Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations

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Abstract

We consider nonnegative solutions of a parabolic equation in a cylinder $D \times I$, where $D$ is a noncompact domain of a Riemannian manifold and $I = (0, T)$ with $0 < T \leq \infty$ or $I = (\infty, 0)$. Under the assumption [SSP] (i.e., the constant function 1 is a semismall perturbation of the associated elliptic operator on $D$), we establish an integral representation theorem of nonnegative solutions: In the case $I = (0, T)$, any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0, T])$, where $\partial_M D$ is the Martin boundary of $D$ for the elliptic operator; and in the case $I = (\infty, 0)$,


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any nonnegative solution is represented uniquely by the sum of an integral on \( \partial_M D \times (-\infty, 0) \) and a constant multiple of a particular solution. We also show that [SSP] implies the condition [SIU] (i.e., the associated heat kernel is semi-intrinsically ultracontractive).

1 Introduction

This paper is a continuation of [34]. It is concerned with integral representations of nonnegative solutions to parabolic equations and perturbation theory for elliptic operators.

We consider nonnegative solutions of a parabolic equation

\[
(\partial_t + L)u = 0 \quad \text{in} \quad D \times I,
\]

where \( \partial_t = \partial/\partial t \), \( L \) is a second order elliptic operator on a noncompact domain \( D \) of a Riemannian manifold \( M \), and \( I \) is a time interval: \( I = (0, T) \) with \( 0 < T \leq \infty \) or \( I = (-\infty, 0) \).

During the last few decades, much attention has been paid to the structure of all nonnegative solutions to a parabolic equation, perturbation theory for elliptic operators, and their relations. (See [1], [2], [4], [5], [6], [11], [14], [17], [19], [20], [22], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [36], [37], [38], [40], [41], [42].) Among others, Murata [34] has established integral representation theorems of nonnegative solutions to the equation (1.1) under the condition [IU] (i.e., intrinsic ultracontractivity) on the minimal fundamental solution \( p(x, y, t) \) for (1.1). Furthermore, he has shown that [IU] implies [SP] (i.e., the constant function 1 is a small perturbation of \( L \) on \( D \)). It is known ( [30]) that [SP] implies [SSP] (i.e., 1 is a semismall perturbation of \( L \) on \( D \)).

In this paper, we show that [SSP] implies [SIU] (i.e., semi-intrinsic ultracontractivity) and give integral representation theorems of nonnegative solutions to (1.1) under the condition [SSP]. We consider that [SSP] is one of the weakest possible condition for getting “explicit” integral representation theorems.

Now, in order to state our main results, we fix notations and recall several notions and facts. Let \( M \) be a connected separable \( n \)-dimensional smooth manifold with Riemannian metric of class \( C^0 \). Denote by \( \nu \) the Riemannian measure on \( M \). \( T_x M \) and \( TM \) denote the tangent space to \( M \) at \( x \in M \) and the tangent bundle, respectively. We denote by \( \text{End}(T_x M) \) and \( \text{End}(TM) \) the set of endmorphisms in \( T_x M \) and the corresponding bundle, respectively.
The inner product on $TM$ is denoted by $\langle X, Y \rangle$, where $X, Y \in TM$; and $|X| = \langle X, X \rangle^{1/2}$. The divergence and gradient with respect to the metric on $M$ are denoted by $\text{div}$ and $\nabla$, respectively. Let $D$ be a noncompact domain of $M$. Let $L$ be an elliptic differential operator on $D$ of the form

$$Lu = -m^{-1}\text{div}(mA\nabla u) + Vu,$$

(1.2)

where $m$ is a positive measurable function on $D$ such that $m$ and $m^{-1}$ are bounded on any compact subset of $D$, $A$ is a symmetric measurable section on $D$ of End($TM$), and $V$ is a real-valued measurable function on $D$ such that

$$V \in L^p_{\text{loc}}(D, md\nu) \quad \text{for some } p > \max\left(\frac{n}{2}, 1\right).$$

Here $L^p_{\text{loc}}(D, md\nu)$ is the set of real-valued functions on $D$ locally $p$-th integrable with respect to $md\nu$. We assume that $L$ is locally uniformly elliptic on $D$, i.e., for any compact set $K$ in $D$ there exists a positive constant $\lambda$ such that

$$\lambda|\xi|^2 \leq \langle A_x\xi, \xi \rangle \leq \lambda^{-1}|\xi|^2, \quad x \in K, \ (x, \xi) \in TM.$$

We assume that the quadratic form $Q$ on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A\nabla u, \nabla u \rangle + Vu^2)md\nu$$

is bounded from below, and put

$$\lambda_0 = \inf \left\{ Q[u]; u \in C_0^\infty(D), \quad \int_D u^2md\nu = 1 \right\}.$$

Then, for any $a < \lambda_0$, $(L - a, D)$ is subcritical, i.e., there exists the (minimal positive) Green function of $L - a$ on $D$. We denote by $L_D$ the selfadjoint operator in $L^2(D; md\nu)$ associated with the closure of $Q$. The minimal fundamental solution for (1.1) is denoted by $p(x, y, t)$, which is equal to the integral kernel of the semigroup $e^{-tL_D}$ on $L^2(D, md\nu)$.

Let us recall several notions related to [SSP].

[IU] $\lambda_0$ is an eigenvalue of $L_D$; and there exists, for any $t > 0$, a constant $C_t > 0$ such that

$$p(x, y, t) \leq C_t \phi_0(x)\phi_0(y), \quad x, y \in D,$$

where $\phi_0$ is the normalized positive eigenfunction for $\lambda_0$. 
This notion was introduced by Davies-Simon [13], and investigated extensively because of its important consequences (see [7], [8], [9], [10], [12], [23], [24], [31], [34], [42], and references therein). It looks, on the surface, not related to perturbation theory. But it has turned out ([34]) that [IU] implies the following condition [SP] for any $a < \lambda_0$.

**[SP]** The constant function 1 is a small perturbation of $L - a$ on $D$, i.e., for any $\varepsilon > 0$ there exists a compact subset $K$ of $D$ such that

$$
\int_{D \setminus K} G(x, z) G(z, y) m(z) d\nu(z) \leq \varepsilon G(x, y), \quad x, y \in D \setminus K,
$$

where $G$ is the Green function of $L - a$ on $D$.

This condition is a special case of the notion introduced by Pinchover [37]. Recall that [SP] implies the following condition [SSP] (see [30]).

**[SSP]** The constant function 1 is a semismall perturbation of $L - a$ on $D$, i.e., for any $\varepsilon > 0$ there exists a compact subset $K$ of $D$ such that

$$
\int_{D \setminus K} G(x^0, z) G(z, y) m(z) d\nu(z) \leq \varepsilon G(x^0, y), \quad y \in D \setminus K,
$$

where $x^0$ is a fixed reference point in $D$.

This condition [SSP] implies that $L_D$ admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^\infty$ with eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ repeated according to multiplicity; furthermore, for any $j = 1, 2, \cdots$, the function $\phi_j / \phi_0$ has a continuous extension $[\phi_j / \phi_0]$ up to the Martin boundary $\partial_M D$ of $D$ for $L - a$ (see Theorem 6.3 of [38]).

We show in this paper that [SSP] also implies the following condition [SIU].

**[SIU]** $\lambda_0$ is an eigenvalue of $L_D$; and there exist, for any $t > 0$ and compact subset $K$ of $D$, positive constants $A$ and $B$ such that

$$
A \phi_0(x) \phi_0(y) \leq p(x, y, t) \leq B \phi_0(x) \phi_0(y), \quad x \in K, \quad y \in D.
$$

This notion was introduced by Bañuelos-Davis [9], where they called it one half IU. Here we should recall that [IU] implies that for any $t > 0$ there exists a constant $c_t > 0$ such that

$$
c_t \phi_0(x) \phi_0(y) \leq p(x, y, t), \quad x, y \in D.
$$
We see that the same argument as in the proof of Theorem 3.1 in [25] (or the argument in the proof of Theorem 1.2 below) shows that [SIU] implies the following condition [NUP] (i.e., non-uniqueness for the positive Cauchy problem).

[NUP] The Cauchy problem

\[(\partial_t + L)u = 0 \quad \text{in} \quad D \times (0, T), \quad u(x, 0) = 0 \quad \text{on} \quad D \tag{1.3}\]

admits a solution \(u\) with \(u(x, t) > 0\) in \(D \times (0, T)\).

We say that [UP] holds for (1.3) when any nonnegative solution of (1.3) is identically zero. We note that [UP] implies that the constant function 1 is a "big" perturbation of \(L - a\) on \(D\) in some sense (see Theorem 2.1 of [32]).

Fix \(a < \lambda_0\), and suppose that [SSP] holds. Let \(D^* = D \cup \partial_M D\) be the Martin compactification of \(D\) for \(L - a\), which is a compact metric space. Denote by \(\partial_m D\) the minimal Martin boundary of \(D\) for \(L - a\), which is a Borel subset of the Martin boundary \(\partial_M D\) of \(D\) for \(L - a\). Here, we note that \(\partial_M D\) and \(\partial_m D\) are independent of \(a\) in the following sense: if [SSP] holds, then for any \(b < \lambda_0\) there is a homeomorphism \(\Phi\) from the Martin compactification of \(D\) for \(L - a\) onto that for \(L - b\) such that \(\Phi|_D = \text{identity}\), and \(\Phi\) maps the Martin boundary and minimal Martin boundary of \(D\) for \(L - a\) onto those for \(L - b\), respectively (see Theorem 1.4 of [30]).

Now, we are ready to state our main results. In the following theorems we assume that [SSP] holds for some fixed \(a < \lambda_0\).

**Theorem 1.1** The condition [SSP] implies [SIU].

**Theorem 1.2** Assume [SSP]. Then, for any \(\xi \in \partial_M D\) there exists the limit

\[
\lim_{D^{*} \ni y \to \xi} \frac{p(x, y, t)}{\phi(y)} \equiv q(x, \xi, t), \quad x \in D, \ t \in \mathbb{R}. \tag{1.4}\]

Here, as functions of \((x, t)\), \(\{p(x, y, t)/\phi(y)\}_y\) converges to \(q(x, \xi, t)\) as \(y \to \xi\) uniformly on any compact subset of \(D \times \mathbb{R}\). Furthermore, \(q(x, \xi, t)\) is a continuous function on \(D \times \partial_M D \times \mathbb{R}\) such that

\[
q > 0 \quad \text{on} \quad D \times \partial_M D \times (0, \infty), \tag{1.5}\]

\[
q = 0 \quad \text{on} \quad D \times \partial_M D \times (-\infty, 0], \tag{1.6}\]

\[
(\partial_t + L)q(\cdot, \xi, \cdot) = 0 \quad \text{on} \quad D \times \mathbb{R}. \tag{1.7}\]
**Theorem 1.3** Assume [SSP]. Consider the equation (1.1) for $I = (0, T)$ with $0 < T \leq \infty$. Then, for any nonnegative solution $u$ of (1.1) there exists a unique pair of Borel measures $\mu$ on $D$ and $\lambda$ on $\partial_M D \times [0, T)$ such that $\lambda$ is supported by the set $\partial_m D \times [0, T)$, and

$$u(x, t) = \int_D p(x, y, t) d\mu(y) + \int_{\partial_M D \times [0, t]} q(x, \xi, t - s) d\lambda(\xi, s) \quad (1.8)$$

for any $(x, t) \in D \times I$.

Conversely, for any Borel measures $\mu$ on $D$ and $\lambda$ on $\partial_M D \times [0, T)$ such that $\lambda$ is supported by $\partial_m D \times [0, T)$ and

$$\int_D p(x^0, y, t) d\mu(y) < \infty, \quad 0 < t < T, \quad (1.9)$$

$$\int_{\partial_M D \times [0, t]} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad 0 < t < T, \quad (1.10)$$

where $x^0$ is a fixed point in $D$, the right hand side of (1.8) is a nonnegative solution of (1.1) for $I = (0, T)$ with $0 < T \leq \infty$.

The proof of this theorem will be given in Sections 4 and 5. It is based upon the abstract integral representation theorem established in [34], without assuming [IU], via a parabolic Martin representation theorem and Choquet’s theorem (see [18], [21], [35]). Its key step is to identify the parabolic Martin boundary.

This theorem is an improvement of Theorem 1.2 of [34]; where the condition [IU], which is more stringent than [SSP], is assumed. It is also an answer to a problem raised in Remark 4.13 of [34]. Note that (1.8) gives explicit integral representations of nonnegative solutions to (1.1) provided that the Martin boundary $\partial_M D$ of $D$ for $L - a$ is determined explicitly. We consider that [SSP] is one of the weakest possible condition for getting such explicit integral representations.

Let us recall that when [UP] holds for (1.3), the structure of all nonnegative solutions to (1.1) for $I = (0, T)$ is extremely simple. Namely, the following theorem holds (see [5]).

**Fact AT** Assume [UP]. Then, for any nonnegative solution $u$ of (1.1) with $I = (0, T)$, there exists a unique Borel measure $\mu$ on $D$ such that

$$u(x, t) = \int_D p(x, y, t) d\mu(y), \quad (x, t) \in D \times I. \quad (1.11)$$
Conversely, for any Borel measure \( \mu \) on \( D \) satisfying (1.9), the right hand side of (1.11) is a nonnegative solution of (1.1) with \( I = (0, T) \).

It is quite interesting that when [UP] holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters in many cases of [NUP].

Finally, we state an integral representation theorem for the case \( I = (-\infty, 0) \).

**Theorem 1.4** Assume [SSP]. Consider the equation (1.1) for \( I = (-\infty, 0) \). Then, for any nonnegative solution \( u \) of (1.1) there exists a unique pair of a nonnegative constant \( \alpha \) and a Borel measure \( \lambda \) on \( \partial M \times (-\infty, 0) \) supported by the set \( \partial m D \times (-\infty, 0) \) such that

\[
\begin{align*}
  u(x, t) = & \quad \alpha e^{-\lambda_0 t} \phi_0(x) + \int_{\partial M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s) \tag{1.12}
\end{align*}
\]

for any \( (x, t) \in D \times (-\infty, 0) \).

Conversely, for any nonnegative constant \( \alpha \) and a Borel measure \( \lambda \) on \( \partial M D \times (-\infty, 0) \) such that it is supported by \( \partial m D \times (-\infty, 0) \) and

\[
\begin{align*}
  \int_{\partial M D \times (-\infty, t)} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad -\infty < t < 0, \tag{1.13}
\end{align*}
\]

the right hand side of (1.12) is a nonnegative solution of (1.1).

This theorem is an improvement of Theorem 6.1 of [34], where [IU] is assumed instead of [SSP].

Here, in order to illustrate a scope of Theorems 1.3 and 1.4, we give a simple example. Further examples will be given in Section 7.

**Example 1.5** Let \( D \) be a domain in \( \mathbb{R}^2 \) with finite area. Then, by Theorem 6.1 of [33], the constant function 1 is a small perturbation of \( L = -\Delta \) on \( D \). Thus Theorems 1.3 and 1.4 hold true for the heat equation

\[
(\partial_t - \Delta) u = 0 \quad \text{in} \quad D \times I.
\]

Note that there exist many bounded planar domains for which the heat semigroup is not intrinsically ultracontractive (see Example 1 of [13] and Section 4 of [9]). Thus, the last assertion of this example is new for such domains.
The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.1, and Theorem 1.2 is proved in Section 3. Sections 4 and 5 are devoted to the proof of Theorem 1.3. In Section 4 we show it in the case of $I = (0, \infty)$. In Section 5 we show it in the case of $I = (0, T)$ with $0 < T < \infty$ by making use of results to be given in Section 4. Theorem 1.4 is proved in Section 6. Finally we shall give two more concrete examples in Section 7 with emphasis on sharpness of concrete sufficient conditions of [SSP].

2 [SSP] implies [SIU]

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 We may and shall assume that $a = 0 < \lambda_0$. Let $G$ be the Green function of $L$ on $D$. For any $t > 0$, put

$$G_t(x, y) = \int_t^{\infty} p(x, y, s) \, ds,$$

$$G^t(x, y) = \int_0^t p(x, y, s) \, ds.$$

Then $G = G_t + G^t$. Let us show that for any $t > 0$ and any compact subset $K$ of $D$ there exists a constant $A > 0$ such that

$$A \phi_0(x) \phi_0(y) \leq p(x, y, t), \quad x \in K, \quad y \in D. \quad (2.1)$$

Fix a compact subset $K$. We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of $K$. Then the same argument as in the proof of Theorem 1.5 of [30] shows that

$$C^{-1} G(x^0, z) \leq \phi_0(z) \leq C G(x^0, z), \quad z \in D \setminus K_1, \quad (2.2)$$

for some constant $C > 0$. Fix $t > 0$, and put

$$\epsilon_t = \frac{1}{2\lambda_0} \left( 1 - e^{-t\lambda_0} \right).$$

By [SSP] and (2.2), there exits a compact subset $K_2 \supset K_1$ such that

$$\int_{D \setminus K_2} \phi_0(z) G(z, y) \, d\mu(z) \leq \epsilon_t \phi_0(y), \quad y \in D \setminus K_2, \quad (2.3)$$
where \(d\mu(z) = m(z)\, d\nu(z)\). Since
\[
\frac{\phi_0(y)}{\lambda_0} = \int_D G(y, z) \, \phi_0(z) \, d\mu(z),
\]
and \(G(y, z) = G(z, y)\), (2.3) yields
\[
\frac{\phi_0(y)}{\lambda_0} \leq \int_{K_2} G_t(z, y) \, \phi_0(z) \, d\mu(z) + \int_{K_2} G^t(z, y) \, \phi_0(z) \, d\mu(z) + \epsilon_t \, \phi_0(y)
\]
(2.4)
for any \(y \in D \setminus K_2\). By Fubini’s theorem,
\[
\int_D G_t(z, y) \, \phi_0(z) \, d\mu(z) = \int_t^\infty ds \int_D p(z, y, s) \, \phi_0(z) \, d\mu(z)
\]
\[
= \int_t^\infty e^{-\lambda_0 s} \, \phi_0(y) \, ds
\]
\[
= \frac{1}{\lambda_0} e^{-\lambda_0 t} \, \phi_0(y).
\]
Thus
\[
\int_{K_2} G_t(z, y) \, \phi_0(z) \, d\mu(z) \leq \frac{1}{\lambda_0} e^{-\lambda_0 t} \, \phi_0(y).
\]
This together with (2.4) implies
\[
\epsilon_t \, \phi_0(y) \leq \int_{K_2} G^t(z, y) \, \phi_0(z) \, d\mu(z).
\]
(2.5)
Choose a compact subset \(K_3\) whose interior includes \(K_2\). By the parabolic Harnack inequality, there exists a constant \(C_1\) depending on \(t, K_2, K_3\) such that
\[
p(z, y, s) \leq C_1 \, p(x, y, 2t),
\]
for any \(x, z \in K_2, y \in D \setminus K_3\), and \(0 < s \leq t\). We have
\[
G^t(z, y) = \int_0^t p(z, y, s) \, ds
\]
\[
\leq C_1 \, t \, p(x^0, y, 2t), \quad z \in K_2, \, y \in D \setminus K_3.
\]
(2.6)
Thus
\[
\int_{K_2} G^t(z, y) \, \phi_0(z) \, d\mu(z) \leq C_1 \, t \int_{K_2} \phi_0(z) \, dz \, p(x^0, y, 2t).
\]
This together with (2.5) implies
\[ \phi_0(y) \leq C_2 p(x^0, y, 2t), \quad y \in D \setminus K_3, \tag{2.7} \]
where
\[ C_2 = \frac{1}{\epsilon_t} C_1 t \int_{K_2} \phi_0(z) d\mu(z). \]
By the parabolic Harnack inequality,
\[ p(x^0, y, 2t) \leq C p(x, y, 3t), \quad x \in K, \ y \in D, \]
for some constant \( C > 0 \). This together with (2.7) yields the desired inequality (2.1). It remains to show that for any \( t > 0 \) and a compact subset \( K \) of \( D \) there exists a constant \( B \) such that
\[ p(x, y, t) \leq B \phi_0(x) \phi_0(y), \quad x \in K, \ y \in D. \tag{2.8} \]
Fix a compact subset \( K \). We may assume that \( x^0 \in K \). Let \( K_1 \subset D \) be a compact neighborhood of \( K \). By the parabolic Harnack inequality there exists a constant \( c > 0 \) such that
\[ c p(x^0, y, t) \leq p(z, y, 2t), \quad z \in K_1, \ y \in D. \]
Thus, for any \( y \in D \),
\[
e^{-2t\lambda_0} \phi_0(y) = \int_D \phi_0(z) p(z, y, 2t) d\mu(z)
\geq \int_{K_1} \phi_0(z) p(z, y, 2t) d\mu(z)
\geq c \left[ \int_{K_1} \phi_0(z) d\mu(z) \right] p(x^0, y, t).
\]
This implies (2.8), since
\[ C_2 p(x^0, y, t) \geq p(x, y, t/2), \quad x \in K, \ y \in D, \]
for some constant \( C > 0 \). (We should note that in proving (2.8) we have only used the consequence of [SSP] that \( \phi_0 \) is a positive eigenfunction.) \( \Box \)

**Remark 2.1** It is an open problem whether [SIU] implies [SSP] or not. Furthermore, the problem whether [SSP] implies [SP] or not in the case \( n > 1 \) is still open.
3 Parabolic Martin kernels

In this section we prove Theorem 1.2. Throughout the present section we assume [SSP]. We may and shall assume that \( a = 0 < \lambda_0 \). Let \( G \) be the Green function of \( L \) on \( D \). For any \( 0 < \delta < t \), put

\[
G^t_\delta(x,y) = \int_\delta^t p(x,y,s) \, ds. \tag{3.1}
\]

We denote by \( \partial_M D \) the Martin boundary of \( D \) for \( L \). In order to prove Theorem 1.2, we need two lemmas.

**Lemma 3.1** Let \( \xi \in \partial_M D \). Suppose that a sequence \( \{y_n\}_{n=1}^\infty \subset D \) converges to \( \xi \), and there exists the limit

\[
\lim_{n \to \infty} \frac{G^t_\delta(z,y_n)}{\phi_0(y_n)} \phi_0(y_n) = w(z,t), \quad z \in D. \tag{3.2}
\]

Then

\[
\lim_{n \to \infty} \int_D G(x,z) \frac{G^t_\delta(z,y_n)}{\phi_0(y_n)} d\mu(z) = \int_D G(x,z) w(z,t) d\mu(z) \tag{3.3}
\]

for any \( x \in D \), where \( d\mu(z) = m(z)d\nu(z) \).

**Proof** Fix \( x \in D \). Let \( K_1 \subset D \) be a compact neighborhood of \( x \). By [SSP], there exists a constant \( C > 0 \) such that

\[
C^{-1} \phi_0(y) \leq G(x,y) \leq C \phi_0(y), \quad y \in D \setminus K_1. \tag{3.4}
\]

Let \( \epsilon > 0 \). Then there exists a compact subset \( K \supset K_1 \) such that

\[
\int_{D \setminus K} G(x,z) \frac{G(z,y)}{G(x,y)} d\mu(z) < \frac{\epsilon}{3C}, \quad y \in D \setminus K.
\]

Thus, for \( n \) sufficiently large,

\[
\int_{D \setminus K} G(x,z) \left[ \frac{G^t_\delta(z,y_n)}{\phi_0(y_n)} \right] d\mu(z) \leq \int_{D \setminus K} G(x,z) \left[ \frac{C G(z,y_n)}{G(x,y_n)} \right] d\mu(z) < \frac{\epsilon}{3}.
\]
By Fatou’s lemma,
\[ \int_{D \setminus K} G(x, z) w(z, t) \, d\mu(z) \leq \frac{\epsilon}{3}. \]

By Theorem 1.1, there exist constants \( A_1 \) and \( A_2 \) such that
\[ A_1 \phi_0(x) \phi_0(y) \leq p(x, y, \delta) \leq A_2 \phi_0(x) \phi_0(y), \quad x \in K, \ y \in D. \]

Then, for any \( t > \delta \), the semigroup property yields
\[ A_1 e^{-\lambda_0(t-\delta)} \phi_0(x) \phi_0(y) \leq p(x, y, t) \leq A_2 e^{-\lambda_0(t-\delta)} \phi_0(x) \phi_0(y) \quad (3.5) \]
for any \( x \in K, y \in D \). Thus there exists a constant \( B > 0 \) such that for any \( n \)
\[ \frac{G_\delta(z, y_n)}{\phi_0(y_n)} \leq B \phi_0(z), \quad z \in K. \]

Then Lebesgue’s dominated convergence theorem yields
\[ \lim_{n \to \infty} \int_K G(x, z) \left[ \frac{G_\delta(z, y_n)}{\phi_0(y_n)} \right] \, d\mu(z) = \int_K G(x, z) w(z, t) \, d\mu(z). \]

Therefore, for \( n \) sufficiently large,
\[ \left| \int_D G(x, z) \left[ \frac{G_\delta(z, y_n)}{\phi_0(y_n)} \right] \, d\mu(z) - \int_D G(x, z) w(z, t) \, d\mu(z) \right| < \epsilon. \]
This shows (3.3). □

By Lemma 6.1 of [38], it follows from [SSP] that there exists the limit
\[ \lim_{D \ni y \to \xi} G_D(y, z) \phi_0(y) = h(\xi, z), \quad (\xi, z) \in \partial M D \times D, \quad (3.6) \]
and \( h \) is a positive continuous function on \( \partial M D \times D \). From this we show the following lemma.

**Lemma 3.2** Under the same assumptions as in Lemma 3.1, one has
\[ \int_D h(\xi, z) G_\delta^*(z, x) \, d\mu(z) = \lim_{n \to \infty} \int_D \frac{G(y_n, z)}{\phi_0(y_n)} G_\delta^*(z, x) \, d\mu(z) \]
\[ = \int_D G(x, z) w(z, t) \, d\mu(z) \quad (3.7) \]
for any \( x \in D \).
Proof Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of $x$. By Theorem 1.1, (3.4) and (3.5), there exists a constant $C_1 > 0$ such that

$$C_1 G(z, x) \leq G_\delta^t(z, x) \leq G(z, x), \quad z \in D \setminus K_1.$$ 

Let $\epsilon > 0$. By [SSP], there exists a compact subset $K \supset K_1$ such that

$$\int_{D \setminus K} \left[ \frac{G(y_n, z)}{\phi_0(y_n)} \right] G_\delta^t(z, x) \, d\mu(z) < \frac{\epsilon}{3}, \quad (3.8)$$

for $n$ sufficiently large. By Fatou’s lemma,

$$\int_{D \setminus K} h(\xi, z) \, G_\delta^t(z, x) \, d\mu(z) \leq \frac{\epsilon}{3}. \quad (3.9)$$

On the other hand, for any sufficiently large $n$

$$\left[ \frac{G(y_n, z)}{\phi_0(y_n)} \right] G_\delta^t(z, x) \leq C_2, \quad z \in K,$$

where $C_2$ is a positive constant. By Lebesgue’s dominated convergence theorem,

$$\lim_{n \to \infty} \int_{K} \frac{G(y_n, z)}{\phi_0(y_n)} G_\delta^t(z, x) \, d\mu(z) = \int_{K} h(\xi, z) \, G_\delta^t(z, x) \, d\mu(z). \quad (3.10)$$

Combining (3.8), (3.9) and (3.10), we get the first equality. It remains to show the second equality of (3.7). By Fubini’s theorem and the symmetry

$$p(x, y, t) = p(y, x, t),$$

we have

$$\int_{D} G(y_n, z) \, G_\delta^t(z, x) \, d\mu(z) = \int_{0}^{\infty} dr \int_{\delta}^{t} ds \, p(y_n, x, r + s) = \int_{D} G(x, z) \, G_\delta^t(z, y_n) \, d\mu(z).$$

This together with Lemma 3.1 implies the second equality. \qed

Proof of Theorem 1.2 Let $\{y_j\}_{j=1}^{\infty} \subset D$ be any sequence converging to $\xi \in \partial_M D$. Put

$$u_j(x, t) = \frac{p(x, y_j, t)}{\phi_0(y_j)} \quad \text{for } t > 0, \quad u_j(x, t) = 0 \quad \text{for } t \leq 0. \quad (3.11)$$
Since [SIU] holds, it follows from the parabolic Harnack inequality and local a priori estimates for nonnegative solutions to parabolic equations (see [6] and [16]) that there exists a subsequence \( \{u_{j_k}\}_{k=1}^{\infty} \) such that \( u_{j_k} \) converges, as \( k \to \infty \), uniformly on any compact subset of \( D \times \mathbb{R} \) to a solution \( u \) of the equation

\[
(\partial_t + L) u = 0 \text{ in } D \times \mathbb{R}
\]

satisfying \( u > 0 \) on \( D \times (0, \infty) \) and \( u = 0 \) on \( D \times (-\infty, 0] \). Thus, in order to prove Theorem 1.2, it suffices to show that the limit function \( u \) is independent of \( \{y_{j_k}\}_{k=1}^{\infty} \) and uniquely determined by \( \xi \). Let \( \{y_j\}_{n=1}^{\infty} \) and \( \{y'_j\}_{n=1}^{\infty} \) be two sequences in \( D \) converging to \( \xi \). Define \( u_j \) by (3.11), and \( u'_j \) by (3.11) with \( y_j \) replaced by \( y'_j \). Suppose that \( \{u_j\}_{j=1}^{\infty} \) and \( \{u'_j\}_{j=1}^{\infty} \) converge to \( u \) and \( u' \), respectively. For any \( t > \delta > 0 \), put

\[
w(z, t) = \int_\delta^t u(z, s) \, ds, \quad w'(z, t) = \int_\delta^t u'(z, s) \, ds.
\]

Then we have

\[
\lim_{n \to \infty} \frac{G^t_\delta(z, y_n)}{\phi_0(y_n)} = w(z, t), \quad \lim_{n \to \infty} \frac{G^t_\delta(z, y'_n)}{\phi_0(y'_n)} = w'(z, t).
\]

By Lemma 3.2,

\[
\int_D G(x, z) \, w(z, t) \, d\mu(z) = \int_D h(\xi, z) \, G^t_\delta(z, x) \, d\mu(z) = \int_D G(x, z) \, w'(z, t) \, d\mu(z).
\]

Thus \( w(x, t) = w'(x, t) \), which implies \( u(x, t) = u'(x, t) \). This completes the proof of Theorem 1.2.

\[\square\]

4 Integral representations; the case \( I = (0, \infty) \)

In this section we prove Theorem 1.3 in the case \( T = \infty \).

We first state an abstract integral representation theorem which holds without [SSP]. For \( x \in D \) and \( r > 0 \), we denote by \( B(x, r) \) the geodesic ball in the Riemannian manifold \( M \) with center \( x \) and radius \( r \). Let \( x^0 \) be a reference point in \( D \). Choose a nonnegative continuous function \( a \) on \( D \) such
that $a(x) = 1$ on $B(x^0, r^0)$ and $a(x) = 0$ outside $B(x^0, 2r^0)$ for some $r^0 > 0$ with $B(x^0, 3r^0) \subset D$. Choose a nonnegative continuous function $b$ on $\mathbb{R}$ such that $0 < b(t) < e^{\gamma t}$ on $(1, \infty)$ for some $\gamma < \lambda_0$, and $b(t) = 0$ on $(-\infty, 1]$. Denote by $\beta$ the measure defined by $d\beta(x, t) = a(x)b(t)m(x)\,d\nu(x)dt$. For any nonnegative measurable function $u$ on $Q = D \times (0, \infty)$, we write
\[
\beta(u) = \iint_Q u(x, t)\,d\beta(x, t).
\]
Denote by $P(Q)$ the set of all nonnegative solutions of (1.1) with $I = (0, \infty)$, and put
\[
P_\beta(Q) = \{u \in P(Q) ; \beta(u) < \infty\}.
\]
Note that for any $u \in P(Q)$ there exists a function $b$ as above such that $\beta(u) < \infty$; thus $P(Q) = \bigcup_\beta P_\beta(Q)$. Furthermore, the parabolic Harnack inequality shows that if $\beta(u) = 0$, then $u = 0$. Now, let us define the $\beta$-Martin boundary $\partial^\beta_M Q$ of $Q$ with respect to $\partial_t + L$ along the line given in [21] and [18]. Put
\[
p(x, t; y, s) = p(x, y, t - s), \quad t > s, \quad x, y \in D,
\]
\[
p(x, t; y, s) = 0, \quad t \leq s, \quad x, y \in D.
\]
Define the $\beta$-Martin kernel $K_\beta$ by
\[
K_\beta(x, t; y, s) = \frac{p(x, t; y, s)}{\beta(p(\cdot; y, s))}, \quad (x, t), \ (y, s) \in Q,
\]
where $\beta(p(\cdot; y, s)) = \int_0^\infty p(z, r; y, s)\,d\beta(z, r)$. Note that $\beta(p(\cdot; y, s)) < \infty$ for any $(y, s) \in Q$, since $0 < b(t) < e^{\gamma t}$ on $(1, \infty)$ for some $\gamma < \lambda_0$. Let $\{D_j\}_{j=1}^\infty$ be an exhaustion of $D$ such that each $D_j$ is a domain with smooth boundary, $D_j \Subset D_{j+1} \Subset D$, $\bigcup_{j=1}^\infty D_j = D$, and $B(x^0, 3r^0) \Subset D_1$. Put $Q_j = D_j \times (1/j, j)$. For $Y = (y, s)$, $Z = (z, r) \in Q$, let
\[
\delta_\beta(Y, Z) = \sum_{j=1}^\infty 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y) - K_\beta(X; Z)|}{1 + |K_\beta(X; Y) - K_\beta(X; Z)|}.
\]
Then we see that $\delta_\beta$ is a metric on $Q$, and the topology on $Q$ induced by $\delta_\beta$ is equivalent to the original topology of $Q$. Denote by $Q^{\beta*}$ the completion of $Q$ with respect to the metric $\delta_\beta$. Put $\partial^\beta_M Q = Q^{\beta*} \setminus Q$. A sequence $\{Y^k\}_{k=1}^\infty$ in $Q$
is called a fundamental sequence if \( \{Y_k\}_{k=1}^{\infty} \) has no point of accumulation in \( Q \) and \( \{K_\beta(\cdot; Y_k)\}_{k=1}^{\infty} \) converges uniformly on any compact subset of \( Q \) to a nonnegative solution of (1.1) with \( I = (0, \infty) \). By the local a priori estimates for solutions of (1.1), for any \( \Xi \) nonnegative solution of (1.1) with \( \partial_t + L \) the parabolic Harnack inequality, we have a fundamental sequence \( K_\beta(\cdot; \Xi) \) of (1.1) and a fundamental sequence \( \{Y_k\}_{k=1}^{\infty} \) in \( Q \)

\[
\lim_{k \to \infty} \sum_{j=1}^{\infty} 2^{-j} \sup_{x \in Q_j} \frac{|K_\beta(X; Y^k) - K_\beta(X; \Xi)|}{1 + |K_\beta(X; Y^k) - K_\beta(X; \Xi)|} = 0.
\]

Thus the metric \( \delta_\beta \) is canonically extended to \( Q^{\beta*} \). Furthermore, \( Q^{\beta*} \) becomes a compact metric space, since by the parabolic Harnack inequality, any sequence \( \{Y_k\}_{k=1}^{\infty} \) with no point of accumulation in \( Q \) has a fundamental subsequence. We call \( K_\beta(\cdot; \Xi) \), \( \partial^\beta_M Q \) and \( Q^{\beta*} \) the \( \beta \)-Martin kernel, \( \beta \)-Martin boundary and \( \beta \)-Martin compactification for \( (Q, \partial_t + L) \), respectively. Note that \( \beta(K_\beta(\cdot; \Xi)) \leq 1 \) by Fatou’s lemma; and so \( K_\beta(\cdot; \Xi) \in P_\beta(Q) \). A nonnegative solution \( u \in P_\beta(Q) \) is said to be minimal if for any nonnegative solution \( v \leq u \) there exists a nonnegative constant \( C \) such that \( v = Cu \). Put

\[
\partial^\beta_m Q = \left\{ \Xi \in \partial^\beta_M Q ; K_\beta(\cdot; \Xi) \text{ is minimal and } \beta(K_\beta(\cdot; \Xi)) = 1 \right\},
\]

which we call the minimal \( \beta \)-Martin boundary for \( (Q, \partial_t + L) \).

Observe that \( D \times [0, \infty) \) is embedded into \( Q^{\beta*} \), and \( D \times \{0\} \subset \partial^\beta_M Q \). Indeed, with \( y \in D \) fixed, for any sequence \( \{Y_k\}_{k=1}^{\infty} \) in \( Q \) with \( \lim_{k \to \infty} Y_k = (y, 0) \) we have \( \lim_{k \to \infty} K_\beta(x, t; Y_k) = p(x, t; y, 0) / \beta(p(\cdot; y, 0)) \); furthermore, \( K_\beta(\cdot; y, 0) \neq K_\beta(\cdot; z, 0) \) if \( y \neq z \). We also note that any sequence \( \{Y_k = (y^k, s^k)\}_{k=1}^{\infty} \) in \( Q \) with \( \lim_{k \to \infty} s^k = \infty \) is a fundamental sequence, since \( \lim_{k \to \infty} K_\beta(\cdot; Y_k) = 0 \). We denote by \( \varpi \) the point in \( \partial^\beta_M Q \) corresponding to the Martin kernel which is identically zero : \( K_\beta(\cdot; \varpi) = 0 \). Put

\[
\mathcal{L}^\beta_m Q = \partial^\beta_m Q \setminus (D \times \{0\} \cup \{\varpi\}).
\]

We obtain the following abstract integral representation theorem in the same way as in the proof of Theorem 2.1 and Lemma 2.2 of [34].

**Theorem 4.1** For any \( u \in P_\beta(Q) \), there exists a unique pair of finite Borel measures \( \kappa \) on \( D \) and \( \lambda \) on \( \partial^\beta_m Q \setminus (D \times \{0\}) \) such that \( \lambda \) is supported by the set \( \mathcal{L}^\beta_m Q \),

\[
u(x, t) = \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))} \, d\kappa(y) + \int_{\mathcal{L}^\beta_m Q} K_\beta(x, t; \Xi) \, d\lambda(\Xi) \quad (4.1)
\]
for any \((x, t) \in Q\), and
\[
\beta(u) = \kappa(D) + \lambda(L^\beta_m Q).
\]
(4.2)

Furthermore, the function
\[
v(x, t) = u(x, t) - \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))} d\kappa(y)
\]
is a nonnegative solution of the equation
\[
(\partial_t + L)v = 0 \quad \text{in} \quad D \times \mathbb{R}
\]
such that \(v = 0\) on \(D \times (-\infty, 0]\).

Conversely, for any finite Borel measures \(\kappa\) on \(D\) and \(\lambda\) on \(\partial_M Q\setminus(D \times \{0\})\) such that \(\lambda\) is supported by the set \(L^\beta_m Q\), the right hand side of (4.1) belongs to \(P_\beta(Q)\).

We put
\[
P^0_\beta(Q) = \left\{ v \in P_\beta(Q); \lim_{t \downarrow 0} v(x, t) = 0 \text{ on } D \right\}.
\]

We show Theorem 1.3 on the basis of Theorem 4.1. To this end it suffices to show (1.8) for \(u \in P^0_\beta(Q)\). The key step in the proof is to identify \(L^\beta_m Q\).

Under the condition [SSP], we shall show that \(L^\beta_m Q = \partial M D \times [0, \infty)\). In the remainder of this section we assume [SSP]. We may and shall assume that \(a = 0 < \lambda_0\).

**Lemma 4.2** For any domains \(U\) and \(W\) with \(U \Subset W \Subset D\), there exist positive constants \(C\) and \(\alpha\) such that
\[
p(x, y, t) \leq Cf(t)\phi_0(x)\phi_0(y), \quad x \in U, \ y \in D \setminus W, \ t > 0,
\]
(4.3)
where \(f(t) = e^{-\alpha/t}\) for \(0 < t < 1\), and \(f(t) = e^{-\lambda_0 t}\) for \(t \geq 1\). Furthermore,
\[
q(x, \xi, t) \leq Cf(t)\phi_0(x), \quad x \in U, \ \xi \in \partial_M D, \ t > 0,
\]
\[
G(x, y) \leq C\phi_0(x)\phi_0(y), \quad x \in U, \ y \in D \setminus W,
\]
(4.4)
(4.5)
where \(G\) is the Green function of \(L\) on \(D\).

This lemma is shown in the same way as Lemmas 4.2 and 4.4 of [34].

Let \(K(x, \xi)\) be the Martin kernel for \(L\) on \(D\) with reference point \(x^0 \in D\), i.e., \(K(x^0, \xi) = 1, \ \xi \in \partial_M D\). The following lemma gives a relation between \(K\) and \(q\).
Lemma 4.3 For any $\xi \in \partial_M D$,
\[
\lim_{D \ni y \to \xi} G(x, y) = \int_0^\infty q(x, \xi, t) \, dt, \quad x \in D, \quad (4.6)
\]
\[
K(x, \xi) = \frac{\int_0^\infty q(x, \xi, t) \, dt}{\int_0^\infty q(x_0, \xi, t) \, dt}, \quad x \in D. \quad (4.7)
\]

This lemma is shown in the same way as Lemma 4.5 of [34].

Lemma 4.4 Let $\xi, \eta \in \partial_M D$, $0 \leq s, r < \infty$ and $C > 0$. If
\[
q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad (x, t) \in Q,
\]
then $\xi = \eta$, $s = r$ and $C = 1$.

Proof Since $q(x, \xi, \tau) > 0$ for $\tau > 0$ and $q(x, \xi, \tau) = 0$ for $\tau \leq 0$, we obtain that $s = r$. Thus $q(x, \xi, \tau) = q(x, \eta, \tau)$. This together with (4.7) implies that $K(\cdot, \xi) = K(\cdot, \eta)$ on $D$. Hence $\xi = \eta$, and so $C = 1$.

Now, let $\beta$ be a measure on $Q = D \times (0, \infty)$ as described in the beginning of this section: $d\beta(x, t) = a(x)b(t)m(x) \, dv(x) \, dt$. The following proposition determines the $\beta$-Martin boundary $\partial_\beta M$, $\beta$-Martin compactification $Q^\beta$, and $\beta$-Martin kernel $K_\beta$ for $(\partial_t + L, Q)$. Recall that $p(x, t; y, s) = p(x, y, t - s)$ and $K_\beta(\cdot; y, s) = p(\cdot; y, s)/\beta(p(\cdot; y, s))$. We write
\[
q(x, t; \xi, s) = q(x, \xi, t - s)
\]
for $\xi \in \partial_M D$ and $0 \leq s < \infty$.

Proposition 4.5 (i) The $\beta$-Martin boundary $\partial_\beta M$ of $Q$ for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0, \infty)$ and the one point set $\{\varpi\}$:
\[
\partial_\beta M = D \times \{0\} \cup \partial_M D \times [0, \infty) \cup \{\varpi\}. \quad (4.8)
\]
In particular, $\partial_\beta M$ does not depend on $\beta$.

(ii) The $\beta$-Martin compactification $Q^\beta$ of $Q$ for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0, \infty)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of $\varpi$ is given by the family $\{\varpi\} \cup D^* \times (N, \infty)$, $N > 1$. In particular, $Q^\beta$ does not depend on $\beta$. 

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The $\beta$-Martin kernel $K_\beta$ is given as follows: For $(x, t) \in Q$,

$$K_\beta(x, t; y, 0) = \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}, \quad (y, 0) \in D \times \{0\},$$

(4.9)

$$K_\beta(x, t; \xi, s) = \frac{q(x, t; \xi, s)}{\beta(q(\cdot; \xi, s))}, \quad (\xi, s) \in \partial_M D \times [0, \infty),$$

(4.10)

and $K_\beta(x, t; \omega) = 0$.

This proposition is shown in the same way as Proposition 4.8 of [34].

**Lemma 4.6** Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. Then there exists a finite Borel measure $\gamma$ on $\partial_M D$ supported by $\partial_m D$ such that

$$q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s) d\gamma(\eta).$$

(4.11)

**Proof** For reader’s convenience, we give a sketch of the proof for the case $s = 0$. (For details, see the proof of Lemma 4.10 of [34].) By the elliptic Martin representation theorem, there exists a unique finite Borel measure $\mu$ on $\partial_M D$ supported by $\partial_m D$ such that

$$K(x, \xi) = \int_{\partial_m D} K(x, \eta) d\mu(\eta).$$

This together with (4.7) implies

$$\int_0^\infty q(x, \xi, t) dt = \int_{\partial_m D} \left( \int_0^\infty q(x, \eta, t) dt \right) d\gamma(\eta),$$

(4.12)

where $d\gamma(\eta) = [H(x^0, \xi)/H(x^0, \eta)] d\mu(\eta)$ with

$$H(x, \eta) = \int_0^\infty q(x, \eta, t) dt.$$

For $\alpha > 0$, denote by $G_\alpha$ the Green function of $L + \alpha$ on $D$. By the resolvent equation and [SSP], we then have

$$\int_0^\infty e^{-\alpha t} q(x, \eta, t) dt$$

(4.13)

$$= \int_0^\infty q(x, \eta, t) dt - \alpha \int_D G_\alpha(x, z) \left( \int_0^\infty q(z, \eta, t) dt \right) m(z) d\nu(z),$$
for any $\eta \in \partial_M D$. By combining (4.12) and (4.13), we get

$$
\int_0^\infty e^{-\alpha t} \left( \int_{\partial_m D} q(x, \eta, t) \, d\gamma(\eta) \right) \, dt = \int_0^\infty e^{-\alpha t} q(x, \xi, t) \, dt.
$$

Thus the Laplace transforms of $q(x, \xi, t)$ and $\int_{\partial_m D} q(x, \eta, t) \, d\gamma(\eta)$ coincide; and so (4.11) holds.

\[\square\]

**Lemma 4.7** Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. Then $q(\cdot; \xi, s)$ is not minimal.

**Proof** For reader’s convenience, we give a proof. We have (4.11). Suppose that $q(\cdot; \xi, s)$ is minimal. Then, along the line given in the proof of Lemma 12.12 of [15], we obtain from (4.11) that the support of $\gamma$ consists of a single point. Thus, for some $\eta \in \partial_m D$ and constant $C$

$$
q(\cdot; \xi, s) = Cq(\cdot; \eta, s).
$$

Hence, by Lemma 4.4, $\xi = \eta$; which is a contradiction. \[\square\]

**Lemma 4.8** Let $(\xi, s) \in \partial_m D \times (0, \infty)$. Then $q(\cdot; \xi, s)$ is minimal if and only if $q(\cdot; \xi, 0)$ is minimal.

**Proof** Assume that $q(\cdot; \xi, 0)$ is minimal. Suppose that a nonnegative solution $u$ of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, s)$ on $Q$. Put $v(x, t) = u(x, t + s)$. Then $v(\cdot) \leq q(\cdot; \xi, 0)$. Thus $v(\cdot) = Cq(\cdot; \xi, 0)$ for some constant $C$. Hence $u(x, t) = Cq(x, t; \xi, s)$ for $t > s$, and $u(x, t) = 0 = Cq(x, t; \xi, s)$ for $t \leq s$. This shows that $q(\cdot; \xi, s)$ is minimal. Next, assume that $q(\cdot; \xi, s)$ is minimal. Suppose that a nonnegative solution $u$ of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, 0)$ on $Q$. Put $v(x, t) = u(x, t - s)$ for $t > s$, and $v(x, t) = 0$ for $0 < t \leq s$. Then $v(\cdot) \leq q(\cdot; \xi, s)$. Thus $v(\cdot) = Cq(\cdot; \xi, s)$ for some constant $C$. Hence $u(x, t) = Cq(x, t; \xi, 0)$. This shows that $q(\cdot; \xi, 0)$ is minimal. \[\square\]

By Theorem 4.1 and Lemmas 4.7 and 4.8, we have the following proposition.

**Proposition 4.9** There exists a Borel subset $R$ of $\partial_M D$ such that

$$
R \subset \partial_m D, \quad \mathcal{L}_m^\beta Q = R \times [0, \infty),
$$

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for any $u \in P^0_\beta(Q)$ there exists a unique Borel measure $\lambda$ on $\partial_M D \times [0, \infty)$ which is supported by $R \times [0, \infty)$ and satisfies

$$u(x, t) = \int_{R \times [0, \infty)} q(x, \xi, t - s) d\lambda(\xi, s) \quad (x, t) \in Q. \quad (4.14)$$

**Lemma 4.10** Let $(\xi, s) \in \partial_m D \times [0, \infty)$. Then $q(\cdot; \xi, s)$ is minimal.

**Proof** Suppose that $q(\cdot; \xi, 0)$ is not minimal. Then $\xi \notin R$ and

$$q(x, \xi, t) = \int_{R \times [0, \infty)} q(x, \eta, t - s) d\lambda(\eta, s)$$

for some Borel measure $\lambda$. We have

$$K(x, \xi) \int_0^\infty q(x^0, \xi, t) dt = \int_0^\infty q(x, \xi, t) dt = \int_{R \times [0, \infty)} d\lambda(\eta, s) K(x, \eta) \int_0^\infty q(x^0, \eta, t) dt.$$ 

Thus

$$K(x, \xi) = \int_R K(x, \eta) d\Lambda(\eta)$$

for some Borel measure $\Lambda$. But $\xi \in \partial_m D \setminus R$ and $R \subset \partial_m D$. This contradicts the uniqueness of a representing measure in the elliptic Martin representation theorem. Hence $q(\cdot; \xi, 0)$ is minimal; which together with Lemma 4.8 shows Lemma 4.10.

**Completion of the proof of Theorem 1.3 in the case $I = (0, \infty)$** By Lemma 4.10, $R = \partial_m D$ and

$$\mathcal{L}^3_m Q = \partial_m D \times [0, \infty).$$

Thus Proposition 4.9 shows Theorem 1.3.

**5 Proof of Theorem 1.3; the case $0 < T < \infty$**

In this section we prove Theorem 1.3 in the case $0 < T < \infty$ by making use of the results in Section 4. To this end, the following proposition plays a crucial role.
Proposition 5.1 Let $\xi \in \partial_M D$ and $0 \leq s < r < \infty$. Then
\[ \int_D p(x, y, t - r)q(y, r; \xi, s)d\mu(y) = q(x, t; \xi, s), \quad x \in D, \ t > r, \] (5.1)
where $d\mu(y) = m(y)d\nu(y)$

Proof We first show (5.1) for $\xi \in \partial_m D$. Define $u(x, t)$ by
\[ u(x, t) = \begin{cases} q(x, t; \xi, s), & 0 < t \leq r, \\ \int_D p(x, y, t - r)q(y, r; \xi, s)d\mu(y), & r < t < \infty. \end{cases} \] (5.2)
(We call $u$ the minimal extension of $q$ from $t = r$.) Then we see that $u$ is a nonnegative solution of $(\partial_t + L)u = 0$ in $D \times (0, \infty)$ such that $u(\cdot) \leq q(\cdot; \xi, s)$ on $D \times (0, \infty)$. By Lemma 4.10, $u(\cdot) = Cq(\cdot; \xi, s)$ for some constant $C$. But $u(x, t) = q(x, t; \xi, s)$ for $0 < t \leq r$. Thus $C = 1$, and so $u(\cdot) = q(\cdot; \xi, s)$.

Next, let $\xi \notin \partial_m D$. By Lemma 4.6, there exists a finite Borel measure $\gamma$ on $\partial_M D$ supported by $\partial_m D$ such that
\[ q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s) d\gamma(\eta). \] (5.3)
Thus
\[ \int_D p(x, y, t - r)q(y, r; \xi, s)d\mu(y) = \int_{\partial_m D} \left( \int_D p(x, y, t - r)q(y, r; \eta, s)d\mu(y) \right) d\gamma(\eta) = \int_{\partial_m D} q(x, t; \eta, s) d\gamma(\eta) = q(x, t; \xi, s). \]

This proves (5.1).

Lemma 5.2 Let $\xi, \eta \in \partial_M D$, $0 \leq s, r < T$ and $C > 0$. If
\[ q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \ 0 < t < T, \] (5.4)
then $\xi = \eta$, $s = r$ and $C = 1$. 

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\textbf{Proof} Choose \( u \) such that \( \max(r, s) < u < T \), and construct minimal extensions of both sides of (5.4) from \( t = u \). Then, by (5.1) we have
\[ q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \ 0 < t < \infty. \]
By Lemma 4.4, this implies that \( \xi = \eta, \ s = r \) and \( C = 1 \). \hfill \Box 

Now, let \( \beta \) be a measure on \( Q = D \times (0, T) \) defined by
\[ d\beta(x, t) = a(x) b(t) m(x) d\nu(x) dt. \]
Here \( a(x) \) is a nonnegative continuous function on \( D \) as described in the beginning of Section 4, and \( b(t) \) is a nonnegative continuous function on \( \mathbb{R} \) such that \( b(t) > 0 \) on \((T/2, T)\) and \( b(t) = 0 \) on \( \mathbb{R} \setminus (T/2, T) \). Let \( K_\beta(\cdot; \Xi) \), \( \partial^3_M Q \), \( \partial^3_M D \), and \( Q^{3*} \) be the \( \beta \)-Martin kernel, \( \beta \)-Martin boundary, minimal \( \beta \)-Martin boundary, and \( \beta \)-Martin compactification for \((Q, \partial_t + L)\) with \( Q = D \times (0, T) \), respectively. The following proposition is an analogue of Proposition 4.5, and is shown in the same way.

**Proposition 5.3**

(i) The \( \beta \)-Martin boundary \( \partial^3_M Q \) of \( Q \) for \( \partial_t + L \) is equal to the disjoint union of \( D \times \{0\} \), \( \partial_M D \times [0, T) \) and the one point set \( \{\varpi\} \):
\[ \partial^3_M Q = D \times \{0\} \cup \partial_M D \times [0, T) \cup \{\varpi\}. \tag{5.5} \]
In particular, \( \partial^3_M Q \) does not depend on \( \beta \).

(ii) The \( \beta \)-Martin compactification \( Q^{3*} \) of \( Q \) for \( \partial_t + L \) is homeomorphic to the disjoint union of the topological product \( D^* \times [0, T) \) and the one point set \( \{\varpi\} \), where a fundamental neighborhood system of \( \varpi \) is given by the family \( \{\varpi\} \cup D^* \times (T - \varepsilon, T), \ 0 < \varepsilon < T/2 \). In particular, \( Q^{3*} \) does not depend on \( \beta \).

(iii) The \( \beta \)-Martin kernel \( K_\beta \) is given as follows: For \( (x, t) \in Q \),
\[ K_\beta(x, t; y, 0) = \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}, \quad (y, 0) \in D \times \{0\}, \tag{5.6} \]
\[ K_\beta(x, t; \xi, s) = \frac{q(x, t; \xi, s)}{\beta(q(\cdot; \xi, s))}, \quad (\xi, s) \in \partial_M D \times [0, T), \tag{5.7} \]
and \( K_\beta(x, t; \varpi) = 0 \).
Lemma 5.4 Let \((\xi, s) \in (\partial M \setminus \partial_m D) \times [0, T)\). Then \(q(\cdot; \xi, s)\) is not minimal.

**Proof** Suppose that \(q(\cdot; \xi, s)\) is minimal. Then we obtain from (5.3) that

\[
q(x, \xi, t - s) = Cq(x, \eta, t - s), \quad x \in D, \quad 0 < t < T,
\]

for some \(\eta \in \partial_m D\) and \(C > 0\). By Lemma 5.2, this is a contradiction. \(\square\)

Lemma 5.5 Let \((\xi, s) \in \partial_m D \times [0, T)\). Then \(q(\cdot; \xi, s)\) is minimal.

**Proof** Let \(u\) be a nonnegative solution of \((\partial_t + L)u = 0\) in \(Q\) such that \(u(\cdot) \leq q(\cdot; \xi, s)\) in \(Q\). For \(r \in (s, T)\), let \(u_r\) be the minimal extension of \(u\) from \(t = r\). By Proposition 5.1,

\[
u_r(x, t) \leq q(x, t; \xi, s), \quad x \in D, \quad t > 0.
\]

By Lemma 4.10, there exists a constant \(C_r\) such that \(u_r(x, t) = C_r q(x, t; \xi, s)\) for \(t > 0\). But \(u_r(x, t) = u(x, t)\) for \(0 < t < r\). Thus \(C_r\) is independent of \(r\); and so \(u(\cdot) = C q(\cdot; \xi, s)\) in \(Q\) for some constant \(C\). \(\square\)

**Completion of the proof of Theorem 1.3 in the case \(0 < T < \infty\)**

Put

\[
\mathcal{L}_m^\beta = \partial_m^\beta D \setminus (D \times \{0\} \cup \{\omega\}).
\]

By Proposition 5.3, Lemmas 5.4 and 5.5, we get

\[
\mathcal{L}_m^\beta = \partial_m D \times [0, T).
\]

Thus, Theorem 2.1 of [34] which is an analogue of Theorem 4.1 completes the proof. \(\square\)

### 6 Integral representations; the case \(I = (-\infty, 0)\)

In this section we prove Theorem 1.4. We begin with the following proposition, which can be shown in the same way as in the proof of Theorem 1 of [9] (see also [39]).
Proposition 6.1 Assume [SIU]. Then
\[
\lim_{t \to \infty} e^{\lambda_0 t} \frac{p(x, y, t)}{\phi_0(x) \phi_0(y)} = 1 \quad \text{uniformly in } (x, y) \in K \times D \tag{6.1}
\]
for any compact subset $K$ of $D$.

In the rest of this section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. By Theorem 1.1, we have the following corollary of Proposition 6.1.

Corollary 6.2 Assume [SSP]. Then, for any compact subset $K$ of $D$ and $N > 1$,
\[
\lim_{s \to -\infty} e^{\lambda_0 s} \frac{p(x, y, t - s)}{e^{\lambda_0 t} \phi_0(y)} = e^{-\lambda_0 t} \phi_0(x) \quad \text{uniformly in } (x, y, t) \in K \times D \times (-N, 0).
\]

Lemma 6.3 The solution $e^{-\lambda_0 t} \phi_0(x)$ is minimal.

Proof Suppose that $e^{-\lambda_0 t} \phi_0(x)$ is not minimal. Then, in view of Corollary 6.2, the same argument as in the proof of Theorem 1.3 shows that for any nonnegative solution $u$ of the equation
\[
(\partial_t + L)u = 0 \quad \text{in } Q = D \times (-\infty, 0)
\]
there exists a unique Borel measure $\lambda$ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_m D \times (-\infty, 0)$ such that
\[
u(x, t) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q.
\]
Thus
\[
e^{-\lambda_0 t} \phi_0(x) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q, \tag{6.2}
\]
for such a measure $\lambda$. Now, fix $x$. It follows from Theorems 1.1 and 1.2 that for any $\delta > 0$ there exists a positive constant $C_\delta$ such that
\[
C_\delta^{-1} \leq \frac{q(x, \xi, \tau)}{e^{-\lambda_0 \tau} \phi_0(x)} \leq C_\delta, \quad \tau \geq \delta, \quad \xi \in \partial_M D. \tag{6.3}
\]
By (4.4),
\[ q(x, \xi, \tau) \leq C e^{-\alpha/\tau} \phi_0(x), \quad \xi \in \partial_M D, \ 0 < \tau < 1, \tag{6.4} \]
for some positive constants \( \alpha \) and \( C \). By (6.2) and (6.3),
\[ e^{\lambda_0} \phi_0(x) \geq \int_{\partial_M D \times (-\infty, -2)} C_1^{-1} e^{-\lambda_0(1-s)} d\lambda(\xi, s). \]
Thus
\[ \int_{\partial_M D \times (-\infty, -2)} e^{\lambda_0 s} d\lambda(\xi, s) \leq C_1 \phi_0(x). \tag{6.5} \]

For \( t < -2 \) and \( 0 < \delta < 1 \), we have
\[ \phi_0(x) = \int_{\partial_M D \times \{(t-\delta, t]\}} e^{\lambda_0(t-s)} q(x, \xi, t-s) e^{\lambda_0 s} d\lambda(\xi, s). \tag{6.6} \]

In view of (6.4) and (6.5), we choose \( \delta \) so small that the integral on \( \partial_M D \times (t-\delta, t) \) of the right hand side of (6.6) is smaller than \( \phi_0(x)/3 \). Then, in view of (6.3) and (6.5), we choose \( t < -2 \) with \(|t|\) being so large that the integral on \( \partial_M D \times (-\infty, t-\delta] \) of the right hand side of (6.6) is smaller than \( \phi_0(x)/3 \). This is a contradiction. \( \square \)

**Completion of the proof of Theorem 1.4**

By virtue of Corollary 6.2 and Lemma 6.3, the same argument as in the proof of Theorem 1.3 shows Theorem 1.4. \( \square \)

### 7 Examples

In this section we give two examples in order to illustrate a scope of Theorem 1.3. Throughout this section \( L_0 \) is a uniformly elliptic operator on \( \mathbb{R}^n \) of the form
\[ L_0 u = -\sum_{i,j=1}^{n} \partial_i \left( a_{ij}(x) \partial_j u \right), \]
where \( a(x) = [a_{ij}(x)]_{i,j=1}^{n} \) is a symmetric matrix-valued measurable function on \( \mathbb{R}^n \) satisfying, for some \( \Lambda > 0 \),
\[ \Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad x, \xi \in \mathbb{R}^n. \]
Let $V(x)$ be a measurable function in $L^\infty_{\text{loc}}(\mathbb{R}^n)$, and $L = L_0 + V(x)$ on $D = \mathbb{R}^n$.

**Theorem 7.1** Suppose that there exist a positive constant $c < 1$ and a positive continuous increasing function $\rho$ on $[0, \infty)$ such that

$$c \left[ \rho(|x|) \right]^2 \leq V(x) \leq \left[ \rho(|x|) \right]^2, \quad x \in \mathbb{R}^n, \quad (7.1)$$

$$c \rho \left( r + \frac{c}{\rho(r)} \right) \leq \rho(r), \quad r \geq 0. \quad (7.2)$$

Assume that

$$\int_1^\infty \frac{dr}{\rho(r)} < \infty. \quad (7.3)$$

Then 1 is a small perturbation of $L$ on $\mathbb{R}^n$. Thus Theorem 1.3 holds true.

**Remark.** Compare this theorem with a non-uniqueness theorem of [26].

**Proof.** We first note that (7.2) yields

$$c \rho(r) \leq c \rho \left( r - \frac{c}{\rho(r)} + \frac{c}{c \rho(r)} \right) \leq c \rho \left( r - \frac{c}{\rho(r)} \right), \quad r \geq \frac{c}{\rho(0)},$$

since $\rho$ is increasing. We show the theorem by using the same approach as in the proof of Theorem 5.1 of [31]. Put $b = c^{-2}$ and

$$\ell = \inf \{ j \in \mathbb{Z}; \rho(0) < b^j \}.$$

For $k \geq \ell$, put $r_k = \sup \{ r \geq 0; \rho(r) \leq b^k \}$. By the continuity of $\rho$ and (7.3), $\rho(r_k) = b^k$ and $\lim_{k \to \infty} r_k = \infty$. By (7.2),

$$\rho(r_k + cb^{-k}) \leq c^{-1} \rho(r_k) = b^{1/2} b^k < b^{k+1} = \rho(r_{k+1}).$$

Thus $r_k + cb^{-k} < r_{k+1}$ for $k \geq \ell$. Define a positive continuously differentiable increasing function $\tilde{\rho}$ on $[0, \infty)$ as follows: Put $\tilde{\rho}(r) = b^j$ for $r \leq r_\ell$,

$$\tilde{\rho}(r) = b^{k+1} \quad \text{for} \quad r_k + cb^{-k} \leq r \leq r_{k+1} \quad (k \geq \ell);$$

and $\tilde{\rho}(r) = \rho_k(r)$ for $r_k \leq r \leq r_k + cb^{-k}$ ($k \geq \ell$) by choosing a continuously differentiable function $\rho_k$ on $[r_k, r_k + cb^{-k}]$ such that

$$\rho_k(r_k) = b^k, \quad \rho_k'(r_k) = 0, \quad \rho_k(r_k + cb^{-k}) = b^{k+1}, \quad \rho_k'(r_k + cb^{-k}) = 0,$$
and

\[ 0 \leq \rho_k'(r) \leq B b^{2k}, \quad r_k \leq r \leq r_k + cb^{-k}, \]

for some constant \( B > 0 \) independent of \( k \). Then we have

\[ C^{-1} \leq \frac{\tilde{\rho}(r)}{\rho(r)} \leq C, \quad 0 \leq \tilde{\rho}'(r) \leq C \rho(r)^2, \quad r \geq 0, \tag{7.4} \]

for some positive constant \( C \). Introduce a Riemannian metric \( g = (g_{ij})_{i,j=1}^n \) by \( g_{ij} = \tilde{\rho}(|x|)^2 \delta_{ij} \). Then \( M = \mathbb{R}^n \) with this metric \( g \) becomes a complete Riemannian manifold. Furthermore, by (7.2) and (7.4), \( M \) has the bounded geometry property (1.1) of [4]. The associated gradient \( \nabla \) and divergence \( \text{div} \) are written as

\[ \nabla = \tilde{\rho}(|x|)^{-2} \nabla^0, \quad \text{div} = \tilde{\rho}(|x|)^{-n} \circ \text{div}^0 \circ \tilde{\rho}(|x|)^n, \]

where \( \nabla^0 \) and \( \text{div}^0 \) are the standard gradient and divergence on \( \mathbb{R}^n \). Put

\[ \mathcal{L} = \tilde{\rho}(|x|)^{-2} L, \quad m(x) = \tilde{\rho}(|x|)^{2-n}, \quad A(x) = [a_{ij}(x)]_{i,j=1}^n, \quad \gamma(x) = \tilde{\rho}(|x|)^{-2} V(x). \]

Then

\[ \mathcal{L} u = -\frac{1}{m} \text{div} (m A \nabla u) + \gamma \]

\[ = -\text{div} (A \nabla u) - \langle \frac{1}{m} A \nabla^0 m, \nabla u \rangle^0 + \gamma, \]

where \( \langle \cdot, \cdot \rangle^0 \) is the standard inner product on \( \mathbb{R}^n \). Since the inner product \( \langle \cdot, \cdot \rangle \) associated with the metric \( g \) is written as

\[ \langle X, Y \rangle = \langle \tilde{\rho}^2 X, Y \rangle^0, \]

we have

\[ \mathcal{L} u = -\text{div} (A \nabla u) - \langle \tilde{\rho}^{-2} A \nabla^0 m, \nabla u \rangle^0 + \gamma. \tag{7.5} \]

By (7.4),

\[ |\nabla^0 m(x)| \leq C_3 |n - 2| \tilde{\rho}(|x|) m(x). \]

From this we have

\[ \langle \tilde{\rho}^{-2} A \nabla^0 m, \tilde{\rho}^{-2} A \nabla^0 m \rangle \leq \tilde{\rho}^{-2} \Lambda^2 (C_3 |n - 2| \tilde{\rho})^2 \]

\[ \leq \{ \Lambda (C_3 |n - 2|) \}^2. \]
By (7.1) and (7.4),
\[ cC^{-2} \leq \gamma(x) \leq C^2. \]
Thus the operator \( \mathcal{L} - cC^{-2}/2 \) has the Green function; and \( \mathcal{L} \) belongs to the class \( \mathcal{D}_M(\theta, \infty, \epsilon) \) introduced by Ancona [4], where
\[ \theta = \max \left( \Lambda, \Lambda(C^3 |n - 2|), C^2 \right), \quad \epsilon = cC^{-2}/2. \]

Put
\[ \mathcal{L}_2 = \tilde{\rho}(|x|)^{-2} (L + 1) = \mathcal{L} + \tilde{\rho}(|x|)^{-2}. \]
In order to apply the results of [4], we proceed to estimate \( \tilde{\rho}(|x|)^{-2} \). Let \( d(x) \) be the Riemannian distance \( \text{dist}(0, x) \) from the origin 0 to \( x \), and put
\[ \psi(r) = \int_0^r \tilde{\rho}(s) \, ds. \]
Then we see that \( d(x) = \psi(|x|) \). Denote by \( \psi^{-1} \) the inverse function of \( \psi \), and put
\[ \Phi(s) = \left[ \tilde{\rho} \left( \psi^{-1}(s) \right) \right]^{-2}, \quad s \geq 0. \]
Then
\[ 0 < \tilde{\rho}(|x|)^{-2} = \Phi \left( d(x) \right), \quad x \in M. \]
Furthermore,
\[
\int_0^\infty \Phi(s) \, ds = \int_0^\infty \Phi(\psi(r)) \tilde{\rho}(r) \, dr = \int_0^\infty \frac{dr}{\tilde{\rho}(r)} \leq C \int_0^\infty \frac{dr}{\rho(r)} \, dr < \infty.
\]
Hence, by virtue of Corollary 6.1, Theorems 1 and 2 of [4], \( \tilde{\rho}(|x|)^{-2} \) is a small perturbation of \( \mathcal{L} \) on the manifold \( M \). That is, for any \( \varepsilon > 0 \) there exists a compact subset \( K \) of \( D = M \) such that
\[
\int_{D \setminus K} H(x, z) \tilde{\rho}(|z|)^{-2} H(z, y) \tilde{\rho}(|z|)^n \, dz \leq \varepsilon H(x, y), \quad x, y \in D \setminus K,
\]
where \( dz \) is the Lebesgue measure on \( \mathbb{R}^n \), and \( H(x, z) \) is the Green function of \( \mathcal{L} \) on \( D \) with respect to the measure \( \tilde{\rho}(|z|)^n \, dz \). Denote by \( G(x, z) \) the Green function of \( L \) on \( D \) with respect to the measure \( dz \). Since \( \mathcal{L} = \tilde{\rho}(|x|)^{-2} L \), we have
\[
H(x, z) = G(x, z) \tilde{\rho}(|z|)^{2-n}
\]
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Thus
\[
\int_{D \setminus K} \frac{G(x,z)\tilde{\rho}(|z|)^{(2-n)-2} G(z,y) \tilde{\rho}(|y|)^{2-n} \tilde{\rho}(|z|)^{n} dz}{G(x,y)\tilde{\rho}(|y|)^{2-n}} \leq \varepsilon G(x,y)\tilde{\rho}(|y|)^{2-n}
\]
for any \(x, y \in D \setminus K\). Hence 1 is a small perturbation of \(L\) on \(\mathbb{R}^n\). □

**Remark.** A sufficient condition for (7.2) is the following: \(\rho\) is a positive differentiable function on \([0, \infty)\) satisfying
\[
0 \leq \rho'(r)\rho(r)^{-2} \leq C, \quad r \geq 0,
\]
for some positive constant \(C\). Indeed, from (7.6) we have
\[
X(\delta) \equiv \rho \left( r + \frac{\delta}{\rho(r)} \right) \rho(r)^{-1} \leq \exp[C\delta X(\delta)], \quad r \geq 0, \quad \delta > 0.
\]
Put \(\delta = (2Ce)^{-1}\), and let \(\gamma \in (1, e)\) be the solution of the equation
\[
\exp[X/2e] = X.
\]
Then we get \(1 \leq X(\delta) \leq \gamma\). Thus (7.2) holds with \(c = \min(\delta, 1/\gamma)\).

The condition (7.3) is sharp, since Theorem 6.2 of [17] yields the following uniqueness theorem.

**Theorem 7.2** Suppose that there exists a positive continuous increasing function \(\rho\) on \([0, \infty)\) such that
\[
|V(x)| \leq \rho(|x|)^2, \quad x \in \mathbb{R}^n.
\]
Assume that
\[
\int_1^\infty \frac{dr}{\rho(r)} = \infty.
\]
Then \([\text{UP}]\) holds. Thus Fact AT holds true.

**7.2.** Throughout this subsection we assume that \(D\) is a bounded domain of \(\mathbb{R}^n\). Let \(L\) be an elliptic operator on \(D\) of the form
\[
L = \frac{1}{w(x)} L_0,
\]
where \(w\) is a positive measurable function on \(D\) such that \(w, w^{-1} \in L^\infty_{\text{loc}}(D)\).
Theorem 7.3  Let $D$ be a Lipschitz domain. Suppose that there exists a positive function $\psi$ on $(0, \infty)$ such that $s^2 \psi(s)$ is increasing and
\[ w(x) \leq \psi(\delta_D(x)), \quad x \in D, \tag{7.9} \]
where $\delta_D(x) = \text{dist}(x, \partial D)$. Assume that
\[ \int_0^1 s \psi(s) ds < \infty. \tag{7.10} \]
Then $1$ is a small perturbation of $L$ on $D$. Thus Theorem 1.3 holds true.

Remark. (i) The first assertion of this theorem is implicitly shown in [17] (see Theorem 7.11 and Remark 7.12 (ii) there).
(ii) The Lipschitz regularity of the domain $D$ is assumed only for the Hardy inequality to hold for any function in $C_0^\infty(D)$. Thus, for this theorem to hold, it suffices to assume (for example) that $D$ is uniformly $\Delta$-regular John domain or a simply connected domain of $\mathbb{R}^2$ (see [3], [4]).

Proof of Theorem 7.3  For $x \in D$, put
\[ D_x = \left\{ y \in D; |x - y| < \frac{\delta_D(x)}{2} \right\}. \]
Then
\[ \frac{1}{2} \delta_D(x) \leq \delta_D(y) \leq \frac{3}{2} \delta_D(x), \quad y \in D_x. \]
Thus
\[ \delta_D(x)^2 w(y) \leq 4 \delta_D(y)^2 \psi(\delta_D(y)) \leq 4 \left( \frac{3}{2} \delta_D(x) \right)^2 \psi \left( \frac{3}{2} \delta_D(x) \right). \]
Put $\Psi(s) = 9s^2 \psi((3/2)s)$. Then $\Psi(s)$ is increasing, and satisfies
\[ \delta_D(x)^2 \left( \sup_{y \in D_x} w(y) \right) \leq \Psi(\delta_D(x)), \quad \int_0^1 \frac{\Psi(s)}{s} ds < \infty. \]
Hence, by virtue of Proposition 9.2, Theorem 9.1’ and Corollary 6.1 of [4], $w$ is a small perturbation of $L_0$ on $D$. This implies that $1$ is a small perturbation of $L$ on $D$. \qed

The condition (7.10) is sharp, since Theorem 7.8 and Lemma 7.6 of [17] yield the following uniqueness theorem.
Theorem 7.4 Suppose that there exists a positive continuous increasing function \( \psi \) on \((0, \infty)\) such that

\[
\frac{c\psi(\delta_D(x))}{\psi(\delta_D(x))} \leq w(x) \leq \psi(\delta_D(x)), \quad x \in D
\] (7.11)

for some positive constant \( c \), and

\[
\nu \leq \frac{\psi(\eta s)}{\psi(s)} \leq \nu^{-1}, \quad s > 0, \quad \frac{1}{2} \leq \eta \leq 2,
\] (7.12)

for some positive constant \( \nu \). Assume

\[
\int_0^1 \left[ \psi(s) \left( \inf_{s \leq r \leq 1} r^2 \psi(r) \right) \right]^{\frac{1}{2}} ds = \infty.
\] (7.13)

Then [UP] holds. Thus Fact AT holds true.

References


