = \rho(w)$. We need to show that $t \in T_{-r'}(x) \neq \varnothing$ for some $r' \geq r$ and some $t \in T$. Lemma 2 and $r = \rho(w)$ imply that there exist $i, j_1, j_2, \dots, j_{r-1}$ such that $1 \leq i < j_1 < j_2 < \cdots < j_{r-1} < k \leq n$ and $w(i) > w(j_l) > w(k)$ for all $1 \leq l \leq r-1$. Let us write this situcation in one-line notation as

$$w = \cdots w(i) \cdots w(j_1) \cdots w(j_2) \cdots w(j_{r-1}) \cdots w(k) \cdots .$$

Now we see $w(i)$ the maximum among these $r+1$ numbers on the left, $w(k)$ the minimum on the right and precisely $r-1$ elements $m(j_l)$ in the middle. Note that the right weak reduction $w \longrightarrow ws_m$ changes only *adjacent* two numbers and remains the rest in one-line notation, i.e., $x(m) = w(m+1), x(m+1) = w(m)$ and $x(a) = w(a)$ for $a \neq m, m+1$. So at most two of $w(i), w(j_1), w(j_2), \dots, w(j_{r-1}), w(k)$ move and the rest stay in the same position under the reduction $w \longrightarrow ws_m = x$. To be more precise, we check several case as below.

(1) If $m \leq i-2$, then $w(a) = x(a)$ for all $a \geq i$. Hence $t_{ik} \in T_{-r}(x)$.
(2) If $m = i-1$, then $x(i-1) = w(i), x(i) = w(i-1)$ (and $x(a) = w(a)$ otherwise). Thus

$$\begin{cases} t_{i-1,k} \in T_{-r}(x) & \text{if } w(i-1) < w(k), \\ t_{i-1,k} \in T_{-(r+1)}(x) & \text{if } w(i-1) > w(k). \end{cases}$$

(3) If $m = i$, then $x(i) = w(i-1), x(i-1) = w(i)$. Note that $w(j_1) < w(i)$ by the assumption and $w(i) < w(i+1)$ since $w \longrightarrow ws_i$. Hence $j_1 \neq i+1$ so that $t_{i+1,k} \in T_{-r}(x)$.
(4) If $i+1 \leq m \leq k-2$, then $t_{ik} \in T_{-r}(x)$ (it does not matter whether $m = j_l$ for some l or not).

Three more cases ($m = k-1, m = k, m \geq k+1$) are symmetric to one of the previous cases. ☐

Remark 6.1.

(1) It is interesting (and probably possible) to generalize Theorem 2 for other Coxeter groups.
(2) Theorem 2 is not true in Bruhat order as $3214 \longrightarrow 3412$ but $\rho(3214) = 2 > 1 = \rho(3412)$.

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Department of Mathematics, the University of Tennessee, Knoxville, TN 37996
E-mail address: kobayashi@math.utk.edu