A criterion for summability of ladder heights by means of the potential function

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Abstract

Consider a recurrent random walk of i.i.d. increments on the one dimensional integer lattice. Let $F(x)$ be the common distribution function of increment variables. It is proved that for the ascending ladder variables to have finite expectations it is necessary and sufficient that

$$\sum_{x=1}^{\infty}[a(x)+a(-x)](1-F(x))<\infty,$$

where $a(x)$ denotes the potential function of the walk. It gives another proof of the integrability condition due to Chow [1]. [1]

1 Introduction and Statements of Results

Let $S_n = X_1+\cdots+X_n$ be a random walk on $\mathbf{Z}$ where the increments $X_1, X_2, \ldots$ are independent and identically distributed random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbf{Z}$. Let X be a random variable having the same law as X_1. We suppose throughout the paper that the walk S_n is irreducible and recurrent. The first (strict) ascending ladder height Z is defined by

$$Z = S_\tau \qquad \text{where} \qquad \tau = \inf\{n > 0 : S_n > 0\}$$

([3]: XII.1). Because of recurrence of the walk Z is a proper random variable whose distribution is concentrated on positive integers $x = 1, 2, \ldots$. The dual variable $\hat{Z}$, the first (negative) height of the descending ladder, is defined by obvious analogy. If $\sigma^2 := EX^2 < \infty$, then $EZ < \infty$ (E indicates integration by P as usual), whereas if $\sigma^2 = \infty$, either $EZ = \infty$ or $E|\hat{Z}| = -\infty$ (cf [4]: Section 17, [2]:Theorem 8.4.7). In this paper we are interested in a criterion for $EZ < \infty$ in the case $\sigma^2 = \infty$.

Put $p^n(x) = P[S_n = x]$, $p(x) = p^1(x)$ and define the potential function

$$a(x) = \sum_{n=0}^{\infty}[p^n(0) - p^n(-x)];$$

the series on the right side is convergent and $a(x)/|x| \to 1/\sigma^2$ as $|x| \to \infty$ (cf. Spitzer [4]:P28.8, P29.2). The main result of this paper is

Theorem 1.1 *$EZ < \infty$ if and only if*

$$\sum_{x=1}^{\infty}[a(x)+a(-x)]P[X > x] < \infty. \tag{1.1}$$

[1] *key words*: ladder height, potential function,
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The necessity of the condition (1.1) in Theorem 1.1 is verified in [8] (see also [6]). The proof of the sufficiency is quite independent from that of necessity. The necessity is a consequence of a certain relation between a and f_r that is relatively easy to derive, and found and exploited in [7] in the cse $\sigma^2 < \infty$ and extended in [8] to the case $\sigma^2 = \infty$. On the other hand the proof of its sufficiency is somewhat involved. The crux of the proof is to show the next lemma. Let $\psi(t)$ be the characteristic function of X:

$$\psi(t) = E[e^{itX}].$$

Put

$$\phi_+(t) = E[1 - e^{itX} + itX; X > 0].$$

Note that $\Im\phi_+(\theta)/t \geq 0$ and $1 - \psi(t) = E[1 - e^{itX} + itX] = \phi_+(t) + \phi_-(t)$ with ϕ_- defined analogously.

Lemma 1.1 *Let $\sigma^2 = \infty$. Then the condition that for some $\varepsilon > 0$*

$$\int_0^\varepsilon \Re\frac{1}{1-\psi(t)}\frac{\Im\phi_+(t)}{t}\,dt < \infty, \tag{1.2}$$

implies

$$\frac{\phi_+(t)}{1-\psi(t)} \longrightarrow 0 \quad \text{as} \quad t \to 0. \tag{1.3}$$

Owing to the representation

$$a(x) + a(-x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-\cos xt}{1-\psi(t)}dt \tag{1.4}$$

(cf.[4]), the conditions (1.1) and (1.2) are the same as one immediately recognizes. Lemma 1.1 then make it possible to reduce the problem to the case when the walk is right continuous (i.e. the case $P[X > 1] = 0$) as is carried out in [6] and this paper is fully devoted to the proof of Lemma 1.1.

In view of (1.4) one derives from Theorem 1.1 the Chow 's integrability criterion for summability of Z, (1.6) below. It is noted that if $\sigma_-^2 := E[X^2; X < 0] < \infty$, then $E|\hat{Z}| < \infty$ and hence both EZ and σ^2 are finite or infinite simultaneously and that because of recurrence the divergence $E[X; X < 0] = -\infty$ entails $E[X; X > 0] = \infty$, hence $EZ = \infty$. Thus only the case when $\sigma_-^2 = \sigma_+^2 = \infty$ and $E|X| < \infty$ is in question. Put for real $x > 0$

$$\begin{cases} \ell_+(x) = P[X > x], \\ \ell_-(x) = P[X < -x] \end{cases} \quad \text{and} \quad \ell(x) = \ell_-(x) + \ell_+(x) = P[\,|X| > x] \tag{1.5}$$

and for $x \geq 0$,

$$m(x) = \int_0^x dy \int_y^\infty \ell(u)du \quad \text{and} \quad m_-(x) = \int_0^x dy \int_y^\infty \ell_-(u)du.$$

We may suppose that $E|X| < \infty$ so that m and m_- are defined as continuously differentiable functions. Chow [1] shows for recurrent random walks on $\mathbf{R}$ that for $EZ < \infty$ to hold it is necessary and sufficient that

$$\int_1^\infty \frac{x^2 P[X \in dx]}{\int_0^\infty (y \wedge x) y P[-X \in dy]} < \infty. \tag{1.6}$$

Owing to Lemma 1.1 it is not hard to see that (1.1) is equivalent to the condition

$$\int_1^\infty \frac{dx}{m(x)x^2}\int_0^x y^2 \ell_+(y)dy < \infty, \tag{1.7}$$

which turns out to be equivalent to (1.6) ([6], Proposition 2.2). Thus our result gives Chow's criterion.

Applying the criterion (1.7) one may determine whether EZ is finite or not from the asymptotic behavior of the tails $\ell_+(x) = P[Y > x]$ and $\ell_-(x) = P[Y < -x]$ as $x \to \infty$. Roughly speaking, if these two tails are suitably balanced, then both EZ and $E\hat{Z}$ are infinite under $\sigma^2 = \infty$, and otherwise one of them is finite. E.g., if $\ell_+(x)/\ell(x) = O((\log x)^{-(1+\delta)})$ as $x \to \infty$ for some $\delta > 1$, then $EZ < \infty$; in fact if there exists a positive function $\varepsilon(x)$ of $x \geq 1$ such that $x\varepsilon(x)$ is non-decreasing and

$$\int_1^\infty \frac{\varepsilon(y)}{y} dy < \infty \quad \text{and} \quad \frac{\ell_+(x)}{\ell(x)} \leq \varepsilon(x) \quad \text{for all sufficiently large } \ x,$$

then the condition (1.7) holds since $\int_0^x y^2 P[X > y]dy \leq x\varepsilon(x) \int_0^x y\ell(y)dy \leq x\varepsilon(x)m(x)$, so that $EZ < \infty$. An analogous sufficient condition for $EZ = \infty$ may be obtained under some regularity of $\ell(x)$: e.g., if $\varepsilon(x)/x$ is non-increasing and

$$\int_1^\infty \frac{\varepsilon(y)}{y} dy = \infty \quad \text{and} \quad \frac{\ell_+(x)}{\ell(x)} \geq \varepsilon(x) \quad \text{for all sufficiently large } \ x,$$

then $EZ = \infty$, provided that $\int_0^x y^3\ell(y)dy/x^2 m(x)$ is bounded away from zero as $x \to \infty$.

2 Proof of Lemma 1.1

This section consisits of three subsections. In the first subsection we bring in the notation and state Lemma 1.1 in terms of it. In the second and third ones we prove several lemmas to be used in the proof of Lemma 1.1, which is given in the fourth subsection. Throughout this section we suppose $E|X| < \infty$, which gives rise to no loss of generality as already remarked.

2.1. Let ℓ, ℓ_+, ℓ_- be functions defined in (1.5) and define functions $\alpha_+, \alpha_-, \beta_+$ and β_- by

$$\alpha_\pm(t) = \int_0^\infty \ell_\pm(x) \sin tx \, dx, \quad \beta_\pm(t) = \int_0^\infty \ell_\pm(x)(1 - \cos tx) \, dx,$$

and put

$$\alpha(t) = \alpha_-(t) + \alpha_+(t) \quad \text{and} \quad \beta(t) = \beta_+(t) + \beta_-(t).$$

In view of (1.4) the summability condition in Theorem 1.1 is rephrased as

$$\int_0^1 \Re \frac{1}{1 - \psi(t)} \beta_+(t) dt < \infty. \tag{2.1}$$

(For: the difference of the integral defining $\beta_+(t)$ from the corresponding sum is $O(\alpha_+(t)t + t^2)$.) On integrating by parts

$$\Re\psi(t) = \int_0^\infty (1 - \cos tx) d(-\ell(x)) = t\alpha(t)$$

and

$$\Im\psi(t) = \int_0^\infty \sin tx \, d(-\ell_+(x) + \ell_-(x)) = t[\beta_+(t) - \beta_-(t)]$$

so that

$$1 - \psi(t) = t\alpha(t) + it[\beta_+(t) - \beta_-(t)],$$

hence

$$\Re \frac{1}{1 - \psi(t)} = \frac{\alpha(t)}{\alpha^2(t) + (\beta_+(t) - \beta_-(t))^2} \cdot \frac{1}{t}.$$

It is noted that $\alpha_\pm(t)$ and $\beta_\pm(t)$ are all positive for $t > 0$; $\alpha_\pm$ are odd and $\beta_\pm$ even; by Fatou's lemma $\liminf_{t\to 0} \alpha(t)/t = \liminf t^{-2} \int_0^\infty (1-\cos tx) d(-\ell(x)) \geq \frac{1}{2}\sigma^2$ and similarly $\liminf \beta(t)/t^2 \geq \frac{1}{6}E[|X|^3]$. Since there arises the difference $\beta_+ - \beta_-$ in the denominator on the right side above, the condition (2.1) clearly entails that

$$\int_0^1 \frac{\alpha(t)\beta_+(t)}{\alpha^2(t)+\beta^2(t)} \cdot \frac{dt}{t} < \infty. \tag{2.2}$$

Lemma 2.1 *If $\sigma^2 = \infty$ and (2.2) holds, then*

$$\lim_{t\to 0} \frac{|\alpha_+(t)| + \beta_+(t)}{|\alpha_-(t)| + \beta_-(t)} = 0. \tag{2.3}$$

Lemma 2.1 shows that the apparently weaker condition (2.2) is actually equivalent to (2.1). Since $|1-\psi(t)| \geq t\alpha_-(t) + |t\beta_-(t)| - |t\beta_+(t)|$, Lemma 1.1 follows from Lemma 2.1.

2.2. Given a non-increasing function $\ell(x)$ on the half real line $x \geq 0$ such that $\lim_{x\to\infty} \ell(x) = 0$ and $\int_0^\infty \ell(x)dx < \infty$ we define

$$c(x) = \int_0^x y\ell(y)dy,$$
$$\eta(x) = \int_x^\infty \ell(y)dy,$$
$$m(x) = \int_0^x \eta(y)dy,$$

and

$$h(x) = \int_0^x y[\ell(y) - \ell(\pi x + y)]dy.$$

By integration by parts

$$m(x) = c(x) + x\eta(x). \tag{2.4}$$

In this subsection we prove several lemmas that concern certain interrelations among these functions and will be used for the proof of Lemma 2.1 in the next subsection .

If there exists a constant $A > 0$ such that

$$h(x) \geq Ac(x) \quad \text{for} \quad x \geq 1, \tag{2.5}$$

then all the arguments in the rest of this section are much simplified. Unfortunately the condition (2.5) may fail to hold in general. To cope with the situation the following lemma will be used.

Lemma 2.2 *If $\delta \in (0,1)$, $x > 0$ and $c(x) \geq \delta m(x)$, then it holds that*

$$h(x) \geq [1 - (1-\delta)(2\pi\delta)^{-1}]c(x);$$

in particular for $\delta = 1/3$ this implies $h(x) \geq \frac{2}{3}c(x)$.

Proof. Integrating by parts we obtain

$$h(x) = c(x) - \int_0^x [\eta(\pi x + u) - \eta(\pi x + x)]du.$$

Owing to monotonicity and convexity of η the integral on the right side is at most

$$\frac{x}{2}[\eta(\pi x) - \eta(\pi x + x)] \leq \frac{x}{2\pi}[\eta(x) - \eta(\pi x + x)] \leq \frac{1}{2\pi} x\eta(x).$$

Hence

$$h(x) \geq c(x) - (2\pi)^{-1} x\eta(x).$$

The last premise of the lemma may be rewritten as $x\eta(x) \leq \delta^{-1}(1-\delta)c(x)$, in view of (2.4), and substitution gives the required inequality.

Proposition 2.1 *If $\int_0^\infty y\ell(y)dy = \infty$, then*

$$\int_1^\infty \frac{h(y)\eta(y)}{m^2(y)}dy = \infty.$$

Proof. We consider three cases separately.

CASE 1. Suppose that there exists $x_0 \geq 1$ such that $c(x) \geq \frac{1}{3}m(x)$ for $x \geq x_0$, which entails $h(x) \geq \frac{2}{3}c(x) \geq \frac{2}{9}m(x)$ owing to Lemma 2.2. Since $m' = \eta$ (the prime denotes the differentiation),

$$\int_{x_0}^{x_1} \frac{h(y)\eta(y)}{m^2(y)}dy \geq \frac{2}{9}\int_{x_0}^{x_1} \frac{m'}{m}dy = \frac{2}{9}\log\frac{m(x_1)}{m(x_0)} \tag{2.6}$$

for $x_1 > x_0$. The hypothesis of the proposition therefore implies $\lim_{x_1\to\infty} m(x_1) = \infty$, and hence its conclusion.

CASE 2. Suppose that there exists $x_0 \geq 1$ such that $c(x) \leq \frac{2}{3}m(x)$ for $x \geq x_0$. In this case we employ the identity

$$\int_{x_0}^{x_1} \frac{h\eta}{m^2}dy = \int_{x_0}^{x_1} \frac{hm'}{m^2}dy = \Big[-\frac{h(y)}{m(y)}\Big]_{y=x_0}^{x_1} + \int_{x_0}^{x_1}\frac{dh(y)}{m(y)} \quad (x_1 > x_0).$$

(The last integral is a Stieltjes integral.) Since ℓ is non-increasing, $dh(x) \geq x[\ell(x) - \ell((\pi+1)x)]dx$. Now use the supposition of the case in the form $x\eta(x) \geq \frac{1}{3}m(x)$ to see that

$$\begin{aligned}\int_{x_0}^{x_1}\frac{dh(y)}{m(y)} &\geq \frac{1}{3}\int_{x_0}^{x_1}\frac{\ell(y)-\ell((\pi+1)y)}{\eta(y)}dy\\ &\geq \frac{1}{3}\int_{x_0}^{x_1}\Big[\frac{\ell(y)}{\eta(y)} - \frac{\ell((\pi+1)y)}{\eta((\pi+1)y)}\Big]dy\\ &= \frac{1}{3}\Big[\log\frac{\eta(x_0)}{\eta(x_1)} - \frac{1}{\pi+1}\log\frac{\eta((\pi+1)x_0)}{\eta((\pi+1)x_1)}\Big].\end{aligned}$$

Using in turn the fact that η is decreasing, the inequalities $\frac{1}{3}m(x) \leq x\eta(x) \leq m(x)$ and the fact that m is increasing, we observe that for any $k > 1$

$$\frac{\eta(kx_0)}{\eta(kx_1)} < \frac{\eta(x_0)}{\eta(x_1)}\cdot\frac{\eta(x_1)}{\eta(kx_1)} \leq \frac{\eta(x_0)}{\eta(x_1)}\frac{3km(x_1)}{m(kx_1)} < 3k\frac{\eta(x_0)}{\eta(x_1)}.$$

Hence, on employing $h/m \leq 1$

$$\int_{x_0}^{x_1}\frac{h(y)\eta(y)}{m^2(y)}dy \geq -1 + \frac{1}{3}\Big[\frac{\pi}{\pi+1}\log\frac{\eta(x_0)}{\eta(x_1)} - \frac{\log[3(\pi+1)]}{\pi+1}\Big]. \tag{2.7}$$

The right side diverging to infinity as $x_1 \to \infty$, the assertion of the proposition is proved.

CASE 3. Let the conditions of CASES 1 and 2 be violated, Then there exists an increasing sequence (x_n) such that $c(x_{2n})/m(x_{2n}) = 2/3$, $c(_{2n+1})/m(x_{2n+1}) = 1/3$ and $c \geq \frac{1}{3}m$ on each interval $[x_{2n}, x_{2n+1}]$. Integrating the equality

$$\frac{d}{dx}\log\frac{c(x)}{m(x)} = \frac{c'(x)}{c(x)} - \frac{m'(x)}{m(x)}, \tag{2.8}$$

one obtains

$$-\log 2 = \int_{x_{2n}}^{x_{2n+1}}\frac{c'}{c}dy - \int_{x_{2n}}^{x_{2n+1}}\frac{m'}{m}dy \geq -\int_{x_{2n}}^{x_{2n+1}}\frac{m'}{m}dy,$$

so that

$$\log[m(x_{2n+1})/m(x_{2n})] \geq \log 2.$$

Since the inequality (2.6) of CASE 1 is valid with the interval $[x_{2n}, x_{2n+1}]$ in place of $[x_0, x_1]$, we conclude that

$$\int_1^\infty \frac{h(y)\eta(y)}{m^2(y)}dy \ge \sum \int_{x_{2n}}^{x_{2n+1}} \frac{h(y)\eta(y)}{m^2(y)}dy = \infty.$$

This accomplishes the proof of Proposition 2.1. $\Box$

2.3 Let $\ell(x)$, $\eta(x)$, $m(x)$ etc. be as above. Define functions $\alpha(t)$ and $\beta(t)$ of $-\pi < t < \pi$ by

$$\alpha(t) = \int_0^\infty \ell(u)\sin tu\, du \quad \text{and} \quad \beta(t) = \int_0^\infty \ell(u)(1-\cos tu)\, du.$$

(These are consistent to the notation of the subsection 2.1.)

Since $\alpha(t) \le \int_0^{\pi/t} \ell(u)\sin tu\, du$, we have the upper bound

$$\alpha(t) \le tc(1/t). \tag{2.9}$$

The lower bound of $\alpha(t)$ is problematic, but at least it follows that

$$\begin{aligned} \alpha(t) \;&\ge\; \int_0^{2\pi/t} \ell(u)\sin tu\, du \ge \int_0^{1/t} [\ell(u) - \ell(\pi/t+u)]\sin tu\, du \\ &\ge\; (5/6)th(1/t), \end{aligned} \tag{2.10}$$

where the inequality $\sin 1 \ge \frac{5}{6}$ is applied. Similarly, we see

$$2\eta(1/t) \ge \int_{1/t}^\infty \ell(u)(1-\cos tu)du = (1-\cos 1)\eta(1/t) + \int_{1/t}^\infty \eta(u)t\sin tu\, du \ge (1-\cos 1)\eta(1/t).$$

where the last inequality is due to convexity of η. Since $1 - \cos 1 > 11/24$, this gives

$$\beta(t) \ge \int_{1/t}^\infty \ell(u)(1-\cos tu)du \ge \frac{11}{24}\eta(1/t). \tag{2.11}$$

Noting

$$\int_0^{1/t} \ell(u)(1-\cos tu)du \le \frac{1}{2}t^2\int_0^{1/t}\ell(u)u^2du \le \frac{1}{2}tc(1/t),$$

and recalling the bound (2.9), we obtain

$$\alpha(t) + \beta(t) \le 2tm(1/t). \tag{2.12}$$

Applying these inequality with $\ell(x) = P[|X| > x]$ we obtain the next lemma as an immediate corollary of Proposition 2.1.

Lemma 2.3 *If $\sigma^2 = \infty$ and $\ell(x) = P[|X| > x]$, then*

$$\int_0^1 \frac{\alpha(t)\beta(t)}{\alpha^2(t)+\beta^2(t)} \cdot \frac{dt}{t} = \infty.$$

We conclude this subsection by giving the two trite lemmas for convenience of later citation. Bring in a function of $x > 0$ by

$$\tilde{c}(x) = \frac{1}{x}\int_0^x u^2\ell(u)du.$$

Lemma 2.4 (i) $\frac{d}{dx}[m(x)/x] = -c(x)/x^2 < 0$.

(ii) $c(x) \ge \frac{1}{2}x^2\ell(x)$, *or what is the same thing*, $c'(x)/c(x) \le 2/x$.

(iii) $\frac{2}{3}c(x) \le h(x) + \tilde{c}(x) \le 2c(x)$.

Proof. (i) is obtained by differentiating the identity $m/x = c/x + \eta$. (ii) is immediate from the definition of c. By the following inequalities

$$\tilde{c}(x) \geq 3^{-1}x^2\ell(x) \quad \text{and} \quad h(x) \geq c(x) - 2^{-1}x^2\ell(x)$$

one obtains $h(x) + \tilde{c}(x) \geq c(x) - 6^{-1}x^2\ell(x)$, of which the right side is not larger than $(2/3)c(x)$ owing to the first inequality of (ii), showing the first inequality of (iii). The second one is trivial. □

Lemma 2.5 (i) $\frac{3}{10}m(1/t)t \leq \alpha(t) + \beta(t) \leq 2m(1/t)t$.

(ii) $$\frac{\alpha(t)+\beta(t)}{7t} \leq \frac{\alpha(s)+\beta(s)}{s} \quad \textit{and} \quad \alpha(s)+\beta(s) \leq 7[\alpha(t)+\beta(t)] \quad \textit{if} \quad 0 < s < t.$$

Proof. In (2.11) take account of the contribution of the integral on $(0, 1/t)$, which may be added to the middle and right-most members. With the help of $1 - \cos y \geq (1-\cos 1)y^2$ $(0 < y < 1)$ we then find that

$$\beta(t) \geq \frac{11}{24}\Big[t\tilde{c}(1/t) + \eta(1/t)\Big],$$

and, using this together with (2.10) and Lemma 2.4 (iii), we deduce that

$$\alpha(t) + \beta(t) \geq \frac{11}{24}\Big[t\tilde{c}(1/t) + th(1/t) + \eta(1/t)\Big] \geq \frac{3}{10}tm(1/t).$$

This verifies the first inequality of (i). The second one is the same as (2.12). (ii) is readily inferred from (i) by noting that $\frac{10}{3} \cdot 2 < 7$ and that $m(x)$ is increasing for the first inequality and $m(x)/x$ is decreasing for the second. □

2.4 Proof of Lemma 2.1. We are given two non-decreasing functions $\ell_\pm(x)$ that tend to zero as $x \to \infty$. Let $\alpha_\pm(t), \beta_\pm(t)$, $\eta_\pm(x)$ etc. be the associated functions. We shall apply the results of the preceding subsection with $\ell_\pm$ in place of ℓ. It is noted that our consideration may be restricted to positive t in view of the symmetry properties of functions $\alpha_\pm$ and $\beta_\pm$.

Let the hypothesis of Lemma 2.1 be true. If $\sigma_+ < \infty$, then $\sigma_- = \infty$ and $\alpha_+(t)+\beta_+(t) = O(t)$ so that the conclusion of Lemma 2.1 obviously follows. Hence we suppose $\sigma_+ = \infty$. It follows that

$$\liminf_{t\downarrow 0} \frac{\alpha_+(t)+\beta_+(t)}{\alpha_-(t)+\beta_-(t)} = 0, \tag{2.13}$$

for otherwise $\int_0^\pi [\alpha^2+\beta^2]^{-1}\alpha\beta_+ dt/t \geq \varepsilon \int_0^\varepsilon [\alpha_+^2 + \beta_+^2]^{-1}\alpha_+\beta_+ dt/t$ for some $\varepsilon > 0$ but the last integral is infinite owing to Lemma 2.3. Suppose that the conclusion of Lemma 2.1 does not hold, so that there exists $0 < p \leq 1$ such that

$$\limsup_{t\downarrow 0} \frac{\alpha_+(t)+\beta_+(t)}{\alpha_-(t)+\beta_-(t)} > p > 0.$$

Combined with (2.13) this implies that for any positive constant $\nu < 1$ there exists a sequence of positive numbers $t_1 > t_2 > t_3 > \cdots$ such that

$$\begin{cases} \alpha_+(t_{2n}) + \beta_+(t_{2n}) &= p[\alpha_-(t_{2n}) + \beta_-(t_{2n})], \\ \alpha_+(t_{2n+1}) + \beta_+(t_{2n+1}) &= \nu p[\alpha_-(t_{2n+1}) + \beta_-(t_{2n+1})]. \end{cases} \tag{2.14}$$

We shall show that this leads to a contradiction. Our strategy will be to derive from the two equalities in (2.14) some lower bounds of the integrals

$$J_n := \int_{t_{2n-1}}^{t_{2n+1}} \frac{\alpha(t)\beta_+(t)}{\alpha^2(t)+\beta^2(t)} \cdot \frac{dt}{t} \tag{2.15}$$

thath are bounded away from zero if ν is chosen small enough, contradicting the assumption (2.2).

To simplify the notation we fix n arbitrarily and write $r = t_{2n}$ and take s from one of the two intervals $(r, t_{2n-1}]$ (the upper half) and $[t_{2n+1}, r)$ (the lower half), so that

$$\alpha_+(r) + \beta_+(r) = p[\alpha_-(r) + \beta_-(r)]; \tag{2.16}$$

$$\alpha_+(s) + \beta_+(s) = \nu p[\alpha_-(s) + \beta_-(s)]; \tag{2.17}$$

$$\alpha_+(t) + \beta_+(t) > \nu p[\alpha_-(t) + \beta_-(t)] \qquad \text{for t between } r \text{ and } s \tag{2.18}$$

(the way which interval we choose will be specified later). From the last inequality it follows that for t in the same interval as above,

$$\frac{\alpha(t)}{\alpha^2(t) + \beta^2(t)} \geq \delta_{\nu p} \frac{\alpha_+(t)}{\alpha_+^2(t) + \beta_+^2(t)}, \tag{2.19}$$

where $\delta_{\nu p}$ is a positive constant that depends only on νp.

Put

$$\lambda_t = \frac{c_+(1/t)}{m_+(1/t)}.$$

The proof of (2.15) is given in two cases $\lambda_r < 1/2$ and $\geq 1/2$ separately.

Case $\lambda_r < 1/2$. In this case s is taken from the lower interval so that $s < r$. Using (2.10) through (2.12) it is easy to see that

$$\frac{\alpha_+(t)\beta_+(t)}{\alpha_+^2(t) + \beta_+^2(t)} \geq A \frac{h_+(1/t)\eta_+(1/t)}{m_+^2(1/t)t},$$

where A is some universal constant (it may be 1/20). In view of the inequality (2.19), if

$$x = 1/r \quad \text{and} \quad y = 1/s,$$

then

$$J_n \geq \delta_{\nu p} \int_s^r \frac{\alpha_+(t)\beta_+(t)}{\alpha_+^2(t) + \beta_+^2(t)} \cdot \frac{dt}{t} \geq C_{\nu p} \int_x^y \frac{h_+(u)\eta_+(u)}{m_+^2(u)} du, \tag{2.20}$$

where $C_{\nu p} = A\delta_{\nu p}$. By (2.7) of the subsection 3.2 with $[x, y]$ replacing $[x_0, x_1]$ therein, we have the following lower bound of the integral above

$$\int_x^y \frac{h_+(u)\eta_+(u)}{m_+^2(u)} du \geq -1 + \frac{1}{3(\pi + 1)} \Big(\pi \log \frac{\eta_+(x)}{\eta_+(y)} - \log[3(\pi + 1)] \Big).$$

First we consider the subcase when

$$\lambda_t \leq 2/3 \quad \text{for} \quad s \leq t \leq r. \tag{2.21}$$

Employing, in turn, (2.17), Lemma 2.5 (ii) and (2.16) we infer that

$$\alpha_+(s) + \beta_+(s) \leq 7\nu p[\alpha_-(r) + \beta_-(r)] = 7\nu[\alpha_+(r) + \beta_+(r)].$$

This together with Lemma 2.5 (i) and the inequality $\lambda_r < 1/2$ shows that

$$\begin{aligned} \eta_+(y) = \eta_+(1/s) \leq s m_+(1/s) &\leq \tfrac{10}{3}[\alpha_+(s) + \beta_+(s)] \\ &\leq \tfrac{10}{3} \cdot 7\nu[\alpha_+(r) + \beta_+(r)] \leq \tfrac{20}{3} \cdot 7\nu m_+(x)/x \leq 100\nu\eta_+(x). \end{aligned}$$

Thus if $\nu \leq e^{-7}/100$, then $J_n/C_{\nu p} \geq -1 + 2\pi/(\pi + 1) = (\pi - 1)/(\pi + 1) > 0$.

Next let (2.21) be violated. Then we can choose two numbers $s_1 < r_1$ from (s, r) so that

$$\lambda_{r_1} = 1/2, \quad \lambda_{s_1} = 2/3 \quad \text{and} \quad 1/3 < \lambda_t \leq 2/3 \quad \text{for} \quad s_1 \leq t \leq r_1. \tag{2.22}$$

Let $x_1 = 1/r_1$ and $y_1 = 1/s_1$. By integrating the equality (2.8) over the interval $[x_1, y_1]$ one derives from the first two equalities in (2.22) the bound

$$\log \frac{4}{3} \leq \int_{x_1}^{y_1} \frac{c_+'(u)}{c_+(u)} du.$$

Substitution from the inequality $c_+'(u)/c_+(u) = u\ell_+(u)/c_+(u) \leq 2/u$ into the integral above therefore gives

$$\log(y_1/x_1) \geq 2^{-1} \log(4/3).$$

On the other hand , we have $m_+'(u)/m_+(u) = \eta_+(u)/m_+(u) \geq \frac{1}{3}u^{-1}$ by virtue of the last inequality of (2.22) , so that

$$\int_{x_1}^{y_1} \frac{m_+'(u)}{m_+(u)} du \geq \frac{1}{3} \log \frac{y_1}{x_1}.$$

Owing to the last condition of (2.22) we can apply Lemma 2.2 to have $h_+(u) \geq \frac{2}{3}c_+(u) \geq \frac{2}{9}m_+(u)$. Now, on using (2.20) together with the inequalities derived above

$$\begin{aligned} J_n &\geq C_{\nu p} \int_{x_1}^{y_1} \frac{h_+(u)\eta_+(u)}{m_+^2(u)} du \\ &\geq \frac{2C_{\nu p}}{9} \int_{x_1}^{y_1} \frac{m_+'(u)}{m_+(u)} du \geq \frac{C_{\nu p}}{27} \log \frac{4}{3} \end{aligned}$$

as required.

Case $\lambda_r \geq 1/2$. In this case we take s from the upper half interval $(t_{2n}, t_{2n-1}]$. Note that (2.20) in the preceding case is valid (but with the roles of s and r reversed). If there exists $t_0 \in [r, s]$ such that $\lambda_{t_0} \leq 1/3$, then the same argument given above shows that $J_n \geq m$ for some universal constant $m > 0$.

Hence we may suppose

$$\lambda_t \geq 1/3 \quad (r \leq t \leq s). \tag{2.23}$$

Using the first inequality in Lemma 2.5 (ii) we obtain first

$$\frac{\alpha_+(r) + \beta_+(r)}{r} = \frac{p[\alpha_-(r) + \beta_-(r)]}{r} \geq \frac{p[\alpha_-(s) + \beta_-(s)]}{7s} = \frac{\alpha_+(s) + \beta_+(s)}{7\nu s},$$

and then, putting $x = 1/s$ and $y = 1/r$,

$$m_+(x) \leq \frac{10}{3} \cdot \frac{\alpha_+(s) + \beta_+(s)}{s} \leq \frac{10}{3} \cdot 7\nu \cdot \frac{\alpha_+(r) + \beta_+(r)}{r} \leq \frac{140}{3} \nu m_+(y).$$

Now if $\nu \leq 1/140$ so that $m_+(y)/m_+(x) \geq 3$, we infer

$$\begin{aligned} J_n \geq \delta_{\nu p} \int_r^s \frac{\alpha_+\beta_+}{\alpha_+^2 + \beta_+^2} \cdot \frac{dt}{t} \geq C_{\nu p}' \int_r^s \frac{\beta_+}{\alpha_+ + \beta_+} \cdot \frac{dt}{t} &\geq C_p'' \int_x^y \frac{\eta_+(u)}{m_+(u)} du \\ &= C_p'' \log \frac{m_+(y)}{m_+(x)} \geq C_p'' \log 3, \end{aligned}$$

where (2.23) is used for the second inequality. This accomplishes the proof of Lemma 2.1.

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