A RECURRENCE FOR THE COMPLEX ROOK NUMBERS ON FERRERS BOARDS

MASATO KOBAYASHI

ABSTRACT. We define α-rook numbers from more general rook placements with the weight on a Ferrers board. We prove a recurrence for them with a smaller board.

1. BOARD, ROOK PLACEMENT AND WEIGHT

Definition 1.1 (Board)**.** A *board* B is a subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ for some $n, m \in \mathbb{N}$. An i-th column of B is a set $B_i = \{(x, y) \in B \mid x = i\}$ and a j-th row is $B^j = \{(x, y) \in B \mid y = j\}$. Let $h_j = |B^j|$. A board B is said to be a *Ferrers board* if $h_1 \leq h_2 \leq \cdots \leq h_n$.

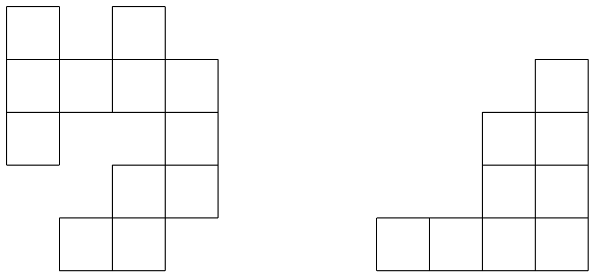

FIGURE 1. Board, Ferrers board

Definition 1.2 (Rook placement)**.** By *a placement of k nontaking rooks*(or simply k rooks placement) on B we mean a subset $P = \{(x_1, y_1), \ldots, (x_k, y_k)\} \subset B$ of size k such that $x_s = x_t$ or $y_s = y_t$ implies $s = t$.

Example 1.3 (FIGURE 1,2)**.** Intuitively, we can regard B as cells on xy-plane. Each cell corresponds to an element of B. A circle indicates a rook, i.e. an element of P. There are no two rooks in the same row nor column.

Unless otherwise specified, below B is a Ferrers board in $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and α is a complex number.

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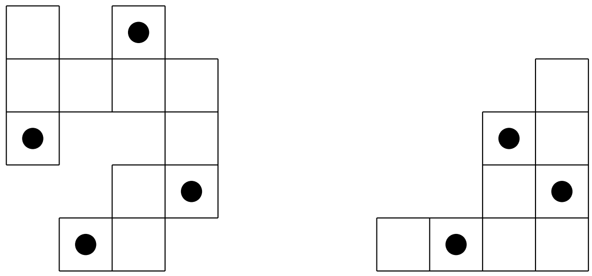
Figure 2. Rook placements



Definition 1.4 (α-weight)**.** Hereafter we consider more general rook placements P on B. We can place more than one rook in each row and at most one rook in each column. (Only $x_s = x_t$ implies $s = t$.) Let $P^j = P \cap B^j$ and $u_j = |P^j|$ the number of rooks in a j-th row on B. Define the α-weight of P^j to be

$$\mathrm{wt}(P^j) = \begin{cases} 1 & \text{if } 0 \leq u_j \leq 1 \\ \alpha(2\alpha-1)(3\alpha-2)\dots((u_j-1)\alpha-(u_j-2)) & \text{if } u_j \geq 2. \end{cases}$$

The weight of the rook placement P is $\mathrm{wt}(P) = \prod_{j=1}^{m} \mathrm{wt}(P^j)$.

Remark 1.5. We can think of the rook placement in Definition 1.2 as a special case (for $\alpha = 0$) of the above rook placement. If $\alpha = 0$ and $u_j \geq 2$, then $\mathrm{wt}(P^j) = 0$ and hence $\mathrm{wt}(P) = 0$. This means that such rook placements P do not contribute anything to the α-rook numbers, which we will define soon. For $\alpha = i \in \mathbb{N}$, rook placements with the i-weight is equivalent to "i-creation rook placements" defined in [4].

2. α-rook numbers

Definition 2.1 (α-rook numbers)**.** Set

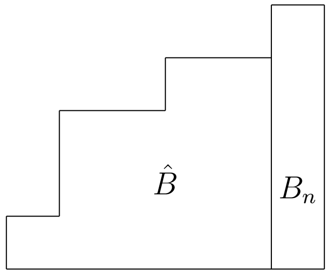
$$r_k^{(\alpha)}(B) = \sum_{P:k \text{ rooks placement on } B} \mathrm{wt}(P).$$

This is the number of ways to place exactly k rooks on B with the α-weight. In particular, $r_0^{(\alpha)}(B) = 1$, and $r_1^{(\alpha)}(B) = |B|$.

Theorem 2.2 (A Recurrence for the α-rook numbers)**.** *Let B be a Ferrers board with column heights $h_1 \leq h_2 \leq \dots \leq h_n$. Let $\hat{B}$ be a board with the last column B_n deleted. Then for all $1 \leq k \leq n$, we have*

$$r_k^{(\alpha)}(B) = r_k^{(\alpha)}(\hat{B}) + (h_n + (\alpha-1)(k-1))\, r_{k-1}^{(\alpha)}(\hat{B}).$$

FIGURE 3. $\hat{B}$ and B_n



Proof. This theorem in fact should follow from [4, α-factorization theorem, p.526]. However, it is difficult to see because of induction. Let Q be a k rooks placement on B. Either $Q \cap B_n = \varnothing$ or $|Q \cap B_n| = 1$. For the first case, Q is also a k rooks placement on $\hat{B}$. A sum of weight of such Q's equals exactly $r_k^{(\alpha)}(\hat{B})$. For the second case, let $P = Q \cap \hat{B}$ be a $k-1$ rooks placement on $\hat{B}$. Say j is an index such that the last rook sits in $B_n \cap B^j$. So write $Q = P(j)$. Now

$$P(j)^i = \begin{cases} P^i & \text{if } i \neq j \\ P^j \cup \{(n,j)\} & \text{if } i = j \end{cases}$$

For $1 \le u \le k-1$, let $m_u = \#\{\ell \,|\, 1 \le \ell \le m,\ |\hat{B}^\ell \cap P| = u\}$ the number of rows of P with exactly u rooks. Therefore, $\sum_{u=1}^{k-1} u m_u = k-1$ is the total number of rooks on $\hat{B}$. Set $u_j = |\hat{B}^j \cap P|$ the number of rooks in $\hat{B}^j$.

If $u_j \ge 1$, then $\mathrm{wt}(P(j)^j) = \alpha(2\alpha-1)\ldots((u_j-1)\alpha-(u_j-2)) \times (u_j\alpha-(u_j-1)) = \mathrm{wt}(P^j) \times (u_j\alpha - (u_j-1))$. It follows that

$$\begin{aligned} \mathrm{wt}(P(j)) &= \prod_{i=1}^{m} \mathrm{wt}(P(j)^i) \\ &= \mathrm{wt}(P(j)^j) \prod_{i \neq j} \mathrm{wt}(P(j)^i) \\ &= \mathrm{wt}(P^j)(u_j\alpha - (u_j-1)) \prod_{i \neq j} \mathrm{wt}(P^i) \\ &= \mathrm{wt}(P)(u_j\alpha - (u_j-1)). \end{aligned}$$

If $u_j = 0$, then $\mathrm{wt}(P(j)) = \mathrm{wt}(P)$. There are $h_n - \sum_{u=1}^{k-1} m_u$ such j's (open rows left).

We have

$$\begin{aligned}\sum_{j=1}^{m}\mathrm{wt}(P(j)) &= \sum_{u_j\geq 1}\mathrm{wt}(P(j)) + \sum_{u_j=0}\mathrm{wt}(P(j))\\ &= \sum_{u=1}^{k-1} m_u \mathrm{wt}(P)(u\alpha-(u-1)) + \mathrm{wt}(P)(h_n - \sum_{u=1}^{k-1} m_u)\\ &= \mathrm{wt}(P)\{h_n - \sum m_u + (\alpha-1)\sum u m_u + \sum m_u\}\\ &= \mathrm{wt}(P)\{h_n + (\alpha-1)(k-1)\}.\end{aligned}$$

We conclude that

$$\begin{aligned}r_k^{(\alpha)}(B) &= \sum_{Q:k \text{ rooks placements on } B} \mathrm{wt}(Q)\\ &= \sum_{Q\,\cap\, B_n=\varnothing}\mathrm{wt}(Q) + \sum_{|Q\,\cap\, B_n|=1}\mathrm{wt}(Q)\\ &= r_k^{(\alpha)}(\hat{B}) + \sum_{P=Q\,\cap\,\hat{B}}\sum_{j=1}^{m}\mathrm{wt}(P(j))\\ &= r_k^{(\alpha)}(\hat{B}) + (h_n+(\alpha-1)(k-1))\sum_{P=Q\,\cap\,\hat{B}}\mathrm{wt}(P)\\ &= r_k^{(\alpha)}(\hat{B}) + (h_n+(\alpha-1)(k-1))\, r_{k-1}^{(\alpha)}(\hat{B}).\end{aligned}$$

□

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Department of Mathematics, the University of Tennessee, Knoxville, TN 37996
E-mail address: kobayashi@math.utk.edu