ENUMERATION OF EDGES ON BRUHAT GRAPHS OF THE
SYMMETRIC GROUPS

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Abstract. The Bruhat graph is a powerful tool to study combinatorics of
Bruhat order in Coxeter groups. This article solves two problems on enumeration
of edges on Bruhat graphs on the symmetric groups. First, we show existence
of permutations $x$ such that there exists a one-to-one correspondence between
descent edges and ascent ones of $x$. Second, the length of longest descent edges
is monotonously increasing in weak order, but not in Bruhat order.

1. Introduction

Coxeter systems $(W, S)$ are of great importance in modern mathematics. It has
a graded partial order structure (Bruhat order) with the length function $\ell$ and
Coxeter generators $S$. There is an important characterization that we define this
order to be the transitive closure of the basic relations $w \rightarrow wt, t \in T$ (meaning
$\ell(w) < \ell(wt)$), called reductions where $T$ is the conjugate of $S$. To study partial
order structure in general, we often draw the Hasse diagram. However, for Coxeter
groups, it is much more useful to draw the Bruhat graphs instead due to these
reductions as just mentioned. Its (directed) edges are reductions $w \rightarrow wt$ which
may or may not be a covering relation, that is, Bruhat graphs have many more
dges. It is thus a $|T|$-regular graph, i.e., each vertex is incident to exactly $|T|
edges. Further we can split edges incident to $w$ into two parts: $wt \rightarrow w$ is a
descent edge of $w$ and $w \rightarrow wt$ is an ascent edge. It follows from the fact in
theory of root systems that $\ell(w)$ is equal to the number of descent edges of $w$. For
example, Figure 1 illustrates that the number of decent edges increases one by one
as $w$ gets larger under Bruhat order on $S_3$.

This article concerns two enumeration problems on edges of Bruhat graphs on
the symmetric groups $W = S_n$ (type A):

1. Symmetric Reductions. Suppose $wt \rightarrow w$. As is well-known, $\ell(w) - \ell(wt)$
is always odd so that we can associate to the edge the grade by natural numbers
$(\ell(w) - \ell(wt) + 1)/2$. This grading for edges leads us to the following idea.

If $|T|$ is even, then there exists some $w \in S_n$ such that $\ell(w) = |T|/2$ because
$(S_n, \leq, \ell)$ is a graded poset. Then $w$ has $|T|/2$ descent edges and $|T|/2$ ascent ones.

Date: December 31, 2010.
2000 Mathematics Subject Classification. Primary:20F55; Secondary:51F15.
Key words and phrases. Coxeter groups, symmetric groups, Bruhat order, Bruhat graphs.
Among those, does there exist \( w \) such that edges incident to \( w \) are “symmetric upside down”, i.e., for each \( t \in T \) such that \( wt \rightarrow w \) there exists a unique \( t' \in T \) such that \( w \rightarrow wt' \) and moreover \( \ell(w) - \ell(wt) = \ell(wt') - \ell(w) \)? Theorem 1 asserts that this guess is true whenever \( |T| \) is even.

**Theorem 1.** Let \( n \) be a positive integer \( \geq 4 \). If \( n \equiv 0 \) or \( 1 \) (mod 4) (so that \( |T| = n(n-1)/2 \) is even), then there exist at least two \( x \in S_n \) such that \( r(w) = r(x) = \ell(w) - \ell(wt) = 2r - 1 \) (\( r(wt) = 2r - 1 \)).

2. **Longest descent edge under weak order.** As seen, \( \ell(w) \) is equal to the number of descent edges of \( w \). We can write it as a sum of non-negative functions

\[
\ell(w) = \ell_{-1}(w) + \ell_{-2}(w) + \cdots ,
\]

with respect to our grading. Needless to say, \( \ell(w) \) is monotonously increasing in Bruhat order. However, each part \( \ell_{-r}(w) \) is not necessarily. Is there any more kinds of monotony about descent edges on Bruhat graphs? If we consider the following suborder of Bruhat order, we can prove hidden and little more monotony of descent edges: The right weak order on \( S_n \) is the transitive closure of \( w \rightarrow ws \), \( s \in S \). This order also plays an important role in theory of Coxeter groups [2, Chapter 3]. Theorem 2 shows one difference between Bruhat and weak orders.

**Theorem 2.** Define \( \rho(w) = \max\{r \mid \ell_{-r}(w) > 0\} \). If \( w \leq x \) in the right weak order, then \( \rho(w) \leq \rho(x) \). This is not necessarily true in Bruhat order.

Section 2 provides preliminaries on Bruhat order and Bruhat graphs. Section 3 prepares a lemma for Theorem 1 and Section 4 gives a proof. Similarly Section 5 prepares a lemma for Theorem 2 and Section 6 gives a proof.

2. **Bruhat order and Bruhat graphs**

This section provides preliminaries on Bruhat order and Bruhat graphs on the symmetric groups as in [2, 3].
Definition 2.1. Let $w, x, y \in S_n$.

1. By $w = i_1 \cdots i_n$ we mean $w(k) = i_k$ (one-line notation). For example, $w = 312$ means $w(1) = 3, w(2) = 1$ and $w(3) = 2$.
2. For $i < j$, denote by $t_{ij}$ the transposition switching $i$ and $j$. Let $T = \{ t_{ij} \}$ denote the set of all transpositions.
3. In particular, set $s_i = (i, i + 1)$ and $S = \{ s_1, \ldots, s_{n-1} \}$ (simple reflections).
4. We write $x \rightarrow w$ and say that $w$ is a reduction of $x$ if $x = wt$ for some $t \in T$ and $\ell(w) < \ell(x)$.
5. Define Bruhat order on $S_n$: $w \leq y$ if there exists $x_0, x_1, \ldots, x_m$ in $S_n$ such that $w = x_0, x_m = y$ and $x_i \rightarrow x_{i+1}$ for all $0 \leq i \leq m - 1$.
6. The Bruhat graph of $S_n$ is the one for nodes $w \in S_n$ and for edges $w \rightarrow wt, t \in T$. For each subset $X \subseteq S_n$ we can naturally define the Bruhat subgraph of $X$ in the same way.
7. Define the permutation $w_0$ by $w_0(i) = n - i + 1$ for all $i$.

Next we recall some basic facts needed later for a proof of main theorems.

Fact 2.2. (See [2, Chapter 2])

1. $S_n$ is a graded poset of rank $|T| = \ell(w_0) = n(n - 1)/2$ with the minimum element $e$ and the maximum $w_0$.
2. The group inverse $x \mapsto x^{-1}$ (multiplication $x \mapsto xw_0$) is an order-preserving (reversing) automorphism on $S_n$. In particular, $\ell(x) = \ell(x^{-1}) = \ell(w_0) - \ell(xw_0)$.
3. $\ell(w) = |\{ t \in T \mid wt < w \}|$.

Definition 2.3. Let $w \in S_n$. Then for integers $1 \leq r \leq n - 1$, define

$$T_{-r}(w) = \{ t \in T \mid \ell(w) - \ell(wt) = 2r - 1 \},$$
$$T_{+r}(w) = \{ t \in T \mid \ell(wt) - \ell(w) = 2r - 1 \}.$$

Define $\ell_{-r}(w) = |T_{-r}(w)|, \ell_{+r}(w) = |T_{+r}(w)|$.

3. Lemma for Theorem 1

Before a proof of Theorem 1, we need the following lemma.

Lemma 1. Let $x \in S_n$. Then for each $t \in T$, the following are equivalent:

1. $x \rightarrow xt$,
2. $x^{-1} \rightarrow (xt)^{-1}$,
3. $(xt)w_0 \rightarrow xw_0$. 
Moreover, we have
\[ (xt)^{-1} - x = (xw_0)^{-1} - (xw_0). \]
As a result, the automorphism
\[ x \mapsto x^{-1}(xw_0) \] on \( S_n \) preserves (reverses) not only Bruhat order but also reductions keeping a grade. Hence \( \ell_{-r}(x) = \ell_{-r}(x^{-1}) = \ell_{+r}(xw_0) \) for all \( r \).

Proof. Note that \( y = xt \iff y^{-1} = tx^{-1} = x^{-1}(xt)^{-1} \iff yw_0 = xw_0(w_tw_0) \). Then use Fact 2.2 (2) for the part of keeping a grade.

4. Theorem 1

Theorem 1. Let \( n \) be a positive integer \( \geq 4 \). If \( n \equiv 0 \) or \( 1 \pmod{4} \), then there exist at least two \( x \in S_n \) such that \( \ell_{-r}(x) = \ell_{+r}(x) \) for all \( 1 \leq r \leq n - 1 \).

Proof. The idea is to find \( x_0 \) such that \( x_0w_0x_0 = e \). It then follows that \( x_0 = x_0^{-1}w_0 \) and hence \( \ell_{-r}(x_0) = \ell_{-r}(x_0^{-1}w_0) = \ell_{+r}(x_0^{-1}) = \ell_{+r}(x_0) \) for all \( 1 \leq r \leq n - 1 \) by Lemma 1. Needless to say, we also have \( \ell_{-r}(x_0^{-1}) = \ell_{+r}(x_0^{-1}) \). Note that \( n(n-1)/2 \) is even \( \iff n \equiv 0 \) or \( 1 \pmod{4} \).

(1) First suppose that \( n \equiv 0 \pmod{4} \). Define
\[
x_0(i) = \begin{cases} 
  i + (n/2) & \text{if } 1 \leq i \leq n/4, \\
  -i + (n/2) + 1 & \text{if } (n/4) + 1 \leq i \leq n/2, \\
  -i + (3n/2) + 1 & \text{if } (n/2) + 1 \leq i \leq 3n/4, \\
  i - (n/2) & \text{if } (3n/4) + 1 \leq i \leq n.
\end{cases}
\]

If \( 1 \leq i \leq n/4 \), then \((n/4) + 1 \leq (n/2) - i + 1 \leq n/2\). We have
\[
x_0w_0x_0(i) = x_0\left(w_0\left(i + \frac{n}{2}\right)\right) \\
= x_0\left(n - \left(i + \frac{n}{2}\right) + 1\right) \\
= x_0\left(\frac{n}{2} - i + 1\right) \\
= -\left(\frac{n}{2} - i + 1\right) + \frac{n}{2} + 1 \\
= i.
\]

If \((n/4) + 1 \leq i \leq n/2\), then \((3n/4) + 1 \leq (n/2) + i \leq n\). We have
\[
x_0w_0x_0(i) = x_0\left(w_0\left(-i + \frac{n}{2} + 1\right)\right) \\
= x_0\left(n - \left(-i + \frac{n}{2} + 1\right) + 1\right) \\
= x_0\left(\frac{n}{2} + i\right) \\
= \left(\frac{n}{2} + i\right) - \frac{n}{2} \\
= i.
\]
It is quite analogous to check the other cases.

(2) Next suppose that \( n \equiv 1 \pmod{4} \). Define

\[
x_0(i) = \begin{cases} 
  i + (n + 1)/2 & 1 \leq i \leq (n - 1)/4, \\
  -i + (n + 1)/2 + 1 & (n - 1)/4 + 1 \leq i \leq (n - 1)/2, \\
  (n + 1)/2 & i = (n + 1)/2, \\
  -i + (3n + 3)/2 & (n + 3)/2 \leq i \leq (3n + 1)/4, \\
  i - (n + 1)/2 & (3n + 5)/4 \leq i \leq n.
\end{cases}
\]

Actually construction of this \( x_0 \) is quite similar to the previous one except the middle fixed point. It is similar to verify that \( x_0 w_0 x_0 = e \). So the same conclusion follows.

Finally, we see that \( x_0 \neq x_0^{-1} \) in any case since if \( x_0 = x_0^{-1} \), then \( x_0 = x_0^{-1} w_0 = x_0 w_0 \). But this implies that \( w_0 = e \), a contradiction.

\[\square\]

**Example 4.1.** Let \( n = 8 \) and \( x_0 = 56218734 \). Then \( x_0^{-1} = 43781265 = x_0 w_0 \). For \( n = 9 \) let \( x_0 = 672159834 \). Then \( x_0^{-1} = 438951276 = x_0 w_0 \).

### 5. Lemma for Theorem 2

**Definition 5.1.** A reduction of the form \( w \rightarrow ws, s \in S \) is said to be right weak. Define the right weak order on \( S_n \) by the transitive closure of this relation.

This is also a graded partial order on \( S_n \). In particular, \( w \leq x \) in the right weak order implies \( \ell(w) \leq \ell(x) \), i.e., \( \ell(w) \) is an increasing function. Recall that \( \ell(w) = \ell_{-1}(w) + \ell_{-2}(w) + \cdots \). Notice that each part \( \ell_{-r}(w) \) is not necessarily increasing.

For example, \( 2431 \rightarrow 4231 \) is right weak and \( \ell_{-2}(2431) = 1 > 0 = \ell_{-2}(4231) \). However, as mentioned in the introduction, there is another monotony for descent edges. To be more precise, we need the following definition.

**Definition 5.2.** For \( w \in S_n \), define \( \rho(w) = \max\{r \geq 0 \mid T_{-r}(w) \neq \emptyset\} \).

The following lemma will be essential for Theorem 2.

**Lemma 2.** [1, Observation 1.2 and 5.3] Let \( x \in S_n \) and \( t_{ik} \in T \). Suppose \( x(i) > x(k) \). Then \( t_{ik} \in T_{-r}(x) \) if and only if

\[
r - 1 = \#\{j \mid i < j < k \text{ and } x(i) > x(j) > x(k)\}.
\]

Consequently, \( ws_i \rightarrow w \iff w(i) > w(i + 1) \). We also have \( \ell_{-r}(w) = 0 \) for all \( r \geq n \).

### 6. Theorem 2

**Theorem 2.** If \( w \leq x \) in the right weak order on \( S_n \), then \( \rho(w) \leq \rho(x) \).
Proof. It is little tedious to write a proof in full detail. Instead we give an idea and a sketch of proof. We may assume that \( w \rightarrow w s_m = x \) and moreover \( \rho(w) \geq 1 \). For convenience let \( r = \rho(w) \). We need to show that \( t \in T_{-r'}(x) \neq \emptyset \) for some \( r' \geq r \) and some \( t \in T \). Lemma 2 and \( r = \rho(w) \) imply that there exist \( i, j_1, j_2, \ldots, j_{r-1} \) such that \( 1 \leq i < j_1 < j_2 < \cdots < j_{r-1} < k \leq n \) and \( w(i) > w(j_1) > w(k) \) for all \( 1 \leq l \leq r - 1 \). Let us write this situation in one-line notation as

\[
w = \cdots w(i) \cdots w(j_1) \cdots w(j_2) \cdots w(j_{r-1}) \cdots w(k) \cdots .
\]

Now we see \( w(i) \) the maximum among these \( r + 1 \) numbers on the left, \( w(k) \) the minimum on the right and precisely \( r - 1 \) elements \( m(j_i) \) in the middle. Note that the right weak reduction \( w \rightarrow w s_m \) changes only adjacent two numbers and remains the rest in one-line notation, i.e., \( x(m) = w(m + 1), x(m + 1) = w(m) \) and \( x(a) = w(a) \) for \( a \neq m, m + 1 \). So at most two of \( w(i), w(j_1), w(j_2), \ldots, w(j_{r-1}), w(k) \) move and the rest stay in the same position under the reduction \( w \rightarrow w s_m = x \). To be more precise, we check several case as below.

1. If \( m \leq i - 2 \), then \( w(a) = x(a) \) for all \( a \geq i \). Hence \( t_{i,k} \in T_{-r}(x) \).
2. If \( m = i - 1 \), then \( x(i - 1) = w(i), x(i) = w(i - 1) \) (and \( x(a) = w(a) \) otherwise). Thus

\[
\begin{cases}
t_{i-1,k} \in T_{-r}(x) & \text{if } w(i - 1) < w(k), \\
t_{i-1,k} \in T_{-(r+1)}(x) & \text{if } w(i - 1) > w(k).
\end{cases}
\]

3. If \( m = i \), then \( x(i) = w(i - 1), x(i - 1) = w(i) \). Note that \( w(j_1) < w(i) \) by the assumption and \( w(i) < w(i + 1) \) since \( w \rightarrow w s_i \). Hence \( j_1 \neq i + 1 \) so that \( t_{i+1,k} \in T_{-r}(x) \).
4. If \( i + 1 \leq m \leq k - 2 \), then \( t_{i,k} \in T_{-r}(x) \) (it does not matter whether \( m = j_l \) for some \( l \) or not).

Three more cases \( (m = k - 1, m = k, m \geq k + 1) \) are symmetric to one of the previous cases. \( \square \)

Remark 6.1.

1. It is interesting (and probably possible) to generalize Theorem 2 for other Coxeter groups.
2. Theorem 2 is not true in Bruhat order as \( 3214 \rightarrow 3412 \) but \( \rho(3214) = 2 > 1 = \rho(3412) \).

References

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