The hitting distributions of a line for two dimensional random walks[1]

Kôhei UCHIYAMA

Department of Mathematics, Tokyo Institute of Technology
Oh-okayama, Meguro Tokyo 152-8551
e-mail: uchiyama@math.titech.ac.jp

Abstract For every irreducible random walk on $\mathbf{Z}^2$ with zero mean and finite $2+\delta$ absolute moment ($0 \le \delta < 1$) we obtain fine asymptotic estimates of the probability that the first visit of the walk to the horizontal axis takes place at a specified site of it. [2]

running head: Hitting distributions of a line

1 Introduction and Results

In this paper we consider an irreducible random walk on the square lattice $\mathbf{Z}^2$ having zero mean and finite $2+\delta$ absolute moment ($0 \le \delta < 1$) and obtain fine asymptotic estimates of the hitting distribution of the horizontal axis $\{(s,0) : s \in \mathbf{Z}\}$, when the walk is started from a point $(0,n)$ on the vertical axis, as $|s|+|n| \to \infty$. This distribution, denoted by $H_n(s)$, would play a significant role in the theory of two dimensional random walks, but does not seem to have received sufficient investigation it would deserve since the advent of Donsker's invariance principle. According to that principle the distribution $H_n(s)$, if suitably normalized, is asymptotically equivalent as $n \to \infty$ to the Cauchy distribution $|n|/\pi(n^2+s^2)$. This equivalence however is in the topology of weak convergence of probability measures and does not imply the equivalence in any stronger sense like point-wise comparability even when n tends to infinity with $|s/n|$ bounded away from infinity, let alone when n remains in a finite interval or gets indefinitely large but in a small order of s. In our asymptotic formula the leading term of $H_n(s)$ is determined in all cases of $|s|+|n|$ tending to infinity, entailing a fairly uniform equivalence in a poin-wise level so that even the tail of the probability (i.e., its asymptotic behavior for large s) is in a good agreement with that of the Cauchy distribution (but with the factor $|n|$ replaced by the potential of the one dimensional walk of the vertical component). The proofs are done by Fourier analytic method.

Let $p(x) = p(x_1,x_2)$, $x = (x_1,x_2) \in \mathbf{Z}^2$ be a probability distribution on $\mathbf{Z}^2$ which is aperiodic in the sense that the set $\{x \in \mathbf{Z}^2 : p(x) > 0\}$ is not included in any proper subgroup of $\mathbf{Z}^2$, and satisfies

$$\sum xp(x) = 0 \quad \text{and} \quad \sum |x|^{2+\delta}p(x) < \infty, \tag{1.1}$$

where $0 \le \delta < 1$, and consider the random walk $S_n = (S_n^{(1)}, S_n^{(2)})$ on $\mathbf{Z}^2$ with i.i.d. increments whose one-step transition probability is given by $p(x,y) = p(y-x)$. Denote by P_x the probability law of the walk started at $x \in \mathbf{Z}^2$ and by E_x the expectation by P_x. Let $L = \{x \in \mathbf{Z}^2 : x_2 = 0\}$ (the first coordinate axis). Then $H_n(\cdot)$, the hitting distribution mentioned above, is written as

$$H_n(s) = P_{(0,n)}\Big[S^{(1)}_{\tau(L)} = s\Big],$$

where $\tau(L) = \inf\{n > 0 : S_n \in L\}$, the first positive time when S_n visits L.

Let Q be the covariance matrix of S_1 under P_0 and write $Q(\theta)$, $\theta = (\theta_1,\theta_2) \in \mathbf{R}^2$ for the quadratic form associated to it so that $Q(\theta) = \theta \cdot Q\theta$ and

$$Q(\theta) = E_0[(S_1 \cdot \theta)^2] = \sigma_1^2\theta_1^2 + 2\sigma_{12}\theta_1\theta_2 + \sigma_2^2\theta_2^2$$

[1]This paper will be published in Trans. Amer. Math. Soc.

[2]*key words*: first visited site, asymptotic formula, Fourier analysis, random walk of zero mean and finite variances.
AMS Subject classification (2000): Primary 60G50, Secondary 60J45.

where $\sigma_{12} = E_0[S_1^{(1)} S_1^{(2)}]$ and $\sigma_j = \sqrt{E[|S_1^{(j)}|^2]}$ $(j = 1, 2)$. Set $\sigma = |\det Q|^{1/4}$ and define the norm

$$\|x\| = \sigma \cdot \sqrt{Q^{-1}(x)} = \sigma^{-1}\sqrt{\sigma_2^2 x_1^2 - 2\sigma_{12} x_1 x_2 + \sigma_1^2 x_2^2},$$

where $Q^{-1}(x)$ stands for the quadratic form for the inverse matrix Q^{-1} of Q. Let $a(n)$ $(n \in \mathbf{Z}^2)$ be the potential function of the one dimensional random walk $S_n^{(2)}$:

$$a(n) = \sum_{k=0}^{\infty} \Big(P_0[S_k^{(2)} = 0] - P_0[S_k^{(2)} = -n] \Big),$$

where the series converges and its sum is larger than or equal to $|n|/\sigma_2^2$ (cf. Spitzer [7]:P28.8, P31.1). Put $a^*(n) = a(n)$, $n \in \mathbf{Z} \setminus \{0\}$ and $a^*(0) = 1$. For $s, n \in \mathbf{Z}$ define $\tilde{s}$ by

$$\tilde{s} = \tilde{s}(s, -n) = s + \mu n, \qquad \text{where} \qquad \mu = \sigma_{12}/\sigma_2^2.$$

Put $\lambda = \sigma^2/\sigma_2^2$, so that

$$\|(s, -n)\|^2 = \lambda^{-1}\tilde{s}^2 + \lambda n^2.$$

It is recalled that $0 \le \delta < 1$ in the moment condition (1.1).

Theorem 1.1 *For each $\varepsilon > 0$,*

$$H_n(s) = \frac{1}{\pi} \cdot \frac{|n|}{\|(s, -n)\|^2}\left[1 + o\Big(\frac{1}{|n|^\delta}\Big)\right] \qquad (|n| > \varepsilon|\tilde{s}|,\ |n| \to \infty), \tag{1.2}$$

$$H_n(s) = \frac{1}{\pi}\left(\frac{\sigma_2^2 a^*(n)}{\tilde{s}^2/\lambda} - \frac{\lambda^2|n|^3}{\|(s, -n)\|^2\tilde{s}^2}\right)\left[1 + o\Big(\frac{\log|\tilde{s}|}{|\tilde{s}|^\delta}\Big)\right] \qquad (|\tilde{s}| > \varepsilon|n|,\ |\tilde{s}| \to \infty). \tag{1.3}$$

Here $o(1/|n|^\delta)$ in (1.2) is uniform for $|\tilde{s}| < |n|/\varepsilon$; similarly $o(|\tilde{s}|^{-\delta}\log|\tilde{s}|)$ in (1.3) is uniform for $|n| < |\tilde{s}|/\varepsilon$.

REMARK 1. (i) In the overlapping region $\varepsilon|n| < |\tilde{s}| < \varepsilon^{-1}|n|$ the two formulae (1.2) and (1.3) coincide except for the logarithmic factor in the error term owing to the asymptotic relation $a(n) = \sigma_2^{-2}|n|(1 + o(|n|^{-\delta}))$ as $|n| \to \infty$ (cf. [12]: Corollary 6.1) and the identity

$$\frac{\lambda^2|n|^3}{\|(s, -n)\|^2\tilde{s}^2} = \frac{|n|}{\tilde{s}^2/\lambda} - \frac{|n|}{\|(s, -n)\|^2}. \tag{1.4}$$

(ii) If (1.1) is true for $1 \le \delta < 2$, the formula (1.2) (resp. (1.3)) remains valid if we add to its right side an extra term that is $O(1/n^2)$ (resp. $O(n/s^3)$); in the case when all the third moments of S_1 under P_0 vanish the extra term is simplified and takes on the form

$$-\frac{C^*}{\pi} \cdot \frac{\lambda n^2 - \lambda^{-1}\tilde{s}^2}{\|(s, -n)\|^4} \qquad \left(\text{resp.} \quad -\frac{C^*}{\pi}\Big(\frac{\lambda n^2 - \lambda^{-1}\tilde{s}^2}{\|(s, -n)\|^4} + \frac{\lambda}{\tilde{s}^2}\Big)\right), \tag{1.5}$$

where $C^* = (2\pi)^{-1}\int_{-\pi}^{\pi}\left[\sigma_2^2(1 - \psi(0, l))^{-1} - (1 - \cos l)^{-1}\right]dl$. The constant C^* is non-negative and vanishes if and only if the walk is continuous in the vertical direction (namely $p((x_1, x_2)) = 0$ if $|x_2| \ge 2$) [14]. See Section 9 for more detail.

(iii) In the case $\delta = 0$ the formula (1.3) of Theorem 1.1 does not determine the precise leading term since the error term in the square brackets may be unbounded. To get the error of order $o(1)$ one needs to impose some additional condition (see REMARK 2 below).

(iv) If the walk is symmetric relative to L and continuous in the vertical direction, then the reflection principle (cf. [7], p.155) can be applied to derive the estimates of $H_n(x)$ from those of the potential function of the walk that are given in [2], [9] (also cf. [4]).

(v) That $a^*(n)$ appears in the leading term on the right side of (1.3) may be explained by means of the formula $P_{(0,n)}[\,|S_k^{(2)}| \ge K$ for some $k < \tau(L)\,] = \sigma_2^2 a^*(n)/K + o(n/K)$ as $K \to \infty$ uniformly for $|n| < K$ (cf. [12]): for the hitting site $S_{\tau(L)}$ to be far from the origin the walk is most likely to once pass across a large ordinate level before returning to L.

Write X and Y for $S_1^{(1)}$ and $S_1^{(2)}$, the first and second components of S_1, respectively and P and E for P_0 and E_0, respectively, so that $\sigma_{12} = EXY$ etc. Define $\tilde{X}$ by

$$\tilde{X} = X - \mu Y.$$

A natural sufficient condition for $H_0(s)$ to behave like const$\times\tilde{s}^{-2}$ as $\tilde{s} \to \infty$ turns out to be

$$E[\tilde{X}^2 \log|\tilde{X}|] < \infty, \tag{1.6}$$

where $t \log t = 0$ for $t = 0$. We also write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for a, b real.

Theorem 1.2 *Suppose that (1.6) holds. Then, as* $|s| \vee |n| \to \infty$

$$H_n(s) = \frac{1}{\pi} \cdot \frac{\sigma_2^2 a^*(n)}{\|(s, -n)\|^2}[1 + o(1)]. \tag{1.7}$$

(Here $o(1) \to 0$ *whenever* $|s| \vee |n| \to \infty$.*)*

The next theorem is of interest in the case when (1.6) is violated.

Theorem 1.3 *Let* $\varepsilon > 0$. *Then for* $|n| < |s|$,

$$H_n(s) \quad = \quad \frac{1}{\pi}\left(\frac{\sigma_2^2 a^*(n)}{\tilde{s}^2/\lambda} - \frac{\lambda^2|n|^3}{\|(s,-n)\|^2\tilde{s}^2}\right)(1 + o(1)) + \frac{2a^*(n)}{\lambda\tilde{s}^2}M_\varepsilon(s)(1 + o_b(1)),$$

where

$$M_\varepsilon(s) = E\left[\tilde{X}^2 \log\frac{|\tilde{s}|}{|\tilde{X} - \tilde{s}| \vee |Y| \vee 1}; |\tilde{X} - \tilde{s}| \vee |Y| < \varepsilon|\tilde{s}|\right],$$

and $o(1) \to 0$ *uniformly in* n *and* $o_b(1) \to 0$ *boundedly as* $|s| \to \infty$. *Moreover* $1 + o_b(1) = m(s, n) + o(1)$ *with* $0 \leq m(s, n) \leq 1$ *and* $o(1)$ *as above.*

In the case when X and Y are stochastically independent of each other the leading term in (1.7) can be derived by using the formula $P_{(0,n)}[\tau(L) > k] = a^*(n)\sigma_2/\sqrt{\pi k/2}(1 + o(1))$ $(k \to \infty)$ (cf. §32 of [7]; see [12] for uniformity in n) as well as the local limit theorem applied to $S_n^{(2)}$; this also provides another explanation for the factor $a^*(n)$ to appear in the leading term. Even if X and Y are independent, one needs still to suppose the condition (1.6) to ensure (1.7).

REMARK 2. (i) Consider the case $\delta = 0$. If $|n| > |\tilde{s}|$ the formula (1.2) provides the correct leading term, while if $|\tilde{s}|/(|n| \vee 1) \to \infty$ we have only

$$H_n(s) = \frac{1}{\pi} \cdot \frac{\sigma_2^2 a^*(n)}{\tilde{s}^2/\lambda}\Big[1 + o(\log|\tilde{s}|)\Big] \qquad (|\tilde{s}| > |n|),$$

which cannot be improved: in fact by using Theorem 1.3 one can readily infer that for any increasing function $h(t) > 0$ such that $h(t)/\log t \to 0$ as $t \to \infty$ there exists a probability p such that p satisfies (1.1) (with $\delta = 0$) and the error term in the square brackets above is not bounded by $h(|\tilde{s}|)$.

(ii) In a similar sense the condition (1.6) cannot be relaxed for validity of (1.3) : for any $h(t) > 0$ such as in (i) there exists a probability p satisfying (1.1) such that $E[X^2 h(X)] < \infty$ and for each n, $\limsup_{s\to\infty} H_n(s)s^2 = \infty$. On the other hand the formula (1.7) of Theorem 1.2 may hold for p which fails to satisfy (1.6) if the marginal $P[X = s]$ has sufficiently nice asymptotic behavior as $s \to \infty$: eg., as Theorem 1.3 shows, if $P[X = s] \leq C|s|^3$, then (1.7) is true regardless of condition (1.6). Theorem 1.3 also implies that one-sided moment conditions $E[\tilde{X}^2 \log|\tilde{X}|; \pm\tilde{X} > 1] < \infty$ entail the corresponding one sided asymptotic forms $\sigma^2\pi^{-1}a^*(n)/s^2$ of $H_n(s)$ (as $s \to \pm\infty$).

For higher dimensional walks we have analogous results to Theorem 1.1 as discussed in Section 10. From the estimate of $H_n(s)$ as given in Theorem 1.1 one can derive (cf. [11]) that of the hitting distribution of the negative half of L by considering the one dimensional random walk that is a trace of S_n left on L and applying the theory of ladder processes as found in [7] or in [1]. Such a result in turn would give some precise estimate of the hitting distributions of long segments on L, of which certain upper bounds are obtained by Kesten [5] for simple random walk and by Lawler and Limic [6] for random walks with finite $5/2+\delta$ absolute moment. In a separate paper we compute the asymptotic form of the Green function of the domain $\mathbf{Z}^2 \setminus L$, which is to entail a (less precise) version of Theorem 1.1, but the computation is more involved than these given in this paper and indeed relies on some results in this paper.

We conclude this section with the following result on the integrated tail $E_n(s) = H_n(s)+H_n(s+1)+\cdots$, which is much easier to obtain. The result is stated only for $\delta = 0$.

Theorem 1.4 *As* $\tilde{s}/(|n|\vee 1) \to \infty,\quad E_n(s) = \dfrac{\sigma_2^2 a^*(n)}{\pi\tilde{s}/\lambda}(1+o(1)).$

It is noted that in the case when $\tilde{s}/n$ remains bounded, the invariance principle gives the correct limit value : for each $K>0$, $E_n(s) = \frac{1}{2} - \frac{1}{\pi}\arctan(\tilde{s}/\lambda n) + o(1)$ as $n\to\infty$ with $|\tilde{s}|/n < K$.

2 Preliminary formulae and estimates.

Let $\hat{H}_n(t)$ $(t\in\mathbf{R})$ denote the characteristic function of the probability distribution $H_n(\cdot)$:

$$\hat{H}_n(t) = \sum_{s\in\mathbf{Z}} H_n(s)e^{its}.$$

Define

$$\psi(\theta) = \psi(\theta_1,\theta_2) = E_0\Big[e^{i\theta\cdot S_1}\Big] \qquad (\theta = (\theta_1,\theta_2)\in[-\pi,\pi)^2)$$

and

$$\pi_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-ikl}}{1-\psi(t,l)}dl \qquad (t\neq 0). \tag{2.1}$$

Lemma 2.1 *If* $0<|t|\le\pi$, *then*

$$\hat{H}_n(t) = \frac{\pi_{-n}(t)}{\pi_0(t)} \qquad (n\neq 0) \tag{2.2}$$

and

$$\hat{H}_0(t) = 1 - \frac{1}{\pi_0(t)}. \tag{2.3}$$

The proof of this lemma is standard and postponed until the last section (Appendix (A)).

We must compute

$$H_n(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi_{-n}(t)}{\pi_0(t)}e^{-ist}dt \qquad (n\neq 0). \tag{2.4}$$

In carrying out the computation we suppose that Q is diagonal:

$$Q(t,l) = \sigma_1^2 t^2 + \sigma_2^2 l^2,$$

which gives rise to no loss of generality as will be discussed in Section 9 (see (10.2)) . From (1.1) one obtains

$$1-\psi(t,l) = \frac{1}{2}Q(t,l)(1+o(|t|^\delta + |l|^\delta)) \qquad (|t|\vee|l|\to 0)$$

and then, changing the variable of integration by $u = l/t$, observes that as $t \to 0$,

$$|t|\pi_0(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|t|dl}{1-\psi(t,l)} = \frac{1}{\pi}\int_{-\pi/|t|}^{\pi/|t|} \frac{dl}{Q(1,l)} + o(|t|^{\delta}) \tag{2.5}$$

(see Lemma 2.3 below for evaluation of the error term) and the integral on the right side equals $\pi/\sigma^2 + O(t)$, so that

$$1/\pi_0(t) = \sigma^2|t| + o(|t|^{1+\delta}). \tag{2.6}$$

Our main task for proof of the first formula (1.2) of Theorem 1.1 is to derive from

$$H_n(s) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} \frac{e^{-ist}dt}{\pi_0(t)} \int_{-\pi}^{\pi} \frac{e^{inl}dl}{1-\psi(t,l)} \tag{2.7}$$

the following lemma, whose proof will be given in the next section.

Lemma 2.2 *For each $\varepsilon > 0$, uniformly for $|s| < |n|/\varepsilon$ as $|n| \to \infty$*

$$H_n(s) = \frac{\sigma^2}{(2\pi)^2}\int_{-\pi}^{\pi} |t|\cos st\, dt \int_{-\infty}^{\infty} \frac{2\cos nl\, dl}{Q(t,l)} + o\Big(\frac{|n|^{1-\delta}}{s^2+n^2}\Big). \tag{2.8}$$

Using the following identities for the well-known pair of cosine transforms

$$\frac{\sigma^2|t|}{2\pi}\int_{-\infty}^{\infty} \frac{2\cos nl\, dl}{Q(t,l)} = e^{-\lambda|nt|} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\lambda|nt|}\cos st dt = \frac{2\lambda|n|}{s^2+\lambda^2n^2} \quad \text{with} \quad \lambda = \frac{\sigma_1}{\sigma_2}$$

as well as the trivial estimate $\int_{|t|>\pi} e^{-\lambda|nt|}\cos st\, dt = O(e^{-\lambda|n|})$, we immediately deduce from Lemma 2.2 the formula (1.2) which may be written as

$$H_n(s) = \frac{1}{\pi}\cdot\frac{\lambda|n|}{s^2+\lambda^2n^2}\Big[1 + o\Big(\frac{1}{|n|^{\delta}}\Big)\Big] \qquad (|\tilde{s}| < |n|/\varepsilon). \tag{2.9}$$

In the case $|s| > |n|$ the leading term is to be different from one that comes up above and it will be desirable to arrange the expression on the right side of (2.4) so that an integral that is to become the minor term is separated from one involving the main term. This is achieved by rewriting $\hat{H}_n$ as

$$\hat{H}_n(t) = \frac{\pi_{-n}(t) - \pi_0(t) + a(n)}{\pi_0(t)} + 1 - \frac{a(n)}{\pi_0(t)} \qquad (n \neq 0).$$

Under the condition $E_0[|Y|^2] < \infty$ (remember (X,Y) is written for $(S^{(1)}, S^{(2)})$) we have

$$a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-e^{inl}}{1-\phi(l)}dl \quad \text{where} \quad \phi(l) = \psi(0,l). \tag{2.10}$$

(Cf. [3]; also Appendix of [12].) Set

$$\mathrm{e}_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{1}{1-\psi(t,l)} - \frac{1}{1-\phi(l)}\Big](e^{inl}-1)dl$$

and

$$\rho(t) = \frac{1}{\pi_0(t)}$$

so that $\mathrm{e}_n(t) = \pi_{-n}(t) - \pi_0(t) + a(n)$ and

$$\hat{H}_n(t) = \rho(t)\mathrm{e}_n(t) + 1 - a^*(n)\rho(t) \tag{2.11}$$

(valid also for $n = 0$). We shall show

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(t)e^{-ist}dt = -\frac{\sigma^2}{\pi|s|^2} + o\Big(\frac{\log|s|}{|s|^{2+\delta}}\Big), \tag{2.12}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(t)\mathrm{e}_n(t)e^{-ist}dt = \frac{\lambda^2|n|^3}{\pi\|(s,-n)\|^2s^2} + o\bigg(\frac{n\log|s|}{|s|^{2+\delta}}\bigg). \tag{2.13}$$

The second formula (1.3) of Theorem 1.1 immediately follows from these ones.

The error terms in (2.8), (2.12) and (2.13) are evaluated by integration by parts. In the case $|s| < |n|$ we shall integrate by parts with respect to l to derive the error estimate in (2.8) which is at most $o(1/n)$, whereas in the case $|s| > |n|$ we seek an error estimate of $o(n/s^2)$ or better and thereby need to perform integration by parts twice with respect to t. At the final steps of these processes we shall apply Lemma 11.2 in the former case and Lemma 11.4 in the latter, both of which are derived by standard methods in Fourier analysis. The proofs of Theorems 1.2 and 1.3 involve certain methods that are less standard.

Change of variables as being made in the derivation of (2.5) will be of repeated use in the succeeding sections (sometimes with the roles of t and l reversed) and we here present trite estimates of integrals as involved in (2.5) as the following lemma for convenience of later citation.

Lemma 2.3 *Let δ, α and m are real numbers such that $\delta \geq 0$ and $\alpha > -1$. If $b(t,l)$ is a continuous function on $[-1,1]\times[-1,1]$ such that $b(t,l) = o(|t|^\delta + |l|^\delta)$ as $|t|\vee|l| \to 0$, then as $t \to 0$*

$$\int_0^1 \frac{l^\alpha b(t,l)}{[Q(t,l)]^m}dl = \begin{cases} o(|t|^{\alpha+\delta+1-2m}) & \textit{if} \quad \alpha+\delta+1-2m<0 \\ o(\log|t|) & \textit{if} \quad \alpha+\delta+1-2m=0. \end{cases}.$$

The rest of the paper is organized as follows. The proof of Theorem 1.1 is finished in Section 3 except for two lemmas which are proved in Section 4. In Section 5 we give some detailed estimates of $\mathrm{e}_n(t)$, which are needed for the proof of Theorem 1.2 given in Section 6. Theorems 1.3 and 1.4 are proved in Sections 7 and 8, respectively. Section 9 is devoted to the case when $\delta \geq 1$ in our basic moment condition (1.1). In Section 10 we indicate a way to reduce the problem to the case when Q is diagonal; also briefly discuss the higher dimensional case as mentioned before. Section 11, the last section, is Appendix consisting of (A) the proof of Lemma 2.1 and (B) several lemmas of Fourier analytical nature.

3 Proof of Theorem 1.1

In Sections 3 through 8 we suppose that Q is diagonal: $Q(t,l) = \sigma_1^2t^2 + \sigma_2^2l^2$. The proof of Theorem 1.1 given below is continuation of its outline advanced in the preceding section.

The case $|s| < |n|$. We have only to prove Lemma 2.2, namely to evaluate the difference of the leading term on the right side of (2.8) from $H_n(s)$. Write it as $(2\pi)^{-2}(r+\tilde r)$ so that

$$(2\pi)^2H_n(s) = \sigma^2\int_{-\pi}^{\pi}|t|e^{-ist}dt\int_0^\infty \frac{4\cos nl}{Q(t,l)}dl + r(s,n) + \tilde r(s,n),$$

where

$$r(s,n) = \int_{-\pi}^{\pi}\rho(t)e^{-ist}dt\int_{-\pi}^{\pi}\bigg[\frac{1}{1-\psi(t,l)} - \frac{2}{Q(t,l)}\bigg]e^{inl}dl \tag{3.1}$$

and

$$\tilde r(s,n) = \int_{-\pi}^{\pi}(\rho(t)-\sigma^2|t|)e^{-ist}dt\int_0^{\pi}\frac{4\cos nl}{Q(t,l)}dl - \int_{-\pi}^{\pi}\sigma^2|t|e^{-ist}dt\int_{\pi}^{\infty}\frac{4\cos nl}{Q(t,l)}dl$$

(recall $\rho(t) = 1/\pi_0(t)$). We carry out estimation of r only: that of $\tilde r$ is similar but only simpler because of the absence of a $\sin nl$ part in it (see (3.2) below). On first reversing the order of integration and then integrating by parts (with respect to l)

$$r(s,n) = -\frac{1}{in}\int_{-\pi}^{\pi}e^{inl}\,dl\int_{-\pi}^{\pi}F(t,l)e^{-ist}dt \qquad \text{where} \qquad F(t,l) = \frac{-\partial_l\psi(t,l)\rho(t)}{(1-\psi(t,l))^2} - \frac{4\sigma_2^2l\rho(t)}{[Q(t,l)]^2}$$

(∂_l indicates the partial differentiation). From the readily verified bounds

$$|F(t,l)| = o\Big(\frac{|tl|(|t|^\delta + |l|^\delta)}{(t^2+l^2)^2}\Big) \quad \text{and} \quad |\partial_l F(t,l)| = o\Big(\frac{|t|(|t|^\delta + |l|^\delta)}{(t^2+l^2)^2}\Big) \qquad (|t| \vee |l| \to 0),$$

one employs Lemma 2.3 to deduce that $\int_{-\pi}^{\pi} |F(t,l)|dt = o(|l|^{\delta-1})$ and $\int_{-\pi}^{\pi} |\partial_l F(t,l)|dt = o(|l|^{\delta-2})$. If $\delta > 0$, these two estimates imply the required one, i.e., $\sup_s |r(s,n)| = o(|n|^{-1-\delta})$, according to the last assertion of Lemma 11.2 in Appendix (with the roles of n and s reversed: the convergence in Lemma 11.2 is as $s \to \infty$ and uniform in n in reverse to the present situation).

In the case $\delta = 0$ the same reasoning is inadequate and we need to look at the inner integral in (3.1) more closely. To this end we decompose $r = r_e + r_o$ where

$$\begin{Bmatrix} r_e \\ r_o \end{Bmatrix} = \int_{-\pi}^{\pi} \rho(t) e^{-ist} dt \int_{-\pi}^{\pi} \left[\frac{1}{1-\psi(t,l)} - \frac{2}{Q(t,l)} \right] \begin{Bmatrix} \cos nl \\ i \sin nl \end{Bmatrix} dl. \tag{3.2}$$

Lemma 3.1 $\sup_s (|r_o(s,n)| \vee |r_e(s,n)|) = o(|n|^{-1-\delta})$.

Proof. The case $\delta > 0$ has been dealt with. Let $\delta = 0$. As above we infer first that as $|t| \vee |l| \to 0$

$$\int_{-\pi}^{\pi} |F(t,l)|dt = o(1/l); \qquad \int_{-\pi}^{\pi} |\partial_l F(t,l)|dt = o(1/l^2), \tag{3.3}$$

and then that $\sup_s |r_e| = o(1/n)$ (use the first assertion of Lemma 11.2 of Appendix)

For estimation of r_o we exploit the fact that $\sin nl$ is an odd function of l and to this end put

$$h(t,l) = \frac{\rho(t)}{2} \left[\frac{1}{1-\psi(t,l)} - \frac{1}{1-\psi(t,-l)} \right] = \rho(t) \frac{iE[e^{iXt} \sin Yl]}{(1-\psi(t,l))(1-\psi(t,-l))}$$

so that

$$r_o = 2i \int_0^{\pi} \sin nl dl \int_{-\pi}^{\pi} e^{-ist} h(t,l)\, dt \quad \text{and} \quad \partial_l h(t,l) = \frac{1}{2}[F(t,l) - F(t,-l)].$$

From $EXY = \sigma_{12} = 0$ it follows that $E[e^{iXt} \sin Yl] = o(|tl| + l^2)$. Hence $\int_{-\pi}^{\pi} e^{-ist} h(t,l)\, dt \to 0$ as $l \to 0$ uniformly in s and by integrating by parts

$$r_o = \frac{i}{n} \int_{+0}^{\pi} \cos nl\, dl \int_{-\pi}^{\pi} [F(t,l) - F(t,-l)] dt.$$

Now the required estimate, i.e. $\sup_s |r_o| = o(1/n)$, follows from (3.3) in view of Lemma 11.2. □

Up to now we have shown (1.2), the first formula of Theorem 1.1.

<u>The case $|s| \geq |n|$.</u> We must prove (2.12) and (2.13). For the proof of the latter we set

$$g(t,l) = \rho(t) \left[\frac{1}{1-\psi(t,l)} - \frac{1}{1-\phi(l)} \right], \tag{3.4}$$

so that

$$\rho(t) \mathrm{e}_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t,l)(e^{inl} - 1) dl$$

and carry out simple computations to see that for $t \neq 0$,

$$\partial_t g(t,l) = \frac{\psi' \rho}{(1-\psi)^2} + \frac{(\psi - \phi)\rho'}{(1-\psi)(1-\phi)}$$

and

$$\partial_t^2 g(t,l) = \frac{\big((1-\psi)\psi'' + 2[\psi']^2\big)\rho}{(1-\psi)^3} + \frac{2\psi'\rho'}{(1-\psi)^2} + \frac{(\psi - \phi)\rho''}{(1-\psi)(1-\phi)},$$

where a dash as well as ∂_t denotes (partial) differentiation with respect to t: $\psi' = \partial_t \psi = \partial \psi/\partial t$.

It is easy to see (under the condition $\delta > 0$) that ρ' is bounded, $\mathrm{e}_n(\pm 0) = 0$ and $\mathrm{e}_n'(t) = o(1/t)$ as $t \to 0$ (which will be proved under $\delta = 0$ in Sections 4 and 5); in particular $(\rho \mathrm{e}_n)'(\pm 0) = 0$. Then, performing integration by parts with the help of this last relation as well as of the periodicity of $\rho \mathrm{e}_n$, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(t)\mathrm{e}_n(t)e^{-ist}dt &= -\frac{1}{2\pi s^2}\int_{-\pi}^{\pi}(\rho \mathrm{e}_n)''(t)e^{-ist}dt \\ &= \frac{-1}{4\pi^2 s^2}\int_{-\pi}^{\pi} e^{-ist}dt\int_{-\pi}^{\pi}\partial_t^2 g(t,l)(e^{inl}-1)dl, \end{aligned} \tag{3.5}$$

where because of possible singularity at the origin the outer integral must in general be understood to be an improper integral (which exists since the boundary values $(\rho \mathrm{e}_n)'(\pm 0)$ do).

Let $w(t)$ be a smooth even function such that $w = 1$ in a neighborhood of 0 and $w = 0$ for $|t| > 1$ and set

$$F_1(t,l) = w(t)\partial_t^2\left[\frac{2\sigma^2|t|}{Q(t,l)}\right] \qquad (t \neq 0) \tag{3.6}$$

and make decomposition

$$\partial_t^2 g(t,l) = F_1 + F_2 + F_3 \qquad (t \neq 0), \tag{3.7}$$

with

$$F_2(t,l) = \frac{\rho''(t)}{1-\psi(t,l)} - \frac{\rho''(t)}{1-\phi(l)}$$

and F_3 being the rest $\partial_t^2 g(t,l) - F_1 - F_2$; explicitly

$$F_3 = \frac{\psi''\rho}{(1-\psi)^2} + \frac{2[\psi']^2\rho}{(1-\psi)^3} + \frac{2\psi'\rho'}{(1-\psi)^2} - w(t)\Big(|t|\partial_t^2\frac{2\sigma^2}{Q} - \frac{8\sigma^2\sigma_1^2|t|}{Q^2}\Big). \tag{3.8}$$

In the next section we obtain an explicit form of the contribution of F_1 (Lemma 4.1); according to it the right side of (3.5) can be written as

$$-\frac{\lambda^2|n|^3}{\pi\|(s,-n)\|^2 s^2} - \frac{1}{4\pi^2 s^2}\int_{-\pi}^{\pi} e^{-ist}dt\int_{-\pi}^{\pi}(F_2+F_3)(e^{inl}-1)dl + O(1/s^4). \tag{3.9}$$

Note that $2\sigma^2|t|/Q(t,l)$ appearing in the definition of F_1 is the principal part of $\rho/(1-\psi)$.

Lemma 3.2 *Let $0 \leq \delta < 1$. Then as $|s| \to \infty$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(t)\mathrm{e}_n(t)e^{-ist}dt = -\frac{\lambda^2|n|^3}{\pi\|(s,-n)\|^2 s^2} + o\Big(\frac{n\log|s|}{|s|^{2+\delta}}\Big).$$

Proof. Let $\delta > 0$. In view of the formulae (3.5) and (3.9) it suffices to show that

$$I_k := \int_{-\pi}^{\pi} e^{-ist}dt\int_{-\pi}^{\pi}F_k(t,l)(1-e^{inl})dl = o\Big(\frac{n\log|s|}{|s|^{\delta}}\Big) \qquad (k=2,3). \tag{3.10}$$

It follows that as $t - t' \to 0$ and $t^2 + l^2 \to 0$,

$$F_3(t,l) = \frac{|t|}{[Q(t,l)]^2} \times o\Big(|t|^{\delta} + |l|^{\delta}\Big) \tag{3.11}$$

and

$$|F_3(t,l) - F_3(t',l)| = \frac{|t-t'|}{[Q(t,l)]^2} \times o\Big(|t|^{\delta} + |l|^{\delta}\Big) + \frac{1}{[Q(t,l)]^{3/2}} \times o\Big(|t-t'|^{\delta}\Big).$$

According to Lemma 11.4 (with $\omega = (1,0)$) of Appendix (B) these bounds together yield the required estimate of I_3, provided $\delta > 0$. The case $\delta = 0$ is dealt with at the end of this section.

The estimation of I_2 is done in a similar way if $\delta > 0$: write $I_2 = 2\pi \int e^{-ist}\rho''(t)\mathrm{e}_n(t)dt$, observe that $|\mathrm{e}_n(t)|$ and $|t\mathrm{e}'_n(t)|$ are uniformly bounded by a constant times $|n|$ (we shall give more detailed estimates for both of them in Section 5) and apply the estimates concerning ρ'' given in Lemma 4.2 of Section 4 and you will find the required estimate of I_2 in view of Lemma 11.4 (with $d = 1$). □

The proof of (2.12) is similar but only simpler: one has only to observe that $\int_{-\pi}^{\pi} |t|w(t)e^{-ist}dt = 2\int_0^{\pi} tw(t)\cos st\, dt = -2s^{-2} + O(s^{-N})$ for every $N > 0$ and

$$\int_{-\pi}^{\pi} \Big(\rho(t) - \sigma^2|t|w(t)\Big)e^{-ist}dt = o\Big(\frac{\log|s|}{|s|^{2+\delta}}\Big).$$

In the case $\delta > 0$ the latter is obtained by applying Lemmas 4.2 and 11.4 as in the proof of Lemma 3.2. If $\delta = 0$, we once integrate by parts the above integral and then split the range of the resulting integral at $t = \pm 1/|s|$. The integral on $|t| < 1/|s|$ is $O(1/s^2)$ since ρ' is bounded and the other on $|t| \geq 1/s$ is $o((\log|s|)/s^2)$ owing to $\rho'' = o(1/t)$ (Lemma 4.2). Thus (2.12) has been verified.

For the proof of Lemma 3.2 in the case $\delta = 0$ we can proceed as in the argument just made above (namely we prove that $\int_{|t|<1/s}(\rho\mathrm{e}_n)'e^{-ist}dt = O(1/s)$ and $\int_{|t|\geq 1/s}(\rho\mathrm{e}_n)'e^{-ist}dt = o((\log|s|)/s^2)$ of which the details will be found in the beginning of Section 6; see (6.2) and a similar estimate for F_1 given after it). This completes the proof of Lemma 3.2 and hence the proof of Theorem 1.1.

4 Lemmas on $F_1(t,l)$ and $\rho(t)$

In this section we prove two lemmas that have been applied in the second half of the preceding section, of which all the arguments given in this section are independent. Recall it is supposed that Q is diagonal (so that $\lambda = \sigma^2/\sigma_2^2 = \sigma_1/\sigma/2$). The first lemma concerns the function

$$F_1(t,l) = w(t)\partial_t^2\Big[\frac{2\sigma^2|t|}{Q(t,l)}\Big] = w(t)\frac{(at^2+bl^2)|t|}{[Q(t,l)]^3}, \qquad (t \neq 0),$$

where $a = 4\sigma_1^4\sigma^2$ and $b = -12\sigma^6$ (these values of a, b are of no significance in what follows).

Lemma 4.1 *Uniformly in n, as $|s| \to \infty$,*

$$\frac{1}{4\pi^2 s^2}\int_{-\pi}^{\pi} e^{-ist}\,dt \int_{-\pi}^{\pi} F_1(t,l)(e^{inl}-1)dl = \frac{\lambda^2|n|^3}{\pi\|(s,-n)\|^2 s^2} + O\Big(\frac{1}{s^4}\Big).$$

Proof. Let $n > 0$. Twice differentiate the both sides of the identity

$$\int_{-\pi}^{\pi} \frac{2\sigma^2|t|}{Q(t,l)}(e^{inl}-1)dl = 2\pi e^{-\lambda n|t|} - 2\pi - \int_{\pi}^{\infty} \frac{4\sigma^2|t|}{Q(t,l)}(\cos nl - 1)dl$$

and multiply by $w(t)$ the obtained derivatives, and you see

$$\int_{-\pi}^{\pi} F_1(t,l)(e^{inl}-1)dl = 2\pi(\lambda n)^2 e^{-\lambda n|t|}w(t) - 2\int_{\pi}^{\infty} F_1(t,l)(\cos nl - 1)dl \qquad (t \neq 0). \tag{4.1}$$

Then deduce that as $s \to \infty$

$$\begin{aligned}&\frac{1}{4\pi^2}\int_{-\pi}^{\pi} e^{-ist}\,dt \int_{-\pi}^{\pi} F_1(t,l)(e^{inl}-1)dl\\ &= \frac{\lambda^2 n^3}{\pi\|(s,-n)\|^2} - 4\pi(\lambda n)^2\int_0^{\infty}[1-w(t)]e^{-\lambda nt}\cos st\,dt + O\Big(\frac{1}{s^2}\Big),\end{aligned}$$

which, the last integral being $O(|s|^{-N})$ (since $n^{N+2}\int_0^{\infty}(1-w)e^{-\lambda nt}dt$ is bounded), shows the asymptotic formula of the lemma. □

Remark 3. For $n \neq 0$ the asymptotic formula of Lemma 4.1 is valid and unaltered even if e^{inl} replaces $e^{inl} - 1$. It is noted however that for the derivation of (3.5) the factor $e^{inl} - 1$ plays a significant role and can not be replaced by e^{inl}.

Lemma 4.2 (i) $\rho(t) = \sigma^2|t| + o(|t|^{1+\delta})$, $\rho'(t) = \pm\sigma^2 + o(|t|^\delta)$ *and* $\rho''(t) = o(|t|^{\delta-1})$ *as* $t \to \pm 0$*;*
(ii) *for* $|t_1| < |t_2|$*, as* $t_1 - t_2 \to 0$

$$|\rho''(t_1) - \rho''(t_2)| = o\Big(\frac{|t_1 - t_2|}{|t_1|^{2-\delta}}\Big) + o\Big(\frac{|t_1 - t_2|^\delta}{|t_1|}\Big); \tag{4.2}$$

(iii) *moreover, if*

$$\zeta(t) = -\frac{\rho^2}{2\pi}\int_{-\pi}^{\pi} \frac{\partial_t^2\psi + E[X^2e^{ilY}]}{(1-\psi)^2}dl \tag{4.3}$$

and $\tilde\rho''(t) = \rho''(t) - \zeta(t)$*, then* $\tilde\rho''$ *is differentiable for* $|t| > 0$ *and* $\tilde\rho'''(t) = o(|t|^{\delta-2})$ *as* $t \to 0$

Proof. The first estimate of (i) is the same as (2.6). Similarly to the verification of it one deduces $\pi_0'(t) = -(\sigma^2t^2)^{-1}t/|t| + o(|t|^{\delta-2})$, and substitution from it as well as from $\rho(t) = \sigma^2|t| + o(|t|^{1+\delta})$ into $\rho' = -\rho^2\pi_0'$ gives the desired estimate of ρ'. For verification of the rest of the lemma observe

$$\rho'' = \frac{2(\rho')^2}{\rho} - \rho^2\pi_0'' \quad \text{ and } \quad \pi_0'' = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\psi''}{(1-\psi)^2}dl + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2[\psi']^2}{(1-\psi)^3}dl,$$

where $\psi' = \partial_t\psi$ and $\psi'' = \partial_t^2\psi$. Then as above we see that $\rho^2\pi_0'' \sim 2(\rho')^2/\rho = 2\sigma^2|t|^{-1}(1 + o(1))$ as $t \to 0$, hence $\rho''(t) = o(|t|^{-1})$. Put

$$\theta(t) := \frac{2\rho^2}{\sigma^2|t|^3} + \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma_1^2}{(Q/2)^2}dl - \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{2(\sigma_1^2t)^2}{(Q/2)^3}dl,$$

so that

$$\begin{aligned}\tilde\rho''(t) - \theta(t) \;=\; & \frac{2(\rho')^2}{\rho} - \frac{2\rho^2}{\sigma^2|t|^3} - \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2[\psi']^2}{(1-\psi)^3} - \frac{2(\sigma_1^2t)^2}{(Q/2)^3}\right)dl \\ & + \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{(1-\psi)^2} - \frac{1}{(Q/2)^2}\right)E[X^2e^{ilY}]dl + \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{E[X^2(e^{ilY}-1)]}{(Q/2)^2}dl.\end{aligned}$$

With the help of the estimates $\psi'' = -\sigma_1^2 + o(|t|^\delta + |l|^\delta)$, $\psi' = -\sigma_1^2t + o(|t|^{1+\delta} + |l|^{1+\delta})$ etc. an application of Lemma 2.3 shows that both $\zeta(t)$ and $\tilde\rho''(t) - \theta(t)$ are $o(|t|^{\delta-1})$. It in particular follows that $\tilde\rho''(t) = o(1/t)$, hence $\theta(t) = o(1/t)$, which in turn implies $\theta(t) = O(1)$ and $\theta'(t) = O(1/t)$ since $|t|^3\theta/\rho^2$ is 'analytic' in $|t|$. Consequently $\rho''(t) = o(|t|^{\delta-1})$. Similarly we obtain $\tilde\rho'''(t) = o(|t|^{\delta-2})$.

If $h_l(t) = \psi''(t,l)$, then $h_l(t) - h_l(t') = o(|t-t'|^\delta)$ uniformly in l, and we deduce that $\zeta(t)$ satisfies the property (4.2) in place of ρ''. In view of the estimate $\tilde\rho'''(t) = o(|t|^{\delta-2})$ the function $\tilde\rho''$ also satisfies the same property without the second term on the right side of (4.2). Thus (4.2) is verified. □

5 Estimation of $\mathrm{e}_n(t)$

In this section we consider the case $\delta = 0$ only. The results obtained in this section will be used in the succeeding sections. Set

$$\begin{Bmatrix} h_n^e(t) \\ h_n^o(t)\end{Bmatrix} = \frac{1}{2\pi n}\int_{-\pi}^{\pi}\left[\frac{1}{1-\psi(t,l)} - \frac{1}{1-\phi(l)}\right]\begin{Bmatrix}\cos nl - 1 \\ \sin nl\end{Bmatrix}dl,$$

so that $\mathrm{e}_n(t)/n = h_n^e(t) + ih_n^o(t)$. Recall that $\mathrm{e}_n(t) = \pi_{-n}(t) - \pi_0(t) + a(n)$.

Lemma 5.1 *Put* $f(x) = |x|^{-1}(e^{-|x|} - 1) + 1$ $(= \frac{1}{2!}|x| - \frac{1}{3!}|x|^2 + \cdots)$ *and define* $r_n(t)$ *via*

$$h_n^e(t) = \sigma_2^{-2}f(\lambda nt) + r_n(t).$$

Then $\lim_{t\to 0}\sup_n |r_n(t)| = \lim_{n\to\infty}\sup_t |r_n(t)| = 0$ *and* $r_n(t) = o(nt)$ *as* $nt \to 0$ *and these same estimates hold true both for* $|t|r_n'(t)$ *and for* $t^2r_n''(t)$ *in place of* $r_n(t)$*.*

Proof. By $E[XY]=0$ we have $\psi(t,l)-\phi(l) = -\frac{1}{2}\sigma_1^2 t^2 + t\times o(|t|+|l|)$ and hence

$$\frac{\psi-\phi}{(1-\psi)(1-\phi)} - \frac{-2\lambda^2 t^2}{Q(t,l)l^2} = \frac{t\times o(|t|+|l|)}{(t^2+l^2)l^2} \tag{5.1}$$

as $|t|\vee|l|\to 0$. The required estimate of $r_n(t)$ follows from

$$\frac{1}{\pi}\int_0^\pi \frac{-2\lambda^2t^2}{Q(t,l)l^2}(\cos nl-1)dl = \frac{1}{\pi}\int_0^n dt\int_0^\pi \frac{2\lambda^2t^2}{Q(t,l)}\cdot\frac{\sin tl}{l}dl = \frac{e^{-\lambda nt}-1+\lambda nt}{\sigma_2^2\lambda nt} + O(t^2/n).$$

The estimates of $|t|r_n'(t)$ are obtained by simply observing that the derivative with respect to t of the left side of (5.1) is $o(|t|+|l|)/(t^2+l^2)l^2$ (make a telescopic decomposition of the difference of two ratios in (5.1)). The second derivative is treated similarly. □

The next lemma is an immediate corollary of the preceding lemma.

Lemma 5.2 *Uniformly in $n\geq 1$ and $|t|<\pi$,* (i) $|h_n^e(t)|\asymp |nt|\wedge 1$ *(namely, $C^{-1}(|nt|\wedge 1)\leq |h_n^e(t)|\leq C(|nt|\wedge 1)$ for some constant $C>0$); and* (ii) $|t(h_n^e)'(t)|\asymp |nt|\wedge 1$.

Lemma 5.3

$$\lim_{n\to\infty}\sup_t\left|\int_{-\pi}^{\pi}\frac{1}{1-\psi(t,l)}\frac{\sin nl}{n}dl\right| \quad\longrightarrow\quad 0. \tag{5.2}$$

Moreover $\lim_{t\to 0}\sup_n|h_n^o(t)| = \lim_{n\to\infty}\sup_t|h_n^o(t)| = 0$.

Proof. Denote by $I=I(t,n)$ the integral appearing in (5.2). Then

$$I=\int_{-\pi}^{\pi}\frac{iE[e^{itX}\sin lY]}{(1-\psi(t,l))(1-\psi(t,-l))}\frac{\sin nl}{n}dl.$$

Since $E[XY]=0$, we can write

$$E[e^{itX}\sin lY] = E[e^{itX}(\sin lY - lY)] + E[(e^{itX}-1-itX)Y]l;$$

correspondingly we decompose $I=I_1+I_2$. In view of the integrability

$$\int_{-\pi}^{\pi}\frac{E|\sin lY-lY|}{|l|^3}dl = E\left[\int_{-|Y|\pi}^{|Y|\pi}\frac{|\sin u-u|}{|u|^3}du|Y|^2\right] < 4E[Y^2]<\infty, \tag{5.3}$$

the dominated convergence theorem shows that I_1 converges to zero as $n\to\infty$ uniformly in t; moreover it also follows that

$$I_1\longrightarrow\int_{-\pi}^{\pi}\frac{1}{1-\phi(l)}\frac{\sin nl}{n}dl$$

as $t\to 0$ uniformly in n. The second part I_2 converges to zero as $n\to\infty$ uniformly in $t, \varepsilon<|t|\leq\pi$ for every $\varepsilon>0$; but in the inequality

$$|I_2|\leq C\frac{E|(e^{itX}-1-itX))Y|}{|t|}\int_{-\pi}^{\pi}\frac{|t|l^2}{(t^2+l^2)^2}\left|\frac{\sin nl}{nl}\right|dl,$$

the integral on the right side is uniformly bounded and the fraction before it approaches zero as $t\to 0$, so that the convergence as $n\to\infty$ is uniform in $|t|\leq\pi$. These show the three assertions of the lemma simultaneously. □

Lemma 5.4 $\lim_{t\to 0}\sup_n|t(h_n^o)'(t)|=0$ *and* $\lim_{t\to 0}\sup_n|t^2(h_n^o)''(t)|=0$.

Proof. For the proof of the first half it suffices (owing to skew symmetry in l) to verify that as $t \to 0$

$$\int_{-\pi}^{\pi}\left[\left|E[Xe^{iXt+iYl}]\right|\cdot\left|\frac{1}{(1-\psi)^2}-\frac{4}{Q^2}\right|+\frac{4|E[Xe^{iXt}\sin Yl]|}{[Q(t,l)]^2}\right]|tl|dl \to 0.$$

But this follows from Lemma 2.3 since $E[Xe^{iXt+iYl}] = O(|t|+|l|)$ and $E[Xe^{iXt}\sin Yl] = o(|l|)$ as $|t|+|l| \to 0$). The proof of the second one is similar. □

As a consequence of Lemmas 5.1, 5.3 and 5.4 we have

Lemma 5.5 *If* $f(x) = |x|^{-1}(e^{-|x|}-1)+1$, *then* $\mathrm{e}_n(t)/|n| = \sigma_2^{-2}f(\lambda nt)+r_n(t)+h_n^o(t)$ *with* $r_n(t)+h_n^o(t)$, $t(r_n+h_n^o)'(t)$ *and* $t^2(r_n+h_n^o)''(t)$ *approaching zero as* $t \to 0$ *uniformly in* n. *In particular each of* $\mathrm{e}_n(t)/n$, $t\mathrm{e}_n'(t)/n$ *and* $t^2\mathrm{e}_n''(t)/n$ *tends to zero as* $nt \to 0$ *and are uniformly bounded.*

In the next section we apply the result of this section only for (6.2). It is noted that we have dealt and will deal with the derivatives $\mathrm{e}_n'(t)$, $\mathrm{e}_n''(t)$ but only implicitly in the estimation of the integral of F_3 through the identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}F_3(t,l)(1-e^{inl})dl = \rho(t)\mathrm{e}_n''(t)+2\rho'(t)\mathrm{e}_n'(t)-\frac{1}{2\pi}\int_{-\pi}^{\pi}F_1(t,l)(1-e^{inl})dl,$$

wherein we shall need to more closely look at them in relation to the corresponding parts involved in its last term.

6 Proof of Theorem 1.2

We are to evaluate $H_n(s)$ for $|s|>|n|$ under the condition (1.6), i.e. under $E[X^2\log|X|]<\infty$. Let g, F_1, F_2 and F_3 are functions defined in (3.4) and (3.7). Because of possible singularity at the origin we separate

$$\eta_{s,n}^{\alpha} := \frac{1}{4\pi^2}\int_{|t|<\alpha/|s|}e^{-ist}dt\int_{-\pi}^{\pi}\partial_t^2 g(t,l)(e^{inl}-1)dl$$

from the integral on the right side in (3.5) and write the latter as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(t)\mathrm{e}_n(t)e^{-ist}dt = \frac{-1}{4\pi^2 s^2}\int_{\alpha/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}\partial_t^2 g(t,l)(e^{inl}-1)dl-\frac{\eta_{s,n}^{\alpha}}{s^2}. \tag{6.1}$$

Recalling $\int\partial_t g(e^{inl}-1)dl = 2\pi(\rho\mathrm{e}_n)'$ one integrates by parts back and uses Lemma 5.5 to find that

$$\frac{1}{n}\eta_{n,s}^{\alpha} \to 0 \quad \text{as } s\to\infty \text{ and } \alpha\downarrow 0 \text{ in this order uniformly for } |n|<|s|. \tag{6.2}$$

In the same limit we have also $\int_{|t|<\alpha/|s|}e^{-ist}dt\int_{-\pi}^{\pi}F_1(t,l)(e^{inl}-1)dl \to 0$ and, in view of (3.9),

$$H_n(s) = a^*(n)H_0(s)-\frac{\lambda^2|n|^3}{\pi\|(s,n)\|^2 s^2}-\frac{1}{4\pi^2 s^2}\int_{\alpha/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}(F_2+F_3)(e^{inl}-1)dl+o\left(\frac{n}{s^2}\right).$$

Our task for the rest of the proof consists of proving that for each $\alpha>0$

$$\frac{1}{n}\int_{\alpha/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}(F_2+F_3)(e^{inl}-1)dl \to 0 \quad \text{as } s\to\infty \text{ uniformly for } |n|<|s| \tag{6.3}$$

and of obtaining a correct asymptotic form of H_0 and Lemma 4.1). For both purposes we shall make use of the assumption (1.6), which will be applied via the following lemma.

Lemma 6.1 $E[X^2\log|X|]<\infty$ *if and only if* $\iint_{[-\pi,\pi]^2}E_0\Big[X^2(1-\cos tX)\Big]\frac{dtdl}{t^2+l^2}<\infty$, *and if this is the case, then* $\iint_{[-\pi,\pi]^2}E_0\Big[X^2|1-e^{itX}|\Big]\frac{dtdl}{t^2+l^2}<\infty$.

Proof. The integral $\iint_{[-\pi,\pi]^2}(1-\cos tX)\mathbf{1}(|tX|<1)\dfrac{dtdl}{t^2+l^2}$ is dominated by

$$\int_{-\pi}^{\pi}dl\int_{|t|<1/|X|}\frac{|tX|dt}{t^2+l^2}\le 4\int_0^\infty dv\int_0^1\frac{udu}{u^2+v^2}<\infty,$$

while the same integral but on $|tX|\ge 1$ equals

$$4\mathbf{1}(|X|\ne 0)\int_{1/|X|}^{\pi}\frac{1-\cos tX}{t}dt\int_0^{\pi/t}\frac{du}{1+u^2}\sim C\log|X|$$

as $|X|\to\infty$ (with $C=4\int_0^\infty(1+u^2)^{-1}du$). Thus the equivalence of the first half of the lemma follows. The second half is proved by the same argument that is just advanced. □

Now we proceed into evaluation of H_0, in which we need to cope with the delicate circumstance that there is little information available as to regularity of ρ'' other than $\rho''(t)=o(1/t)$.

Evaluation of H_0. For simplicity let $s>0$. Then, recalling $2\pi H_0(s)=-\int_{-\pi}^{\pi}\rho(t)e^{-ist}dt$ we directly derive

$$2\pi s^2H_0(s)=\Big[\rho'(t)e^{-ist}\Big]_{t=-\alpha/s}^{\alpha/s}+is\int_{|t|<\alpha/s}\rho'(t)e^{-ist}dt+\int_{\alpha/s<|t|<\pi}\rho''(t)e^{-ist}dt \qquad (6.4)$$

independently of the arguments made above. From the relation $\rho'=\sigma^2t/|t|+o(1)$ it follows that as $s\to\infty$ and $\alpha\downarrow 0$ in this order, the first term converges to $2\sigma^2$ and the second one to zero. For the derivation of the desired estimate of H_0 we must prove that for each $\alpha>0$,

$$\lim_{s\to\infty}\int_{\alpha/s<|t|<\pi}\rho''(t)e^{-ist}dt=0. \qquad (6.5)$$

For the proof we make use of the decomposition of ρ'' in Lemma 4.2:

$$\rho''(t)=\tilde\rho''(t)+\zeta(t),\qquad \text{where}\qquad \zeta(t)=-\frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_t^2\psi+E[X^2e^{ilY}]}{(1-\psi)^2}dl. \qquad (6.6)$$

Note that $\tilde\rho''$ is differentiable for $|t|>0$, while $\zeta(t)$ may not. In any case we have

Lemma 6.2 (i) $\int_{-\pi}^{\pi}|\zeta(t)|dt<\infty$ *if* $E[X^2\log|X|]<\infty$ *and* (ii) $\tilde\rho'''(t)=o(|t|^{-2})$ *as* $t\to 0$.

Proof. (i) is immediate from Lemma 6.1. (ii) is obtained in Lemma 4.2 □

Remark to Lemma 6.2. One can show that $\int_{-\pi}^{\pi}|\rho''(t)|dt<\infty$ if $E[X^2\log|X|]<\infty$, which is useful but dispensable; we shall need to use the second assertion of the lemma in several places.

The proof of (6.5) is now given as follows. The integrability of $\zeta(t)$ implies that its contribution vanishes in the limit under consideration and that the improper integral $\int_{-\pi}^{\pi}\tilde\rho''dt$ exists. On using the latter fact together with the bound $\tilde\rho'''=o(1/t^2)$ the contribution of $\tilde\rho''$ also vanishes (see Lemma 11.2 of Appendix).

We can thus conclude that $H_0(s)=\pi^{-1}\sigma^2s^{-2}(1+o(1))$ under $E[X^2\log|X|]<\infty$.

We turn to the proof of (6.3). First consider the contribution of F_3 and break it into two parts:

$$\Theta_I=\int_{\alpha/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}[l^2F_3(t,l)]\frac{1-\cos nl}{l^2}dl,$$

$$\Theta_{II}=\int_{\alpha/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}\Big[lF_3(t,l)\Big]\frac{\sin nl}{l}dl.$$

We have

$$|F_3|\le\frac{|t|}{[Q(t,l)]^2}c(t,l) \qquad (6.7)$$

with $c(t,l)$ bounded and $c(t,l) \to 0$ $(|t|+|l| \to 0)$ and, making use of this, infer that $\Theta_I = n \times o(1)$ as $s \to \infty$ (see Lemma 11.1 of Appendix). The corresponding estimate of Θ_{II} requires the condition (1.6).

Lemma 6.3 *If $E[X^2 \log |X|] < \infty$, then $\Theta_{II}/n \to 0$ as $|s| \to \infty$ uniformly in n.*

Proof. Put

$$h(t,l) = \rho(t) \frac{\partial_t^2 \psi(t,l) + E[X^2 e^{ilY}]}{(1-\psi(t,l))^2}$$

and

$$f_n(t) = \int_{-\pi}^{\pi} [lF_3(t,l) - lh(t,l)] \frac{\sin nl}{nl} dl,$$

so that

$$\Theta_{II} = n \int_{\alpha/s<|t|<\pi} e^{-ist} dt \int_{-\pi}^{\pi} lh(t,l) \cdot \frac{\sin nl}{nl} dl + n \int_{\alpha/s<|t|<\pi} e^{-ist} f_n(t) dt. \qquad (6.8)$$

As before $f_n(t)$ is differentiable for $t \neq 0$ while $h(\cdot, l)$ may not. By the same arguments that prove Lemma 6.2 one verifies that

(i) $\int_{[-\pi,\pi]^2} |lh(t,l)| dt dl < \infty$ if $E[X^2 \log |X|] < \infty$ and

(ii) $\sup_n |f_n(t)| = o(|t|^{-1})$ and $\sup_n |f_n'(t)| = o(|t|^{-2})$ as $t \to 0$.

Taking these into account, we make use of Lemmas 11.1 and 11.2 of Appendix for the first and the second terms, respectively, to conclude that $\Theta_{II} = n \times o(1)$. □

It remains to deal with the contribution of F_2 in (6.3). Suppose $E[X^2 \log^+ |X|] < \infty$. Put

$$F_{21} = \frac{\rho''(t)}{1-\psi(t,l)} \quad \text{and} \quad F_{22} = \frac{\rho''(t)}{1-\phi(l)},$$

so that $F_2 = F_{21} - F_{22}$. In the evaluation of H_0 made above we have verified

$$\int_{\alpha/s<|t|<\pi} e^{-ist} dt \int_{|l|<\pi} F_{22}(1-e^{inl}) dl = 2\pi a(n) \int_{\alpha/s<|t|<\pi} \rho''(t) e^{-ist} dt = n \times o(1). \qquad (6.9)$$

In a similar way the integral involving $F_{21} \sin nl$ is estimated to be $n \times o(1)$ in view of (5.2).

Finally we work with $F_{21}(1-\cos nl)$. We make decomposition $\rho'' = \tilde{\rho}'' + \zeta$ in (6.6). On the one hand the integrability of ζ (Lemma 6.3 (i)) implies, by dominated convergence, that $\int_{-\pi}^{\pi} |\zeta(t)| \cdot |1-\psi|^{-1} dt = o(1/l^2)$ as $l \to 0$ and hence the contribution of ζ is $n \times o(1)$ in view of Lemma 11.1. On the other hand, noting that for some constant C

$$\int_{|l|<\pi} \frac{1}{|1-\psi|} \cdot |1-\cos nl| dl \le Cn; \qquad \int_{|l|<\pi} \left| \frac{\partial_t \psi}{(1-\psi)^2} \right| \cdot |1-\cos nl| dl \le C \frac{n}{|t|},$$

we apply Lemma 11.2 to see that the contribution of $\tilde{\rho}''$ is $n \times o(1)$. We accordingly conclude that $\int_{\alpha/s<|t|<\pi} e^{-ist} dt \int_{-\pi}^{\pi} F_{21}(1-\cos nl) dl = n \times o(1)$ for each $\alpha > 0$ as required.

The proof of (6.3) is now complete. Summarizing the results obtained above we have

Proposition 6.1 *If $E[X^2 \log |X|] < \infty$, then uniformly for $n < |s|$, as $|s| \to \infty$,*

$$\int_{-\pi}^{\pi} \rho(t) e_n(t) e^{-ist} dt = -\frac{\sigma_1^2}{\pi \sigma_2^2} \cdot \frac{|n|^3}{\|(s,-n)\|^2 s^2} + \frac{n}{s^2} \times o(1).$$

Combined with (2.11) and the estimate of $H_0(s)$ already obtained this formula yields

$$H_n(s) = \frac{\sigma_2^2 a^*(n)}{\pi \|(s,n)\|^2} (1+o(1)) \qquad (|s| > n),$$

which together with the first half of Theorem 1.1 finish the proof of Theorem 1.2 (in view of the asymptotic form of $a(n)$ as mentioned in Remark 1 (i)).

7 Proof of Theorem 1.3

Let $s > 0$ throughout this section. First consider H_0. The proof starts from the expression of $2\pi s^2 H_0(s)$ given in (6.4). The sum of the first two terms of it equals

$$\Big[\rho'(t)e^{-ist}\Big]_{t=-\alpha/s}^{\alpha/s} + is\int_{|t|<\alpha/s}\rho'(t)e^{-ist}dt = 2\sigma^2 + \int_{|t|<\alpha/s}\rho''(t)e^{-ist}dt.$$

But $\int_{|t|<\alpha/s}\rho''(t)dt = (\rho'(\alpha/s)-\sigma_2^2)+(-\sigma_2^2-\rho'(-\alpha/s)) \to 0$ and $\int_{|t|<\alpha/s}|\rho''(t)(e^{-ist}-1)|dt \to 0$ as $s\to\infty$ locally uniformly in α, so that

$$2\pi s^2 H_0(s) = 2\sigma^2 + \int_{\alpha/s<|t|<\pi}\rho''(t)e^{-ist}dt + o(1), \tag{7.1}$$

where $o(1)\to 0$ as $s\to\infty$ for each $\alpha>0$. Thus, on letting $\alpha=1$, the assertion of Theorem 3 is paraphrased that

$$\int_{1/s<|t|<\pi}\rho''(t)e^{-ist}dt = \frac{2}{\lambda}M_\varepsilon(s)(1+o(1))+o(1). \tag{7.2}$$

Recall (6.6), i.e., the decomposition $\rho''=\tilde\rho''+\zeta$ made in (iii) of Lemma 4.2, where we observed that $\int_{1/s<|t|<\pi}\tilde\rho''(t)e^{-ist}dt\to 0$ (under the existence of the second moment only); hence (7.2) is reduced to

$$J(s) := \int_{1/s<|t|<\pi}\zeta(t)e^{-ist}dt = \frac{2}{\lambda}M_\varepsilon(s)(1+o(1))+o(1). \tag{7.3}$$

The proof of this relation is given below, of which the method is the same as one devised in [9] for similar formulae.

We write

$$\zeta(t) = \frac{\rho^2(t)}{2\pi}\int_{-\pi}^{\pi}\frac{E[X^2e^{ilY}(e^{itX}-1)]}{(1-\psi(t,l))^2}dl$$

and decompose $\zeta=\zeta_0+\zeta_1+\zeta_2$ where

$$\zeta_0(t) = \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{E[X^2e^{ilY}e^{itX}(1-w(Xt))]}{(1-\psi)^2}dl; \qquad \zeta_1(t) = -\frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{E[X^2e^{ilY}(1-w(Xt))]}{(1-\psi)^2}dl$$

and

$$\zeta_2(t) = \frac{\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{E[X^2e^{ilY}(e^{itX}-1)w(Xt)]}{(1-\psi)^2}dl.$$

Here w is a smooth function introduced just prior to (3.6). We may suppose that $w(t)=1$ for $|t|<1/2$.

Similarly to the proof of Lemma 6.1 we see that

$$\int_{-\pi}^{\pi}|\zeta_2(t)|dt \le C\int_0^1 du\int_0^\infty\frac{u^2|e^{iu}-1|}{(u^2+v^2)^2}dv<\infty,$$

which in view of the Riemann-Lebesgue lemma implies that

$$\int_{1/s<|t|<\pi}\zeta_2(t)e^{-ist}dt = o(1). \tag{7.4}$$

Note that $|w'(Xt)Xt|$ as well as $1-w(Xt)$ is bounded by a constant times the indicator function $\mathbf{1}_{\{|Xt|>1/2\}}$. On observing that $\zeta_1(t)=o(1/t)$ and $\zeta_1'(t)=o(1/t^2)$, an integration by parts gives

$$\int_{1/s<|t|<\pi}\zeta_1(t)e^{-ist}dt = o(1).$$

We are left with ζ_0. Split the range of integration into three parts according as $|X-s|\ge\varepsilon s$, $|X-s|<\varepsilon s$ or $X=s$ and call J_1, J_2 and J_3, respectively, their contributions to $\int_{1/s<|t|<\pi}\zeta_0(t)e^{-ist}dt$.

We then integrate by parts with respect to t by factorizing the integrand as $e^{it(X-s)}\times$(the other) to deduce that for each $\varepsilon > 0$, as $s \to \infty$

$$\begin{aligned}|J_1| &= \frac{1}{2\pi}\bigg|\int_{1/s<|t|<\pi}\rho^2 dt\int_{-\pi}^{\pi}\frac{E[X^2 e^{it(X-s)}(1-w(Xt))e^{ilY};|X-s|\geq\varepsilon s]}{(1-\psi)^2}dl\bigg|\\ &\leq CE\bigg[\frac{sX^2}{|X-s|};|X-s|\geq\varepsilon s,|X|\geq\frac{s}{2}\bigg]+C\int_{1/s}^{\pi}E\bigg[\frac{X^2}{|X-s|};|X-s|\geq\varepsilon s,|X|\geq\frac{1}{2t}\bigg]\frac{dt}{t^2}\\ &\longrightarrow 0.\end{aligned}$$

Here for the inequality we have made use of the relations $\rho(\pi) = \rho(-\pi)$, $w(\pm X\pi) = 1$ for $X \neq 0$, $s^{-2}\int_{-\pi}^{\pi}(s^{-2}+l^2)^{-2}dl = O(s)$ and $t\int_{-\pi}^{\pi}(t^2+l^2)^{-2}dl = O(1/t^2)$.

We may suppose that $\varepsilon < \frac{1}{4}$ and $w(x) = 0$ if $|x| \geq 3/4$, so that on the event $|X - s| < \varepsilon s$, $1 - w(Xt) = 1$ for $|t| \geq 1/s$. Taking this into acount we write J_2 in the form

$$J_2 = E\bigg[X^2\int_{1/s<|t|<\pi}\frac{e^{it(X-s)}}{|t|}h(t)dt;\ 1\leq|X-s|<\varepsilon s\bigg], \tag{7.5}$$

where

$$h(t) = h(t,Y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|t|\rho^2 e^{ilY}}{(1-\psi)^2}dl.$$

Split the range of the integral under the expectation symbol in (7.5) according as $|(X - s)t| > 1$ or ≤ 1 and call J_{21} and J_{22} the corresponding parts of J_2.

By integration by parts

$$|J_{21}| \leq E\bigg[X^2\int_{1/|X-s|<|t|<\pi}\bigg|\int_t^{(\mathrm{sign}\,t)\infty}\frac{e^{i(X-s)u}}{|u|}du\,h'(t)\bigg|dt;\ 1\leq|X-s|<\varepsilon s\bigg]+o(1), \tag{7.6}$$

where $\mathrm{sign}\,t = t/|t|$. Since $|\int_{|t|}^{\infty}e^{\pm i|X-s|u}u^{-1}du| \leq C/|(X-s)t|$ and $|h'(t)| \leq C/|t|$, the outer integral under the expectation is bounded, hence J_{21} is $o(1)$ (as $s \to \infty$).

For evaluation of J_{22} we further decompose it into three parts in the same way as we did J but by means of Y in place of $X - s$ and call J_{221}, J_{222} and J_{223} those that correspond to $|Y| \geq \varepsilon s$, $0 < |Y| < \varepsilon s$ and $Y = 0$, respectively. The same method that is applied to J_1 verifies $J_{221} = o(1)$. In what follows we prove that the remaining parts, i.e., J_{222}, J_{223} and J_3, together constitute $M_\varepsilon(s)$. For simplicity we consider the double integral involved in J_{222} only on the first quadrant $t > 0, l > 0$, and let $\tilde{J}_{222}$ denote the corresponding one fourth of it, so that

$$\tilde{J}_{222} = \frac{1}{2\pi}E\bigg[X^2\int_{1/s}^{1/|X-s|}dt\int_0^{\pi}\frac{\rho^2 e^{i[t(X-s)+lY]}}{(1-\psi)^2}dl;\ 1\leq|X-s|<\varepsilon s, 1\leq|Y|<\varepsilon s\bigg].$$

By means of the indefinite integral $\int_l^{\infty}u^{-1}e^{iYu}du$ we integrate by parts as in (7.6) to see that the upper limit π of the inner integral may be replaced by $\alpha/|Y|$ with an arbitrarily small $\alpha > 0$; also $1/|X - s|$ may be replaced by $\alpha/|X - s|$. These in turn allow $e^{i[t(X-s)+lY]}$ in the integrand to be replaced by 1. Now observing that

$$\int_{1/s}^{1/|X-s|}dt\int_0^{1/|Y|}\frac{\rho^2 dl}{(1-\psi)^2} = \int_{1/s}^{1/|X-s|}g(Yt)\frac{dt}{t}(1+o(1)) \quad \text{where} \quad g(x) = \int_0^{1/|x|}\frac{4\sigma^4 du}{[Q(1,u)]^2}, \tag{7.7}$$

that $g(x) \to \pi/\lambda$ $(x \to 0)$ and $g(x) < C/|x|$ $(|x| > 1)$ and then that as $(|X - s| \vee |Y|)/s \to 0$, the double integral above takes on the form $(\pi/\lambda)\log[s/(|X - s| \vee |Y|)] \cdot (1 + o(1))$, we obtain

$$J_{222} = \frac{2}{\lambda}E\bigg[X^2\log\frac{s}{|X-s|\vee|Y|};1\leq|X-s|<\varepsilon s, 1\leq|Y|<\varepsilon s\bigg](1+o(1))+o(1).$$

In the same way we also obtain

$$J_{223} = \frac{2}{\lambda}E\bigg[X^2\log\frac{s}{|X-s|};1\leq|X-s|<\varepsilon s, Y=0\bigg](1+o(1))+o(1);$$

$$J_3 = s^2 E\Big[\int_{1/s<|t|<\pi} \frac{h(t)}{|t|}dt;\ X=s\Big] = \frac{2}{\lambda}E\Big[X^2\log\frac{s}{|Y|\vee 1};X=s,|Y|<\varepsilon s\Big](1+o(1))+o(1).$$

Thus we conclude $J_{222}+J_{223}+J_3 = 2\lambda^{-1}M_\varepsilon(s)(1+o(1))+o(1)$. The proof of (7.2) is complete.

The case $n\neq 0$. In the proof of Theorem 1.2 the term Θ_I and the second half of Θ_{II} in (6.8) are disposed of without using the extra moment condition; the contribution of F_{22} given in (6.9) cancels out with that involved in $a(n)H_0(s)$ (see (7.1)). The remaining terms among those arising from F_2+F_3 are the first half of Θ_{II} in (6.8) and the contribution of F_{21}, both of which we must examine. As for F_{21} the factor $\rho''(t)$ may be replaced by $\zeta(t)$. From these observations it follows that

$$H_n(s) = \frac{1}{\pi}\Big(\frac{\sigma_2^2 a^*(n)}{s^2/\lambda} - \frac{\lambda^2|n|^3}{\|(s,-n)\|^2 s^2}\Big)[1+o(1))] + \frac{J^A+J^B}{2\pi s^2} \qquad (|s|>\varepsilon|n|,\ |s|\to\infty),$$

where

$$J^A = -\int_{1/s<|t|<\pi}[\mathrm{e}_n(t)-a(n)]\zeta(t)e^{-ist}dt$$

$$J^B = \frac{-i}{2\pi}\int_{1/s<|t|<\pi}e^{-ist}dt\int_{-\pi}^{\pi}\rho(t)\frac{E[X^2e^{iYl}(e^{iXt}-1)]}{(1-\psi)^2}\sin nl dl$$

and $o(1)$ is uniform in n. The estimation of J^A and J^B proceeds parallel to that of J and we only point out what modification is needed. To this end it is convenient to write ζ^A, J^A_k etc. for the corresponding functions.

First we prove that

$$J^B = |n|M_\varepsilon(s)\times o(1). \tag{7.8}$$

We must show that the convergence is uniform in n. Those for which this matters are the convergence in (7.4) with ζ_2^B in place of ζ_2 and the estimation of $J^B_{222}+J^B_{222}+J^B_3$. The former one, for which the Riemann-Lebesgue lemma is used, can be disposed of by applying Lemma 11.1. For the latter we observe that first $1-\psi$ may be replaced by $\frac{1}{2}Q$ and then e^{iYl} by $i\sin Yl$; hence it suffices to verify

$$E\Big[X^2\int_{1/s}^{1/|X-s|\vee 1}dt\int_0^{1/|Y|}\frac{\rho l\sin Yl}{[Q(t,l)]^2}dl;\ |X-s|<\varepsilon s, 0<|Y|<\varepsilon s\Big] = o(1).$$

This however is obvious from $\int_0^1 dt\int_0^\alpha (t^2+l^2)^{-2}tl^2dl = \frac{1}{2}\arctan\alpha \le \alpha/2$ $(\alpha>0)$. Thus (7.8) has been proved.

Since both $\mathrm{e}_n(t)/n$ and $t\mathrm{e}_n'(t)/n$ are uniformly bounded and $\mathrm{e}_n(t)/n\to 0$ as $t\to 0$, the very same arguments leading to (7.3) verifiy that $s\to\infty$

$$J^A = 2\lambda^{-1}a(n)M_\varepsilon(s)(1+o_b(1)),$$

but here uniformity of $o_b(1)$ is not claimed (and not true in general). To have some uniform estimate we use the decomposition $\mathrm{e}_n(t) = |n|[f(\lambda nt)/\sigma_2^2 + r_n(t) + h_n^o(t)]$ given in Lemma 5.5. According to the estimates stated therein the contribution of $r_n(t)+h_n^o(t)$ is $o(1)$. We then infer from (7.3) that

$$J^A = \frac{2}{\lambda}\Big(a(n)-\frac{|n|}{\sigma_2^2}\Big)M_\varepsilon(s)(1+o(1)) + \int_{1/s<|t|<\pi}\frac{|n|(1-f(\lambda nt))}{\sigma_2^2}\zeta(t)e^{-ist}dt + o(1). \tag{7.9}$$

On arguing as in the case of J_{222} (we have the additional factor $|n|(1-f(\lambda nt)) = (1-e^{-\lambda|nt|})/\lambda|t|$ in the second integral in (7.7)) the last integral may be written in the form

$$\frac{2|n|}{\lambda\sigma_2^2}E\Big[X^2\log\frac{s}{|X-s|\vee|Y|\vee n};1\le|X-s|\vee|Y|<\varepsilon s\Big](1+o(1))+o(1). \tag{7.10}$$

Finally use the bound $a(n)\ge|n|/\sigma_2^2$ ([7]:P31.1) and note that the expectation above is not larger than $M_\varepsilon(s)$ to obtain the second assertion of Theorem 1.3. Thus its proof is complete.

8 Proof of Theorem 1.4

Although Theorem 1.4 can be derived from Theorem 1.3 by elementary computations, here is given a direct proof. We are to make estimation of the integrated tail $E_n(s) = H_n(s) + H_n(s+1) + \cdots$ in the case $\delta = 0$. For each $\alpha > 0$ define a probability, c_α say, on $\mathbf{Z}$ by

$$c_\alpha(0) = \frac{1}{\pi\alpha} - \frac{2e^{-2\pi\alpha}}{1-e^{-2\pi\alpha}}; \qquad c_\alpha(s) = \frac{1}{\pi}\cdot\frac{\alpha}{\alpha^2+s^2} \qquad s \neq 0$$

and put $C_\alpha(s) = c_\alpha(s) + c_\alpha(s+1) + \cdots$. Using the Poisson summation formula one sees that

$$\hat{c}_\alpha(t) := \sum_s c_\alpha(s)e^{ist} = \sum_s e^{-\alpha|t+2\pi s|} - \frac{2e^{-2\pi\alpha}}{1-e^{-2\pi\alpha}};$$

hence

$$C_\alpha(s) = \frac{\alpha}{\pi s} + O\Big(\frac{\alpha}{(\alpha^2+s^2)^2}\Big) \quad \text{as} \quad \frac{\alpha}{s} \to 0 \quad \text{and} \quad \hat{c}_\alpha(t) = e^{-\alpha|t|} + r_\alpha(t)$$

with $r_\alpha(t)$ being differentiable arbitrarily many times, $r_\alpha(0) = r'_\alpha(0) = 0$ and $r''_\alpha(t)$ bounded for $\alpha \geq 1$, $|t| < \pi$. In view of (2.11) and Lemma 5.5, $\hat{H}_n(t) = 1 - a^*(n)\rho(t) + o(nt)$ as $nt \to 0$, in particular $|\hat{H}_n(t) - \hat{c}_{\sigma^2 a^*(n)}(t)|/t$ is integrable about the origin. Also, as in the proof of Lemma 5.2, one observes that $\hat{H}'_n(t) = O(n)$ and $\hat{H}''_n(t) = O(n/t)$ $(t \neq 0)$. Now we take $\alpha = \sigma^2 a^*(n)$ and compare E_n with C_α:

$$E_n(s) - C_{\sigma^2 a^*(n)}(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\hat{H}_n(t) - \hat{c}_{\sigma^2 a^*(n)}(t)}{1-e^{-it}} e^{-ist}dt.$$

Splitting the range of integration at $|t| = K/s$ and making integration by parts for the integral over $K/s \leq |t| \leq \pi$ one infers that the integral on the right side may be written as

$$-\frac{1}{s}\int_{K/s<|t|<\pi} \frac{\hat{H}'_n(t)}{1-e^{-it}} e^{-ist}dt - \frac{2\sigma^2 a^*(n)}{s}\int_{K/s}^{\pi} \frac{e^{-\sigma^2 a^*(n)t}}{t}\cos st\,dt + o\Big(\frac{|n|\vee 1}{s}\Big)$$

as $s/(|n|\vee 1) \to \infty$ for each $K > 1$. On integrating by parts once more the first integral is dominated by a constant times $|n|/K$. On the other hand changing the variable shows that the second integral is dominated by $1/K$. These together verify that as $s/(|n| \vee 1) \to \infty$, $E_n(s) - \sigma^2 a^*(n)/\pi s = o((|n| \vee 1)/s)$ as desired.

9 The case $\delta \geq 1$

In this section we consider the case when the moment condition (1.1) holds for some $\delta \geq 1$. Main results are given in (9.5) and (9.7) where all the third moments are supposed to vanish.

Make decomposition $\pi_0(t) = I(t) + II(t)$, where

$$I(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2dl}{Q(t,l)} \qquad \text{and} \qquad II(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\frac{1}{1-\psi(t,l)} - \frac{2}{Q(t,l)}\Big]dl.$$

Then

$$I(t) = \frac{2\lambda}{\pi|t|\sigma_1^2}\Big(\frac{\pi}{2} - \arctan\frac{\lambda|t|}{\pi}\Big) = \frac{1}{\sigma^2|t|} - \frac{2}{(\sigma_2\pi)^2} + a_2|t|^2 + \cdots. \tag{9.1}$$

For evaluation of $II(t)$ we set $f(t,l) = \psi(t,l) - 1 + \frac{1}{2}Q(t,l)$ and further decompose

$$II(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(0,l)}{(1-\psi(t,l))Q(t,l)}dl + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t,l)-f(0,l)}{(1-\psi(t,l))Q(t,l)}dl. \tag{9.2}$$

Suppose $1 \le \delta < 2$. Then, the first term on the right side can be written as $C^*/\sigma_2^2 + 2/(\sigma_2\pi)^2 + o(|t|^{\delta-1})$, where we have used the identity

$$C^* - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\sigma_2^2}{1-\phi(l)} - \frac{2}{l^2}\right]dl = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{2}{l^2} - \frac{1}{\cos l}\right]dl = -\frac{2}{\pi^2}$$

(the first equality is simply by definition of C^* given in REMARK 1 (ii)). On the other hand

$$f(t,l) - f(0,l) = -\frac{i}{6}E[X^3]t^3 - \frac{i}{2}E[X^2Yt^2l + XY^2tl^2] + o(|t|^{2+\delta} + |t||l|^{1+\delta}),$$

as $|t| + |l| \to 0$, which together with $\int_0^\infty [Q(1,u)]^{-2}du = \pi/4(\sigma_1\sigma)^2$ and $\int_0^\infty u^2[Q(1,u)]^{-2}du = \pi/4(\sigma_2\sigma)^2$ shows that the second term in (9.2) equals $-\frac{i}{6}(\sigma_1\sigma)^{-2}E[X^3] - \frac{i}{2}(\sigma_2\sigma)^{-2}E[XY^2])\,\mathrm{sign}\,t + o(|t|^{\delta-1})$ in view of Lemma 2.3. It is worth noting that the term lead by $E[X^2Y]$ contributes only a constant multiple of $|t|$ (the next order term of the expansion) since it is an odd function of l. In any way, combined with (9.1), these estimates give

$$\sigma^2\pi_0(t) = |t|^{-1} + \lambda C^* - i((6\sigma_1^2)^{-1}E[X^3] + (2\sigma_2^2)^{-1}E[XY^2])\,\mathrm{sign}\,t + o(|t|^{\delta-1}),$$

or what is the same thing,

$$\rho(t) = \sigma^2|t| - \sigma_1^2 C^* t^2 + iC^\sharp t|t| + o(|t|^{\delta+1}). \tag{9.3}$$

where $C^\sharp = (6\lambda)^{-1}E[X^3] + 2^{-1}\lambda E[XY^2]$. One easily obtains that $\rho'''(t) = o(|t|^{\delta-2})$ ($t \ne 0$) and then that

$$H_0(s) = \frac{\sigma^2}{\pi s^2} - \frac{2C^\sharp}{s^3} + o\left(\frac{\log|s|}{|s|^{2+\delta}}\right) \qquad (|s| \to \infty).$$

(If $\delta > 2$, the error term may be replaced by $O(|s|^{-4})$.

Now we turn to the integral (2.7). The essential contributions to it of the terms $-\sigma_1^2 C^* t^2$ and $iC^\sharp|t|t$ in (9.3) are given by

$$\frac{-\sigma_1^2 C^*}{4\pi^2}\iint_{[-\pi,\pi)^2}\frac{t^2 w(t)}{1-\psi(t,l)}e^{i(-st+nl)}dtdl = -\frac{C^*}{\pi}\cdot\frac{\lambda n^2 - \lambda^{-1}\tilde{s}^2}{\|(s,-n)\|^4} + o\left(\frac{\log(|n| \vee |\tilde{s}| \vee 2)}{\|(s,-n)\|^3}\right) \tag{9.4}$$

and $$\frac{iC^\sharp}{4\pi^2}\iint_{[-\pi,\pi)^2}\frac{t|t|w(t)}{1-\psi(t,l)}e^{i(-st+nl)}dtdl = \frac{C^\sharp}{\pi(\lambda\sigma)^2}\cdot\frac{2|n\tilde{s}|}{\|(s,-n)\|^4} + o\left(\frac{\log(|n| \vee |\tilde{s}| \vee 2)}{\|(s,-n)\|^3}\right)$$

(as $\|(s,-n)\| \to \infty$ for both formulae), respectively. Here we truncate the integrand by $w(t)$ since the functions t^2 and $t|t|$ are not periodic.

Suppose that $|n| > \varepsilon|\tilde{s}|$ and $\delta \ge 1$. Apart from the leading term there arises a rational function of $(\tilde{s}, n)$ similar to the above ones plus an error term, which together may be written in the form $\{s,n\}^4\|(s,-n)\|^{-6} + o(n^{-2-\delta})$ if $\delta < 2$, where $\{s,n\}^4$ represents a homogeneous polynomial of (s,n) of degree 4 with coefficients that are linear combinations of the third moments $E[X^jY^{3-j}]$ ($j = 0,1,2,3$).

If $\delta \ge 2$, one obtains the next order estimate and in the case when all the third moments vanish the result is given in a rather simple form:

$$H_n(s) = \frac{1}{\pi}\cdot\frac{|n|}{\|(s,-n)\|^2} - \frac{C^*}{\pi}\cdot\frac{\lambda n^2 - \lambda^{-1}\tilde{s}^2}{\|(s,-n)\|^4} + O\left(\frac{1}{n^3}\right) \qquad (|n| > \varepsilon|\tilde{s}|,\ |n| \to \infty). \tag{9.5}$$

For the case $|\tilde{s}| > \varepsilon|n|$ we examine the contribution of $-\sigma_1^2 C^* t^2$ to the integral (2.13). To this end we follow the arguments given in the second half of Section 3 with ρ replaced by $-\sigma_1^2 C^* t^2$. Suppose Q to be diagonal and consider the integral corresponding to that involving F_1. Then similarly to the proof of Lemma 4.1 one finds that

$$\frac{1}{4\pi^2 s^2}\int_{-\pi}^{\pi}e^{-ist}dt\int_{-\pi}^{\pi}w(t)\partial_t^2\left[\frac{-\sigma_1^2 C^* t^2}{Q(t,l)}\right](e^{inl}-1)dl = -C^*\left(\frac{\lambda n^2 - \lambda^{-1}s^2}{\pi\|(s,-n)\|^4} + \frac{\lambda}{\pi s^2}\right) + O(|s|^{-N}) \tag{9.6}$$

for each $N > 0$. On the other hand on arguing with $\rho + \sigma_1^2 C^* t^2$ in place of ρ the error term in (2.13) becomes $o((n \log |s|)/|s|^{3+\delta})$ if $1 \le \delta \le 2$ plus a term involving the third moments as its coefficient; the latter is of the order magnitude $O(n/|s|^3)$.

Again suppose that $\delta > 2$ and all the third moments vanish. Then for $|\tilde{s}| > \varepsilon |n|$, as $|\tilde{s}| \to \infty$

$$H_n(s) = \frac{1}{\pi}\Big(\frac{\sigma_2^2 a^*(n)}{\tilde{s}^2/\lambda} - \frac{\lambda^2 |n|^3}{\|(s,-n)\|^2 \tilde{s}^2}\Big) - \frac{C^*}{\pi}\Big(\frac{\lambda n^2 - \lambda^{-1}\tilde{s}^2}{\|(s,-n)\|^4} + \frac{\lambda}{\tilde{s}^2}\Big) + O\Big(\frac{n}{|s|^4}\Big). \tag{9.7}$$

(If $\delta = 2$, we need the logarithmic factor in the numerator in the O term as in (9.4).) In view of the expansion $\sigma_2^2 a^*(n) = |n| + C^* - (\operatorname{sign} n) E[Y^3]/3\sigma_2^2 + O(1/n)$ (cf. Appendix of [12]) the two formulae (9.5) and (9.7) are consistent. The last O terms in them seem to represent the correct order (i.e. not replaced by a smaller order term) in general. (This is the case at least for simple random walk for which $a(n) = |n|$ and the second order term can be explicit.)

Proof of (9.7) is outlined below. First note that the third order term of the expansion of $\rho(t)$ takes on the form $a|t|^3$ (apart from λ and σ the coefficient a involves only the two fourth moments $E[X^4]$ and $E[X^2Y^2]$; $E[Y^4]$ is absorbed in C^* and both $E[X^3Y]$ and $E[XY^3]$ contribute only the next order term). On the one hand we compute $I_3 := \int |t|^3 \mathrm{e}_n(t) e^{-ist} dt$ in a similar way to the above and find that $I_3 = O(n/s^4)$. On the other hand under the present assumptions on the moments we have $|\partial_t^2 F_3| \le |Q|^{-3/2}$, the contribution to the double integral in (3.9) of its even (w.r.t. l) part is directly evaluated to be $O(n/s^2)$ (hence $O(n/s^4)$ as a whole). For the odd part use the last assertion of Lemma 11.4 to have the same bound. As for the function F_2 the factor ρ'' in it may be understood to be $o(t^{\delta-1})$ since the terms up to the order of magnitude $O(t^3)$ may be considered as being subtracted from ρ. Hence its contribution is negligible and (9.7) is concluded.

10 The case $d \ge 3$ and reduction to the diagonal case.

This section concerns the random walk on $\mathbf{Z}^d$ for $d \ge 2$. In the first half of it we exhibit some algebraic manipulations which reduce our problem to that in the case when Q is diagonal and in the second some higher dimensional analogue of what is given in Section 2.

Let X and Y be, respectively, $\mathbf{Z}^{d-1}$-valued and $\mathbf{Z}$-valued random variables of zero mean and finite variance. Let R be the covariance matrix of X, $\gamma = E[YX]$ $(\in \mathbf{R}^{d-1})$ and $\sigma_d^2 = E[Y^2]$, so that the covariance matrix of $S_1 := (X, Y)$ and its quadratic form are given by

$$Q = \left(\begin{array}{c|c} R & \gamma \\ \hline {}^t\gamma & \sigma_d^2 \end{array}\right),$$

and $Q(\theta, l) = R(\theta) + 2(\gamma\cdot\theta)l + \sigma_d^2 l^2$ $(\theta \in \mathbf{R}^{d-1},\ l \in \mathbf{R})$, respectively. Set

$$\mu = \gamma/\sigma_d^2 \quad \text{and} \quad \tilde{X} = X - \mu Y$$

and let $\tilde{R}$ be the covariance matrix of random $(d-1)$-vector $X - \mu Y$. Each component of $\tilde{X}$ is perpendicular to Y; $\tilde{R}$ is the $(d-1)$-dimensional symmetric matrix whose quadratic form is given by

$$\tilde{R}(\theta) = E[(\tilde{X}\cdot\theta)^2] = R(\theta) - \sigma_d^2(\mu\cdot\theta)^2 \qquad (\theta \in \mathbf{R}^{d-1}),$$

so that $\tilde{R} = R - \sigma_d^2 \mu\, {}^t\mu$ and

$$Q(\theta, l) = E[(\tilde{X}\cdot\theta + Y\tilde{l})^2] = \tilde{R}(\theta) + \sigma_d^2 \tilde{l}^2 \quad \text{where} \quad \tilde{l} = l + \mu\cdot\theta.$$

$\tilde{R}$ is positive definite. Indeed, for $\theta \ne 0$, taking $l = -\mu\cdot\theta$, we have $\tilde{R}(\theta) = Q(\theta, l) > 0$.

Denote by $\tilde{Q}$ the covariance matrix of $(\tilde{X}, Y)$ and let A be the $d \times d$ matrix which transforms (θ, l) into $(\theta, \mu\cdot\theta + l)$, and A^{-1} its inverse:

$$\tilde{Q} = \left(\begin{array}{c|c} \tilde{R} & 0 \\ \hline 0 & \sigma_d^2 \end{array}\right), \quad A = \left(\begin{array}{c|c} I & 0 \\ \hline {}^t\mu & 1 \end{array}\right), \quad {}^tA^{-1} = \left(\begin{array}{c|c} I & -\mu \\ \hline 0 & 1 \end{array}\right),$$

where I stands for the unit matrix. Then

$$Q(\theta,l) = \tilde{Q}(\theta,\tilde{l}) = \tilde{Q}(A[\theta,l]),$$

where $A[\theta,l]$ denotes the action of A on the d-column vector $[\theta,l]$ consisting of a $(d-1)$-vector θ and a number l; hence $Q = {}^tA\tilde{Q}A$, $\det Q = \det\tilde{Q}$ and

$$Q^{-1}(x,k) = \tilde{Q}^{-1}(\tilde{x},k) \qquad \text{where} \quad \tilde{x} = x - \mu k. \tag{10.1}$$

Now we consider a random walk S_n on $\mathbf{Z}^d$ with the law of S_1 under P_0 equal to that of (X,Y) and the corresponding hitting distribution $H_n(x), (x,0) \in L_d$ of the hyper plane $L_d = \{(x,0) : x \in \mathbf{Z}^{d-1}\}$. Let $\psi(\theta,l) = E[e^{iX\cdot\theta+iYl}]$ $((\theta,l) \in T_{d-1}\times[-\pi,\pi),\ T_{d-1} = [-\pi,\pi]^{d-1})$. The functions $\pi_n(\theta)$ of $\theta \in T_d \setminus \{0\}$ are defined in the same way as in the case $d = 2$ and the obvious analogue of Fourier representation of $H_n(x)$ by it is valid:

$$\pi_k(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{ikl}dl}{1-\psi(\theta,l)};\quad H_n(x) = \frac{1}{(2\pi)^{d-1}}\int_{T_{d-1}}\frac{\pi_{-n}(\theta)}{\pi_0(\theta)}e^{-ix\cdot\theta}d\theta \quad (n \neq 0),$$

and similarly for H_0. It therefore follows that if $\tilde{\psi}(\theta,l) = E[e^{i\tilde{X}\cdot\theta+iYl}]$ $(= \psi(\theta, l-\mu\cdot\theta)\,)$ and

$$\tilde{\pi}_k(\theta) = \frac{1}{2\pi}\int_{\mu\cdot\theta-\pi}^{\mu\cdot\theta+\pi}\frac{e^{-ikl}}{1-\tilde{\psi}(\theta,l)}dl = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-ikl}}{1-\tilde{\psi}(\theta,l)}dl \qquad (\theta \neq 0)$$

(note that $\tilde{\psi}(\theta,\cdot)$ is periodic), then $\pi_k(\theta) = \tilde{\pi}_k(\theta)e^{ik\mu\cdot\theta}$ so that

$$H_n(x) = \frac{1}{(2\pi)^{d-1}}\int_{T_{d-1}}\frac{\tilde{\pi}_{-n}(\theta)}{\tilde{\pi}_0(\theta)}e^{-i(x+n\mu)\cdot\theta}d\theta \qquad (n \neq 0). \tag{10.2}$$

By virtue of the last formula we may suppose that $Q(\theta,l)$ is of the form $R(\theta)+\sigma_d^2 l^2$ for evaluation of $H_n(x)$, in particular for the proofs of Theorems 1 and 2 (see below for more details).

Let $d \geq 3$. The principal term of H_n in the higher dimensions is derived by using the classical formula

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\frac{|\theta|}{|\theta|^2+l^2}e^{-ix\cdot\theta}e^{inl}dld\theta &= \frac{\pi}{(2\pi)^{d-1}}\int_{\mathbf{R}^{d-1}}e^{-|n||\theta|}e^{-ix\cdot\theta}d\theta \\ &= \frac{\Gamma(d/2)}{2\pi^{d/2}}\cdot\frac{|n|}{(n^2+|x|^2)^{d/2}}\end{aligned} \tag{10.3}$$

([8], p.6). Write $\tilde{l} = \mu\cdot\theta + l$ so that $Q(\theta,l) = \tilde{Q}(\theta,\tilde{l}) = \tilde{R}(\theta) + \sigma_d^2\tilde{l}^2$. Then

$$\begin{aligned}\pi_0(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dl}{1-\psi} &= \frac{1}{2\pi}\int_{-\mu\cdot\theta-\pi}^{-\mu\cdot\theta+\pi}\frac{dl}{1-\psi(\theta,l)} \\ &\sim \frac{1}{\pi}\int_{-\mu\cdot\theta-\pi}^{-\mu\cdot\theta+\pi}\frac{dl}{Q(\theta,l)} = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d\tilde{l}}{\tilde{Q}(\theta,\tilde{l})} = \frac{1}{\sigma_d\sqrt{\tilde{R}(\theta)}} + O(1).\end{aligned}$$

Consequently

$$\rho(\theta) = \sigma_d\sqrt{\tilde{R}(\theta)} + o(|\theta|).$$

By simple changes of variables we derive from this and (10.2) that for $x \in \mathbf{R}^{d-1}$,

$$\begin{aligned}H_n(x) &\sim \frac{\sigma_d}{(2\pi)^d}\int_{T^{d-1}}\sqrt{\tilde{R}(\theta)}\,e^{-i(x+n\mu)\cdot\theta}d\theta\int_{-\pi}^{\pi}\frac{2e^{inl}dl}{\tilde{R}(\theta)+\sigma_d^2 l^2} \\ &\sim \frac{1}{(2\pi)^d\det\tilde{R}^{-1/2}}\int_{\mathbf{R}^{d-1}}|\theta|e^{-i\sqrt{\tilde{R}}^{-1}(x+n\mu)\cdot\theta}d\theta\int_{-\infty}^{\infty}\frac{2e^{inl/\sigma_d}dl}{|\theta|^2+l^2}.\end{aligned}$$

Hence by (10.3) and (10.1) (note that $\tilde{Q}^{-1}(x+n\mu,-n) = Q^{-1}(x,-n)$),

$$\begin{aligned} H_n(x) &\sim \frac{\Gamma(d/2)}{\pi^{d/2}\sqrt{\det Q}} \cdot \frac{|n|}{[(n/\sigma_d)^2 + \tilde{R}^{-1}(x+n\mu)]^{d/2}} \\ &= \frac{\Gamma(d/2)}{\pi^{d/2}} \cdot \frac{|n|}{\|(x,-n)\|^d}. \end{aligned}$$

This gives a correct asymptotic form of $H_n(x)$ as $|n| \to \infty$. If $|n|$ remains finite, $|n|$ in the numerator must be replaced by $\sigma_d^2 a^*(n)$ as in the two-dimensional case. Thus

Theorem 10.1 *If the random walk S_n on $\mathbf{Z}^d$ is irreducible and has zero mean and finite variances, then*

$$H_n(x) = \frac{\Gamma(d/2)}{\pi^{d/2}} \cdot \frac{\sigma_d^2 a^*(n)}{\|(x,-n)\|^d}(1+o(1)).$$

For verification, in the case, $n \geq |x|$ the proof given in Section 3 straightforwardly applies to the higher dimensional case. In the case $n < |x|$ the arguments made for the two dimensions are only simplified. The analogues of the two lemmas of Section 4 are verified by essentially the same proofs; the lemmas of Section 5 are almost obviously extended to the dimensions $d \geq 3$. It is noted that if ∇ denotes the gradient operator with respect to $\theta \in \mathbf{R}^{d-1}$ and $\omega = x/|x|$, then as $\theta \to 0$

$$(\omega\cdot\nabla)\rho(\theta) = \sigma_d \frac{\omega\cdot\tilde{R}\theta}{[\tilde{R}(\theta)]^{1/2}} + o(1) \quad \text{and} \quad (\omega\cdot\nabla)^2\rho(\theta) = \sigma_d \frac{\tilde{R}(\omega) - (\omega\cdot\tilde{R}\theta)^2/\tilde{R}(\theta)}{[\tilde{R}(\theta)]^{1/2}} + o\Big(\frac{1}{|\theta|}\Big);$$

in particular $(\omega\cdot\nabla)^2\rho(\theta)$ is integrable on $|\theta| < 1$, which trivializes the argument given for the estimate of H_0 in section 6 and makes the moment condition $E[|\tilde{X}|^2 \log^+ |\tilde{X}|] < \infty$ dispensable.

11 Appendix

(A) We give a proof of (2.2) and (2.3), namely $\hat{H}_n(t) = \pi_{-n}(t)/\pi_0(t)$ $(n \neq 0)$ and $\hat{H}_0(t) = 1 - 1/\pi_0(t)$, respectively. Let $p^n(x,y) = P_x[S_n = y]$ and define for $|z| < 1$

$$\pi_k(t,z) = \sum_{\nu=0}^{\infty}\sum_{s\in\mathbf{Z}} p^\nu((0,0),(s,k))e^{its}z^\nu.$$

Then $\sum_k \pi_k(t,z)e^{ikl} = (1 - z\psi(t,l))^{-1}$ and hence for $t \neq 0$,

$$\pi_k(t,r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ikl}dl}{1 - r\psi(t,l)} \quad \longrightarrow \quad \pi_k(t) \qquad \text{as} \quad r \uparrow 1.$$

It is convenient to bring in the joint distribution

$$f_n(\tau,s) = P_{(0,n)}[\tau(L) = \tau,\ S_{\tau(L)} = (s,0)]$$

and its Abel-Fourier series

$$\Phi_n(t,z) = \sum_{\tau=1}^{\infty}\sum_{s\in\mathbf{Z}} f_n(\tau,s)e^{its}z^\tau.$$

Using $p^\nu((0,n),(s,0)) = \sum_{\tau=1}^{\nu}\sum_{j\in\mathbf{Z}} f_n(\tau,j)p^{\nu-\tau}((j,0),(s,0))$ $(\nu \geq 1)$ and making a routine computation, one then finds that for $t \in \mathbf{R}$ and $|z| < 1$,

$$\begin{aligned} \pi_{-n}(t,z) &= \sum_{\nu=0}^{\infty}\sum_{s\in\mathbf{Z}} p^\nu((0,n),(s,0))e^{its}z^\nu \\ &= \delta_{n,0} + \pi_0(t,z)\Phi_n(t,z), \end{aligned}$$

or what is the same thing,

$$\Phi_n(t,z) = \frac{\pi_{-n}(t,z)}{\pi_0(t,z)} \quad (n \neq 0); \quad \Phi_0(t,z) = 1 - \frac{1}{\pi_0(t,z)}, \tag{11.1}$$

which, on setting $z = 1$ (for $0 < |t| \le \pi$), reduce to (2.2) and (2.3), respectively.

(B) Here we collect several lemmas concerning Fourier analysis and used in this paper.

Lemma 11.1 *Let $f_n(t,l)$ be a sequence of measurable functions on $(-\pi,\pi]^2$ of the form*

$$f_n(t,l) = a(t,l)\frac{1-\cos nl}{nl^2} \qquad or \qquad f_n(t,l) = b(t,l)\frac{\sin nl}{nl} \tag{11.2}$$

$(n = 1,2,\ldots)$. *Suppose that $\int_{-\pi}^{\pi} |a(t,l)|dt$ is bounded and tends to zero as $l \to 0$ in the first case and $b(t,l)$ is integrable on $[-\pi,\pi]^2$ in the second case. Then*

$$\int_{-\pi}^{\pi} e^{ist}dt \int_{-\pi}^{\pi} f_n(t,l)dl \to 0 \quad as \quad |s| \to \infty \quad uniformly\ in\ \ n. \tag{11.3}$$

Proof. Put $g_{n,\varepsilon}(t) = \int_{\varepsilon<|l|<\pi} f_n(t,l)dl$ and $h_{n,\varepsilon}(t) = \int_{-\varepsilon}^{\varepsilon} f_n(t,l)dl$. Then from the assumptions on f_n of the lemma it follows that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g_{n,\varepsilon}(t)|dt = 0 \ \text{ for each } \ \varepsilon > 0; \quad \text{and} \qquad \sup_n \int_{-\pi}^{\pi} |h_{n,\varepsilon}(t)|dt \longrightarrow 0 \ \text{ as } \ \varepsilon \downarrow 0.$$

which together show that $\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f_n(t,l)dl \right| dt = 0$, reducing the assertion to be proved to the Riemann-Lebesgue lemma. □

Remark to Lemma 11.1. (i) The pair of two function forms in (11.2) may appear ill-matched but actually they are well on balance if the supposed conditions on $a(t,l)$ and $b(t,l)$ in the lemma are taken into account. Indeed, on the one hand the function $(1-\cos nl)/nl^2$ (Fejér's kernel) approaches π times Dirac's delta function as $n \to \infty$; on the other hand $\sin nl/nl$ is π/n times Dirichlet's kernel and the factor $1/n$ is apparantly superfluous in comparison to the former one, which however is needed for counterweighing the possible singularity of $\int b(t,l)dt$ at $l = 0$ that duly arises: in our application of Lemma 10.1 these two function forms come up in pair (see the definitions of Θ_I and Θ_{II} in Section 6), where we encounter the situation such that a priori we know only $\int |b(t,l)|dt = o(1/l)$.

(ii) If only boundedness of $\int_{-\pi}^{\pi} |a(t,l)|dt$ is assumed, the convergence in (11.3) still holds boundedly but it is not necessarily uniform. Eg., if $a(t,l) = |l|/(t^2+l^2)$, then $\int_{-\pi}^{\pi} |a(t,l)|dt$ is bounded, whereas the double integral in (11.3), taking on the form

$$\frac{4}{n}\int_0^{\pi} \cos st\, dt \int_0^{\pi} \frac{(1-\cos nl)}{(t^2+l^2)l}dl = 2\pi \int_0^{n\pi} e^{-|s/n|u}\frac{1-\cos u}{u^2}du + O\Big(\frac{1}{s}\Big),$$

does not uniformly approach zero as $s \to \infty$.

Lemma 11.2 *Let $f_n(t)$ be a sequence of continuous functions on the (half-open) interval* $(0,1]$ *such that as $t \to 0$*

$$\sup_n |f_n(t)| = o(t^{-1}), \tag{11.4}$$

and that f_n are continuously differentiable in $t > 0$ and satisfies

$$\sup_n |f_n'(t)| = O(t^{-2}) \tag{11.5}$$

or more generally $\sup_n \int_t^1 |df_n(u)| = O(1/t)$ $(t \to 0)$. *Then for each* $\alpha > 0$,

$$\int_0^1 f_n(t)\sin st\,dt \to 0 \quad \textit{and} \quad \int_{\alpha/s}^1 f_n(t)\cos st\,dt \to 0 \quad \textit{as} \quad s \to \infty \quad \textit{uniformly in} \quad n. \tag{11.6}$$

If in addition, the improper integrals $\int_{+0}^{\varepsilon} f_n(t)dt$ *exist and approach zero as* $\varepsilon \downarrow 0$ *uniformly in* n, *then*

$$\int_{+0}^1 f_n(t)\cos st\,dt \to 0 \quad \textit{as} \quad s \to \infty \quad \textit{uniformly in} \quad n.$$

Moreover, if $0 < \delta < 1$ *and the right sides in (11.4) and (11.5) are replaced by* $o(t^{\delta-1})$ *and* $O(t^{\delta-2})$, *respectively, then* $\int_0^1 f_n(t)e^{ist}dt = o(|s|^{-\delta})$ *as* $|s| \to \infty$ *uniformly in* n.

Proof. From (11.4) it follows that $\sup_n \int_{\alpha/s}^{M/s} |f_n(t)|dt = o(1)$ for each $M > \alpha$ and

$$\sup_n \int_0^{M/s} |f_n(t)t|dt = o(1/s) \quad \text{for each } M > 0. \tag{11.7}$$

On the other hand integrating by parts yields that as $s \to \infty$ and $M \to \infty$ in this order

$$\int_{M/s}^1 f_n(t)e^{ist}\,dt = \frac{1}{is}[f_n(t)e^{ist}]_{t=M/s}^1 - \frac{1}{is}\int_{M/s}^1 e^{ist}df_n(t) = O(1/M) \to 0.$$

Combining these yields the relation (11.6). The last assertion is verified in the same way.

For the second assertion of the lemma, using the supposition of it, we observe $\int_{+0}^{\alpha/s} f_n(t)\cos st dt = \int_0^{\alpha/s} f_n(t)(\cos st - 1)dt + o(1)$, but the first term is also $o(1)$ owing to (11.7). □

Lemma 11.3 *Let* Λ *be any parameter set and* $\{f_\lambda(t)\}$ *a family of functions on* $(0,\infty)$ *with parameter* $\lambda \in \Lambda$ *such that*

$$\sup_\lambda |f_\lambda(1)| < \infty \qquad \textit{and} \qquad \sup_\lambda \int_0^{\varepsilon} |f_\lambda(t)|dt \to 0 \qquad \textit{as} \qquad \varepsilon \to 0 \tag{11.8}$$

and that for each $\varepsilon > 0$, *the total variations of* f_λ *on* $[\varepsilon,\infty)$ *are bounded:* $\sup_\lambda \int_\varepsilon^\infty |df_\lambda(t)| < \infty$. *Then*

$$\int_0^M f_\lambda(t)e^{ist}\,dt \to 0 \quad \textit{as} \quad |s| \to \infty \quad \textit{uniformly in} \quad \lambda \in \Lambda \textit{ and } M > 1.$$

Proof. Integrating by parts we obtain $\limsup_{s\to\infty} \sup_{\lambda,M} \left| \int_\varepsilon^M f_\lambda(t)e^{ist}dt \right| = 0$ for each $\varepsilon > 0$. The assertion of the lemma then follows from the second condition in (11.8). □

Lemma 11.4 *Let* ω *be a unit vector in* $\mathbf{R}^d$ *and* $f(\theta)$ *a continuous function on* $0 < |\theta| < 2$, $\theta \in \mathbf{R}^d$ *such that for some constants* $0 < \delta < 1$, K *and* M_ω

$$|f(\theta)| \le K|\theta|^{\delta-d} \qquad (|\theta| < 2), \tag{11.9}$$

$$|f(\theta) - f(\theta + u\omega)| \le M_\omega u^\delta |\theta|^{-d} \qquad \textit{for} \qquad 0 < u < \theta\cdot\omega.$$

Then

$$\left| \int_{|\theta|\le 1} f(\theta)e^{-ir\omega\cdot\theta}d\theta \right| \le C_\omega r^{-\delta}\log r \qquad (r > 2\pi), \tag{11.10}$$

where the constant C_ω *may be taken as* $A\lambda_d(K + M_\omega)[(1-\delta)\delta]^{-1}$ *with a universal constant* A *and a constant* λ_d *depending only on* d. *If* f *satisfies the first condition (11.9) and is differentiable for* $\theta \ne 0$ *with* $|\omega\cdot\nabla f(\theta)| = O(|\theta|^{\delta-d-1})$, *then the right side of (11.10) may be replaced by* $C'r^{-\delta}$.

Moreover if $f(\theta) = K/|\theta|^d$ (i.e., (11.9) with $\delta = 0$) and

$$|f(\theta) - f(\theta + u\omega)| \leq M_\omega |\theta|^{-d} |\log u|^{-1} \qquad \textit{for} \qquad 0 < u < \theta \cdot \omega \leq 1/2,$$

then

$$\left| \int_{|\omega\cdot\theta|\geq\alpha/r, |\theta|\leq 1} f(\theta) e^{-ir\omega\cdot\theta} d\theta \right| \leq A\lambda_d (K + M_\omega).$$

Proof. Use the argument given in the beginning of §3, Chapter II of [16]. The details are omitted since the proof is the same as for Lemma 21 of [10].

Acknowledgments. The author wishes to thank the referee for his critical comments and constructive suggestions to the original manuscript, which stimulated and helped the author to revise the paper to make the proofs more transparent and better organized.

References

[1] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, John Wiley and Sons, Inc. (1965)

[2] Y. Fukai and K. Uchiyama, Potential kernel for two-dimensional random walk, Ann. Probab. **24** (1992) 1979-1992.

[3] W. Hoeffding, On sequences of sums of independent random vectors, Fourth Berleley Symp. on Stat. and Probab., Vol II (1961), 213-226.

[4] T. Kazami and K. Uchiyama, Random walks on periodic graphs, to appear in TAMS.

[5] H. Kesten, Hitting probabilities of random walks on $\mathbf{Z}^d$, Stoch. Proc. Appl. **25** (1987) 165-184.

[6] G. F. Lawler and V. Limic, The Beurling estimate for a class of random walks, Elec. Jour. Probab. **9** (2004) 846-861.

[7] F. Spitzer, Principles of Random Walks, Van Nostrand, Princeton, 1964.

[8] E. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, 1971.

[9] K. Uchiyama, Green's functions for random walks on $\mathbf{Z}^d$, Proc. London Math. Soc. **77** (1998), 215-240.

[10] K. Uchiyama, Asymptotic estimates of Green's functions and transition probabilities for Markov additive processes, Elec. J. Probab. **12** (2007), 138-180.

[11] K. Uchiyama, The hitting distributions of a half real line for two dimensional random walks. to appear in Arkiv för Mat.

[12] K. Uchiyama, The hitting time of a single point for random walks. (2008), http://www.math.titech.ac.jp/ tosho/Preprints/index-j.html.

[13] K. Uchiyama, The hitting distribution of line segments for two dimensional random walks. (in preparation)

[14] K. Uchiyama, One dimensional lattice random walks killed at a point or on a line. (preprint)

[15] K. Uchiyama, The random walks on the upper half plane. preprint.

[16] A. Zygmund, Trigonometric series, vol. 1, 3rd ed., Cambridge Univ. Press (2002)