A GENERALIZATION OF KAZHDAN-LUSZTIG μ-TRANSPORT

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Abstract. This article gives a proof of a generalization of results by Kazhdan-Lusztig on μ-coefficient (term of the highest degree) of Kazhdan-Lusztig polynomials.

1. Introduction

This article gives a proof of a small generalization of μ-transport by Kazhdan-Lusztig. We assume basic knowledge of Coxeter groups and Bruhat order as in [1, Chapters 1, 2] or [2]. First we prepare notation. Fix a Coxeter system (W, S) with the length function ℓ and Bruhat order $\leq$. Define $D_L(w) = \{s \in S \mid \ell(ws) < \ell(w)\}, D_R(w) = D_L(w^{-1})$ and $\ell(u, w) = \ell(w) - \ell(u)$. For $s, t \in S$ with $s \neq t$, denote the (group-theoretic) order of st by $m(s, t)$ ($\in \{2, 3, \dots, \} \cup \{\infty\}$). The following property will be important in the proof of Theorem.

Proposition 1.1. [1, Proposition 2.2.7, Lifting Property] *Let $u, w \in W$, $s \in S$. Suppose $u < w, s \in D_R(w) \setminus D_R(u)$. Then we have $u \leq ws$ and $us \leq w$.*

Definition 1.2. By W_I we mean the parabolic subgroup of W generated by $I \subseteq S$. wW_I means the right coset.

Fact 1.3. [1, Chapter 2] Each wW_I has the representative (denoted by w^I) of minimal length.

Definition 1.4. Let $I = \{s, t\} \subseteq S$ such that $3 \leq m(s, t) < \infty$. For $x \in wW_I$, define $h_{st}(x) = \ell(x^I, x)$.

Notice that $wW_{\{s,t\}}$ is a dihedral interval of length $m(s, t)$ so that $h_{st}(x)$ indicates a position (height) of x in $wW_{\{s,t\}}$.

2. Kazhdan-Lusztig polynomials

Fact 2.1. [1, Chapter 5] Given (W, S), there exists a unique family of polynomials $\{P_{uw}(q) \mid u, w \in W\} \subseteq \mathbf{N}[q]$ such that

(1) $P_{uw}(q) = 0$ if $u \not\leq w$,
(2) $P_{uw}(q) = 1$ if $u = w$,
(3) $\deg P_{uw}(q) \leq (\ell(u, w) - 1)/2$ if $u < w$,
(4) If $s \in D_R(w)$, then $P_{uw} = P_{us,w}$.

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(5) Define

$$\mu(u,w) = \begin{cases} \text{the coefficient of } q^{(\ell(u,w)-1)/2} \text{ of } P_{uw} & \text{if } \ell(u,w) \text{ is odd,} \\ 0 & \text{if } \ell(u,w) \text{ is even.} \end{cases}$$

If $s \in D_R(w)$, then we have

$$P_{uw} = q^{1-c}P_{us,ws} + q^c P_{u,ws} - \sum_{u \leq zs < z < ws} q^{\ell(z,w)/2}\mu(z,ws)P_{uz}$$

where $c = 1$ if $s \in D_R(u)$ and $c = 0$ otherwise.

(there is a left version of (4), (5)). We call $\{P_{uv}\}$ the *Kazhdan-Lusztig polynomials.*

We state its important properties (see [1, Chapter 5]).

Fact 2.2. If $u < w$ and $\ell(u,w) \leq 2$ then $P_{uw} = 1$.

Fact 2.3. If $u < w$, $\ell(u,w) \geq 3$ and $\mu(u,w) \neq 0$, then $D_L(w) \subseteq D_L(u)$ and $D_R(w) \subseteq D_R(u)$.

3. μ-TRANSPORT

Kazhdan and Lusztig proved invariance of μ under some conditions.

Fact 3.1 (μ-transport [p.175, Theorem 4.2])**.** Let $u, w \in W$, $s, t \in S$ such that $u < w, m(s,t) = 3$, $s \in D_R(u) \cap D_R(w)$ and $t \notin D_R(u) \cup D_R(w)$. Suppose further $h_{st}(u) = h_{st}(w) = 2$. Then $\mu(u,w) = \mu(us,ws)$.

Our aim is to prove a generalization of this. Before that, we need two results.

Lemma 3.2. *Let $u \leq w$ and $s \in D_R(u) \cap D_R(w)$. If $u \not\leq ws$, then $P_{uw} = P_{us,ws}$.*

Proof. Apply the Kazndah-Lusztig induction for (u,w) with $s \in D_R(u) \cap D_R(w)$:

$$P_{uw} = P_{us,ws} + qP_{u,ws} - \sum_{u \leq zs < z < ws} q^{\ell(z,w)/2}\mu(z,ws)P_{uz}$$

If $u \not\leq ws$, then the second and third terms are both zero (for the third term, there cannot exist such z). Hence $P_{uw} = P_{us,ws}$. □

Fact 3.3. If $x < y, \ell(x,y) = 2$ in W, then the interval $\{z \mid x \leq z \leq y\}$ has precisely two atoms.

Theorem is a generalization of Fact 3.1 (underlined parts are different).

Theorem. *Let $u, w \in W$, $s, t \in S$ such that $u < w$, $\underline{3 \leq m(s,t) < \infty}$, $s \in D_R(u) \cap D_R(w)$ and $t \notin D_R(u) \cup D_R(w)$. Suppose further $\underline{h_{st}(u) = m - 1 \text{ and } h_{st}(w) = 2}$. Then at least one of $\ell(z,ws), \ell(u,z)$ is 1.*

(1) *If $\ell(z,ws) = 1, \ell(u,z) \geq 3$ then $\mu(u,w) = \mu(us,ws) + \mu(ut,ws) - \mu(u,wst)$.*
(2) *If $\ell(z,ws) \geq 3, \ell(u,z) = 1$ then $\mu(u,w) = \mu(us,ws)$.*

(3) *If* $\ell(z, ws) = 1, \ell(u, z) = 1$ *then*

$$\mu(u, w) = \begin{cases} \mu(us, ws) & \text{if } m = 3, \\ \mu(us, ws) + 1 & \text{if } m \geq 4. \end{cases}$$

Proof. If $\ell(u, w) = 1 = \ell(us, ws)$ then $P_{u,w} = 1 = P_{us,ws}$ so that $\mu(u, w) = 1 = \mu(us, ws)$. If $\ell(u, w) = \ell(us, ws)$ is even, then $\mu(u, w) = 0 = \mu(us, ws)$. Hence we may assume that $\ell(u, w) \geq 3$ and $\ell(u, w)$ is odd.

If $u \not\leq ws$, then $P_{u,w} = P_{us,ws}$ by Lemma 3.2. This implies $\mu(u, w) = \mu(us, ws)$. Next assume that $u \leq ws$. Consider a coefficient of degree $(\ell(u, w) - 1)/2$ of each term of the right hand side of

$$P_{uw} = P_{us,ws} + qP_{u,ws} - \sum_{u \leq zs < z < ws} q^{\ell(z,w)/2} \mu(z, ws) P_{uz}.$$

The first one is $\mu(us, ws)$. Second, since $t \in D_R(ws)$ (because $h_{st}(w) = m - 1$) and $t \notin D_R(u)$, we have $P_{u,ws} = P_{ut,ws}$ with $\ell(ut, ws) = \ell(u, w) - 2$. So $qP_{ut,ws}$ produces the coefficient $1 \cdot \mu(ut, ws)$ of degree $1 + (\ell(ut, ws) - 1)/2 = (\ell(u, w) - 1)/2$. Thrid, we only need to consider terms such that $\mu(z, ws) \neq 0$ and $\mu(u, z) \neq 0$ because $q^{\ell(z,w)/2} \mu(z, ws) P_{uz}$ has the degree at most $\ell(z, w)/2 + (\ell(u, z) - 1)/2 = (\ell(u, w) - 1)/2$. If $\mu(z, ws) \neq 0$, $\mu(u, z) \neq 0$ and moreover $\ell(z, ws) \geq 3, \ell(u, z) \geq 3$ then $t \in D_R(ws) \subseteq D_R(z) \subseteq D_R(u)$, a contradiction. Thus at least one of $\ell(z, ws), \ell(u, z)$ is 1. We check the following three cases.

(1) $\ell(z, ws) = 1, \ell(u, z) \geq 3$. since $t \notin D_R(u)$, $D_R(z) \subseteq D_R(z)$ and $t \notin D_R(u)$ $z < ws$ we have $zt \leq ws$ by Lifting Property for $(z, ws), t$. Note that $\ell(zt) = \ell(z) + 1 = \ell(ws)$. So $zt \leq ws$ must be an equality. Hence $z = wst$. This term produces the coefficient $-\underbrace{\mu(wst, ws)}_{1} \mu(u, wst) = -\mu(u, wst)$.

(2) $\ell(z, ws) \geq 3, \ell(u, z) = 1$. since $t \in D_R(ws) \subseteq D_R(z), t \notin D_R(u)$ and $u < z$ we have $ut \leq z$ by Lifting Property for $(u, z), t$. Note that $\ell(ut) = \ell(u) + 1 = \ell(z)$. So $ut \leq z$ must be an equality. The $z = ut$ term produces the coefficient $-\mu(ut, ws)$.

(3) $\ell(z, ws) = 1, \ell(u, z) = 1$. since $t \in D_R(ws) \setminus D_R(u), u < ws$, we have $(u <)ut \leq ws$ and also $u \leq wst (< ws)$ by Lifting Property. Note that $\ell(u, ws) = 2$ and clearly $ut \neq wst$. By Fact 3.3, $z = ut$ or wst. Now observe that

$$\begin{cases} s \in D_R(ut) & \text{if } m = 3, \\ s \notin D_R(ut) & \text{if } m \geq 4, \end{cases}$$

and $s \notin D_R(wst)$. Recall that z must satisfy $zs < z$. Thus at most one z produces the coefficient $-\underbrace{\mu(z, ws)\mu(u, z)}_{\text{both are 1 because of length}} = -1$ as

$$\mu(u, w) = \begin{cases} \mu(us, ws) + \underbrace{\mu(ut, ws)}_{1} - 1 = \mu(us, ws) & \text{if } m = 3, \\ \mu(us, ws) + \underbrace{\mu(ut, ws)}_{1} = \mu(us, ws) + 1 & \text{if } m \geq 4. \end{cases}$$

□

References

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