The first hitting time of a single point for random walks

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Abstract

This paper concerns the first hitting time τ_0 of the origin for random walks on d-dimensional integer lattice with zero mean and a finite $2+\delta$ absolute moment ($0 \le \delta \le 2$). We derive detailed asymptotic estimates of the probabilities $P_x[\tau_0 = n]$ as $n \to \infty$ that are valid uniformly in x, the position at which the random walks start.

Introduction

Let $S_n^x = x + X_1 + \cdots + X_n$ be a random walk on the d-dimensional square lattice $\mathbf{Z}^d$ starting at x where the increments X_j are i.i.d. random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbf{Z}^d$. Let X be a random variable having the same law as X_1 and $\psi(\theta)$ the characteristic function of X: $\psi(\theta) = Ee^{iX\cdot\theta}$, $\theta \in T^d$, where T^d stands for the d-dimensional torus $\mathbf{R}^d/(2\pi\mathbf{Z})^d \cong [-\pi,\pi)^d$ and E indicates the expectation by P. Throughout the paper we suppose unless explicitly stated otherwise that the distribution of X is strongly aperiodic in the sense of Spitzer [9], i.e., $|\psi(\theta)| < 1$ for $\theta \in T^d \setminus \{0\}$ (which imposes no essential restriction) and that

$$EX = 0 \quad \text{and} \quad E|X|^{2+\delta} < \infty, \tag{0.1}$$

where δ is a constant such that $0 \le \delta \le 2$.

This paper concerns the asymptotic evaluation of the hitting-time distribution

$$f_x(k) = P[\tau_0^x = k] \qquad (k = 1, 2, \ldots)$$

as $k \to \infty$, where $\tau_0^x = \inf\{n > 0 : S_n^x = 0\}$, the first time that S_n^x hits the origin after time 0 ($\inf \emptyset = \infty$), which plays a fundamental role in the theory of random walk and its applications. We derive asymptotic formulae of $f_x(k)$ with certain bounds for error terms valid uniformly in x for each dimension $d = 1, 2, \ldots$: under $\delta = 0$, in particular, the asymptotic form is determined in any parabolic region $x^2 \le ak$. For the computation of $f_x(k)$ we use the Fourier analytic method as in [8].

When the walk is started at the origin there are several results. In Kesten [8] it is proved, among many other things, that if the walk is one-dimensional and strongly aperiodic, and satisfies that for some $1 \le \alpha \le 2$, $|\theta|^{-\alpha}(1-\psi(\theta))$ converges to a positive constant, C say, as $\theta \to 0$, then the asymptotic form of $f_0(k)$ is $C_\alpha k^{-2+1/\alpha}(1+o(1))$ if $1 < \alpha \le 2$ (with $C_\alpha = (\alpha - 1)\sin(\pi/\alpha)C^{1/\alpha}/\Gamma(1/\alpha)$) and $\pi C[k(\lg k)^2]^{-1}(1+o(1))$ if $\alpha = 1$; in the particular case $\alpha = 2$ this implies in our setting that if $d = 1$, $f_0(k) = \sqrt{E|X|^2/2\pi}\, k^{-3/2}(1+o(1))$. For the two dimensional walk satisfying (0.1) with $\delta = 0$ Jain and Pruitt show that $f_0(k) = c[k(\lg k)^2]^{-1}(1+o(1))$ (Section 4 of [7]). (This result actually follows from Kesten's one (for $\alpha = 1$), the latter being based only on an estimate of the characteristic function of f_0 (see Remark at the end of the subsection **4.1**). The proof of [7] is rather probabilistic and quite different from Kesten's proof.). Combined with the ratio limit theorem ([8], [9]) these give the asymptotic form of the tail $P[\tau_0^x > k]$ (for each x) in the cases $d = 1$ and 2 (with $\delta = 0$) but there seems to be no results on estimation of f_x uniformly valid for x except for a few special cases. Recently Y. Hamana ([6]) has proved that for the simple random walk, $f_0(2k) = \pi[k \lg^2 k]^{-1}[1 + O((\lg\lg k)/\lg k)]$ if $d = 2$, $f_0(2k) = c_d k^{-d/2}[1 + O((\lg k)^{-\alpha})]$ if $d \geq 3$ (c_d is a certain positive constant and α may be arbitrary) and applied these results to the study of the range of the pinned walk. (In [5] the error term is improved to $O(k^{-5/8})$ for $d = 3$.)

For $d \geq 2$ we study in a separate paper [13] the random variable $Z_n = \sharp\{S_1^0, S_2^0, \ldots, S_n^0\}$, the number of sites visited by $S^0_\cdot$ until the n-th step. The expectation EZ_n is equal to $e_0 n + \sum_{k=0}^{n-1} F_k$ where $F_k = \sum_{j>k} f_0(j)$ and $e_0 = 1 - F_0$ and readily computed (upto $O(\lg\lg n)$ if $d = 2$ and $O(1)$ if $d \geq 3$) from the estimates of $f_0(k)$ obtained in this paper. In [13] we are interested in the conditional expectation $E[R_n | S_n^0 = x]$, i.e. the expectation under the law of the random walk bridge, of which the asymptotic evaluation, being made by means of Fourier analytic method, depends on several subsidiary results from the present paper. For $d = 1$ the estimate of $f_x(k)$ is effectively used to compute the transition probability of one dimensional walk killed at the origin which is studied in another paper [14].

1 Statements of Results

Let S_n^x, X, $\psi(\theta)$ and $f_x(k)$ be as in Introduction and suppose the condition (0.1) to hold true with some $0 \leq \delta \leq 2$ unless otherwise is stated explicitly. Set $p^n(x) = P[S_n^0 = x]$, $p(x) = p^1(x)$, $a(x) = \sum_{n=0}^{\infty}[p^n(0) - p^n(-x)]$ (cf. [5], [9] for convergence of the series) and

$$a^*(x) = \mathbf{1}_{\{0\}}(x) + a(x) = 1 + \sum_{n=1}^{\infty}[p^n(0) - p^n(-x)],$$

where $\mathbf{1}_{\{0\}}(x) = 1$ or 0 according as $x = 0$ or $x \neq 0$. Denote by Q the covariance matrix of X and by $Q(\theta)$ its quadratic form: $Q(\theta) = E(X \cdot \theta)^2 = \theta \cdot Q\theta$ ($\theta \in \mathbf{R}^d$). For $x \in \mathbf{Z}^d$ put

$$\tilde{x} = Q^{-1/2}x.$$

If $d = 1$, let $\sigma^2 = E|X|^2$ so that $\tilde{x} = x/\sigma$.

The following notations will be used: $\operatorname{sgn} t = t/|t|$ ($t \neq 0$); $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ ($a, b \in \mathbf{R}$); $|\theta|$ denotes the Euclidean length of $\theta \in \mathbf{R}^d$, $\theta^2 = |\theta|^2$; for functions g and G of a variable ξ, $g(\xi) = O(G(\xi))$ means that there exists a constant C such that $|g(\xi)| \leq C|G(\xi)|$ whenever ξ ranges over a specified set; $\lg^+ a = \lg(a \vee 1)$ ($a \geq 0$) and

$$|x|_+ = |x| \vee 1 \ \ (x \in \mathbf{Z}^d).$$

1.1. Here we consider the one dimensional case.

Theorem 1.1 *Let $d = 1$. Then, uniformly in $x \in \mathbf{Z}$, as $k \to \infty$*

$$f_x(k) = \frac{\sigma}{\sqrt{2\pi}} \frac{a^*(x)}{k^{3/2}}\Big(1 + o\Big(\frac{1}{k^{\delta/2}}\Big) + O\Big(\frac{|x|_+^2}{k}\Big)\Big). \tag{1.1}$$

The estimate given in Theorem 1.1 is poor in the case $x^2 > k$, when the second error term (represented by O symbol) is not smaller than the principal part. The following theorem is complementary in this respect. If $\delta \geq 1$ and $d = 1$, define constants β_3 and C^* by

$$\beta_3 = \frac{1}{6}E[X^3] \quad \text{and} \quad C^* = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{\sigma^2}{1 - \psi(\theta)} - \frac{1}{1 - \cos\theta}\Big]d\theta. \tag{1.2}$$

The integral above is understood to be the principal value; the imaginary part vanishes and the real part is absolutely convergent (see (3.5) in Section 3). For convenience sake we put $\beta_3 = C^* = 0$ if $\delta < 1$.

Theorem 1.2 *Let $d = 1$ and δ (in the moment condition (0.1)) be any non-negative number less than three. Then, as $k \wedge |x| \to \infty$*

$$f_x(k) = \frac{|\tilde{x}|e^{-\tilde{x}^2/2k}}{\sqrt{2\pi}\, k^{3/2}}\Big[1 + \frac{\mathrm{P}_1(\tilde{x}^2/k) + (\operatorname{sgn} x)\beta_3 \mathrm{P}_2(\tilde{x}^2/k)}{\sigma|\tilde{x}|} + \frac{\mathrm{P}_3(\tilde{x}^2/k)}{k}\Big] + o\Big(\frac{1}{|x|^{2+\delta}}\Big),$$

where

$$\mathrm{P}_1(z) = C^*(1-z), \quad \mathrm{P}_2(z) = -\sigma^{-2}(2 - 5z + z^2),$$

and P_3 is a certain polynomial of degree 2 or 3 according as $\beta_3 = 0$ or $\beta_3 \neq 0$.

Theorems 1.1 and 1.2 together entails the following result.

Corollary 1.1 *Uniformly in x as $k \to \infty$*

$$f_x(k) = \frac{\sigma a^*(x)}{\sqrt{2\pi} k^{3/2})} e^{-\tilde{x}^2/2k} + o\Big(\frac{|x|_+}{k^{3/2}} \wedge \frac{1}{|x|_+^2}\Big).$$

REMARK 1. The error term in Theorem 1.2 can be replaced by $o(k^{-1-\delta/2})$ if $0 \le \delta < 1$ (but not if $\delta \ge 1$), by $o(|x|^{-j}k^{-1-(\delta-j)/2})$ if $j \le \delta < j+1$ $(j = 1, 2)$ and by $O(|x|^{-1}k^{-3/2})$ if $\delta > 2$.

REMARK 2. (i) Suppose that $1 \le \delta < 2$. According to [14] (Appendix) $\sigma^2 a(x) = |x| + C^* - 2\sigma^{-2}\beta_3 \operatorname{sgn} x + o(|x|^{1-\delta})$ if $d = 1$. (ii) If $p(x) = 0$ either for all $x \le -2$ (LC) or for all $x \ge 2$ (RC), then $C^* = 2\sigma^{-2}|\beta_3|$ (the converse is also true [14]). This is a consequence of the asymptotic form of $a(x)$ just mentioned together with the fact ([9]:P30.3) that $a(x) = |x|/\sigma^2$ for all $x > 0$ in the case (LC) and for all $x < 0$ in the case (RC). If both (LC) and (RC) are the case, we have $C^* = \beta_3 = 0$ and both $\mathrm{P}_1(z)$ and $\mathrm{P}_2(z)$ disappear from the formula of Theorem 1.2 and Corollary 1.1 (otherwise $C^* > 0$, provided that $E|X|^3 < \infty$).

REMARK 3. The random walk on $\mathbf{Z}^d$ with $P[X = \omega] = 1/2d$ for all $\omega \in \{\pm 1, \dots, \pm i\}$ is said simple. The simple random walk is not strongly aperiodic; it being periodic with period $d_\circ = 2$. (The period is the smallest integer r such that $p^{rn}(0) > 0$ for all sufficiently large n). In general, if X is aperiodic but not strongly aperiodic, then the walk has period $d_\circ > 1$ and we obtain the correct formula for $f_x(k)$ by simply multiplying by the factor

$$d_\circ \mathbf{1}(p^k(-x) \neq 0) \tag{1.3}$$

the leading term of the formulae obtained under strong-aperiodicity assumption, where $\mathbf{1}(\mathcal{S})$ is 1 or 0 according as the statement $\mathcal{S}$ is true or false.

For the one dimensional simple random walk we have a simple explicit expression of $f_x(k)$ (cf. [2]: III.4), from which, with the help of Stirling's formula, one deduces that uniformly for $x \neq 0$ with $k + x$ even, as $x^4/k^3 \to 0$

$$f_x(k) = \frac{2|x|}{\sqrt{2\pi}\, k^{3/2}} e^{-x^2/2k}\Big[1 + \frac{1}{k}\mathrm{P}\Big(\frac{x^2}{k}\Big) + O\Big(\frac{x^2}{k^3} + \frac{x^6}{k^5}\Big)\Big],$$

where $\mathrm{P}(z) = -\frac{1}{4} + \frac{1}{2}z - \frac{1}{12}z^2$. (It is also remarked that $f_0(2k) = f_1(2k-1)$.)

1.2. Next consider the case $d = 2$. In order to have a formula more or less parallel to that of Theorem 1.1 we introduce the function

$$W(\lambda) = \int_0^\infty \frac{e^{-\lambda u} du}{[\lg u]^2 + \pi^2} \qquad (\lambda > 0).$$

We also bring in the constants

$$c_1 = \frac{1}{(2\pi)^2}\int_{T^2} \Re\Big(\frac{1}{1-\psi(\theta)} - \frac{2}{Q(\theta)}\Big) d\theta \quad (\le \infty),$$

$$c_2 = \frac{\lg(a/2)}{2\pi|Q|^{1/2}} + \frac{1}{(2\pi)^2}\int_{\{Q>a\}\cap T^2} \frac{2}{Q(\theta)}d\theta,$$

where T^2 is the two dimensional torus (as in Introduction), $|Q| = \det Q$ and a is chosen so small that $\{\theta : Q(\theta) < a\} \subset T^2$, and define

$$c_\circ = 2\pi|Q|^{1/2}\,(c_1 + c_2) \quad \text{if } E[X^2 \lg^+ |X|] < \infty.$$

The negative part of the integrand of the integral defining c_1 is integrable so that c_1 is well defined, whereas c_1 itself (possibly $+\infty$) is finite if and only if $E[X^2 \lg^+ |X|] < \infty$; and c_2 does not depend on the choice of a. If $Q(\theta)$ is of the form $\sigma^2\theta^2$, then, on examining the proof of Proposition 12.3 of [9],

$$\pi\sigma^2 c_2 = \lg(\pi\sqrt{2}) - 2\varepsilon(2)/\pi,$$

where $\varepsilon(2) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^2 = 0.9159\ldots$ (Cataran's constant) (see also REMARK 6 below).

In the case $E[X^2 \lg^+ |X|] = \infty$ we put $c_\circ = 2\pi|Q|^{1/2}\,c_2$ for convenience sake.

Theorem 1.3 *Let $d = 2$. Then, uniformly in $x \in \mathbf{Z}^2$, as $k \to \infty$*

$$f_x(k) = 2\pi|Q|^{1/2}\,a^*(x)\,e^{c_\circ}W(e^{c_\circ}k)\Big[1 + o\Big(k^{-\delta/2}\Big)\Big] + O\Big(\frac{|x|_+^2}{k^2\lg k}\Big).$$

REMARK 4. $\lambda W(\lambda)$ admits the following asymptotic expansion in powers of $1/\lg\lambda$:

$$\lambda W(\lambda) = \frac{1}{(\lg\lambda)^2} - \frac{2\gamma}{(\lg\lambda)^3} - \frac{\frac{1}{2}\pi^2 - 3\gamma^2}{(\lg\lambda)^4} + \cdots$$

valid in the both limits as $\lambda \to \infty$ and as $\lambda \downarrow 0$, where $\gamma = -\int_0^\infty(\lg u)e^{-u}du$ (Euler's constant). The Fourier representation of $W(\lambda)$ takes the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda u}}{\lg(-iu)}du = \begin{cases} W(\lambda) & (\lambda > 0), \\ -e^{\lambda} & (\lambda < 0), \end{cases} \tag{1.4}$$

as is readily derived by Cauchy's theorem (cf. Appendix of [15]).

According to the rule (1.3), a remedy for periodic walks, Theorem 1.3 has the following corollary (note that $c_\circ$ is well defined under irreducibility of the walk).

Corollary 1.2 *For simple random walk on $\mathbf{Z}^2$, it holds that $c_\circ = \lg 8$ (see* REMARK 6 *below) and uniformly for $x = (x_1, x_2)$ with $x_1 + x_2 - k$ even,*

$$f_x(k) = 16\pi a^*(x)W(8k) + O\Big(\frac{|x|_+^2}{k^2\lg k}\Big).$$

The asymptotic form of $f_x(k)$ as $|x|$ becomes large comparably to $\sqrt{k}$ is provided not by Theorem 1.3 but by Theorems 1.4 and 1.5 given below. (But it should be kept in mind that for $|x|$ much larger than $\sqrt{k}$, the trivial upper bound $f_x(k) \le p^k(x)$ may provide fairly nice estimates for such x (see REMARK 10 below).)

Theorem 1.4 *Let $d = 2$. Then as $k \wedge |x| \to \infty$, in general*

$$f_x(k) = \frac{2\lg|\tilde{x}|}{k(\lg k)^2}\,e^{-\tilde{x}^2/2k} + \frac{1}{k\lg k}\cdot o\Big(1 \wedge \frac{\sqrt{k}}{|x|}\Big);$$

and if $\delta > 0$,

$$f_x(k) = \frac{\lg(\frac{1}{2}e^{c_\circ}\tilde{x}^2)}{k(\lg(e^{c_\circ}k))^2}e^{-\tilde{x}^2/2k} + \begin{cases} \dfrac{2\gamma\lg(k/\tilde{x}^2)}{k(\lg k)^3} + O\Big(\dfrac{1}{k(\lg k)^3}\Big) & \text{for } \tilde{x}^2 < k, \\ O\Big(\dfrac{|\lg(\tilde{x}^2/k)|_+^2}{x^2(\lg k)^3}\Big) & \text{for } \tilde{x}^2 \ge k. \end{cases} \tag{1.5}$$

REMARK 5. If $d = 2$, $a(x)$ has the asymptotic form $(\pi|Q|^{1/2})^{-1}(\lg|\tilde{x}|)(1+o(1))$ as $|x| \to \infty$, ensuring the consistency between Theorems 1.3 and 1.4. The second term on the right side of (1.5) in its first case is significant for properly evaluating the probability $\sum_{j\le k} f_x(j) = P[\tau_0^x \le k]$ so as to have its correct asymptotic form that turns out to be $(\lg k)^{-1}\int_0^k u^{-1}e^{-x^2/2u}du(1+o(1))$ as $k \to \infty$ valid uniformly at least for $|x| \le \sqrt{3k\lg\lg k}$ (see [15] for more details).

In the formula (1.5) the estimate does not depend on $\delta > 0$, although the estimate is best possible for $x^2 < k$. This is because the bottle neck for the estimate comes from a term that does not depend on $\psi(\theta)$ except via the constant $c_\circ$ (see Lemma 4.5). The situation becomes different if $f_x(k)$ is compared with the corresponding Brownian object

$$q_r(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K_0(|x|\sqrt{-i2u})}{K_0(r\sqrt{-i2u})}e^{-itu}du, \tag{1.6}$$

where $q_r(t,x)$ is the density for Brownian hitting time of $D(r)$ the disc of radius $r>0$ centered at the origin (see (6.2) below). The result is stated only in the case $\delta = 2$ (see Subsection **4.3** for more details). Define the constant b_3 to be 1 if $\delta \ge 1$ and at least one of third moments of X does not vanish, and to be 0 otherwise.

Theorem 1.5 *Let $d = 2$, $\delta = 2$ and $r_\circ = \sqrt{2}e^{-(\gamma+c_\circ/2)}$. Then as $k\wedge|x| \to \infty$*

$$f_x(k) = q_{r_\circ}(k,\tilde{x}) + O\Big(\frac{1}{k\lg k}\Big(\frac{b_3}{|x|}+\frac{1}{|x|^2}\Big)\Big).$$

REMARK 6. Of the radius $r_\circ$ defined in Theorem 1.5 we have another formula

$$\pi\sqrt{|Q|}\,a(x) = \lg(|\tilde{x}|/r_\circ) + o(1) \qquad \text{as} \quad |x| \to \infty, \tag{1.7}$$

provided $E[|X|^2\lg^+|X|] < \infty$. (This is proved in [9] under $E[|X|^{2+\delta}] < \infty$; see also [10] and [3].) The function $\lg(|x|/r_\circ)$ equals apart from a constant factor the corresponding potential of the process $Q^{1/2}B_t$ killed on the ellipse $Q^{1/2}D(r_\circ)$ where B_t is the standard two-dimensional Brownian motion. For the two dimensional simple random walk we know that $a(x) = 2\pi^{-1}\lg(\sqrt{8}e^\gamma|x|_+) + (8(x_1x_2)^2 - x^4)/6\pi x^6 + O(1/|x|^4)$ (cf. [3]). Comparing this with (1.7) we find that $r_\circ = 1/\sqrt{8}e^\gamma$ and $c_\circ = \lg 8$.

Asymptotic form of the distribution function $P[\tau_0^x \le n] = \sum_{k=1}^n f_x(k)$ is easily computed from Theorem 1.3 if x^2 is much smaller than n (it is sharper than one given in the next theorem if $x^2 < n/(\log n)^2$), while the corresponding computation based on Theorem 1.4 is somewhat complicated. The result becomes as follows.

Theorem 1.6 *Let $d = 2$ and $\delta > 0$ and write $\xi = |x|/\sqrt{n}$. Then as $n\wedge|x| \to \infty$,*

$$P[\tau_0^x \le n] = D(e^{c_\circ}n, \xi^2/2) + \frac{1}{(\lg n)^3}\times\begin{cases} O(|\lg\xi|_+) & \text{for } x^2 < n,\\ O(|\lg\xi|_+^2/\xi^2) & \text{for } x^2 \ge n,\end{cases} \tag{1.8}$$

where

$$D(t,\alpha) = \frac{1}{\lg t}\Big[1 - \frac{\gamma}{\lg t}\Big]\int_\alpha^\infty \frac{e^{-u}}{u}du - \frac{1}{[\lg t]^2}\int_1^\infty \frac{e^{-\alpha u}}{u}\lg\Big(1-\frac{1}{u}\Big)du, \quad \alpha > 0.$$

The following upper bounds are obtained as a corollary of Theorems 1.1 through 1.4.

Corollary 1.3 *For some constant C,*

$$f_x(k) \le \begin{cases} C\Big(\dfrac{|x|_+}{k^{3/2}} \wedge \dfrac{1}{|x|_+^2}\Big) & (d=1), \\ \dfrac{C}{k \lg k}\Big(\dfrac{\lg(|x| \vee 2)}{\lg k} \wedge \dfrac{\sqrt{k}}{|x|_+}\Big) & (d=2). \end{cases}$$

1.3. Suppose $d \ge 3$ and let e_x be the probability that the random walk starting at x escapes the origin after time 0:

$$e_x = P[S_k^x \ne 0 \text{ for all } k \ge 1],$$

and $G(x) = \sum_{n=0}^{\infty} P[S_n^0 = x]$. It holds that

$$1 - e_x = \frac{G(-x)}{G(0)} \ (x \ne 0) \ \text{ and } \ e_0 = \frac{1}{G(0)} \tag{1.9}$$

(cf. [9]) and $G(x) = O(1/|x|_+)$ for $d = 3$ (see [10] for $d \ge 4$).

Theorem 1.7 *Let $d \ge 3$. Then uniformly in $x \in \mathbf{Z}^d$, as $k \to \infty$*

$$\begin{aligned} f_x(k) \;&=\; e_0\Big[\,e_x p^k(0) - p^k(0) + p^k(-x)\,\Big] \qquad\qquad (1.10) \\ &+ \frac{1}{k^{1+d/2}} \times \begin{cases} O\Big(|x|_+ \wedge \dfrac{k}{|x|_+}\Big) + o\Big(k^{(1-\delta)/2}\Big) & \text{if } d = 3, \\ (\lg k) \times O\Big(1 \wedge \dfrac{k^{1/2}}{|x|_+}\Big) & \text{if } d = 4, \\ O\Big(1 \wedge \Big(\dfrac{k^{1/2}}{|x|_+}\Big)^{d-3}\Big) & \text{if } d \ge 5. \end{cases} \end{aligned}$$

REMARK 7. If $x = o(\sqrt{k})$, the leading term in (1.10) may be written as $e_0 e_x p^k(0)[1 + O(x^2/k)]$; in view of (1.9) it is also written as $e_0 p^k(-x)[1 + O(|x|^{-d+2})]$ for $0 < |x| < M\sqrt{k}$ (for each $M > 1$), provided that $|S_1^0|$ satisfies a sufficient moment condition so that $G(x) = O(|x|_+^{-d+2})$. It is noted that $a^*(x) = \mathbf{1}_{\{0\}}(x) + G(0) - G(-x) = e_x G(0)$, so the factor $a^*(x)$ persists to appear in the leading term (for the case $x^2/k = o(1)$) in all dimensions.

For $d \ge 4$ the error estimate in Theorem 1.7 results from rather crude evaluation of the 'error term', whereas for $d = 3$ we need to perform some detailed computation, which actually gives the following theorem.

Theorem 1.8 *Let $d = 3$ and $r_\circ = 1/2\pi|Q|^{1/2}G(0)$. Then uniformly in $x \in \mathbf{Z}^3$, as $k \to \infty$*

$$\begin{aligned} f_x(k) \;&=\; \frac{r_\circ e^{-\tilde{x}^2/2k}}{k\sqrt{2\pi\, k}}\left[e_x + \frac{r_\circ|\tilde{x}|}{k} + \frac{b_3}{\sqrt{k}}\mathrm{P}_3\Big(\frac{-\tilde{x}}{\sqrt{k}}\Big) + \frac{O(1 + x^4/k^2)}{k}\right] \\ &+ \frac{1}{k^{3/2}}\left[b_3 O\Big(\frac{1}{k} \wedge \frac{1}{|x|_+^2}\Big) + o\Big(\frac{1}{k^{\delta/2}} \wedge \frac{k}{|x|_+^{2+\delta}}\Big)\right], \end{aligned} \tag{1.11}$$

where $\mathrm{P}_3(z)$ is the odd polynomial of degree 3 that appears in the Edgeworth expansion of $p^k(x)$; b_3 is defined as in Theorem 1.5 (i.e., b_3 is 0 if all the third moments of X vanish and it is 1 otherwise).

REMARK 8. The asymptotic form of $f_x(k)$ given in Corollary 1.1 is in good accordance with $(2\pi)^{-1/2}|x|t^{-3/2}e^{-x^2/2t}$, the density of corresponding distribution of the standard one dimensional

Brownian motion started at $x \in \mathbf{R}$. In the higher dimensions $d \geq 2$ let $\mathbf{t}_r^{(d)}$ denote the Brownian hitting time of the ball of radius $r > 0$ centered at the origin. The probability $f_x(k)$ is to be compared with the density of the distribution of $\mathbf{t}_r^{(d)}$. For $d = 3$ it holds (see Appendix B) that for $|x| > r$,

$$P_x[\mathbf{t}_r^{(3)} \in dt]/dt = \frac{re^{-(|x|-r)^2/2t}}{t\sqrt{2\pi t}}\Big(1 - \frac{r}{|x|}\Big). \tag{1.12}$$

Taking $r = r_\circ$ exhibits a close similarity between this formula and that of Theorem 1.8: if $\delta = 2$, the latter may be written as

$$f_x(k) = \frac{r_\circ e^{-(|\tilde{x}|-r_\circ)^2/2k}}{k\sqrt{2\pi\, k}}\Big[e_x + \frac{b_3}{\sqrt{k}}\mathrm{P}_3\Big(\frac{-\tilde{x}}{\sqrt{k}}\Big)\Big] + O\Big(\frac{1 \wedge (k/|x|_+^2)^2}{k^{5/2}}\Big);$$

note also that $e_x = 1 - |\tilde{x}|^{-1} r_\circ + o(1/|x|_+)$.

Remark 9. The proofs of Theorems stated above largely depend on the expansion of the characteristic function: $e^{it}\psi(\theta) = 1 + it - \frac{1}{2}Q(\theta) + o(|\theta|^{2+\delta}) + O(t^2 + |t|\theta^2)$ (if $0 \leq \delta < 1$), and can be adapted for finding the asymptotic form of the hitting distribution to the first coordinate axis $x_1 = |x|e_1$ for the random walks that is biased to the direction e_1, the present problem of hitting time being the extreme case where the first coordinate of the walk deterministically increases by one at each step. (Cf. [11] for the unbiased case.)

Remark 10. When x^2/k is large the trivial bound $f_x(k) \leq p^k(x)$ may be useful as noted previously. For example, from the theorems above together with the estimate

$$p^k(x) = \frac{e^{-\tilde{x}^2/2k}}{|Q|^{1/2}(2\pi k)^{d/2}}\Big[1 + O\Big(\sum_{1\leq j\leq\delta}\frac{1 + (x^2/k)^{1+j/2}}{k^{j/2}}\Big)\Big] + o\Big(\frac{1}{k^{d/2}}\Big[\frac{1}{k^{1+\delta/2}} \wedge \frac{k}{|x|^{2+\delta}}\Big]\Big) \tag{1.13}$$

valid under (0.1) with any $\delta \geq 0$ (cf. [12]) one can readily deduce that as $|x| \to \infty$

$$\max_{k\geq 1} f_x(k) \sim \begin{cases} [(3/e)^{3/2}/\sqrt{2\pi}\,]|\tilde{x}|^{-2} & \text{if } \ d = 1, \\ 8e^{-2}(|\tilde{x}|^2 \lg|x|)^{-1} & \text{if } \ d = 2,\ \delta > 0, \\ \big[(d/2\pi e)^{d/2}/|Q|^{1/2}G(0)\big]|\tilde{x}|^{-d} & \text{if } \ d \geq 3,\ \delta = d-2 \end{cases}$$

(the bound (1.13) is applied for $k < \tilde{x}^2/3\lg\lg|x|$ if $d = 2$ and for $k < \tilde{x}^2/6\lg|x|$ if $d \geq 3$).

The rest of the paper is organized as follows. In Section 2 we shall provide some preliminary formulae and lemmas which will be applied throughout Sections 3, 4 and 5, where we shall give proofs of Theorems above for the cases $d = 1$, $d = 2$ and $d \geq 3$, respectively. The final section consists of two appendices: we shall prove a lemma of Fourier analytic nature in the first one and give a simple comment on the formula (1.12) in the second.

2 Preliminary formulae and lemmas

Set

$$\pi_x(t) = \frac{1}{(2\pi)^d}\int_{T^d}\frac{e^{-ix\cdot\theta}}{1 - e^{it}\psi(\theta)}d\theta \qquad (t \neq 0, x \in \mathbf{Z}^d)$$

and

$$\hat{f}_x(t) = \sum_{k=1}^{\infty} f_x(k)e^{ikt}.$$

Since $p^n(x) = (2\pi)^{-d}\int_{T^d}[\psi(\theta)]^n e^{-ix\cdot\theta}d\theta$, we have

$$\pi_x(t) = \lim_{r\uparrow 1}\sum_{n=0}^{\infty} p^n(x)e^{itn}r^n.$$

Substituting from the identity $p^n(-x) = \sum_{k=1}^n f_x(k)p^{n-k}(0)$ and making usual manipulation for the convolution sum, we infer that for $t \neq 0$, $\pi_{-x}(t) = \delta_{0x} + \hat{f}_x(t)\pi_0(t)$, or on solving for $\hat{f}_x(t)$,

$$\hat{f}_x(t) = -\frac{\delta_{0x}}{\pi_0(t)} + \frac{\pi_{-x}(t)}{\pi_0(t)}; \tag{2.1}$$

in particular

$$\hat{f}_0(t) = 1 - \frac{1}{\pi_0(t)}.$$

Note that $\pi_0(t)$ is smooth in $t \in T^1 \setminus \{0\}$ owing to the strong aperiodicity; also that since f_x is a probability supported on the positive integers we have three expressions of $f_x(k)$:

$$f_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}_x(t)e^{-ikt}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_x(t)\left\{\begin{array}{c} \cos kt \\ -i\sin kt \end{array}\right\} dt \tag{2.2}$$

among which we may choose one that is suitable to each occasion.

Bring in the functions $R_1(t,\theta)$ and $R_2(t,\theta)$ by

$$R_1 = \frac{1}{1-e^{it}\psi(\theta)} - \frac{1}{-it+1-\psi(\theta)} \quad \text{and} \quad R_2 = \frac{1}{-it+1-\psi(\theta)} - \frac{1}{-it+\frac{1}{2}Q(\theta)}$$

so that

$$\frac{1}{1-e^{it}\psi(\theta)} = \frac{1}{-it+\frac{1}{2}Q(\theta)} + R_1 + R_2. \tag{2.3}$$

Observe that $(-it+1-\psi) - (1-e^{it}\psi) = \frac{1}{2}t^2 + (-it+\frac{1}{2}t^2)(-it+1-\psi) + O(t^3)$ to have

$$R_1 = \frac{\frac{1}{2}t^2}{[1-e^{it}\psi(\theta)][-it+1-\psi(\theta)]} - \frac{it}{1-e^{it}\psi(\theta)} + O\Big(\frac{t^2}{|t|+\theta^2}\Big); \tag{2.4}$$

also

$$R_2 = \frac{\psi(\theta)-1+\frac{1}{2}Q(\theta)}{(-it+\frac{1}{2}Q(\theta))^2} + \frac{\psi(\theta)-1+\frac{1}{2}Q(\theta)}{-it+\frac{1}{2}Q(\theta)}R_2. \tag{2.5}$$

The next lemma (or its variants), stating elementary results, will be repeatedly used throughout the proofs of Theorems 1.1 to 1.8.

Lemma 2.1 *Let j and k be real constants such that $j>0$ and $2k>-d$ and put $\alpha = j-k-d/2$. Then as $t \to 0\pm$*

$$\int_{T^d} \frac{[\frac{1}{2}Q(\theta)]^k d\theta}{(-it+\frac{1}{2}Q(\theta))^j} = \left\{\begin{array}{lll} C^I_{\pm}|t|^{-\alpha} + O(1) & \textit{if} & \alpha>0 \\ \frac{1}{2}A\lg|t|^{-1} & \textit{if} & \alpha=0 \\ C^{II} + \eta(t) & \textit{if} & \alpha<0 \end{array}\right. \tag{2.6}$$

with

$$A = \frac{2^{d/2}|S^{d-1}|}{|Q|^{1/2}}, \quad C^I_{\pm} = A\int_0^\infty \frac{r^{2k+d-1}dr}{(\mp i + r^2)^j}, \quad C^{II} = \int_{T^d}\Big[\frac{1}{2}Q(\theta)\Big]^{k-j}d\theta$$

($|S^{d-1}|$ *is the hyper-area of $d-1$ dimensional unit sphere if $d \geq 2$ and $|S^0| = 2$) and for $\alpha<0$,*

$$\eta(t) = \left\{\begin{array}{lll} O(|t|^{|\alpha|}) & \textit{if} & \alpha>-1 \\ O(t\lg|t|) & \textit{if} & \alpha=-1 \\ O(t) & \textit{if} & \alpha<-1. \end{array}\right.$$

Proof. Denote by $V(t)$ the integral to be estimated. In the case $\alpha > 0$ the change of variable (scaling θ by $\sqrt{|t|}$) then shows that

$$V(t) = \frac{1}{|t|^\alpha} \int_{T^d/\sqrt{|t|}} \frac{[\frac{1}{2}Q]^k d\theta}{(-i \operatorname{sgn} t + \frac{1}{2}Q)^j} = \frac{1}{|t|^\alpha} \int_{\mathbf{R}^d} \frac{[\frac{1}{2}Q]^k d\theta}{(-i \operatorname{sgn} t + \frac{1}{2}Q)^j} + O(1).$$

In the case $\alpha = 0$ we have only to replace the right most member above by $A \lg |t|^{-1/2}$.

Now consider the case $\alpha < 0$. It follows that $V(0) = C^{II} < \infty$ and $\eta(t) = V(t) - V(0) = \int_0^t V'(u)du$. The required estimates of η are then obtained by the result just proved (with j replaced by $j+1$) since it yields that $V'(t) = O(|t|^{-\alpha-1})$ if $0 < -\alpha < 1$, $V'(t) = O(\lg |t|)$ if $-\alpha = 1$ and $V'(t) = O(1)$ if $-\alpha > 1$. The proof of the lemma is complete.

In the first case of Lemma 2.1 the integral is unbounded and the order of growth as $t \to 0$ is found out by scaling the variable of integration by $\sqrt{|t|}$, while in the third case the integral is bounded and the convergence is trivial. In Lemma 2.1 the results are exhibited only on typical integrals of which there are many variants we shall encounter in the proofs of the main Theorems. In dealing with such variants, we shall refer to Lemma 2.1 even if it is not directly applicable.

When $d = 2$ we shall need to evaluate integrals such that

$$I_j^s(k) := \int_0^a \frac{g(t)}{t^j |\lg t|^p} \sin kt \, dt \quad \text{and} \quad I_0^c(k) := \int_0^a \frac{g(t)}{t^j |\lg t|^p} \cos kt \, dt \qquad (j = 0, 1)$$

where p is a positive constant, a is a constant from the unit open interval $(0,1)$ and g is twice differentiable in $t > 0$. The way of computation involved in the proof of the following lemma will also be employed throughout the paper.

Lemma 2.2 *Let $0 \le \alpha < 1$, a constant and suppose that $g(a) = 0$, $g = O(t^\alpha)$ and $g'(t) = O(t^{\alpha-1})$ as $t \downarrow 0$. Then $I_1^s(k) = O(1/k^\alpha (\lg k)^p)$ (for $k \ge 2$); and if $\alpha > 0$, then $I_1^c = o(1/k^\alpha (\lg k)^p)$ (as $k \to \infty$) and if $p > 1$ and $\alpha = 0$, then $I_1^c = O(1/(\lg k)^{p-1})$. If $g''(t) = O(t^{\alpha-2})$ in addition, then $I_0^s(k) = O(1/k^{1+\alpha} (\lg k)^p)$ and $I_0^c(k) = O(1/k^{1+\alpha}(\lg k)^p)$.*

Proof. Splitting the range of integration and integrating by parts one obtains that

$$|I_0^s(k)| \le \int_0^{1/k} \frac{|g(t)|}{|\lg t|^p} \sin kt \, dt + \frac{|g(1/k)|}{k (\lg k)^p} + \frac{1}{k} \left| \int_{1/k}^a \left(\frac{d}{dt} \frac{g(t)}{(-\lg t)^p} \right) \cos kt \, dt \right|.$$

On using $\sin kt < kt$ the first integral on the right side is evaluated to be $O(1/k^{1+\alpha}(\lg k)^p)$. Integrating by parts once more shows that the second one is $O(1/k^\alpha (\lg k)^p)$, hence the required estimate. Estimations of $I_1^s(k)$ and $I_j^c(k)$ are made in a similar way (for I_1^c take ε/k with a small $\varepsilon > 0$ as the point at which the integral is split).

The following argument or its modifications will also made throughout the paper. For simplicity we consider $f_0(n)$ of the case $d = 2$. Let $w = w(t)$ be a function on $\mathbf{R}$ that is even, smooth, equal to 1 in a neighborhood of the origin and identically zero for $|t| \ge 1$. Employing (2.1) we write the first equality in (2.2) in the form

$$2\pi f_0(k) = -\int_{-\pi}^{\pi} w(t) \frac{e^{-ikt}}{\pi_0(t)} dt - \int_{-\pi}^{\pi} (1 - w(t)) \frac{e^{-ikt}}{\pi_0(t)} dt \qquad (k \ne 0).$$

Since $(1 - w(t))/\pi_0(t)$ is smooth (differentiable arbitrary times) and periodic function, the second integral is a rapidly decreasing function of k. On using (2.3) the principal part of $1/\pi_0(t)$ takes on the form $-C/[\lg(-it) + c_\circ]$ ($C = 2\pi |Q|^{1/2}$) as we shall see in Section 4. Writing $h(t)$ for the remainder term and further decomposing the first integral above we find that

$$\begin{aligned} 2\pi f_0(k) &= \int_{-\infty}^{\infty} \frac{Ce^{-ikt}}{\lg(-it) - c_\circ} dt - \int_{-\infty}^{\infty} (1 - w(t)) \frac{Ce^{-ikt}}{\lg(-it) - c_\circ} dt \\ &\quad - \int_{-\pi}^{\pi} w(t) h(t) e^{-ikt} dt - \int_{-\pi}^{\pi} (1 - w(t)) \frac{e^{-ikt}}{\pi_0(t)} dt. \qquad (2.7) \end{aligned}$$

The first integral represent the principal part. The second one as well as the last one is rapidly decreasing. Thus our task is to evaluate the third integral to reasonable accuracy.

3 The case d=1

3.1. Let $d = 1$. We use the letter l ($|l| \le \pi$) instead of θ for Fourier parameter. Let $R_1(t, l)$ and $R_2(t, l)$ be the functions introduced in the preceding section and define $\lambda(t)$ by

$$\lambda(t) = -\frac{1}{2\pi}\int_{\mathbf{R}\setminus[-\pi,\pi)} \frac{dl}{-it + \frac{1}{2}Q(l)} + \frac{1}{2\pi}\int_{-\pi}^{\pi}(R_1(t,l) + R_2(t,l))dl$$

so that

$$\pi_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dl}{-it + \frac{1}{2}Q(l)} + \lambda(t).$$

From the formula $\int_0^\infty (x^2 + a^2)^{-1}dx = \pi/2a$ valid if $\Re a > 0$ ($\Re$ indicates the real part) we derive for $n = 1, 2, \ldots$

$$\int_{-\infty}^{\infty} \frac{du}{(-it + u^2)^n} = \frac{\pi}{2}\frac{A_n}{(\sqrt{-it}\,)^{2n-1}}, \tag{3.1}$$

where $A_1 = 1, A_2 = \frac{1}{2}$ and in general $A_n = 2^{-n+1}(2n-3)(2n-5)\cdots 1/(n-1)!$. This with $n = 1$ and a simple change of variable of integration give

$$\pi_0(t) = \frac{1}{\sigma\sqrt{-2it}} + \lambda(t) \qquad (t \neq 0). \tag{3.2}$$

Moreover

$$\lambda(t) = \sigma^{-2}C^* + o\Big(|t|^{(\delta-1)/2}\Big) + O\Big(|t|^{1/2}\Big) \quad (t \to 0) \quad \text{ if } \quad 0 \le \delta \le 2, \tag{3.3}$$

as will be verified shortly. Here C^* is a (real) constant which may be arbitrary if $\delta < 1$ (since then it may be absorbed in the first error term); in the case $\delta \ge 1$ it is given by

$$\frac{C^*}{\sigma^2} = -\frac{2}{\sigma^2\pi^2} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(l) - 1 + \frac{1}{2}Q(l)}{\frac{1}{2}(1-\psi(l))Q(l)}\,dl, \tag{3.4}$$

where the last integral (understood to be the principal value at 0: see (3.5) below) appears as the limit as $t \to 0$ of $\int_{-\pi}^{\pi} R_2 dl$ (it comes up as a constant in the third case of Lemma 2.1 if $\delta > 1$). This constant agrees with that defined just before Theorem 1.2 owing to the identity $\int_0^\pi \Big[(1-\cos l)^{-1} - 2l^{-2}\Big]dl = [-\cot y + y^{-1}]_{y=0+}^{\pi/2} = 2/\pi$.

The proof of (3.3) is performed as follows. In the expression defining $\lambda(t)$ the first integral defines a smooth function of t which is of the form $a_0 + a_1 t + \cdots$ with $a_0 = -2/(\sigma\pi)^2$, and we have only to examine the second integral, of which one observes, using (2.4), that the contribution of R_1 to $\lambda(t)$ is $O(\sqrt{|t|})$ (actually of the form $3\pi(2\sqrt{2|Q|})^{-1}(1 - i\,\mathrm{sgn}\,t)\sqrt{|t|} + O(|t|^{3/2})$).

Let $\delta < 1$. Then $\psi - 1 + \frac{1}{2}Q = o(|l|^{2+\delta})$ and an application of Lemma 2.1 (the first case) deduces that $\int_{-\pi}^{\pi} R_2 dl = o(|t|^{(\delta-1)/2})$, which implies (3.3).

In the case $\delta = 1$, we need to verify the convergence of $\int_{-\pi}^{\pi} R_2 dl$ to $\int_{-\pi}^{\pi} R_2(0, l)dl$ as $t \to 0$. To this end we have only to deal with the first term of the expression (2.5) of R_2, the second one being bounded. By symmetry $E[\sin Xl]$ involved in ψ then can be deleted from the integrand. Now the dominated convergence theorem concludes that the integral thus modified converges to the constant

$$\int_{-\pi}^{\pi} \frac{\psi(l) - 1 + \frac{1}{2}Q(l)}{(\frac{1}{2}Q(l))^2}dl = E\Big[|X|^3\int_{-|X|\pi}^{|X|\pi} \frac{\cos u - 1 + \frac{1}{2}u^2}{(\frac{1}{2}Q(u))^2}du\Big] < \infty. \tag{3.5}$$

Combined with the bounds $|E[\sin Xl]| = O(l^3)$ and $\psi(l) - \psi(-l) = O(l^3)$ this in particular verifies the existence of the integral in (3.4).

If $1 < \delta < 2$, then $\psi - 1 + \frac{1}{2}Q = -i\beta_3 l^3 + o(|l|^{2+\delta})$, and (3.3) follows from (2.5) and Lemma 2.1 (the third case). It is readily seen that $\lambda(t) = \sigma^{-2}C^* + O(\sqrt{|t|})$ if $\delta = 2$. Thus (3.3) has been proved.

We write down the estimate (3.3) in the following form

$$\frac{1}{\pi_0(t)} = \sigma\sqrt{-2it} + i2C^* t + o\Big(|t|^{(1+\delta)/2}\Big) + O(|t|^{3/2}) \qquad (t \to 0). \tag{3.6}$$

3.2. Let $0 \le \delta < 2$. By applying Lemma 2.1 again (the first case only) we obtain that as $t \to 0$,

$$(d/dt)^j \lambda(t) = o(|t|^{(\delta-1)/2}|t|^{-j}) \quad \text{for} \quad j = 1, 2, 3; \tag{3.7}$$

especially

$$\pi_0'(t) = \frac{i}{\sigma(\sqrt{-2it}\,)^3}\Big(1 + o(|t|^{\delta/2})\Big), \tag{3.8}$$

in view of (3.2). By (2.1) $\hat{f}_0(t) = 1 - 1/\pi_0(t)$ and if $\eta(t)$ is defined by

$$\hat{f}_0'(t) = \frac{\pi_0'(t)}{[\pi_0(t)]^2} = \frac{i\sigma}{\sqrt{-2it}} + i2C^* + \eta(t),$$

we infer from (3.6), (3.7) and (3.8) that $(d/dt)^j \eta(t) = o(|t|^{(\delta-1)/2})|t|^{-j}$ for $j = 0, 1, 2$.

Let $w(t)$ be a smooth cutoff function introduced at the end of Section 2. Then by Fourier inversion and integration by parts

$$\begin{aligned} f_0(k) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_0(t)\cos kt\, dt \\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi} \hat{f}_0'(t)\frac{\sin kt}{k}dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\hat{f}_0'(t)\frac{\sin kt}{k}dt + \varepsilon(k) \\ &= -\frac{\sigma}{\pi k}\int_{-\pi}^{\pi}\frac{iw(t)}{\sqrt{-2it}}\sin kt\, dt - \frac{1}{\pi k}\int_{-\pi}^{\pi} w(t)\eta(t)\sin kt\, dt + \varepsilon(k) \\ &= K_1 + K_2 + \varepsilon(k) \qquad \text{(say)}. \end{aligned}$$

Here, as well as in what follows, $\varepsilon(k)$ denotes any function approaching zero faster than k^{-N} as $k \to \infty$ for all N that needs not be the same at each occurrence. On using $\int_0^\infty t^{-1/2}\sin t\, dt = \sqrt{\pi/2}$ and $1/\sqrt{-2it} = (1 + i\,\mathrm{sgn}\,t)/2\sqrt{|t|}$,

$$K_1 = \frac{\sigma}{\pi k}\int_{-\pi}^{\pi}\frac{-i\sin kt}{\sqrt{-2it}}w(t)dt = \frac{\sigma}{\sqrt{2\pi}}\cdot\frac{1}{k\sqrt{k}} + \varepsilon(k). \tag{3.9}$$

For $0 \le \delta < 1$, the estimation of K_2 is carried out as in the proof of Lemma 2.2 and it is found that $K_2 = o((1/\sqrt{k})^{3+\delta})$. In the case $1 \le \delta < 2$ we perform integration by parts once more and use $\eta''(t) = o(|t|^{(-5+\delta)/2})$ to have the desired estimate.

If $\delta = 2$, the error term is easily evaluated to be $O(k^{-5/2})$. Thus we have shown

Proposition 3.1 *For $0 \le \delta \le 2$, as $k \to \infty$*

$$f_0(k) = \frac{\sigma}{\sqrt{2\pi}}\cdot\frac{1}{k\sqrt{k}}\Big(1 + o\Big(\frac{1}{|k|^{\delta/2}}\Big) + O\Big(\frac{1}{k}\Big)\Big).$$

3.3. In this subsection Theorem 1.1 is proved when $0 < \delta < 1$. (The case $1 \le \delta \le 2$ will be treated in the next subsection.) Recalling $\hat{f}_x(t) = \pi_{-x}(t)/\pi_0(t)$ we introduce

$$\mathrm{e}_x(t) := \pi_{-x}(t) - \pi_0(t) + a(x)$$

so that

$$\hat{f}_x(t) = \frac{\mathrm{e}_x(t)}{\pi_0(t)} + 1 - \frac{a^*(x)}{\pi_0(t)}.$$

The integral representation $a(x) = (2\pi)^{-1}\int_{\pi}^{\pi}(1-\psi)^{-1}(1-e^{ixl})dl$ (cf. [4], Appendix of [14]) yields

$$\mathrm{e}_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{1-e^{it}\psi(l)} - \frac{1}{1-\psi(l)}\Big)(e^{ixl}-1)\,dl.$$

We make the decomposition $(2\pi)\mathrm{e}_x(t) = \mathrm{c}_x(t) + i\,\mathrm{s}_x(t)$, where

$$\mathrm{c}_x(t) = \int_{-\pi}^{\pi}\Big(\frac{1}{1-e^{it}\psi(l)} - \frac{1}{1-\psi(l)}\Big)(\cos xl - 1)dl$$

$$\mathrm{s}_x(t) = \int_{-\pi}^{\pi}\Big(\frac{1}{1-e^{it}\psi(l)} - \frac{1}{1-\psi(l)}\Big)\sin xl\,dl.$$

Lemma 3.1 *There exists a constant C such that*

$$|\mathrm{c}_x(t)| \le Cx^2\sqrt{|t|}, \quad |\mathrm{c}'_x(t)| \le Cx^2/\sqrt{|t|}, \quad |\mathrm{c}''_x(t)| \le Cx^2/|t|^{3/2}.$$

Proof. Writing

$$\mathrm{c}_x(t) = \int_{-\pi}^{\pi}\frac{(e^{it}-1)\psi(l)}{1-e^{it}\psi(l)}\cdot\frac{\cos xl - 1}{(1-\psi(l))}dl \tag{3.10}$$

one applies Lemma 2.1 to see that $|\mathrm{c}_x(t)| = Cx^2|t|\int_{-\pi}^{\pi}(|t|+l^2)^{-1}\,dl \le C'x^2\sqrt{|t|}$, hence the first bound of the lemma. Differentiating the defining expression of c_x we see

$$\mathrm{c}'_x(t) = ie^{it}\int_{-\pi}^{\pi}\frac{\psi(l)(\cos xl - 1)}{[1-e^{it}\psi(l)]^2}\,dl,$$

which together with $|1-e^{it}\psi| \ge C^{-1}(|t|+l^2)$ yields the second one in a similar way to the above. The third one is derived in a similar way.

Lemma 3.2 *Let $0 \le \delta < 1$. Uniformly in $x \ne 0$, as $t \to 0$*

$$|\mathrm{s}_x(t)|/|x| = o(|t|^{\delta/2}), \quad |\mathrm{s}'_x(t)|/|x| = o(|t|^{\delta/2}/|t|), \quad |\mathrm{s}''_x(t)|/|x| = o(|t|^{\delta/2}/|t|^2).$$

Proof. Write

$$\begin{aligned} I_x(t) := \frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin xl\,dl}{1-e^{it}\psi(l)} &= \frac{1}{2}\int_{-\pi}^{\pi}\Big[\frac{1}{1-e^{it}\psi(l)} - \frac{1}{1-e^{it}\psi(-l)}\Big]\frac{\sin xl}{x}\,dl \\ &= ie^{it}\int_{-\pi}^{\pi}\frac{lE[\sin Xl]}{(1-e^{it}\psi(l))(1-e^{it}\psi(-l))}\cdot\frac{\sin xl}{xl}\,dl. \end{aligned} \tag{3.11}$$

Putting $\Lambda(l) = E[\sin Xl]/l^3$ so that $lE[\sin Xl] = l^4\Lambda(l)$ one observes that $\Lambda(l)$ is integrable and then applies the dominated convergence theorem to see that $\mathrm{s}_x(t)/x = I_x(t) - I_x(0) \to 0$ as $t \to 0$ uniformly in x. We also have $\Lambda(l) = o(|l|^{\delta-1})$ and $|1-e^{it}\psi(l)| \ge C^{-1}(|t|+l^2)$ and, employing Lemma 2.1 (the first case), obtain $\mathrm{s}_x(t)/x = o(|t|^{\delta/2})$. Thus the first estimate of the lemma has been verified. The other two are readily seen by differentiating the last expression of $I_x(t)$ and applying the estimates just obtained together with the inequality $|1-e^{it}\psi(l)| \ge C^{-1}|t|$.

In the case $0 \le \delta < 1$ Theorem 1.1 is proved by the same argument as made for Proposition 3.1 with the help of it as well as of Lemmas 3.1 and 3.2. The details are omitted.

3.4. We prove Theorem 1.1 in the case $1 \le \delta \le 2$. We need to make more detailed estimation of c_x and s_x than we have made above. We continue to suppose $x \ne 0$.

Lemma 3.3 *Let $\delta \geq 1$. Then uniformly in x, as $k \to \infty$*

$$\int_{-\pi}^{\pi} \frac{c_x(t)}{\pi_0(t)} e^{-ikt} dt = O\Big(\frac{x^3}{k^2\sqrt{k}}\Big).$$

Proof. Rewrite the expression of c_x in (3.10) in the form

$$c_x(t)/x^2 = -i2\pi t \pi_0(t)/\sigma^2 + r_1(t) + r_2(t),$$

where

$$r_1 = \int_{-\pi}^{\pi} \frac{(e^{it}-1)\psi(l) - it}{1 - e^{it}\psi(l)} \cdot \frac{\cos xl - 1}{x^2(1-\psi(l))} dl$$

and

$$r_2 = \int_{-\pi}^{\pi} \frac{it}{1 - e^{it}\psi(l)} \Big[\frac{\cos xl - 1}{x^2(1-\psi(l))} + \frac{1}{\sigma^2}\Big] dl.$$

It is immediate that $|r_1| \leq (|t|/|x|) \int_0^\infty (1-\cos u)u^{-2} du$ and similar bounds for derivatives hold, yielding

$$r_1 = O(t/x), \quad r_1' = O(1/x), \quad r_1'' = O(1/xt), \quad r_1''' = O(1/xt^2).$$

Here the condition $\delta \geq 1$ is needed for (and only for) the last estimate. Also observe that

$$\begin{aligned} r_2 &= it\int_{-\pi}^{\pi} \frac{\cos xl - 1 + \frac{1}{2}(xl)^2}{x^2(1-e^{it}\psi)(1-\psi)} dl - it\int_{-\pi}^{\pi} \frac{\psi - 1 + \frac{1}{2}Q}{(1-e^{it}\psi)(1-\psi)\sigma^2} dl \\ &= O(xt) + O(t), \end{aligned} \tag{3.12}$$

where the first and second terms in the last line represent the corresponding ones in the preceding line and we have used the integrability $\int_0^\infty |\cos l - 1 + \frac{1}{2}l^2| l^{-4} dl < \infty$ for the first integral and (3.5) for the second (in the case $\delta = 1$), and similarly that

$$|r_2'| \leq C|x|, \quad |r_2''| \leq C|x/t|, \qquad |r_2'''| \leq C|x/t^2|.$$

Now it is easy to see

$$\begin{aligned} \int \frac{c_x}{\pi_0} e^{-ikt} dt = \frac{1}{ik}\int \Big(\frac{c_x}{\pi_0}\Big)' e^{-ikt} dt &= \frac{x^2}{ik}\int \Big(\frac{r_1 + r_2}{\pi_0}\Big)' e^{-ikt} dt \\ &= -\frac{x^2}{k^2}\int \Big(\frac{r_1 + r_2}{\pi_0}\Big)'' e^{-ikt} dt. \end{aligned}$$

The integrand of the last integral is $O(x/\sqrt{|t|})$ and in the same way as in the proof of Lemma 2.2 the integral itself is shown to be $O(x/\sqrt{k})$. Thus we conclude the assertion of the lemma.

Lemma 3.4 *Let $1 \leq \delta \leq 2$. Then uniformly in x, as $k \to \infty$*

$$\int_{-\pi}^{\pi} \frac{\mathrm{s}_x(t)}{\pi_0(t)} e^{-ikt} dt = o\Big(\frac{x}{k^{(3+\delta)/2}}\Big) + O\Big(\frac{x^2}{k^2\sqrt{k}}\Big).$$

Proof. Recalling (3.11) and $\mathrm{s}_x/x = I_x(t) - I_x(0)$ we write s_x/x in the form

$$\frac{\mathrm{s}_x(t)}{x} = ie^{it}\int_{-\pi}^{\pi} \frac{F(t,l)l^4\Lambda(l)}{(1-e^{it}\psi(l))(1-e^{it}\psi(-l))(1-\psi(l))(1-\psi(-l))} \cdot \frac{\sin xl}{xl} dl, \tag{3.13}$$

where $\Lambda(l) = E[\sin Xl]/l$ and

$$F(t,l) = (1-\psi(l))(1-\psi(-l)) - (1-e^{it}\psi(l))(1-e^{it}\psi(-l)).$$

One has the expansion

$$F = t^2 + it[2-\psi(l)-\psi(-l)] + O(t^2l^2 + |t||l|^4) = t^2 + itQ(l) + O(t^2l^2 + |t||l|^{2+\delta}) \tag{3.14}$$

and those of the derivatives $\partial_t^j F$; the contribution of the error terms of F in (3.14) to the integral above is readily seen to be $O(|t|^{(1+\delta)/2})$. In the denominator of the integrand in (3.13) the first two factors $1-e^{it}\psi(\pm l)$ and the remaining two $1-\psi(\pm l)$ may then be replaced by $-it+Q(l)/2$ and $Q(l)/2$, respectively, the error caused by the replacement being negligible. We wish further replace $\sin xl/xl$ by 1. The error by this replacement is shown to be $O(xt)$ in the same way as r_2 is estimated in the preceding proof but this time using $\int_0^\infty |\sin u - u|u^{-3}du < \infty$. Finally note that $\Lambda(l) = \beta_3 + o(|l|^{\delta-1})$. These considerations then lead to

$$\frac{\mathrm{s}_x(t)}{x} = \frac{i4\beta_3}{\sigma^4}\int_{-\pi}^{\pi} \frac{t^2+itQ(l)}{(-it+Q(l)/2)^2}\,dl + O(|xt|) + o(|t|^{\delta/2}), \tag{3.15}$$

provided that $1 \le \delta < 2$. Here the last error term comes from the replacement of $\Lambda(l)$ by β_3. If $\delta = 2$, it may be replaced by $O(t)$ (hence superfluous), as assured by the inequality $\int E|\sin Xl - Xl + \frac{1}{6}(Xl)^3||l|^{-5}dl \le CE|X|^4$.

Differentiating the last expression of I_x in (3.11), we derive in the same way as above that for $t \ne 0$,

$$\frac{\mathrm{s}'_x(t)}{x} = -2\beta_3\int_{-\pi}^{\pi} \frac{l^4}{(-it+Q(l)/2)^3}\,dl + O(|x|) + o(|t|^{(\delta-2)/2}). \tag{3.16}$$

From the formula (3.1) (with $n = 1,2,3$) it follows that for any complex numbers α, β,

$$\int_{-\infty}^{\infty} \frac{\alpha t^2 + \beta t u^2}{(-it+u^2)^2}du = A_{\alpha,\beta}\sqrt{-i2t} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\alpha t u^2 + \beta u^4}{(-it+u^2)^3}du = B_{\alpha,\beta}\frac{1}{\sqrt{-i2t}}$$

where $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ are certain complex numbers whose values are not important for our present purpose. A simple computation then deduces from (3.15), (3.16), (3.2) and (3.8) that

$$\left(\frac{\mathrm{s}_x}{x\pi_0}\right)'(t) = \alpha_\circ + r(t)$$

with some complex number $\alpha_\circ$ and the remainder term $r(t) = O\big(x\sqrt{|t|}\big) + o(|t|^{(\delta-1)/2})$, so that

$$\int_{-\pi}^{\pi} \frac{\mathrm{s}_x(t)}{\pi_0(t)}e^{-ikt}dt = \frac{x}{ik}\int_{-\pi}^{\pi} r(t)e^{-ikt}dt = O\Big(\frac{x^2}{k^{5/2}}\Big) + o\Big(\frac{x}{k^{(\delta+3)/2}}\Big),$$

where the estimation of the last integral is carried out by estimating the derivatives r' and r'' as that of the corresponding one in the preceding proof. The proof of the lemma is complete.

The asymptotic formula of (i) of Theorem 1.1 is immediate from Lemmas 3.3 and 3.4 and Proposition 3.1.

3.5. Here we give a proof of Theorem 1.2. Let $w = w(l)$ be a cutoff function as introduced in the subsection **3.2** (but of the argument l instead of t) and define

$$\lambda_x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-w)e^{ixl}dl}{1-e^{it}\psi(l)} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1-w)e^{ixl}dl}{-it+\frac{1}{2}Q(l)} + \frac{1}{2\pi}\int_{-\pi}^{\pi}(R_1+R_2)we^{ixl}dl, \tag{3.17}$$

so that

$$\pi_x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos xl\,dl}{-it+\frac{1}{2}Q(l)} + \lambda_x(t). \tag{3.18}$$

The last integral can be explicitly computed (see (3.21)). The first and second terms on the right side of (3.17) make only a negligible contribution to $f_x(k)$ (cf. Lemma 6.1 in Appendix

A for the first one). The integral $\int_{-\pi}^{\pi} R_1 w e^{ixl} dl$ is much easier to evaluate than $\int_{-\pi}^{\pi} R_2 w e^{ixl} dl$, and we concentrate on the latter in what follows (except in the final several lines of this proof). Decomposing

$$R_2 = \frac{\psi - 1 + \frac{1}{2}Q}{(-it + \frac{1}{2}Q(l))^2} + \frac{(\psi - 1 + \frac{1}{2}Q)^2}{(-it + \frac{1}{2}Q(l))^3} + \left[\frac{\psi - 1 + \frac{1}{2}Q}{-it + \frac{1}{2}Q(l)}\right]^2 R_2,$$

we write

$$\int_{-\pi}^{\pi} R_2 w e^{ixl} dl \;=\; \int_{-\pi}^{\pi} \frac{-4i\beta_3 l^3 w e^{ixl}\, dl}{(-2it + l^2)^2\,\sigma^4} + \int_{-\pi}^{\pi} \frac{4\sigma^2 b_4 l^4(-2it + l^2) - 8\beta_3^2 l^6}{(-2it + l^2)^3} w e^{ixl}\, \frac{dl}{\sigma^7} + r_x(t), \tag{3.19}$$

where $b_4 = \frac{1}{24}E[X^4]$ if $\delta = 2$, $b_4 = 0$ otherwise. The evaluation of the contribution to $f_x(k)$ of the two integrals on the right side will be made by rather explicit computations as given shortly.

It is easy to see that $r_x(t) = o(|t|^{(\delta-1)/2})$ if $\delta < 1$ and for $j = 1$ and 2,

$$r_x(t) = |x|_+^{-j} \times o(|t|^{(\delta-1-j)/2}) \qquad (j \le \delta < j+1)$$

(perform integration by parts j times) and $|r_x(t)|, |r_x'(t)| \le C|x|_+^{-1}$ if $\delta > 2$, which lead to the error estimates mentioned in Remark 1.

As for the error estimate of Theorem 1.2 remember that we have the three expressions of $f_x(k)$ given in (2.2). Here we use the last one of them. The estimate that we need to verify may accordingly be written as

$$\int_{-\pi}^{\pi} \frac{r_x(t)}{\pi_0(t)} \sin kt\, dt = o(|x|^{-2-\delta}). \tag{3.20}$$

(Note that the results obtained in the preceding subsection are valid for any choice of the three.) For the proof of (3.20) we are to employ Lemma 6.1 in Appendix A. Let m be the non-negative integer such that $m - 1 < \delta \le m$ and perform integration by parts $m + 1$ times for the integral that defines $r_x(t)$. An application of Lemma 6.1 with $\nu = \delta - m + 1$ then will lead to the desired estimate. Eg. if $m = 1$, a typical term that arises after the integrating by parts is a constant multiple of

$$I(x) := \frac{1}{x^2}\int_{-\pi}^{\pi} \sqrt{-i2t}\, \sin kt\, dt \int_{-\pi}^{\pi} \frac{v(l)}{(-it + \frac{1}{2}Q(l))^2} e^{ixl} dl \quad \text{with} \quad v(l) = \psi''(l) + \sigma^2.$$

One sees that $v(l) - v(l') = o(|l - l'|^\delta)$ and applies Lemma 6.1 with $\nu = \delta$ ($\alpha = 1/2, \beta = 0, j = 2$) to find the bound $o(|x|^{-2-\delta})$ for $I(x)$ as required.

The rest of the proof of Theorem 1.2 consists of elementary calculus based on the following formulae: for $\alpha > 0$ and $y > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos yl}{-i2\alpha t + l^2} dl = \frac{\pi}{\sqrt{-i2\alpha t}} e^{-y\sqrt{-i2\alpha t}} \tag{3.21}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{-i2\alpha t}} e^{-y\sqrt{-i2\alpha t}} e^{-ikt} dt = \begin{cases} \dfrac{\sqrt{2\pi}}{\sqrt{\alpha k}} e^{-\alpha y^2/2k} & (k > 0) \\ 0 & (k < 0). \end{cases} \tag{3.22}$$

The latter formula is the Laplace inversion of the well known formula for the resolvent kernel of the one-dimensional Brownian motion ([1], p.146 (27)).

Now, applying (3.21) and (3.22) successively, we find that for $\alpha > 0$, $k > 0$ and $y \in \mathbf{R} \setminus \{0\}$,

$$
\begin{aligned}
T = T_k(y,\alpha) := \int_{-\infty}^{\infty} e^{-ikt}dt \int_{-\infty}^{\infty} \frac{e^{iyl}dl}{-i2\alpha t + l^2} &= \int_{-\infty}^{\infty} \frac{\pi}{\sqrt{-i2\alpha t}} e^{-|y|\sqrt{-i2\alpha t}} e^{-ikt}dt \\
&= \frac{\pi\sqrt{2\pi}}{\sqrt{\alpha k}} e^{-\alpha y^2/2k}.
\end{aligned}
$$

Hence

$$
\int_{-\infty}^{\infty} \sqrt{-i2\alpha t}\, e^{-ikt}dt \int_{-\infty}^{\infty} \frac{e^{iyl}dl}{-i2\alpha t + l^2} = -\mathrm{sgn}\, y\, \partial_y T = \frac{\pi\sqrt{2\alpha\pi}\,|y|}{k\sqrt{k}} e^{-\alpha y^2/2k}, \tag{3.23}
$$

$$
\int_{-\infty}^{\infty} ite^{-ikt}dt \int_{-\infty}^{\infty} \frac{e^{iyl}dl}{-i2\alpha t + l^2} = -\frac{1}{2\alpha}\partial_y^2 T = \Big(1 - \frac{\alpha y^2}{k}\Big)\frac{\pi\sqrt{2\pi}}{2k\sqrt{\alpha k}} e^{-\alpha y^2/2k},
$$

of which the first and second formulae give the principal term and the polynomial P_1, respectively, in the expansion of $f_x(k)$ in Theorem 1.2, in view of (3.6) as well as (; note that the integral $\int_{\mathbf{R}\setminus[-\pi,\pi)} \rho e^{-ikt}dt \int_{-\infty}^{\infty}(-i2t + l^2)^{-1}e^{iyl}dl$ is also a ngl. term.

Keeping (3.21) in mind we derive from (3.23) first

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sqrt{-i2t}\, e^{-ikt}dt \int_{-\infty}^{\infty} \frac{ile^{iyl}dl}{-i2\alpha t + l^2} &= -\alpha^{-1/2}\partial_y\Big(\mathrm{sgn}\, y\, \partial_y T\Big) \\
&= \mathrm{sgn}\, y \left(1 - \frac{y^2}{k}\alpha\right)\frac{\pi\sqrt{2\pi}}{k\sqrt{k}} e^{-\alpha y^2/2k},
\end{aligned}
$$

and then

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sqrt{-i2t}\, e^{-ikt}dt \int_{-\infty}^{\infty} \frac{il^3e^{iyl}dl}{(-i2\alpha t + l^2)^2} &= \mathrm{sgn}\, y \Big[-\alpha\partial_\alpha(\alpha^{-1/2}\partial_y^2 T) - \alpha^{-1/2}\partial_y^2 T\Big] \\
&= \mathrm{sgn}\, y \left(1 - \frac{5y^2}{2k}\alpha + \frac{y^4}{2k^2}\alpha^2\right)\frac{\pi\sqrt{2\pi}}{k\sqrt{k}} e^{-\alpha y^2/2k},
\end{aligned}
$$

and you see that the last formula evaluates the contribution of the first integral on the right side of (3.19), giving the polynomial P_2 in the expansion of Theorem 1.2 (note that we can replace e^{-ikt} by $-2i\sin kt$ in all the formulae given above in view of (3.22)). Those of the second one in (3.19) and of R_1 are evaluated in a similar way to yield the term involving P_3 (as for the former one insert $w(l)$ in the left side integral in the last formula, which gives rise to only a negligible term on its right side, and then differentiations by y or α give relevant formulae). This completes the proof of Theorem 1.2.

4 The case $d = 2$

This section, concerning solely to the case $d = 2$, consists of three subsections. In the first subsection $\pi_0(t)$ is evaluated and the asymptotic estimate of $f_0(k)$ is given. The second one is devoted to the evaluation of $\pi_x(t)$. The proofs of Theorems 1.4, 1.5 and 1.6 are completed in the third one.

4.1. Since $(-it + \frac{1}{2}Q(\theta))^{-1}$ is not integrable on $\{\theta \in \mathbf{R}^2\}$ we proceed somewhat differently from the case $d = 1$.

Suppose $E[|X|^2 \lg^+ |X|] < \infty$. From (2.3) we deduce as in the case $d = 1$ that

$$
\pi_0(t) = \frac{1}{(2\pi)^2}\int_{T^2} \frac{1}{-it + \frac{1}{2}Q(\theta)} d\theta + c_1 + \lambda(t), \tag{4.1}
$$

where

$$
c_1 = \frac{1}{(2\pi)^2}\int_{T^2} R_2(0,\theta)d\theta = \frac{1}{(2\pi(\theta))^2}\int_{T^2} \frac{\psi(\theta) - 1 + \frac{1}{2}Q(\theta)}{(1-\psi(\theta))\frac{1}{2}Q(\theta)} d\theta > -\infty
$$

and $\lambda(t) = (2\pi)^{-2} \int_{T^2} [(R_1 + R_2)(t,\theta) - R_2(0,\theta)]d\theta$. We write

$$\lambda(t) = \frac{1}{(2\pi)^2} \int_{T^2} \bigg(\frac{[t^2 + it(1-\psi+\frac{1}{2}Q)](\psi - 1 + \frac{1}{2}Q)}{\frac{1}{2}(-it+1-\psi)(-it+\frac{1}{2}Q)(1-\psi)Q} + R_1 \bigg) d\theta. \tag{4.2}$$

The present moment condition guarantees that $c_1 < \infty$ as is verified in the same way as in (3.5). It follows that

$$\lambda(t) = o(|t|^{\delta/2}) + O(t \lg |t|) \tag{4.3}$$

(the first (second) error term is superfluous if $\delta = 2$ (respectively if $\delta < 2$); the contribution of R_1 is $O(t \lg |t|)$, which the first integrand in (4.2) also contributes if $\delta = 2$) and

$$\lambda'(t) = o(|t|^{\delta/2-1}) + O(\lg |t|);$$

$$(d/dt)^j \lambda(t) = o(|t|^{\delta/2-j}) + O(|t|^{-(j-1)}) \quad (j = 2,3). \tag{4.4}$$

Splitting T^2, the range of integration, into two parts by the curve $\{Q(\cdot) = a\}$ with $a > 0$ chosen arbitrarily so far as $\{Q(\cdot) \le a\} \subset T^2$, we obtain

$$\int_{T^2} \frac{1}{-it + \frac{1}{2}Q(\theta)} d\theta = \frac{2\pi}{|Q|^{1/2}} \int_0^{a/2} \frac{du}{-it+u} + \int_{\{Q>a\}\cap T^2} \frac{1}{-it+\frac{1}{2}Q(\theta)} d\theta,$$

of which the first integral on the right side equals $\lg(-it + a/2) - \lg(-it) = -\lg(-it) + \lg(a/2) + O(t)$ so that $(2\pi)^{-2}$ times the integral on the left side above may be written as $-\lg(-it)/2\pi|Q|^{1/2} + c_2 + \eta(t)$ with the constant c_2 introduced in Section 1 and a smooth function $\eta(t)$ which vanishes at $t = 0$. Thus, with $c_\circ = 2\pi\sqrt{|Q|}\,(c_1 + c_2)$ (also introduced in Section 1) and $\tilde{\lambda}(t) = \lambda(t) + \eta(t)$,

$$\pi_0(t) = \frac{-\lg(-it) + c_\circ}{2\pi|Q|^{1/2}} + \tilde{\lambda}(t). \tag{4.5}$$

Define $h(t)$ via

$$\frac{1}{\pi_0(t)} = \frac{-2\pi|Q|^{1/2}}{\lg(-it) - c_\circ}\bigg(1 - \frac{\tilde{\lambda}(t)}{\pi_0(t)}\bigg) = \frac{-2\pi|Q|^{1/2}}{\lg(-it) - c_\circ} + h(t). \tag{4.6}$$

Employing (4.3) and (4.4), which are satisfied by $\tilde{\lambda}$ in place of λ, we then see that for $j = 0,1,2$,

$$\frac{d^j}{dt^j} h(t) = o\bigg(\frac{|t|^{\delta/2-j}}{(\lg |t|)^2}\bigg) + O\bigg(\frac{t^{1-j}}{\lg |t|}\bigg) \tag{4.7}$$

and, proceeding as in the subsection **3.2**) (or rather by (2.7)), that

$$f_0(k) \;=\; \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2\pi|Q|^{1/2}}{\lg(-it) - c_\circ} \cos kt\, dt - \frac{1}{\pi}\int_{-\pi}^{\pi} h(t) w(t) \cos kt\, dt + \varepsilon(k).$$

On changing the variable of integration the first term on the right side may be written as

$$|Q|^{1/2} e^{c_\circ} \int_{-\infty}^{\infty} \frac{1}{\lg(-it)} \cos(e^{c_\circ} kt)\, dt,$$

which equals $2\pi|Q|^{1/2} e^{c_\circ}\Big[W(e^{c_\circ}k) - e^{-e^{c_\circ}k}\Big]$ as is easily deduced from the identity (1.4) (cf. [15] Appendix). The second term is easily evaluated by integrating by parts (cf. Lemma 2.2) and we can conclude that if $E[X^2 \lg^+ |X|] < \infty$,

$$f_0(k) = 2\pi|Q|^{1/2} e^{c_\circ} W(e^{c_\circ}k) + \frac{o(k^{-\delta/2})}{k(\lg k)^2} + O\bigg(\frac{1}{k^2 \lg k}\bigg). \tag{4.8}$$

Without assuming the condition $E[X^2 \lg^+ |X|] < \infty$ it holds that if

$$g(t) = \int_{T^2} R_2(t,\theta)d\theta = \int_{T^2}\Big[\frac{1}{-it+1-\psi(\theta)} - \frac{1}{-it+\frac{1}{2}Q(\theta)}\Big]d\theta,$$

then

$$g_e(t) = o(\lg|t|) \quad \text{and} \quad g_o(t) = o(1), \tag{4.9}$$

as $t \to 0$, where g_e and g_o denote the even and odd parts of g, respectively. In fact the odd part of the integrand takes on the form

$$\frac{it\Big([\frac{1}{2}Q(\theta)]^2 - [1-\psi(\theta)]^2\Big)}{|-it+1-\psi(\theta)|^2|-it+\frac{1}{2}Q(\theta)|^2}$$

and an application of Lemma 2.1 shows the second relation of (4.9); the first one is shown in the same way. Similarly we obtain $g'(t) = o(1/t)$ and $g''(t) = o(1/t^2)$. The integral $\int_{T^2} R_1 d\theta$ is negligible in comparison with g. With the term $c_1+\lambda(t)$ in (4.1) replaced by $(2\pi)^{-1}\int_{T^2}(R_1+R_2)d\theta$ and with $c_\circ$ by $2\pi|Q|^{1/2}c_2$ the functions $h(t)$ and $\tilde\lambda(t)$ defined via (4.6) and (4.5), respectively, satisfy (4.7) (with $\delta = 0$) for $j = 1,2$ and $\tilde\lambda^{(j)}(t) = o(1/t^j)$ $(j > 0)$; on the other hand, for the even and odd parts of $h(t)$ we have

$$h_e(t) = o\Big(\frac{1}{\lg|t|}\Big) \quad \text{and} \quad h_o(t) = O\Big(\frac{1}{(\lg|t|)^2}\Big). \tag{4.10}$$

On using Lemma 2.2 the same argument as above shows (4.8) with the new $c_\circ$.

Thus the asymptotic formula of Theorem 1.3 have been verified for $x = 0$.

Remark. The proof of (4.8) for the case $\delta = 0$ given above is essentially the same as that in [8] given to the one dimensional result mentioned in Introduction (the case $\alpha = 1$). The imbedded walk that consists of traces on the horizontal axis of our walk on $\mathbf{Z}^2$ is a one dimensional walk whose characteristic function is $|Q|^{1/2}|t|(1+o(1))$ as $t \to 0$ ([11]), so that according to Kesten's result its hitting time distribution $f_0(k) = \pi|Q|^{1/2}[k(\lg k)^2]^{-1}(1+o(1))$. It may be worth noticing that this asymptotic form differs from the one for the two dimensional walk itself only by the factor $1/2$ despite the fact that the time spent outside the horizontal axis is not counted for the imbedded walk.

4.2. Define $\mathrm{e}_x(t)$ as in the case $d = 1$, namely

$$\begin{aligned}\mathrm{e}_x(t) &= \pi_{-x}(t) - \pi_0(t) + a(x)\\ &= \frac{1}{(2\pi)^2}\int_{T^2}\Big(\frac{1}{1-e^{it}\psi(\theta)} - \frac{1}{1-\psi(\theta)}\Big)(e^{ix\cdot\theta}-1)\,d\theta,\end{aligned}$$

so that $\hat f_x(t) = \mathrm{e}_x(t)/\pi_0(t) + 1 - a^*(x)/\pi_0(t)$, and $\mathrm{c}_x(t)$ and $\mathrm{s}_x(t)$ analogously to those given in **3.3** so that $2\pi\mathrm{e}_x = \mathrm{c}_x + i\mathrm{s}_x$.

Lemma 4.1 *There exists a constant C such that for $0 < |t| < 1/2$,*

(i) $|\mathrm{c}_x(t)| \le Cx^2|t|\lg|t|^{-1}$ *and* $|\mathrm{c}'_x(t)| \le Cx^2\lg|t|^{-1}$,

(ii) $|\mathrm{c}''_x(t)| \le C|x|^2/|t|$ *and* $|\mathrm{c}'''_x(t)| \le C|x|^2/|t|^2$,

(iii) $\mathrm{c}'_x(t)/i = a(t,x)\lg|t|^{-1} + ib(t,x)\,\mathrm{sgn}\,t$

with the functions a and b both even in t and dominated by Cx^2 (in absolute value).

Proof. From the expression of c_x corresponding to (3.10) we have

$$|\mathrm{c}_x(t)| \le C_1\int_{T^2}\frac{|t|(1-\cos x\cdot\theta)d\theta}{|-it+\frac{1}{2}Q(\theta)|Q(\theta)} \le C_2x^2|t|\lg|t|^{-1}. \tag{4.11}$$

where for the last inequality we have dominated $1 - \cos x \cdot \theta$ by $x^2\theta^2$ and applied Lemma 2.1 (the second case). Thus the first bound of (i) is verified.

Differentiate the defining expression of c_x we see that

$$\mathrm{c}'_x(t) = \int_{T^2} \frac{i(1-\cos x\cdot\theta)d\theta}{[-it+1-\psi(\theta)]^2} + \int_{T^2} \partial_t R_1(1-\cos x\cdot\theta)d\theta$$

On employing (2.4) (where the derivative of its error term is $O(t/(t+\theta^2)))$ as readily seen) and the inequality $1-\cos x\cdot\theta \leq |x||\theta|$ the second integral is evaluated to be $O(|x|)$. The first one being evaluated as above, this verifies not only the second bound of (i) but also (iii). For the proof of (ii) we have only to observe the bound

$$|\mathrm{c}''_x(t)| \leq C_1 \int_{T^2} \frac{1-\cos x\cdot\theta}{(|t|+\theta^2)^3} d\theta \leq C_2 x^2/|t|,$$

and a similar one for $\mathrm{c}'''_x(t)$. The proof of Lemma 4.1 is complete.

Lemma 4.2 *Let $0 \leq \delta < 1$. As $t \to 0$, uniformly in $x \neq 0$*

$$|\mathrm{s}_x(t)| = o(|t|^{\delta/2}), \quad |\mathrm{s}'_x(t)| = o(|x||t|^{(\delta-1)/2}), \quad |\mathrm{s}''_x(t)| = o(|x||t|^{(\delta-3)/2}).$$

Proof. The proof of the first bound is the same as that of Lemma 3.2 except that we have $|\sin x\cdot\theta|$ dominated by 1 (instead of $|x\cdot\theta|$). For estimation of s'_x we differentiate the analogue for s_x of the expression of I_x given in (3.11) to see that for any $\varepsilon > 0$,

$$\begin{aligned} |\mathrm{s}'_x(t)| &\leq C_1 \int_{T^2} \frac{|E[\sin X\cdot\theta]\sin x\cdot\theta|}{(|t|+\theta^2)^3} d\theta + C_2 \\ &\leq \varepsilon|x||t|^{(\delta-1)/2} \int_{R^2} \frac{|\theta|^{3+\delta}d\theta}{(1+|\theta|)^6} + C(\varepsilon) \end{aligned}$$

for some positive constant $C(\varepsilon)$ depending on ε but not on x nor on t, showing the second bound. The third one is proved in the same way. The proof of the lemma is complete.

In the second half of the subsection **4.1** it is notified that the bounds for the derivatives of $h^{(j)}$ and $\tilde{\lambda}^{(j)}(t)$ $(j > 0)$ derived in its first half are valid without assuming $E[X^2 \lg|X|] < \infty$. Taking this as well as (4.10) into account we infer from Lemmas 4.1 and 4.2 the following

Corollary 4.1 *Uniformly in $x \in \mathbf{Z}$, as $t \to 0$*

$$\pi'_x(t) = -(2\pi|Q|^{1/2}\,t)^{-1} + o\Big(|t|^{\delta/2-1}(1+|x||t|^{1/2})\Big) + O\Big(|x|_+^2 \lg|t|\Big).$$

In what follows of this section any estimates are insignificant unless $k \to \infty$, so k is understood large unless the contrary is explicitly stated.

Lemma 4.3

$$\int_{-\pi}^{\pi} \frac{\mathrm{c}_x(t)}{\pi_0(t)} e^{-ikt} dt = O\Big(\frac{x^2}{k^2 \lg k}\Big).$$

Proof. Write $g(t)$ for $\mathrm{c}_x(t)/\pi_0(t)$. First we verify that

$$g'(t) = \tilde{a}(t,x) + \tilde{b}(t,x)\,(\mathrm{sgn}\,t)/\lg|t|^{-1} \qquad (0 < |t| < 1/2), \tag{4.12}$$

where both $\tilde{a}$ and $\tilde{b}$ are even in t and bounded by Cx^2. To this end we employ the estimate of $h(t)$ in (4.10) together with Lemma 4.1 (iii) to see that $c'_x(t)/\pi_0(t)$ may be written in the same form as the right side of (4.12). On the other hand, using the estimates $\pi_0(t) = C\lg|t| + O(1)$ and

$\pi_0' = O(1/t)$ as well as the bound of $c_x(t)$ in Lemma 4.1 (i), one infers that $|c_x(t)\pi_0'(t)/\pi_0^2(t)| \leq Cx^2/\lg|t|^{-1}$. Thus (4.12) holds true.

Integrating by parts (once / twice), splitting the range of integration at $t = \pm 1/k, \pm\varepsilon$ and letting $\varepsilon \downarrow 0$ with the help of $\lim_{\varepsilon\downarrow 0}[g'(\varepsilon) - g'(-\varepsilon)] = 0$, which follows from (4.12), one obtains

$$\int_{-\pi}^{\pi} g(t)e^{-ikt}dt = \frac{1}{(ik)^2}\Big[\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|t|\leq 1/k} e^{-ikt}dg'(t) + \int_{1/k<|t|\leq\pi} g''(t)e^{-ikt}dt\Big]. \qquad (4.13)$$

The last integral is easily evaluated to be $O(x^2/\lg k)$ by applying the bounds

$$|g''(t)| \leq Cx^2/|t|\lg|t|^{-1}, \quad |g'''(t)| \leq Cx^2/t^2\lg|t|^{-1} \quad (0 < |t| < 1/2),$$

which follow from Lemma 4.1 and the bounds $\pi_0^{(j)}(t) = O(t^{-j})$, $(j \geq 1)$. The limit on the right side of (4.13) is bounded by

$$|g'(1/k) - g'(-1/k)| + \int_{|t|<1/k} |1 - e^{-ikt}||g''(t)|dt \leq \frac{2C\|\tilde{b}\|_\infty x^2}{\lg k} + Cx^2 k\int_0^{1/k}\frac{2dt}{\lg|t|^{-1}}.$$

The integral in the right-most member being $O(1/k\lg k)$, this concludes the assertion of the lemma.

Lemma 4.4 *If* $1 \leq \delta \leq 2$,

$$\int_{-\pi}^{\pi}\frac{\mathrm{s}_x(t)}{\pi_0(t)}e^{-ikt}dt = O\Big(\frac{|x|}{k^2\lg k}\Big).$$

Proof. We proceed as in the proof of Lemma 4.1 starting with a two dimensional analogue of (3.13) (instead of (3.10)) or with (3.11) (for derivatives) to see that

$$|\mathrm{s}_x(t)| \leq C_1|t|\int_{T^2}\frac{|\sin x\cdot\theta|}{(|t|+\theta^2)|\theta|}\,d\theta$$

and similar bounds for the derivatives, which reduce to

$$\mathrm{s}_x(t) = O(|x|t\lg|t|^{-1}), \quad \mathrm{s}_x''(t) = O(|x|/t) \quad \text{and} \quad \mathrm{s}_x'''(t) = O(|x|/t^2)$$

(for $0 < |t| < 1/2$). Further employing (3.15) (of which only the term involving itQ is relevant here) we also deduce (as in the proof of Lemma 4.1 (iii)) that

$$\begin{aligned}\mathrm{s}_x'(t) &= \int_{T^2}\frac{-i2E[\sin X\cdot\theta]}{(1-e^{it}\psi(\theta))(1-e^{it}\psi(-\theta))Q(\theta)}\sin x\cdot\theta\,d\theta + O(|x|)\\ &= ia(t,x)\lg|t|^{-1} + b(t,x)\,\mathrm{sgn}\,t,\end{aligned}$$

where a and b are even in t and bounded by $C|x|$ (see the proof of (iii) of Lemma 4.1). By these bounds we derive that of the lemma as in the proof of Lemma 4.3.

Proof of Theorem 1.3. The case $1 \leq \delta < 2$ is immediate from the last two lemmas (together with the result on $f_0(k)$ in **4.1**). For $0 \leq \delta < 1$, the same argument as made in the proof of Lemma 4.3 deduces from Lemma 4.2 that

$$\int_{-\pi}^{\pi}\frac{\mathrm{s}_x(t)}{\pi_0(t)}e^{-ikt}dt = o\Big(\frac{|x|}{k^{(3+\delta)/2}\lg k}\Big), \qquad (4.14)$$

which in turn shows the asserted estimate of Theorem 1.3 in view of Lemma 4.3 and the inequality $|x|/\sqrt{k} \leq \lg|x|/\lg k$ $(3 \leq x^2 \leq k)$. The case $\delta = 2$ is similarly dealt with. The proof of Theorem 1.3 is complete.

4.3. We complete the proofs of Theorems 1.4, 1.5 and 1.6. Recalling (2.1) through (2.3) and (4.6) we have

$$f_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\frac{-2\pi|Q|^{1/2}}{\lg(-it)-c_\circ}+h(t)\Big)\pi_{-x}(t)e^{-ikt}dt,$$

where $h = h(t)$ is defined via (4.6) (see the second half of **4.1** in the case $E[X^2\lg^+|X|]=\infty$). Let $x \neq 0$ and define

$$q_x(k) = -\frac{2\pi|Q|^{1/2}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{-ikt}dt}{\lg(-it)-c_\circ}\int_{\mathbf{R}^2}\frac{e^{ix\cdot\theta}d\theta}{-it+\frac{1}{2}Q(\theta)}$$

and

$$r_x(t) = \frac{1}{(2\pi)^2}\int_{T^2}[R_1(t,\theta)+R_2(t,\theta)]w(|\theta|)e^{ix\cdot\theta}\,d\theta.$$

Then, making truncation and decomposition as in (3.17) and (3.18) (but here the truncation is not only for θ but also for t) and using (4.7) one deduces that as $|x|\wedge k\to\infty$,

$$\begin{aligned} f_x(k) &= q_x(k)+\frac{1}{2\pi}\int_{-\pi}^{\pi}h(t)\pi_{-x}(t)w(t)e^{-ikt}dt \\ &\quad -|Q|^{1/2}\int_{-\pi}^{\pi}\frac{r_x(t)w(t)e^{-ikt}dt}{\lg(-it)-c_\circ}+o\Big(\frac{1}{x^{2+\delta}k\lg k}\Big). \end{aligned} \tag{4.15}$$

One can write $q_x(k)$ in the form

$$q_x(k) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{e^{-ikt}dt}{\lg(-it)-c_\circ}\int_{\mathbf{R}^2}\frac{e^{i\tilde{x}\cdot\theta}d\theta}{-it+\frac{1}{2}\theta^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{K_0(|\tilde{x}|\sqrt{-i2t})}{-\lg\sqrt{-ie^{-c_\circ}t}}e^{-ikt}dt.$$

The following lemma is proved in [15].

Lemma 4.5 *As $k\wedge|x|\to\infty$*

$$q_x(k) = \begin{cases} \dfrac{\lg(\frac{1}{2}e^{c_\circ}\tilde{x}^2)}{k(\lg(e^{c_\circ}k))^2}e^{-\tilde{x}^2/2k}+\dfrac{2\gamma\lg(k/x^2)}{k(\lg k)^3}+O\Big(\dfrac{1}{k(\lg k)^3}\Big) & \textit{for } \tilde{x}^2<k, \\ \dfrac{\lg(\frac{1}{2}e^{c_\circ}\tilde{x}^2)}{k(\lg(e^{c_\circ}k))^2}e^{-\tilde{x}^2/2k}+O\Big(\dfrac{1+[\lg(\tilde{x}^2/k)]^2}{x^2(\lg k)^3}\Big) & \textit{for } \tilde{x}^2\geq k. \end{cases} \tag{4.16}$$

For the proof of Theorems 1.4 and 1.5 it is needed to evaluate the two integrals in (4.15) and we prove the following estimates (i) through (iii) valid whenever $k\wedge|x|\to\infty$.

(i) If $E[\,|X|^2\lg^+|X|\,]<\infty$, then

$$H := \int_{-\pi}^{\pi}h(t)\pi_{-x}(t)w(t)e^{-ikt}dt = o\Big(\frac{1}{|x|k^{(\delta+1)/2}(\lg k)^2}\Big) \qquad \text{for} \quad 0\leq\delta<2.$$

In general, $\quad H = o\Big(\dfrac{1}{|x|k^{1/2}\lg k}\Big).$

(ii)

$$R := \int_{-\pi}^{\pi}\frac{r_x(t)w(t)e^{-ikt}dt}{\lg(-it)-c_\circ} = \frac{1}{|x|k\lg k}\Big[o(k^{(1-\delta)/2})+b_3O(1)\Big] \qquad \text{for} \quad 0\leq\delta<2.$$

(iii)

$$H = O\Big(\frac{1}{|x|k^{3/2}\lg k}\Big) \quad \text{and} \quad R = O\Big(\frac{1}{k\lg k}\Big(\frac{b_3}{|x|}+\frac{1}{|x|^2}\Big)\Big) \qquad \text{if} \quad \delta=2.$$

Proof of (i) *through* (iii). An application of divergence theorem gives

$$\pi_{-x}(t) = \frac{e^{it}}{i|x|(2\pi)^2} \int_{T^2} \frac{|x|^{-1}x \cdot \nabla\psi}{(1-e^{it}\psi)^2} e^{ix\cdot\theta} d\theta \tag{4.17}$$

and using this we deduce that $\pi_{-x}(t) = O(1/|x||t|^{1/2})$ and $\pi'_{-x}(t) = O(1/|x||t|^{3/2})$. Combined with the estimate of h given in (4.7) and (4.10) these yield the bounds of H in (i), in view of Lemma 2.2.

For the proof of (ii) first we see, by using Lemma 2.1, that for $\delta < 1$,

$$r_x(t) = o(|t|^{(\delta-1)/2}/|x|) \quad \text{and} \quad r'_x(t) = o(|t|^{(\delta-3)/2}/|x|). \tag{4.18}$$

Next let $\delta \geq 1$. Then

$$\int \frac{\psi(\theta) - 1 + \frac{1}{2}Q(\theta)}{(-i2t + Q(\theta))^2} \tilde{w}(\theta) e^{ix\cdot\theta}\, d\theta = \frac{b_3 O(1) + o(1)}{|x|}, \tag{4.19}$$

giving the estimate of the essential part of $r_x(t)$, so that

$$r_x(t) = b_3 O(1/|x|) + o(1/|x|). \tag{4.20}$$

The proof of (4.19) may proceed analogously to that of Lemma 2.2: split the range T^2 by means of the circle $|\theta| = 1/|x|$ and apply the divergence theorem twice for the integral on $|\theta| > 1/|x|$, in which the quantity arising in the last step is dominated by a positive multiple of

$$\frac{1}{x^2} \int_{1/|x|<|\theta|<\pi} \frac{b_3 + o(1)}{|-i2t + Q(\theta)|^{3/2}} d\theta \leq \frac{C}{x^2} \int_{1/|x|<|\theta|<\pi} \frac{b_3 + o(1)}{|\theta|^3} d\theta = C\frac{b_3 + o(1)}{|x|}.$$

The first formula of (4.18) does not hold for $\delta > 1$ (we have the third case of Lemma 2.1), but we still have

$$r'_x(t) = o(|t|^{(\delta-3)/2}/|x|) + b_3 O(1/|x|\sqrt{|t|}) \tag{4.21}$$

as is readily seen. Now, (ii) follows from (4.18), (4.20) and (4.21) on using Lemma 2.2.

For (iii), i.e. in case $\delta = 2$, first integrate by parts relative to θ, and then proceed as above.

Proof of Theorem 1.4. In view of (4.15) the assertion is readily deduced from (i), (ii) and Lemma 4.5 if one also employs Theorem 1.3 and the trivial bound $f_x(k) \leq p^k(x)$ (in disposing of the case $x^2 < k/\lg k$ and of the case $x^2 > k(\lg k)$, respectively).

Proof of Theorem 1.5. This follows from (iii) given above and the following lemma.

Lemma 4.6 *If* $r_\circ = \sqrt{2}\, e^{-\gamma - c_\circ/2}$, *then uniformly for* $|x| > r_\circ$, *as* $k \to \infty$

$$q_{r_\circ}(k, \tilde{x}) - q_x(k) = O\Big(\frac{1}{k^2(\lg k)} \wedge \frac{1}{|x|^4 \lg(|x|+1)}\Big).$$

Proof. This is Lemma 4 of [15].

Proof of Theorem 1.6. Let $\xi^2 = x^2/n$. We derive the formula (1.8) from Theorem 1.3 if $\xi^2 < 1/(\lg n)^2$ and from Theorem 1.4 if $\xi^2 \geq 1/(\lg n)^2$. First let $\xi^2 < 1/(\lg n)^2$. Then an elementary computation shows that $1 - D(e^{c_\circ}n, \xi^2/2)$ agrees with $2\lg(|x|/r_\circ)\int_{e^{c_\circ}}^\infty W(u)du$ within the error of magnitude $O((\lg \xi)/(\lg n)^3)$, where $r_\circ$ is given in Lemma 4.6 (see Remark 4 of [15]). Now (1.8) readily follows from Theorem 1.3. Next let $\xi^2 \geq 1$ and integrate the error term in (1.5):

$$\int_2^n \frac{|\lg(x^2/t)|^2 + 1}{x^2(\lg t)^3} dt = O\Big(\frac{|\lg \xi|_+^2}{(\lg n)^3 \xi^2}\Big).$$

In view of the results of [15] (as presented in Appendix (C) of this paper) this combined with Theorem 1.5 shows (1.8). A similar argument applies in the case $1/(\lg n)^2 \leq \xi^2 < 1$. The proof of Theorem 1.6 is finished.

5 The case $d \geq 3$

This section is divided into three subsections. The first two are devoted to the proof of Theorem 1.8 (the case $d = 3$), which is somewhat involved, while the case $d \geq 4$, much simpler, is dealt with in the third. Here it is noted that

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \pi_x(t) e^{-ikt} dt = \begin{cases} p^k(x) & (k \geq 0) \\ 0 & (k < 0), \end{cases} \tag{5.1}$$

which certainly holds in the dimensions $d \geq 3$.

5.1. Let $d = 3$. Details of the proof are quite similar to that for the case $d = 1$ and only main steps of the proof will be indicated.

Since $(1 - \psi)^{-1}$ is integrable over T^d, it is appropriate to subtract the term $(1 - \psi)^{-1}$ from $(1 - e^{it}\psi)^{-1}$ and accordingly convenient to bring in

$$R_3 = R_3(t, \theta) := R_2(t, \theta) - \frac{1}{1 - \psi(\theta)} + \frac{1}{\frac{1}{2}Q(\theta)}$$

so that

$$\frac{1}{1 - e^{it}\psi(\theta)} - \frac{1}{1 - \psi(\theta)} = \frac{it}{(-it + \frac{1}{2}Q(\theta))\frac{1}{2}Q(\theta)} + R_1 + R_3; \tag{5.2}$$

also

$$R_3 = \left[\frac{1}{\frac{1}{2}Q} + \frac{1}{-it + 1 - \psi}\right] \frac{it(\psi - 1 + \frac{1}{2}Q)}{(1 - \psi)(-it + \frac{1}{2}Q)}. \tag{5.3}$$

In what follows we suppose $0 \leq \delta < 2$ unless explicitly stated otherwise; the case $\delta = 2$ is dealt with at the end of the subsection **5.2**.

It follows that $(2\pi)^{-3} \int_{T^3} R_3 d\theta = iC^\circ t + o(|t|^{(1+\delta)/2})$ where $C^\circ = 0$ if $\delta < 1$ and

$$C^\circ = \frac{1}{(2\pi)^3} \int_{T^3} \frac{(\psi - 1 + \frac{1}{2}Q)(1 - \psi + \frac{1}{2}Q)}{[\frac{1}{2}(1 - \psi)Q]^2} d\theta \tag{5.4}$$

if $\delta \geq 1$. For verification we apply the third case of Lemma 2.1 (with $\alpha = (1 - \delta)/2$) if $\delta \neq 1$ and an obvious analogue of (3.5) if $\delta = 1$; likewise, $(2\pi)^{-3} \int_{T^3} R_1 d\theta = -itG(0) + O(|t|^{3/2})$. Taking these into account we make the same manipulation with a cutoff function $w(\theta)$ as before and then apply the formula (3.1) with $n = 1$ to find

$$\pi_0(t) = G(0) - \frac{\sqrt{-i2t}}{2\pi|Q|^{1/2}} - iC_0 t + o(|t|^{(1+\delta)/2}), \tag{5.5}$$

where $C_0 = -C^\circ + G(0)$. A little inspection assures that the first and second derivatives of the error term are $o(|t|^{(\delta-1)/2})$ and $o(|t|^{(\delta-3)/2})$, respectively, so that

$$\pi_0'(t) = \frac{1}{2\pi|Q|^{1/2}} \cdot \frac{i}{\sqrt{-i2t}} - iC_0 + o(|t|^{(\delta-1)/2}). \tag{5.6}$$

and similarly for $\pi_0''(t)$. Combining (5.5) and (5.6) and using $e_0 = 1/G(0)$ one deduces that

$$\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)} = -\hat{f}_0(t) + 1 - e_0 = \frac{e_0^2}{2\pi|Q|^{1/2}}\sqrt{-i2t} - i2C_1 t + o(|t|^{(1+\delta)/2}) \tag{5.7}$$

with $C_1 = -\frac{1}{2}C_0 e_0^2 + [(2\pi)^2|Q|]^{-1} e_0^3$. Taking these and (3.9) into account we can proceed as in the case $d = 1$ to conclude

$$f_0(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}_0'(t) \frac{e^{-ikt}}{ik} dt = \frac{e_0^2}{|Q|^{1/2}} \cdot \frac{1}{(2\pi k)^{3/2}} \big[1 + o(k^{-\delta/2})\big]. \tag{5.8}$$

This formula is true also for $\delta = 2$ if $o(k^{-\delta/2})$ is replaced by $O(1/k)$.

5.2. For computation of $f_x(k)$ we decompose

$$\frac{\pi_{-x}(t)}{\pi_0(t)} = \frac{G(-x)}{\pi_0(t)} + \frac{\pi_{-x}(t) - \pi_{-x}(0)}{G(0)} + \Big(\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)}\Big)(\pi_{-x}(t) - \pi_{-x}(0)),$$

where the identity $G(-x) = \pi_{-x}(0)$ is used. The contribution of the first term $G(-x)/\pi_0(t)$ to $f_x(k)$ with $x \neq 0$ is $-G(-x)f_0(k)$ and that of the second term equals $p^k(-x)/G(0)$ $(k > 0)$. Hence putting

$$m_x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{\pi_0(t)} - \frac{1}{\pi_0(0)}\Big)(\pi_{-x}(t) - \pi_{-x}(0))w(t)e^{-ikt}dt,$$

we have

$$f_x(k) = e_0 p^k(-x) - G(-x)f_0(k) + m_x(k) + O(k^{-N}|x|_+^{-1}) \qquad (x \neq 0) \tag{5.9}$$

where the error term is caused by truncation by means of $w(t)$. Expanding $1/\pi_0(0) - 1/\pi_0(t)$ in a similar way we also have $f_0(k) = e_0^2 p^k(0) + e_0 m_x(k) + O(k^{-N})$.

Now let $d = 3$. For evaluation of $m_x(k)$ we make an exact calculation based on the formula

$$\int_{-\infty}^{\infty} e^{-y\sqrt{-2it}}e^{-ikt}dt = \sqrt{2\pi}\cdot\frac{y}{k\sqrt{k}}e^{-y^2/2k} \qquad (y > 0), \tag{5.10}$$

which follows from (3.22). The result is formulated in the next lemma. Set

$$H(t,x) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \frac{ite^{ix\cdot\theta}d\theta}{(-it + \frac{1}{2}Q(\theta))\frac{1}{2}Q(\theta)}.$$

Lemma 5.1 *For $x \in \mathbf{R}^3$ and $k > 0$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{-i2t}\,H(t,x)w(t)e^{-ikt}dt = \frac{1}{|Q|^{1/2}|x|_+}\cdot\frac{1}{(2\pi k)^{3/2}}\Big[1 - e^{-\tilde{x}^2/2k}\Big(1 - \frac{\tilde{x}^2}{k}\Big)\Big] + \frac{\varepsilon(k)}{|x|_+},$$

where the first term on the right side is understood to be zero if $x = 0$.

Proof. First we compute $(2\pi)^3 H(t,x)$, which may be written as

$$\frac{2\pi}{|Q|^{1/2}}\int_0^{\pi}\sin\alpha\,d\alpha\int_0^{\infty}\frac{i4t}{-i2t + r^2}\cos[|\tilde{x}|r\cos\alpha]\,dr.$$

Applying (3.21) to the inner integral above and then performing the outer integration we find

$$H(t,x) = \frac{1}{2\pi|Q|^{1/2}|\tilde{x}|}\Big(e^{-\sqrt{-2it}\,|\tilde{x}|} - 1\Big) \qquad (x \neq 0) \tag{5.11}$$

and by continuity $H(t,0) = -(2\pi|Q|^{1/2})^{-1}\sqrt{-2it}$. The formula (5.10) as well as (3.9) (and its cosine companion) is now used to verify that for $x \neq 0$,

$$\int_{-\pi}^{\pi}(e^{-\sqrt{-i2t}\,|\tilde{x}|} - 1)\sqrt{-i2t}\,w(t)e^{-ikt}dt = \frac{\sqrt{2\pi}}{k\sqrt{k}}\Big[1 - e^{-\tilde{x}^2/2k}\Big(1 - \frac{\tilde{x}^2}{k}\Big)\Big] + \varepsilon(k),$$

showing the formula of the lemma. For the verification it suffices to see that for $y > 0$, the integral $\int_{-\infty}^{\infty}(1-w(t))\sqrt{-i2t}\,e^{-y\sqrt{-2it}}e^{-ikt}dt$ is $O(k^{-N})$ for any N, but its absolute value is indeed at most $C_N k^{-N}\Big[1 + y^N\int_0^{\infty}e^{-y\sqrt{t}}(1 - w(t/2))dt\Big]$, which is $O(k^{-N})$. The proof of Lemma 5.1 is complete.

The next lemma provides an asymptotic form of $m_x(k)$. It follows from (5.2) that

$$\pi_{-x}(t) - \pi_{-x}(0) = H(t,x) + \frac{1}{(2\pi)^3}\int_{T^3}(R_1+R_3)\tilde{w}e^{ix\cdot\theta}d\theta + \eta_x(t) \tag{5.12}$$

where $\tilde{w} = \tilde{w}(\theta) := w(|\theta|)$ and

$$\eta_x(t) = \frac{1}{(2\pi)^3}\int_{T^3}\frac{(1-\tilde{w})(e^{it}-1)\psi}{(1-e^{it}\psi)(1-\psi)}e^{ix\cdot\theta}d\theta + \frac{it}{(2\pi)^3}\int_{\mathbf{R}^3}\frac{(1-\tilde{w})e^{ix\cdot\theta}d\theta}{(-it+\frac{1}{2}Q)\frac{1}{2}Q}.$$

It is readily seen that the contribution of $\eta_x(t)$ to m_x is $O(k^{-5/2}|x|_+^{-2-\delta})$.

Lemma 5.2 *Uniformly in $x \in \mathbf{R}^3$, as $k \to \infty$*

$$m_x(k) = \frac{e_0^2}{2\pi|Q|(2\pi k)^{3/2}|\tilde{x}|_+}\Big[1 - e^{-\tilde{x}^2/2k}\Big(1-\frac{\tilde{x}^2}{k}\Big)\Big]\mathbf{1}_{\{0\}}(x) + O(k^{-5/2})$$

$$+\begin{cases} o\Big(\dfrac{\sqrt{k}\wedge|x|_+}{k^{2+\delta/2}|x|_+}\Big) & \text{if } \delta<1;\\[2ex] \dfrac{\sqrt{k}\wedge|x|_+}{k^{5/2}}\times o\Big(|x|_+^{-\delta}\lg(|x|^{\delta-1}\vee e)\Big) + b_3 O\Big(\dfrac{k\wedge x^2}{k^{5/2}|x|_+^2}\Big) & \text{if } 1\le\delta<2.\end{cases}$$

Here b_3 is the same as in Theorem 1.8; $o\Big(|x|_+^{-\delta}\lg(|x|^{\delta-1}\vee e)\Big)$ is bounded and approaches zero faster than $|x|^{-\delta}\lg|x|^{\delta-1}$ as $|x|\to\infty$ (uniformly in k).

Proof. Recall (5.7) as well as (5.12) and observe that the preceding lemma gives the leading term. The contribution to $m_x(k)$ of $-i2C_1t$ involved in (5.7) equals

$$\frac{C_1}{2\pi}\int_{-\pi}^{\pi}(-i2t)H(t,x)w(t)e^{-ikt}dt = \frac{C_1}{|Q|^{1/2}(2\pi k)^{3/2}}\Big(\frac{3}{k}-\frac{|\tilde{x}|^2}{k^2}\Big)e^{-\tilde{x}^2/2k} + \text{ngl.term}. \tag{5.13}$$

That of the error term in (5.7) will be absorbed into the estimate of the one coming from R_3. It remains to appraise the contribution of the integral in (5.12) that involves R_1+R_3. First we verify

$$\int_{-\pi}^{\pi}\Big(\frac{1}{\pi_0(t)}-\frac{1}{\pi_0(0)}\Big)w(t)e^{-ikt}dt\int_{T^3}R_1\tilde{w}e^{-ix\cdot\theta}d\theta = O\Big(\frac{k\wedge|x|^2}{k^{5/2}|x|_+^3}\Big).$$

Performing the same computation as before with the help of (5.7) and (2.4) one reduces the verification to that of

$$\frac{1}{k^2}\int_{-\pi}^{\pi}\frac{1}{\sqrt{-i2t}}w(t)e^{-ikt}dt\int_{T^3}\frac{1}{1-e^{it}\psi}e^{-ix\cdot\theta}d\theta = O\Big(\frac{1}{k^{5/2}|x|}\Big). \tag{5.14}$$

This double integral may be rewritten

$$(2\pi)^3\sum_{n=0}^{\infty}p^n(x)\int_{-\pi}^{\pi}\frac{1}{\sqrt{-i2t}}w(t)e^{i(n-k)t}dt$$

and the integral above equals $2\sqrt{2\pi}/\sqrt{k-n}+\varepsilon(k-n)$ if $k-n>0$ and $\varepsilon(n-k)$ if $k-n<0$. The bound (5.14) is then deduced by using the estimate of $p^n(x)$ as given in (1.13) (with $\delta=0$) (cf. [9]: Proof of P26.1).

We have to prove that the same double integral as above but with R_3 replacing R_1 is appraised with the error term given in the formula of the lemma. Denote by $I_x(k)$ this double integral. Then on integrating by parts

$$I_x(k) := \frac{1}{ik}\int_{-\pi}^{\pi}w(t)e^{-ikt}dt\int_{T^3}\partial_t\Big[\Big(\frac{1}{\pi_0(t)}-\frac{1}{\pi_0(0)}\Big)R_3\Big]\tilde{w}e^{-ix\cdot\theta}d\theta.$$

Note that $R_3 - R_2$ is independent of t and integrable on T^3. At first suppose that $b_3 = 0$ if $\delta \geq 1$. Then with the help of $\partial_t^j R_2 = (\psi - 1 + \frac{1}{2}Q) \times O(|t| + |\theta|^2)^{-2-j}$ for $j = 0, 1, 2, \ldots$ and $\psi - 1 + \frac{1}{2}Q = o(|\theta|^{2+\delta})$ we apply Lemma 2.1 (the first case) to deduce that

$$\int_{T^3} \partial_t^j R_2 \tilde{w} e^{ix\cdot\theta} d\theta = o(t^{(1+\delta)/2-j}) \quad \begin{pmatrix} \text{for } j = 1,2,3 & \text{if } \delta < 1 \\ \text{for } j = 2,3 & \text{if } 1 \leq \delta < 2 \text{ and } b_3 = 0 \end{pmatrix}$$

and

$$\int_{T^3} R_3 \tilde{w} e^{ix\cdot\theta} d\theta = \begin{cases} o(|t|^{(1+\delta)/2}),\ |x|_+^{-1} \times o(|t|^{\delta/2}) & \text{if } \delta < 1 \\ t \times o(|x|_+^{1-\delta} \lg(|x|^{\delta-1} \vee e)),\ |x|_+^{-2} \times o(|t|^{(\delta-1)/2}) & \text{if } 1 \leq \delta < 2 \end{cases}$$

(for the latter (with $x \neq 0$) the integration by parts in θ has been applied once if $\delta < 1$ and twice if $\delta \geq 1$ but further application is not allowed in each case; in the cases $\delta = 0, 1$ split the range of the integral with the spherical surface $|\theta| = 1/|x|$ for integrating by parts as in the proof of Lemma 2.2; also, in the case $\delta \geq 1$, we have used an analogue of Lemma 6.1 of Appendix A (cf. [12]: Appendix) as well as the fact that the integral defining C° in (5.4) is absolutely convergent). From these it is inferred that for $\delta < 1$, $I_x(k) = o\Big((\sqrt{k} \wedge |x|_+)\big/k^{2+\delta/2}|x|_+\Big)$ and that for $1 \leq \delta < 2$ with $b_3 = 0$,

$$I_x(k) = \frac{1 \vee \lg |x|_+^{\delta-1}}{k^{5/2}} \times o(|x|_+^{1-\delta}) \ \ (x^2 \leq k) \quad \text{and} \quad = o\Big(\frac{1}{k^{(2+\delta)/2}x^2}\Big) \ \ (x^2 \geq k),$$

which together imply the required estimates. In order to complete the proof we must deal with the integral involving $E[(X\cdot\theta)^3]$ in the case $\delta \geq 1$. An essential one arising after making integration by parts is

$$J_x(k) := \frac{1}{k}\int_{-\pi}^{\pi} \sqrt{-i2t}\, w(t) e^{-ikt} dt \int_{T^3} \frac{E[(X\cdot\theta)^3]\,\tilde{w}(\theta)\sin x\cdot\theta}{(-it + \frac{1}{2}Q)Q^2} d\theta;$$

the variants of this integral that we must actually compute are treated similarly to it. On further integrating by parts in θ (twice) as well as in t this term is evaluated to be $O(1/k^{3/2}|x|^2)$; on observing that the inner integral is bounded uniformly for x and t it is also evaluated to be $O(1/k^{5/2})$. Hence

$$J_x(k) = b_3 O\Big(\frac{k \wedge x^2}{k^{5/2}|x|_+^2}\Big),$$

and this completes the proof of Lemma 5.2.

Proof of Theorem 1.8. First consider the case $\delta < 1$. We have

$$p^k(-x) = \frac{e^{-\tilde{x}^2/2k}}{|Q|^{1/2}(2\pi\, k)^{3/2}} + o\Big(\frac{1}{k^{(3+\delta)/2}} \wedge \frac{1}{\sqrt{k}\,|x|_+^{2+\delta}}\Big), \tag{5.15}$$

$$[2\pi|Q|^{1/2}\,|\tilde{x}|]^{-1} - G(-x) = O(1/|x|^{1+\delta}) + b_3 O(1/|x|^2) \tag{5.16}$$

(cf. [12]). Substitute into the right side of the decomposition (5.9) from (5.8), Lemma 5.2 and (5.15), and you find that for $x \neq 0$,

$$\begin{aligned} f_x(k) \;=\; & \frac{e_0}{|Q|^{1/2}(2\pi\, k)^{3/2}}\Bigg[\Big(1 - \frac{G(-x)}{G(0)} + \frac{e_0|\tilde{x}|}{2\pi|Q|^{1/2}\, k}\Big)e^{-\tilde{x}^2/2k} \\ & \qquad + \Big(\frac{e_0}{2\pi|Q|^{1/2}\,|\tilde{x}|} - \frac{G(-x)}{G(0)}\Big)(1 - e^{-\tilde{x}^2/2k})\Bigg] \\ & + o\Big(\frac{1}{k^{(3+\delta)/2}} \wedge \frac{1}{\sqrt{k}\,|x|^{2+\delta}}\Big) + o\Big(\frac{\sqrt{k} \wedge |x|_+}{k^{2+\delta/2}|x|_+}\Big) \end{aligned} \tag{5.17}$$

(the bottle neck here is the error term in (5.15) for $x^2 < k$ and that involved in $G(-x)f_0(k)$ for $x^2 \ge k$). In view of (5.16) the second term inside the big square brackets is at most

$$(1 \wedge (x^2/k)) \times o(|x|^{-1-\delta}) + b_3 O[(k \wedge x^2)/k|x|^3]. \tag{5.18}$$

Finally use $f_x(k) < p^k(-x)$ to see that the error terms exhibited above other than the first one (i.e. the one in (5.15)) are all superfluous and may be deleted: indeed, for $|\tilde{x}| \le 4\sqrt{k \lg k}$ on the one hand, the former ones are all dominated by the latter (the first one), which, for $|\tilde{x}| > 4\sqrt{k \lg k}$ on the other hand, is dominant over the Gaussian parts on the right sides of (5.17) as well as of (5.15)). These give the formula of Theorem 1.8 if $\delta < 1$.

The case $1 \le \delta < 2$ is dealt with similarly.

For $\delta = 2$ the estimate of $m_x(k)$ given in Lemma 5.2 with $\delta = 1$ is already enough if one takes (5.13) into account. In fact the quantity appearing in it and the third term in the Edgeworth expansion of $p^k(-x)$ together with the two terms of order at most $O(k^{-5/2}|x|^{-1})$, one being $G(-x)$ times the error term for $f_0(k)$ in (5.8) and the other being corresponding to (5.18), constitute the term involving $O(1 + x^4/k^2)$ in the formula of Theorem 1.8. The proof of Theorem 1.8 is complete.

From Lemma 5.2 and the expression of $m_0(k)$ given just after (5.9) it follows that

$$f_0(k) = e_0^2 p^k(0) + o(k^{-2-\delta/2}) + O(k^{-5/2})$$

and substitution into (5.9) yields the formula (1.10) of Theorem 1.7 for $d = 3$.

5.3. It remains to prove (1.10) in the case $d \ge 4$. We start at the formula (5.9) (together with a similar one for $m_0(k)$), which holds true whenever $d \ge 3$. Since in view of (1.9) the leading term of (1.10) may be written as

$$e_0^2\, p^k(0)\mathbf{1}_{\{0\}}(x) + e_0\, p^k(-x) - e_0^2 G(-x) p^k(0),$$

the formula (1.10) for $d \ge 4$ now follows if we prove that for some constant C,

$$|m_x(k)| \le \begin{cases} \dfrac{C \lg k}{k^3}\left(1 \wedge \dfrac{\sqrt{k}}{|x|_+}\right) & \text{if } d = 4 \\ \dfrac{C}{k^3}\left[1 \wedge \left(\dfrac{\sqrt{k}}{|x|_+}\right)^{d-3}\right] & \text{if } d \ge 5. \end{cases} \tag{5.19}$$

For the proof of (5.19) one has only to look at the main part of $\pi_x(t) - \pi_0(0)$ which is a constant times

$$\int_{T^d} \frac{t e^{i\tilde{x}\cdot\theta}\tilde{w}}{(-i2t + \theta^2)\theta^2} d\theta = c_d t \int_0^\pi \sin\alpha \, d\alpha \int_0^1 \frac{r^{d-3}\cos(|\tilde{x}| r \cos\alpha)}{-2it + r^2} w(r) dr + t\eta(t),$$

where $\eta(t)$ is smooth. It is easy to see that if $d = 4$,

$$\left|\partial_t^j [\pi_x(t) - \pi_0(0)]\right| \le \begin{cases} [(|t| \lg |t|) \wedge (\sqrt{|t|}/|x|)]|t|^{-j} & \text{for } j = 0, 1 \\ [|t| \wedge (\sqrt{|t|}/|x|)]|t|^{-j} & \text{for } j = 2, 3, \end{cases}$$

from which the estimate (5.19) for $d = 4$ follows; that for $d \ge 5$ is obtained similarly.

6 Appendices

(A) Let α, β, j and ν be real constants and $v(l)$ a continuous function of $l \ge 0$.

Lemma 6.1 *Suppose that* $2\alpha - 2j + \beta \geq -3$, $2j - \beta - \nu > 1$, $\nu + \beta > -1$, $\alpha - j < -1$, $0 < \nu \leq 1$, $v(0) = 0$ *and*

$$|v(l) - v(l')| \leq C\,|l - l'|^\nu.$$

Then there exists a constant C' *such that for* $x \neq 0$,

$$\Big|\int_0^1 t^\alpha \left\{ \begin{array}{c} \cos kt \\ \sin kt \end{array} \right\} dt \int_0^1 \frac{v(l)l^\beta}{(-it+l^2)^j} e^{ixl} dl\Big| \leq \frac{C'}{|x|_+^\nu} \times \left\{ \begin{array}{l} \lg[(x^2/k) \vee e] \\ 1; \end{array} \right.$$

if $2\alpha - 2j + \beta > -3$, *the logarithmic term above may be replaced by 1.*

Proof. Let $g(t,l) = t^\alpha v(l) l^\beta/(-it+l^2)^j$. Suppose that $x \geq 1$, which gives rise to no loss of generality. We consider the critical case $2\alpha - 2j + \beta = -3$ only; the other case is easy. Then $\int_0^{1/x^2} dt \int_0^1 |g(t,l)| dl \leq C \int_0^{1/x^2} t^{\nu/2-1} dt = O(x^{-\nu})$ and

$$\int_{1/x^2}^1 dt \int_0^{\pi/x} |g(t,l)| dl \leq C \int_{1/x^2}^1 t^\alpha t^{-j} dt \int_0^{\pi/x} l^{\nu+\beta} dl = O(x^{-\nu}). \tag{6.1}$$

Define $h(t) = \int_{\pi/x}^1 g(t,l) e^{ixl} dl$. Then $h(t) = -\int_0^{1+\pi/x} g(t, l+\pi/x) e^{ixl} dl$. Since the upper limit of the inner integrals in (6.1) may be $2\pi/x$ instead of π/x, we have

$$\int_{1/x^2}^1 e^{-ikt} dt \int_{\pi/x}^1 g(t,l) dl = \int_{1/x^2}^1 e^{-ikt} dt \int_{\pi/x}^1 [g(t,l) - g(t, l+\pi/x)] e^{ixl} dl + O(x^{-\nu}).$$

By the hypothesis of the lemma we also have

$$g(t,l) - g(t, l+\pi/x) = \frac{t^\alpha l^\beta}{(-it+l^2)^j} \times \Big[A(x,l) + O\Big(\frac{l^{\nu+1}}{(t+l^2)x}\Big)\Big]$$

for $l > \pi/x$, where A satisfies that $A(x,l) = O(x^{-\nu})$. On scaling the variable l by t as before the contribution of the second term in the square brackets to the last double integral above may be written in the form

$$\int_{1/x^2}^1 t^{(\nu-3)/2} O(1/x) e^{-ikt} dt = O(x^{-\nu}).$$

Up to now e^{-ikt} may be replaced by either of $\cos kt$ or $\sin kt$. We must evaluate the contribution of the first term. To this end suppose $x^2 > k$ in below; the case $x^2 < k$ is easy to deal with. We decompose

$$\Big(\int_{1/x^2}^{1/k} + \int_{1/k}^1\Big) \left\{ \begin{array}{c} \cos kt \\ \sin kt \end{array} \right\} dt \int_{\pi/x}^1 \frac{t^\alpha l^\beta A(x,l)}{(-it+l^2)^j} dl = I + II \qquad \text{(say)}.$$

One observes that the inner integral and its derivative are $O(1/t)$ and $O(1/t^2)$, respectively and then that $II = O(x^{-\nu})$ and that $x^\nu I$ is dominated by a constant multiple of

$$\Big|\int_{1/x^2}^{1/k} \frac{1}{t} \left\{ \begin{array}{c} \cos kt \\ \sin kt \end{array} \right\} dt\Big| \leq \left\{ \begin{array}{l} C \lg[(x^2/k) \vee e] \\ 1. \end{array} \right.$$

This completes the proof of the lemma.

(B) Let $d \geq 2$ and $\mathbf{t}_r^{(d)}$ be as in REMARK 8. Then for $|x| > r > 0$,

$$E_x[\exp\{-\lambda \mathbf{t}_r^{(d)}\}] = \frac{G_\lambda(|x|, r)}{G_\lambda(r,r)} = \frac{K_{d/2-1}(|x|\sqrt{2\lambda})|x|^{1-d/2}}{K_{d/2-1}(r\sqrt{2\lambda})r^{1-d/2}} \qquad (\lambda > 0), \tag{6.2}$$

where G_λ denotes the resolvent kernel for the d-dimensional Bessel process and K_ν is the usual modified Bessel function. For $d = 3$ the Laplace transform is easily inverted to yield the formula

(1.12) (see (5.10)), which also follows from the one dimensional result since the three dimensional Bessel process conditioned on its eventually arriving at r is a one-dimensional Brownian motion.

(C) Here we give an asymptotic estimate of $P_x[\mathbf{t}^{(2)}_{r_\circ} \le t] = \int_0^t q_{r_\circ}(s,x)ds$ for large t. Put

$$\varphi(\alpha) = -\int_1^\infty \frac{e^{-\alpha y}}{y} \lg\Big(1-\frac{1}{y}\Big)\, dy \qquad (\alpha > 0),$$

and

$$A_x(t) = \frac{1}{\lg(e^{c_\circ}t)}\Big[1 - \frac{\gamma}{\lg(e^{c_\circ}t)}\Big]\int_{x^2/2t}^\infty \frac{e^{-u}}{u}du + \frac{\varphi(x^2/2t)}{[\lg(e^{c_\circ}t)]^2}$$

so that $D(e^{c_\circ}n, x^2/2n) = A_x(t)$. (The function $D(t,\alpha)$ is defined in Theorem 1.4.) The following result belongs to [15], but only not explicitly given there.

Theorem 6.1 *Let $\xi = |x|/\sqrt{t}$. Then, uniformly for $|x| > r_\circ$, as $t \to \infty$*

$$P_x[\mathbf{t}^{(2)}_{r_\circ} \le t] = A_x(t) + \frac{1}{(\lg t)^3} \times \begin{cases} O(\lg \frac{1}{2}\xi) & \text{for } x^2 < t \\ O((\lg 2\xi)^2/\xi^2) & \text{for } x^2 \ge t. \end{cases} \qquad (6.3)$$

Proof. Immediate from Lemma 6 and Eq (26) of [15].

REMARK 4. (i) It holds that $\varphi(\alpha) = O(\alpha^{-1}e^{-\alpha}\log\alpha)$ as $\alpha \to \infty$ and $\varphi(\alpha) = \frac{1}{6}\pi^2 + \alpha\lg\alpha + O(\alpha)$ as $\alpha \downarrow 0$.

(ii) On using the identity $\int_1^\infty e^{-u}u^{-1}du + \int_0^1 (e^{-u}-1)u^{-1}du = -\gamma$

$$\int_{\xi^2/2}^\infty \frac{e^{-u}}{u}du = \gamma - \lg(\xi^2/2) - \int_0^{\xi^2/2}\frac{e^{-u}-1}{u}du.$$

With the help of this together with $2\gamma = \lg[2/e^{c_\circ}r_\circ^2]$ we deduce that as $x^2/t \to 0$,

$$A_x(t) = 1 - \frac{2\lg(|x|/r_\circ)}{\lg(e^{c_\circ}t)}\Big[1 - \frac{\gamma}{\lg(e^{c_\circ}t)}\Big] + \frac{\frac{1}{6}\pi^2 - \gamma^2}{[\lg(e^{c_\circ}t)]^2}(1+o(1)) + \frac{\xi^2/2}{\lg(e^{c_\circ}t)}(1+O(\xi^2)).$$

(iii) Integrating the formula of Theorem 1 of [15] leads to

$$P_x[\mathbf{t}^{(2)}_{r_\circ} > t] = \frac{2\lg(|x|/r_\circ)}{\lg(e^{c_\circ}t)}\Big[1 - \frac{\gamma}{\lg(e^{c_\circ}t)} - \frac{\frac{1}{6}\pi^2-\gamma^2}{[\lg(e^{c_\circ}t)]^2} + \cdots\Big] + O\Big(\frac{\xi^2\lg|x|}{(\lg t)^2}\Big) \qquad (x^2 < t),$$

which agrees with the expression of $1 - A_x(t)$ obtained from (ii) apart from the error of magnitude $O(|\lg\frac{1}{2}\xi|/(\lg t)^3)$. The error estimates for $x^2 < t$ in the formula (6.3) and in this one cannot be improved since otherwise they had become inconsistent as is revealed by examining the sum of their principal terms in comparison to the error terms. On equating the error terms the latter formula is sharper than the former if $\xi^2/\lg\xi = o((\lg t)^{-2})$.

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