Two-sided structure of double cosets in Coxeter groups

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Abstract The aim of this article is to show one clear difference between single (left or right) cosets and double cosets in Coxeter groups. Let (W, S) be a Coxeter system and $I \subseteq S$. As is well-known, (say) left cosets wW_I and xW_I are always isomorphic as sets (and moreover as graded posets under right weak order) as long as we concern the same index I. On the contrary, we show that when it comes to double cosets, $W_I w W_J$ and $W_I x W_J$ are not necessarily isomorphic even as sets. As an application, we show that each lower interval has a decomposition by certain double cosets.

Keywords Coxeter groups · Bruhat order · Weak order · Double cosets

Mathematics Subject Classification (2000) MSC 20F55 · 51F15

1 Introduction

Order structure of Coxeter groups (weak orders and Bruhat order) is an important topic as in the recent book [3]. It has been developed involving many other areas such as Hecke algebras, root systems, and particularly singularity of Schubert varieties [1,2,4–8,10,12–14]. Throughout we assume basic knowledge on length function, weak orders and Bruhat order of Coxeter groups as in [3, Chapters 1, 2, 3].

The aim of this article is to show one clear difference between single (left or right) cosets and double cosets in Coxeter systems. To start the discussion, we prepare notation. Throughout this article (W, S) is a finite Coxeter system with length function ℓ and $I, J \subseteq S$. Although we will be working on several kinds of partial orders (weak orders and Bruhat order) on W, the symbol

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$\leq$ indicates Bruhat order unless otherwise specified. We also follow common notation as follows:

$$
\begin{aligned}
T &= \bigcup w^{-1}sw, \\
D_L(w) &= \{s \in S \mid \ell(sw) < \ell(w)\}, \\
D_R(w) &= \{s \in S \mid \ell(ws) < \ell(w)\}, \\
[u, w] &= \{v \in W \mid u \leq v \leq w\}, \\
\ell(u, w) &= \ell(w) - \ell(u), \\
w_0 &: \text{ longest element of } W.
\end{aligned}
$$

Now choose $w, x \in W$ arbitrarily. Then, as is well-known, (say) left cosets wW_I and xW_I are isomorphic as sets (and moreover as graded posets under right weak order) as long as we concern the same index I. However, when it comes to double cosets, W_IwW_J and W_IxW_J are not necessarily isomorphic even as sets (Sections 2, 3). On the other hand, double cosets has many nice order-theoretic properties quite similar to single cosets. As an application, we prove that there exists a decomposition of lower intervals by certain double cosets (Section 4). We will state all results as a theorem (Section 5).

Most of our results rely on the following important property of Bruaht order:

Proposition 11. *Let $u, v \in W$, $s \in S$. Suppose $u \leq v$ and $s \in D_L(v) \setminus D_L(u)$. Then we have $u \leq sv$ and $su \leq v$ [3, Proposition 2.2.7, Lifting property]. Consequently, if $v' \leq v$ and $s \in D_L(v)$, then $sv' \leq v$ (it does not matter whether $s \in D_L(v')$ or not).*

The right version of this property also holds.

2 Double cosets

There are many results on single (left, right) cosets in the literature (for example, [3, Sections 2.4, 2.5]).

Definition 21. Let $I \subseteq S$. By W_I we mean the parabolic subgroup of W generated by I. A subset X in W is a *double coset* if it is of the form

$$X = W_IxW_J = \{wxy \mid w \in W_I, y \in W_J\}$$

for some $x \in W$ and $I, J \subseteq S$.

In other words, W_IxW_J is the set of all elements "moving around" x by simple reflections in I on the left and ones in J on the right. To be more precise, $v \in W_IxW_J$ if and only if there exist $v_0, v_1, \ldots, v_k \in W$, $s_0, s_1, \ldots, s_{k-1} \in S$ such that $v_0 = x, v_k = v$ and for each i, we have $v_{i+1} = s_iv_i$ with $s_i \in I$ or $v_{i+1} = v_is_i$ with $s_i \in J$. Hence we understand interpret W_IxW_J as a connected component in the Hasse diagram of W (with respect to certain two-sided-labeled edges).

Remark 22. Once we fix I and J, then $\cup_{x\in W} W_IxW_J$ (disjoint) forms an equivalent class on W. In addition, double cosets have a nice property as an entire analogy of single cosets as below.

Proposition 23. *Each double coset X has the representative of maximal (minimal) length.*

Proof. Write $X = W_IxW_J$. Since it is a finite set, we can choose a maximal element v. Then we necessarily have $I \subseteq D_L(v)$ and $J \subseteq D_R(v)$ (otherwise, say left, $v < sv \in W_IxW_J$ for some $s \in I$, a contradiction for maximality of v). Now suppose that v' is another maximal element. We want to show that $v = v'$. Since $v, v' \in W_IxW_J$, there exist $w, w' \in W_I$, $y, y' \in W_J$ such that

$v = wxy, v' = w'xy'$. Since $I \subseteq D_L(v)$ and $J \subseteq D_R(v)$, we have $w'w^{-1} \in W_{D_L(v)}$ and $y'y^{-1} \in W_{D_R(v)}$ so that we can write $w'w^{-1} = r_1r_2\cdots r_j$, $r_i \in D_L(v)$ and $y'y^{-1} = s_1s_2\cdots s_l$, $s_k \in D_R(v)$. Then $v' = w'xy' = (w'w^{-1})v(y^{-1}y') = (r_1r_2\cdots r_j)v(s_1s_2\cdots s_l) \leq v$ (use Proposition 11 $j+l$ times). By symmetry we also have $v \leq v'$. Hence $v = v'$. It is similar to show an existence of the representative of minimal length. □

3 Two-sidedness

Each subset of W can be regarded as a poset under several kinds of graded partial orders (left weak, right weak, two-sided weak and Bruhat order). Note that left and weak order may occur at the same time (as suborders of Bruhat order). This section introduces some terminology to describe precise relations of these orders for coverings as well as double cosets.

Definition 31. Let $u \lhd w$ be a covering (i.e., $u < w$ and $\ell(u, w) = 1$) under Bruhat order on W, say $w = tu = ut'$ for $t, t' \in T$. Say this covering is *left weak* if $t \in S$ and *right weak* if $t' \in S$ and *two-sided weak* (or (LR)) if it is both left weak and right weak and *single* if it is left weak or right weak.

Our next task is to extend this idea to double cosets. In the last section, we introduced a double coset as a set of the form $X = W_IxW_J$ for $I, J \subseteq S$. If $J = \varnothing$ at the extreme case, then $X = W_IxW_J = W_Ix$ is an ordinary right coset. However, it is worth mentioning that the double coset W_IxW_J may be equal to W_Ix (as sets) even if $J \neq \varnothing$ (or to xW_J even if $I \neq \varnothing$). For example, the whole W is itself a double coset $W_Sw_0W_S$ and furthermore $W = w_0W_S = W_Sw_0$. Hence there are many ways to express double cosets with certain choice of x and I, J. Notice that the assertion of Proposition 23 does not depend on how to write X. Often it is convenient to take a representative to be the one of maximal (or minimal) length.

Definition 32. Let X be a double coset in W with x the representative of maximal length of X. Say X is *left weak* if $X = W_Ix$ for some $I \subseteq S$ and *right weak* if $X = xW_J$ for some $J \subseteq S$ and *two-sided weak* if it is both left weak and right weak and *single* if it is left weak or right weak and X is *properly double* if it is not single.

Example 33. Observe in $W = S_4$ (type A_3) with simple reflections (adjacent transpositions) $S = \{s_i\}$ and $I = \{s_2\} = J$ that $W_I1324W_J = W_I1324 = 1324W_J = \{1324, 1234\}$ is single while $W_I3412W_J = \{3412, 2413, 3142, 2143\}$ is properly double.

We next look at one more type of a covering which does not occur in any weak orders.

Definition 34. Let $u \lhd w$ be a covering, say $w = tu = ut'$ for $t, t' \in T$. Say it is *flush* (or (F)) if $t \notin S$ and $t' \notin S$.

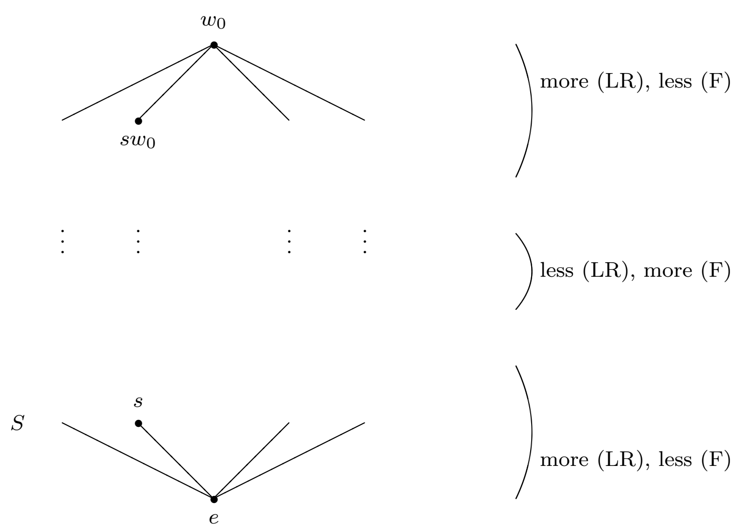
Proposition 35. *If $u \lhd w$ is* (F)*, then $D_L(w) \subseteq D_L(u)$ and $D_R(w) \subseteq D_R(u)$.*

Proof. Choose $s \in D_L(w)$. If $s \notin D_L(u)$, then $u < su \leq w$ (the second inequality is due to Lifting Property) and $\ell(su) = \ell(u) + 1 = \ell(w)$. Hence we must have $su = w = tu$, which implies $s = t$, a contradiction for $t \notin S$. It is similar for right descents. □

At type (F) coverings, u is obtained from w by deleting one simple reflection "in the middle" of a reduced word of w (subword property) so that left and descent descents do not decrease. We may think "F-order" on W as the transitive closure of (F) relation. However, this order is difficult to deal with because it is far from a graded poset unlike weak orders and Bruhat order.

We here try to roughly understand distribution of type of coverings (LR) and (F) in the whole W. Figure 1 illustrates the Hasse diagram of W. e is the unit and the first level (atoms)

Fig. 1 Distribution of (LR), (F) in W.



is S. Clearly for all $s \in S$, $e \lhd s$ is (LR). Since $x \mapsto xw_0$ is an order-reversing isomorphism with weakness preserved on each side, $sw_0 \lhd w_0$ is also (LR) (the right label may be a simple reflection different from s, though). But, type (F) appears less around e and w_0 (indeed only at levels of (co)length ≥ 2). On the contrary, in the middle of W, more (F) appears and less (LR).

4 Flush sets

From now on we fix an element $w \in W$ and concern elements of the lower interval $[e, w]$.

Definition 41. For $w \in W$, define the *flush* and *antiflush* sets of w to be

$$F(w) = \{u \in W \mid u \in [e, w], D_L(w) \subseteq D_L(u) \text{ and } D_R(w) \subseteq D_R(u)\},$$
$$A(w) = \{u \in W \mid u \in [e, w], D_L(u) \cap D_L(w) = D_R(u) \cap D_R(w) = \varnothing\}.$$

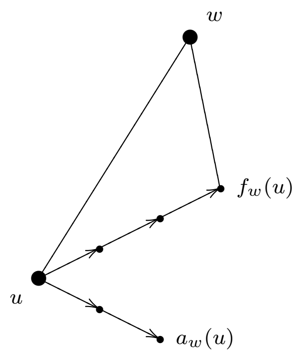
The author took the terminology "flush" from [11]. Some references say (u, w) is a critical pair to mean $u \in F(w)$. "Antiflush" is by the author only here.

Remark 42. (1) They are nonempty since $w \in F(w)$ and $e \in A(w)$. Even though $|[e, w]|$ is increasing with respect to w, it is not that easy to guess the size of $F(w)$ for given w. The extreme case is that $F(w_0)$ is only $\{w_0\}$. Notice that, roughly speaking, as $\ell(u, w)$ gets larger it gets more difficult for $u \in F(w)$ to happen.

(2) Suppose that $u \in F(w), s \in D_L(w), s' \in D_R(w)$. It immediately follows that $s \in D_L(u)$ and $s' \in D_R(u)$. Now we have four coverings $sw \lhd w, ws' \lhd w$, $su \lhd u$ and $us' \lhd u$ at hand. Notice that we cannot say anything more from this information. For example, $su \lhd u$ does not guarantee $u^{-1}su \in D_R(u)$, and $sw = ws'$ does not necessarily imply any of $su = us'$ nor $su \neq us'$. Behavior of such coverings highly depends on positions of u and w in the whole W as mentioned in the last section.

Definition 43. Fix $w \in W$. A double coset of W is said to be *w-lower* if it is of the form $W_{D_L(w)} u W_{D_R(w)}$ for some $u \in [e, w]$. For simplicity, denote this coset by $[u]_w$.

Fig. 2 the lifting-up and lifting-down.



Then for all $x \in [u]_w$ we have $x \le w$ (i.e., the comparability below w does not depend on the choice of the representative of $[u]_w$) because of Lifting Property. These double cosets form an equivalent class on $[e, w]$. In other words, each $u \in [e, w]$ belongs to precisely one w-lower double coset $[u]_w$. Call $[w]_w$ the *top coset* and $[e]_w$ the *bottom coset*. The following proposition is a little genralization of the above argument on the comparability under w.

Proposition 44. *If $u \le v$ and $v \in F(w)$, then for all $x \in [u]_w$, we have $x \le v$.*

Proof. Let $s \in D_L(w)$. Since $v \in F(w)$, we have $s \in D_L(v)$. Since $u \le v$, we have $su \le v$ by Lifting Property. Iterating this argument, we have $x \le v$ for all $x \in [u]_w$. □

Definition 45. For $u \in [e, w]$, define $f_w(u)$ to be the representative of maximal length of $[u]_w$ and $a_w(u)$ the representative of minimal length of $[u]_w$. Let us say $f_w(u)$ $(a_w(u))$ is the two-sided *lifting-up* (*-down*) of u under w.

Note that $f_w(u) \in F(w)$ and $a_w(u) \in A(w)$.

The following proposition shows a two-sided version of the well-known properties for single cosets [3, Sections 2.5 and 2.6].

Proposition 46. *Let $u \in [e, w]$. Then*

(1) $a_w(u) \le u \le f_w(u)$.
(2) $u \in F(w) \iff u = f_w(u) \iff \ell(u) = \ell(f_w(u))$.
(3) $u \in A(w) \iff u = a_w(u) \iff \ell(u) = \ell(a_w(u))$.
(4) $f_w = f_w^2$ *and* $a_w = a_w^2$.
(5) $a_w f_w = a_w$ *and* $f_w a_w = f_w$.
(6) $f_w(u) = \min\{v \in F(w) \mid u \le v\}$.
(7) $a_w(u) = \max\{v \in A(w) \mid v \le u\}$.
(8) *if* $u \le v \le w$, *then* $f_w(u) \le f_w(v)$ *and* $a_w(u) \le a_w(v)$.

Proof. (1)–(5) are clear. For (6), suppose $u \le v' \in F(w)$. By Proposition 44, we have $f_w(u) \le v'$ with $f_w(u) \in F(w)$. This means $f_w(u) = \min\{v \in F(w) \mid u \le v\}$. (7) is similar. For (8), if $u \le v$, then $u \le v \le f_w(v) \in F(w)$. By $f_w(u) \in F(w)$ and (6), we have $f_w(u) \le f_w(v)$. □

Lemma 1. *Let $[u]_w, [v]_w$ be distinct w-lower double cosets. Then $f_w(u) < f_w(v) \iff a_w(u) < a_w(v)$. Interchanging u and v, we also have $f_w(u) > f_w(v) \iff a_w(u) > a_w(v)$. As a consequence of these two statements, we further obtain $f_w(u) \parallel f_w(v) \iff a_w(u) \parallel a_w(v)$ where $\parallel$ means* *an incomparability. Hence $F(w)$ is isomorphic to $A(w)$ as abstract posets under induced Bruhat order from W.*

Proof. Suppose $f_w(u) < f_w(v)$. Then $a_w(f_w(u)) \le a_w(f_w(v))$ since $a_w(-)$ is order-preserving. Now $a_w(u) = a_w(f_w(u)) \le a_w(f_w(v)) = a_w(v)$ and $\le$ must be strict since $[u]_w \ne [v]_w$. Thus we showed $a_w(u) < a_w(v)$. All other statements are shown in the same way. □

Next we prove that each w-lower double coset is in fact an interval.

Lemma 2. *Let $u \in [e, w]$. Then $[u]_w = [a_w(u), f_w(u)]$.*

Proof. The inclusion $\subseteq$ is obvious. For the other inclusion, suppose $x \in [a_w(u), f_w(u)]$, i.e., $a_w(u) \le x \le f_w(u)$. Since $a_w(-)$ and $f_w(-)$ are order-preserving and idempotent, we have $a_w(u) \le a_w(x)$ and $f_w(x) \le f_w(u)$. By Lemma 1, both of these two inequalities must be equalities. Therefore $a_w(u) = a_w(x)$ and $f_w(x) = f_w(u)$, that is, $x \in [u]_w$. □

Definition 47. The *length* of $[u]_w$ is $\ell([u]_w) = \ell(a_w(u), f_w(u))$.

Example 48. Table 1 and Figure 3 show an example of w-lower double cosets and flush sets for $w = 52341$ in $W = S_5$. Observe that the length of w-lower cosets is *not* constant. Moreover, only the bottom coset is single. Observe also that posets $F(w)$ and $A(w)$ are ismorphic to a dihedral interval of length 3 (as abstract posets) but they are embedded in a different manner into $[e, w]$.

$u \in F(w)$	$\ell(u)$	$\ell(a_w(u))$	$\ell([u]_w)$	single/double	Poincaré
$w = 52341$	7	3	4	properly double	$q^3(1+q)^4$
32541	6	2	4	properly double	$q^2(1+q)^4$
52143	6	2	4	properly double	$q^2(1+q)^4$
32154	4	1	3	properly double	$q(1+q)^3$
21543	4	1	3	properly double	$q(1+q)^3$
21354	2	0	2	single	$q^0(1+q)^2$

Table 1 w-lower cosets for $w = 52341$.

Below Lemma 3 gives a formula for $\ell([u]_w)$. Before that we need to mention some notation and facts. For $I \subseteq S$, let $T_I = W_I \cap T$. Note that (W_I, I) is itself a Coxeter system with its reflections T_I. Denote by $w_0(I)$ longest element of W_I.

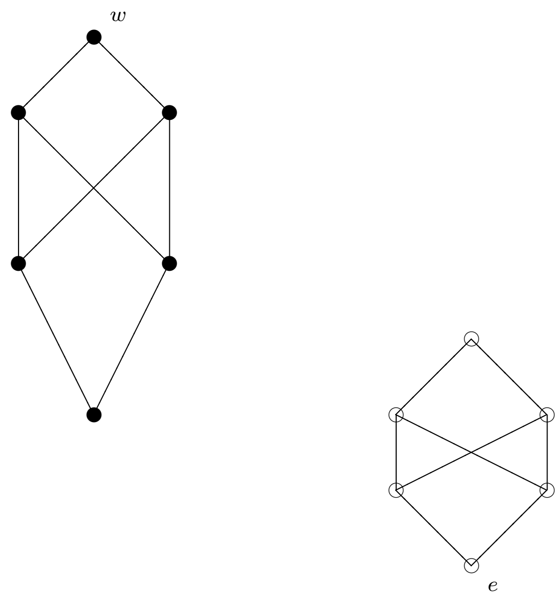
Fact 49.

(1) Let $T_R(w) = \{t \in T \mid wt < w\}$. Then $T_R(uw) = T_R(w) \cup w^{-1}T_R(u)w$ [3, p.23, Exercise 13]. Consequently, taking an intersection of both hand sides with S, we have

$$D_R(uw) = D_R(w) \cup (w^{-1}T_R(u)w \cap S).$$

(2) $T_R(w_0(I)) = T_I$.
(3) For $t \in T_I$, we have $t \in I \iff w_0(I)tw_0(I) \in I$.
(4) Suppose $I \cap D_L(u) = \varnothing$. Then $W_I u$ has the representative $w_0(I)u$ of maximal length so that $\ell(w_0(I)u) = \ell(w_0(I)) + \ell(u)$. Moreover, $I \subseteq D_L(w_0(I)u)$ and $W_I u$ is ismorphic to W_I as posets under left weak order.
(5) If $w < ws, s \in S$, then $D_L(w) \subseteq D_L(ws)$. Consequently, if $w < x$ under right weak order, then $D_L(w) \subseteq D_L(x)$ (See [3, Proposition 3.1.3]).

Fig. 3 $F(w)$ is isomorphic to $A(w)$ as abstract posets for $w = 52341$.



$F(w)$ $A(w)$

Lemma 3. *Let $u \in A(w)$. For simplicity, set $I = D_L(w), J = D_R(w)$ and*

$$J' = J'(u, w) = \{s \in J \mid \text{there does not exist } s' \in I \text{ such that } us = s'u\}.$$

Then $J' = S \setminus D_R(w_0(I)u)$, $f_w(u) = w_0(I)uw_0(J')$ and $\ell([u]_w) = \ell(w_0(I)) + \ell(w_0(J'))$.

Proof. To better understand the proof below, it is helpful to keep in mind two relations $u < w_0(I)u$ in $W_I u$ and $w_0(I)u < w_0(I)uw_0(J')$ in $w_0(I)uW_{J'}$. First we show $J' = S \setminus D_R(w_0(I)u)$. To show $\subseteq$, let $s \in J'$. We suppose $s \in D_R(w_0(I)u)$ toward a contradiction. By Fact 49 (1), we have

$$D_R(w_0(I)u) = D_R(u) \cup (u^{-1}T_R(w_0(I))u \cap S).$$

Since $s \in J$ and $u \in F(w)$, certainly $s \notin D_R(u)$. So $s \in (u^{-1}T_R(w_0(I))u \cap S) = u^{-1}T_I u \cap S$. Write $s = u^{-1}tu$, $t \in T_I$. Note that $t \notin I$ since $s \in J'$. Let $t' = w_0(I)tw_0(I) \in T_I$. Then $t'w_0(I)u$ $(= w_0(I)tu = w_0(I)us) \lhd w_0(I)u$ is a coatom of $W_I u$ under left weak order, i.e. $t' \in I$. But $t' \in I$ implies $t \in I$ (Fact 49 (3)), a contradiction. Thus $s \in S \setminus D_R(w_0(I)u)$. For the other inclusion, suppose $s \in S \setminus D_R(w_0(I)u)$. Then $w_0(I)us \rhd w_0(I)u$. If $s \notin J'$, choose $s' \in I$ such that $us = s'u$. Let $s'' = w_0(I)s'w_0(I) \in I$. Then $w_0(I)us = w_0(I)s'u = s''w_0(I)u \lhd w_0(I)u$, a contradiction for $w_0(I)us \rhd w_0(I)u$. We have finished showing $J' = S \setminus D_R(w_0(I)u)$. Second, we show $f_w(u) = w_0(I)uw_0(J')$. It is enough to show that $w_0(I)uw_0(J') \in F(w)$ since obviously $w_0(I)uw_0(J') \in [u]_w$ and $f_w(u)$ is a unique element of $[u]_w$ which belongs to $F(w)$. We want to show $D_L(w) = I \subseteq D_L(w_0(I)uw_0(J'))$ and $D_R(w) \subseteq D_R(w_0(I)uw_0(J'))$. We see $D_L(w) = I \subseteq D_L(w_0(I)u) \subseteq D_L(w_0(I)uw_0(J'))$ (for the first inclusion, use Fact 49 (4) and for the second, use Fact 49 (5) because $w_0(I)u < w_0(I)uw_0(J')$ is right weak order). Next we show $J = D_R(w) \subseteq D_R(w_0(I)uw_0(J'))$. Let $s \in J$. If $s \in J'$ then $s \in J' \subseteq D_R(w_0(I)uw_0(J'))$ (apply the left version of Fact 49 (4)). So suppose $s \in J$ and $s \notin J'$. Let $s_1 = w_0(J')sw_0(J')$. Then we must have $s_1 \in J \setminus J'$ since $J' \subseteq J, s \in J \setminus J'$ (consider the Coxeter system (W_J, J)). Since $s_1 \notin J'$, choose $s_2 \in I$ such that $us_1 = s_2 u$. Further let $s_3 = w_0(I)s_2 w_0(I) \in I$ (since $s_2 \in I$). Thus $w_0(I)uw_0(J')s = w_0(I)us_1 w_0(J') = w_0(I)s_2 uw_0(J') = s_3 w_0(I)uw_0(J') \lhd w_0(I)uw_0(J')$. The last inequality follows from the result that $I \subseteq D_L(w_0(I)uw_0(J'))$ as just shown. This implies $s \in D_R(w_0(I)uw_0(J'))$. We have shown $f_w(u) = w_0(I)uw_0(J')$. Note that $\ell(f_w(u)) = \ell(w_0(I)uw_0(J')) = \ell(w_0(I)u) + \ell(w_0(J')) = \ell(w_0(I)) + \ell(u) + \ell(w_0(J'))$. Hence conclude that $\ell([u]_w) = \ell(a_w(u), f_w(u)) = \ell(u, f_w(u)) = \ell(w_0(I)) + \ell(w_0(J'))$.

□

Remark 410. There was no specific reason to have considered $u < w_0(I)u$ in $W_I u$ first. If we let

$$I' = I'(u, w) = \{s \in I \mid \text{there does not exist } s' \in J \text{ such that } su = us'\},$$

then we will also obtain $f_w(u) = w_0(I')uw_0(J)$ and $\ell([u]_w) = \ell(w_0(I')) + \ell(w_0(J))$ with exactly the same idea. Note that I' or J' may be empty, which case $[u]_w$ would be single.

5 Final comments

Finally we state all results we have established as a theorem.

Theorem (Decomposition of lower intervals by double cosets)**.** *Let (W, S) be a finite Coxeter system. For each $w \in W$, $[e, w]$ is a disjoint union of w-lower double cosets $\{[u]_w \mid u \in [e, w]\}$. These cosets are indeed Bruhat intervals. There exist two isomorphic posets $F(w)$ and $A(w)$ sitting in $[e, w]$. They form the top and bottom elements of intervals $[u]_w$. However, the length of $[u]_w$ may be different, i.e., w-lower double cosets are not necessarily isomorphic.*

Remark 51.

(1) On each $[u]_w$, $x \mapsto w_0(D_{L(w)})xw_0(D_{R(w)})$ defines a permutation. But its order-theoretic behavior is quite different depending on $[u]_w$ because of a gap of single/doubleness as mentioned many times. In Example 48, $x \mapsto s_1 s_4 x s_1 s_4$ is order-reversing on $[w]_w$ while it is identical on $[e]_w$. On all other cosets, $x \mapsto s_1 s_4 x s_1 s_4$ is just a permutation.

(2) We can easily show that each $[u]_w$ is what is called a smooth interval (palindromic Poincaré polynomial, regular Bruhat graph or trivial Kazhdan-Lusztig polynomial). Thus $[e, w]$ is a disjoint union of smooth intervals. Our decomposition suggests the idea to find a common factor of the Poincaré polynomial of $[e, w]$ (which may not be smooth) as follows: let $\mathcal{P}_{[u,w]}(q) = \sum_{x \in [u,w]} q^{\ell(u,x)}$ be the (truncated) Poincaré polynomials of $[u, w]$. Then $\mathcal{P}_{[e,w]}(q)$ has a common factor $\mathcal{P}_{W_I}(q)$ as

$$\begin{aligned}
\mathcal{P}_{[e,w]}(q) &= \sum_{x \in [e,w]} q^{\ell(x)} \\
&= \sum_{u \in A(w)} q^{\ell(u)} \mathcal{P}_{[u]_w}(q) \\
&= \sum_{u \in A(w)} q^{\ell(u)} \mathcal{P}_{W_I}(q) \mathcal{P}_{W_{J'(u,w)}}(q) \\
&= \underbrace{\mathcal{P}_{W_I}(q)}_{\text{palindromic}} \underbrace{\sum_{u \in A(w)} q^{\ell(u)} \mathcal{P}_{W_{J'(u,w)}}(q)}_{\text{this part may not be palindromic}}
\end{aligned}$$

here $I = D_L(w)$ and $J'(u,w)$ as before. For example, observe that

$$\begin{aligned}\mathcal{P}_{[e,52341]}(q) &= \sum_{u\in A(w)} q^{\ell(u)}\mathcal{P}_{[u]_w}(q)\\ &= q^3(1+q)^4 + 2q^2(1+q)^4 + 2q(1+q)^3 + q^0(1+q)^2\\ &= (1+q)^2\cdot(q^5+4q^4+5q^3+4q^2+2q+1)\end{aligned}$$

is not palindromic (i.e., there are few more elements in an upper half of $[e, 52341]$). It is well-known [9] that $\sum_{x\in[e,w]} q^{\ell(x)}P_{xw}(q)$ (Kazhdan-Lusztig polynomials) is instead palindromic. Hence it is desirable to better understand combinatorics (such as order duality, Möbius function, enumeration, single/doubleness) of $A(w)/F(w)$ to analyze these gaps among double cosets.

(3) We can extend our result to slightly more general situation: let us say $[u,w]$ is a *generalized lower interval* if $u \in A(w)$. Then there exists the same kind of decomposition of $[u,w]$ by double cosets with $[u]_w$ the bottom coset.

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