A stochastic Taylor-like expansion in the rough path theory

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Abstract

In this paper we establish a Taylor-like expansion in the context of the rough path theory for a family of Itô maps indexed by a small parameter. We treat not only the case that the roughness p satisfies $[p] = 2$, but also the case that $[p] \geq 3$. As an application, we discuss the Laplace asymptotics for Itô functionals of Brownian rough paths.

1 Introduction and the main result

Let $\mathcal{V}, \mathcal{W}$ be real Banach spaces and let $X : [0, 1] \to \mathcal{V}$ be a nice path in $\mathcal{V}$. Let us consider the following $\mathcal{W}$-valued ordinary differential equation (ODE);

$$dY_t = \sigma(Y_t)dX_t, \qquad \text{with } Y_0 = 0. \tag{1.1}$$

Here, σ is a nice function from $\mathcal{W}$ to the space $L(\mathcal{V}, \mathcal{W})$ of bounded linear maps. The correspondence $X \mapsto Y$ is called the Itô map and will be denoted by $Y = \Phi(X)$.

In the rough path theory of T. Lyons, the equation (1.1) is significantly generalized. First, the space of geometric rough paths on $\mathcal{V}$ with roughness $p \geq 1$, which contains all the nice paths, is introduced. It is denoted by $G\Omega_p(\mathcal{V})$ and its precise definition will be given in the next section. Then, the Itô map Φ extends to a continuous map from $G\Omega_p(\mathcal{V})$ to $G\Omega_p(\mathcal{W})$. In particular, when $2 < p < 3$ and $\dim(\mathcal{V}), \dim(\mathcal{W}) < \infty$, this equation (1.1) corresponds to a stratonovich-type stochastic differential equation (SDE). (See Lyons and Qian [16] for the facts in this paragraph.)

In many fields of analysis it is quite important to investigate how the output of a map behaves asymptotically when the input is given small perturbation. The Taylor expansion in the calculus is a typical example. In this paper we investigate the behaviour of $\Phi(\varepsilon X + \Lambda)$ as $\varepsilon \searrow 0$ for a nice path Λ and $X \in G\Omega_p(\mathcal{V})$. Slightly generalizing it, we will consider the asymptotic behaviour of $Y^{(\varepsilon)}$, which is defined by (1.2) below, as $\varepsilon \searrow 0$;

$$dY_t^{(\varepsilon)} = \sigma(\varepsilon, Y_t^{(\varepsilon)})\varepsilon dX_t + b(\varepsilon, Y_t^{(\varepsilon)})d\Lambda_t, \qquad \text{with } Y_0^\varepsilon = 0. \tag{1.2}$$

Then, we will obtain an asymptotic expansion as follows; there exist $Y^0, Y^1, Y^2, \ldots$ such that

$$Y^\varepsilon \sim Y^0 + \varepsilon Y^1 + \cdots + \varepsilon^n Y^n + \cdots \qquad \text{as } \varepsilon \searrow 0.$$

We call it a stochastic Taylor-like expansion around a point Λ. Despite its name, this is purely real analysis and no probability measure is involved in the argument.

This kind of expansion in the context of the rough path theory was first done by Aida [1, 2] (for the case where coefficients σ, b are independent of ε, $[p] = 2$, $\mathcal{V}, \mathcal{W}$ are finite dimensional). Then, Inahama and Kawabi [11] (see also [8]) extended it to the infinite dimensional case in order to investigate the Laplace asymptotics for the Brownian motion over loop groups. (The methods in [2] and [11] are slightly different. In [2], unlike in [11], the derivative equation of the given equation is explicitly used. see Introduction of [11].)

The main result (Theorems 4.4 and 4.5) in this paper is to generalize the stochastic Taylor-like expansion in [11]. The following points are improved:

1. The roughness p satisfies $2 \le p < \infty$. In other words, not only the case $[p] = 2$, but also the case $p \ge 3$ is discussed.

2. The coefficients σ and b depend on the small parameter $\varepsilon > 0$. In other words, we treat not just one fixed Itô map, but a family of Itô maps indexed by ε.

3. The base point Λ of the expansion is a continuous q-variational path for any $1 \le q < 2$ with $1/p + 1/q > 1$. In [11], Λ is a continuous bounded variational path (i.e., the case $q = 1$).

4. Not only estimates of the first level paths of $Y^0, Y^1, Y^2, \ldots$, but also estimates of the higher level paths are given.

The organization of this paper is as follows: In Section 2, we briefly recall the definition and basic facts on geometric rough paths. We also prove simple lemmas on continuous q-variational paths ($1 \le q < 2$). In the end of this section we prove a few lemmas, including an extension of Duhamel's principle, for later use.

In Section 3, we first generalize the local Lipschitz continuity of the integration map as the integrand varies (Proposition 3.1). Put simply, the proposition states that the map

$$(f, X) \in C_{b,loc}^{[p]+1}(\mathcal{V}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \mapsto \int f(X)dX \in G\Omega_p(\mathcal{W})$$

is continuous. Here, $C_{b,loc}^{[p]+1}$ denotes the space of $[p]+1$-times Fréchet differentiable maps whose derivatives of order $0, 1, \ldots, [p]+1$ are bounded on every bounded sets. Note that in Lyons and Qian [16], the integrand (or the coefficients of ODE) is always fixed. In the path space analysis, a path on a manifold is often regarded as a current-valued path. This generalization is also necessary for such a viewpoint in the rough path context.

In the latter half of the section, using the above fact, we improve Lyons' continuity theorem (also known as the universal limit theorem) when the coefficient of the ODE varies. (Theorem 3.8 and Corollary 3.9). Put simply, the correspondence

$$(\sigma, X, y_0) \in C_M^{[p]+2}(\mathcal{V}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \times \mathcal{W} \mapsto Z = (X, Y) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$$

is continuous. Here, $Z = (X, Y)$ is the solution of (1.1) with the initial condition replaced with y_0 and $C_M^{[p]+2}(\mathcal{V}, L(\mathcal{V}, \mathcal{W}))$ $(M > 0)$ is a subset of $C_b^{[p]+2}(\mathcal{V}, L(\mathcal{V}, \mathcal{W}))$ (a precise definition is given later).

In Section 4, as we stated above, we prove the main theorems in this paper (Theorems 4.4 and 4.5).

In Section 5, as an application of the expansion in Section 4, we improve the Laplace asymptotics for the Brownian rough path given in [11]. In this paper, we are now able to treat the case where the coefficients of the ODE are dependent on the small parameter $\varepsilon > 0$ (see Remark 5.2).

Remark 1.1 *In Coutin and Qian [6] they showed that, when the Hurst parameter is larger than* $1/4$*, the fractional Brownian rough paths exist and the rough path theory is applicable to the study of SDEs driven by the fractional Brownian motion. In particular, if the Hurst parameter is between* $1/4$ *and* $1/3$*, the roughness satisfies* $[p] = 3$ *and the third level path plays a role. Since Millet and Sanz-Solé [17] proved the large deviation principle for the fractional Brownian rough paths, it is natural to guess that the Laplace asymptotics as in Theorem 5.1 for the fractional Brownian rough paths is also true. However, at the moment of the writing, the author does not have a proof. Note that Baudin and Coutin [4] proved a similar asymptotic problem (the short time asymptotics for finite dimensional, one fixed differential equation) for the fractional Brownian rough paths.*

2 The space of geometric rough paths

2.1 Definition

Let $p \geq 2$ and let $\mathcal{V}$ be a real Banach space. In this section we recall the definition of $G\Omega_p(\mathcal{V})$, the space of geometric rough paths over $\mathcal{V}$. For details, see Lyons and Qian [16].

On the tensor product $\mathcal{V} \otimes \hat{\mathcal{V}}$ of two (or more) Banach spaces $\mathcal{V}$ and $\hat{\mathcal{V}}$, various Banach norms can be defined. In this paper, however, we only consider the projective norm on $\mathcal{V} \otimes \hat{\mathcal{V}}$. The most important property of the projective norm is the following isometrical isomorphism; $L(\mathcal{V} \otimes \hat{\mathcal{V}}, \mathcal{W}) \cong L^2(\mathcal{V}, \hat{\mathcal{V}}; \mathcal{W})$. Here, the right hand side denotes the space of bounded bilinear functional from $\mathcal{V} \times \hat{\mathcal{V}}$ to another real Banach space $\mathcal{W}$. (For definition and basic properties of the projective norm, see Diestel and Uhl [7].)

For a real Banach space $\mathcal{V}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, we set $T^{(n)}(\mathcal{V}) = \mathbb{R} \oplus \mathcal{V} \oplus \cdots \oplus \mathcal{V}^{\otimes n}$. For two elements $a = (a^0, a^1, \dots, a^n), b = (b^0, b^1, \dots, b^n) \in T^{(n)}(\mathcal{V})$, the multiplication and the scalar action are defined as follows;

$$a \otimes b = \big(a^0 b^0, a^1 b^0 + a^0 b^1, a^2 b^0 + a^1 \otimes b^1 + a^0 b^2, \dots, \sum_{i=0}^{n} a^{n-i} \otimes b^i\big),$$
$$ra = (a^0, ra^1, r^2 a^2, \dots, r^n a^n), \qquad r \in \mathbb{R}.$$

Note that, $a \otimes b \neq b \otimes a$ in general. The non-commutative algebra $T^{(n)}(\mathcal{V})$ is called the truncated tensor algebra of degree n. As usual $T^{(n)}(\mathcal{V})$ is equipped with the direct sum norm.

Let $\triangle = \{(s,t) \mid 0 \leq s \leq t \leq 1\}$. We say $X = (1, X^1, \dots, X^{[p]}) : \triangle \to T^{([p])}(\mathcal{V})$ is a rough path over $\mathcal{V}$ of roughness p if it is continuous and satisfies the following;

$$X_{s,u} \otimes X_{u,t} = X_{s,t} \qquad \text{for all } (s,u), (u,t) \in \triangle,$$
$$\|X^j\|_{p/j} := \Big\{\sup_D \sum_{i=1}^{N} |X^j_{t_{i-1},t_i}|^{p/j}\Big\}^{j/p} < \infty \qquad \text{for all } j = 1, \dots, [p].$$

Here, $D = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ runs over all the finite partitions of $[0,1]$. The set of all the rough paths over $\mathcal{V}$ of roughness p is denoted by $\Omega_p(\mathcal{V})$. The distance on $\Omega_p(\mathcal{V})$ is defined by

$$d(X,Y) = \sum_{j=1}^{[p]} \|X^j - Y^j\|_{p/j}, \qquad X, Y \in \Omega_p(\mathcal{V}).$$

With this distance, $\Omega_p(\mathcal{V})$ is a complete metric space. For $X \in \Omega_p(\mathcal{V})$, we set $\xi(X) = \sum_{j=1}^{[p]} \|X^j\|_{p/j}^{1/j}$. It is obvious that $\xi(rX) = |r|\xi(X)$ for $r \in \mathbb{R}$.

Let $\mathrm{BV}(\mathcal{V}) = \{X \in C([0,1], \mathcal{V}) \mid X_0 = 0 \text{ and } \|X\|_1 < \infty\}$ be the space of continuous, bounded variational paths starting at 0. By using the Stieltjes integral, we can define a rough path as follows ($p \geq 2$);

$$X^j_{s,t} := \int_{s<t_1<\cdots<t_j<t} dX_{t_1} \otimes \cdots \otimes dX_{t_j}, \qquad (s,t) \in \triangle, \quad j = 1, \dots, [p].$$

This rough path is called be the smooth rough path lying above $X \in \mathrm{BV}(\mathcal{V})$ and is again denoted by X (when there is no possibility of confusion). The d-closure of the totality of all the smooth rough paths is denoted by $G\Omega_p(\mathcal{V})$, which is called the space of geometric rough paths. This is a complete metric space. (If $\mathcal{V}$ is separable, then $G\Omega_p(\mathcal{V})$ is also separable, which can easily be seen from Corollary 2.3 below.)

2.2 On basic properties of q-variational paths ($1 \leq q < 2$).

Let $1 \leq q < 2$. For a real Banach space $\mathcal{V}$, set

$$C_{0,q}(\mathcal{V}) = \{X \in C([0,1], \mathcal{V}) \mid X_0 = 0 \text{ and } \|X\|_q < \infty\},$$

where $\| \cdot \|_q$ denotes the q-variation norm. When $q = 1$, $\mathrm{BV}(\mathcal{V}) = C_{0,q}(\mathcal{V})$.

Let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$ be a (finite) partition of $[0,1]$. We denote by $\pi_{\mathcal{P}} X$ a piecewise linear path associated with $\mathcal{P}$ (i.e., $\pi_{\mathcal{P}} X_{t_i} = X_{t_i}$ and $\pi_{\mathcal{P}} X$ is linear on $[t_{i-1}, t_i]$ for all i). The following lemma states that $\pi_{\mathcal{P}} : C_{0,q}(\mathcal{V}) \to C_{0,q}(\mathcal{V})$ is uniformly bounded as the partition $\mathcal{P}$ varies.

Lemma 2.1 *Let $1 \le q < 2$. Then, there exists a positive constant $c = c_q$ depending only on q such that $\|\pi_{\mathcal{P}} X\|_q \le c\|X\|_q$ for any partition $\mathcal{P}$ of $[0,1]$.*

Proof. Fix $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$. For a partition $\mathcal{Q} = \{0 = s_0 < s_1 < \cdots < s_M = 1\}$, we set

$$S_{\mathcal{Q}} = \sum_{k=1}^{M} |\pi_{\mathcal{P}} X_{s_i} - \pi_{\mathcal{P}} X_{s_{i-1}}|^q.$$

Suppose that $[t_{i-1}, t_i] \cap \mathcal{Q}$ contains three points (namely, $s_{j-1} < s_j < s_{j+1}$). Then, since $\pi_{\mathcal{P}} X$ is linear on $[s_{j-1}, s_{j+1}]$, we see that

$$|\pi_{\mathcal{P}} X_{s_{j-1}} - \pi_{\mathcal{P}} X_{s_j}|^q + |\pi_{\mathcal{P}} X_{s_j} - \pi_{\mathcal{P}} X_{s_{j+1}}|^q \le |\pi_{\mathcal{P}} X_{s_{j-1}} - \pi_{\mathcal{P}} X_{s_{j+1}}|^q,$$

which implies that $S_{\mathcal{Q}} \le S_{\mathcal{Q} \setminus \{s_j\}}$. Therefore, we have only to consider $\mathcal{Q}$'s such that $|[t_{i-1}, t_i] \cap \mathcal{Q}| \le 2$ for all $i = 1, \ldots, N$.

Let $\mathcal{Q}$ be as such. If $[t_{i-1}, t_i] \cap \mathcal{Q} = \{s_j\}$, then define $\hat{s}_j = t_{i-1}$ except if $s_j = 1$. (If so we set $\hat{s}_j = 1$.) If $[t_{i-1}, t_i] \cap \mathcal{Q} = \{s_j < s_{j+1}\}$, then define $\hat{s}_j = t_{i-1}$ and $\hat{s}_{j+1} = t_i$. Note that $0 = \hat{s}_0 \le \hat{s}_1 \le \cdots \le \hat{s}_M = 1$. Some of $\hat{s}_j$'s may be equal. If so, we only collect distinct $\hat{s}_j$'s and call the collection $\hat{\mathcal{Q}}$. Noting that $S_{\hat{\mathcal{Q}}} \le \|X\|_q^q$ and that

$$|\pi_{\mathcal{P}} X_{s_j} - \pi_{\mathcal{P}} X_{\hat{s}_j}| \le |\pi_{\mathcal{P}} X_{t_i} - \pi_{\mathcal{P}} X_{t_{i-1}}| = |X_{t_i} - X_{t_{i-1}}|,$$

if $s_j \in [t_{i-1}, t_i]$, we see that

$$\begin{aligned} S_{\mathcal{Q}} &= \sum_{j=1}^{M} |\pi_{\mathcal{P}} X_{s_j} - \pi_{\mathcal{P}} X_{s_{j-1}}|^q \\ &\le c_q \Big[\sum_{k=1}^{M} |\pi_{\mathcal{P}} X_{s_j} - \pi_{\mathcal{P}} X_{\hat{s}_j}|^q + \sum_{k=1}^{M} |\pi_{\mathcal{P}} X_{s_{j-1}} - \pi_{\mathcal{P}} X_{\hat{s}_{j-1}}|^q + S_{\hat{\mathcal{Q}}} \Big] \\ &\le c'_q S_{\hat{\mathcal{Q}}} \le c'_q \|X\|_q^q. \end{aligned}$$

Taking supremum over such $\mathcal{Q}$'s, we complete the proof. ∎

Corollary 2.2 *Let $1 \le q < q' < 2$ and $X \in C_{0,q}(\mathcal{V})$. Then,*

$$\lim_{|\mathcal{P}| \to 0} \|X - \pi_{\mathcal{P}} X\|_{q'} = 0.$$

Here, $|\mathcal{P}|$ denotes the mesh of the partition $\mathcal{P}$.

Proof. This is easy from Lemma 2.1 and the fact that $\pi_{\mathcal{P}} X \to X$ as $|\mathcal{P}| \to 0$ in the uniform topology. ▮

For $1 \le q < 2$ and $X \in C_{0,q}(\mathcal{V})$, we set

$$X^j_{s,t} = \int_{s<t_1<\cdots<t_j<t} dX_{t_1} \otimes \cdots \otimes dX_{t_j}, \qquad j = 1, 2, \ldots, [p].$$

Here, the right hand side is the Young integral. As before, $X = (1, X^1, \ldots, X^{[p]})$ satisfies Chen's identity and, if $|X_t - X_s| \le \omega(s,t)^{1/q}$ for some control function ω, then X^j is of finite q/j-variation (i.e., there exists a positive constant C such that $|X^j_{s,t}| \le C\omega(s,t)^{j/q}$). So, $C_{0,q}(\mathcal{V}) \subset \Omega_p(\mathcal{V})$.

By Theorem 3.1.3 in [16], if $X, Y \in C_{0,q}(\mathcal{V})$ satisfy that

$$\begin{aligned} |X_t - X_s|, \, |Y_t - Y_s| &\le \omega(s,t)^{1/q}, \\ |(X_t - X_s) - (Y_t - Y_s)| &\le \varepsilon\omega(s,t)^{1/q}. \end{aligned}$$

for some control function, then there exists a positive constant C depending only on $p, q, \omega(0,1)$ such that

$$|X^j_{s,t} - Y^j_{s,t}| \le C\varepsilon\omega(s,t)^{j/q}, \qquad \text{for all } (s,t) \in \triangle \text{ and } j = 1, \ldots, [p].$$

In particular, the injection $X \in C_{0,q}(\mathcal{V}) \mapsto X = (1, X^1, \ldots, X^{[p]}) \in \Omega_p(\mathcal{V})$ is continuous. Combining this with Corollary 2.2, we obtain the following corollary. (By taking sufficiently small $q'(> q)$.) Originally, $G\Omega_p(\mathcal{V})$ is defined as the closure of $\mathrm{BV}(\mathcal{V})$ in $\Omega_p(\mathcal{V})$. In the the following corollary, we prove that $G\Omega_p(\mathcal{V})$ is also obtained as the closure of $C_{0,q}(\mathcal{V})$.

Corollary 2.3 *Let $1 \le q < 2$ and $p \ge 2$. For any $X \in C_{0,q}(\mathcal{V})$, $\pi_{\mathcal{P}} X \in \mathrm{BV}(\mathcal{V})$ converges to X in $\Omega_p(\mathcal{V})$ as $|\mathcal{P}| \to 0$. In particular, we have the following continuous inclusion; $\mathrm{BV}(\mathcal{V}) \subset C_{0,q}(\mathcal{V}) \subset G\Omega_p(\mathcal{V})$.*

Corollary 2.4 *Let $1 \le q < 2$ and $p \ge 2$ with $1/p + 1/q > 1$. Let $\mathcal{V}$ and $\mathcal{W}$ be real Banach spaces. Then, the following (1) and (2) hold:*
(1). For $X \in G\Omega_p(\mathcal{V})$ and $H \in C_{0,q}(\mathcal{V})$, the natural shift $X + H \in G\Omega_p(\mathcal{V})$ is well-defined. Moreover, it is continuous as a map from $G\Omega_p(\mathcal{V}) \times C_{0,q}(\mathcal{V})$ to $G\Omega_p(\mathcal{V})$.
(2). For $X \in G\Omega_p(\mathcal{V})$ and $H \in C_{0,q}(\mathcal{W})$, $(X, H) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$ is well-defined. Moreover, it is continuous as a map from $G\Omega_p(\mathcal{V}) \times C_{0,q}(\mathcal{W})$ to $G\Omega_p(\mathcal{V} \oplus \mathcal{W})$.

Proof. The shift as a map from $\Omega_p(\mathcal{V}) \times C_{0,q}(\mathcal{V})$ to $\Omega_p(\mathcal{V})$ is continuous. (See Section 3.3.2 in [16].) Therefore, we have only to prove that $X + H \in G\Omega_p(\mathcal{V})$ if $X \in G\Omega_p(\mathcal{V})$. However, it is immediately shown from Corollary 2.3.

The second assertion can be verified in the same way. ▮

Now we consider linear ODEs of the following form: For a given $L(\mathcal{V},\mathcal{V})$-valued path Ω, we set

$$\begin{aligned} dM_t &= d\Omega_t \cdot M_t, \qquad M_0 = \mathrm{Id}_{\mathcal{V}}, &(2.1)\\ dN_t &= -N_t \cdot d\Omega_t, \qquad N_0 = \mathrm{Id}_{\mathcal{V}}. &(2.2)\end{aligned}$$

Here, M and N are also $L(\mathcal{V},\mathcal{V})$-valued. It is well-known that, if $\Omega \in \mathrm{BV}(L(\mathcal{V},\mathcal{V}))$, then unique solutions M, N exist in $\mathrm{BV}(L(\mathcal{V},\mathcal{V}))$ and $M_tN_t = N_tM_t = \mathrm{Id}_{\mathcal{V}}$ for all t. This can be extended to the case of q-variational paths ($1 \le q < 2$) as in the following proposition.

Proposition 2.5 *Let $1 \le q < 2$ and $\Omega \in C_{0,q}(L(\mathcal{V},\mathcal{V}))$. Then, the unique solutions M, N of (2.1) and (2.2) exist in $C_{0,q}(L(\mathcal{V},\mathcal{V})) + \mathrm{Id}_{\mathcal{V}}$. It holds that $M_tN_t = N_tM_t = \mathrm{Id}_{\mathcal{V}}$ for all t. Moreover, if there exists a control function ω such that*

$$\begin{aligned} |\Omega_t - \Omega_s|,\ |\hat{\Omega}_t - \hat{\Omega}_s| &\le \omega(s,t)^{1/q},\\ |(\Omega_t - \Omega_s) - (\hat{\Omega}_t - \hat{\Omega}_s)| &\le \varepsilon\omega(s,t)^{1/q}, \qquad (s,t) \in \triangle,\end{aligned}$$

then, there exists a constant C depending only on q and $\omega(0,1)$ such that

$$\begin{aligned} |M_t - M_s|,\ |\hat{M}_t - \hat{M}_s| &\le C\omega(s,t)^{1/q}, &(2.3)\\ |(M_t - M_s) - (\hat{M}_t - \hat{M}_s)| &\le C\varepsilon\omega(s,t)^{1/q}, \qquad (s,t) \in \triangle. &(2.4)\end{aligned}$$

Similar estimates also hold for N.

Proof. By using the Young integration, set $I^0_t = I^0(\Omega)_t := \mathrm{Id}_{\mathcal{V}}$ and, for $n = 1, 2, \ldots$,

$$I^n_t = I^n(\Omega)_t := \int_{0<t_1<\cdots<t_n<t} d\Omega_{t_n} \cdots d\Omega_{t_1}.$$

If we set $m_t = \sum_{n=0}^{\infty} I^n(\Omega)_t$, then $t \mapsto m_{t-t_0}A$ formally satisfies (2.1) with initial condition replaced with $m_{t_0} = A \in L(\mathcal{V},\mathcal{V})$. Therefore, we will verify the convergence. (Note that in our construction of solutions, the "right invariance" of the given differential equation (2.1) implicitly plays an important role.)

We will prove that, for all $n \in \mathbb{N}$, $0 < T < 1$, and $(s,t) \in \triangle_{[0,T]} = \{(s,t) \mid 0 \le s \le t \le T\}$,

$$|I^n_t - I^n_s| \le K^{n-1}\omega(s,t)^{1/q}, \qquad \text{where } K = \omega(0,T)^{1/q}\bigl(1 + 2^{2/q}\zeta(2/q)\bigr). \tag{2.5}$$

Here, ζ denotes the ζ-function. Obviously, (2.5) holds for $n = 1$.

Suppose that (2.5) holds for n. Recall that $I^{n+1}_t - I^{n+1}_s = \int_s^t d\Omega_u I^n_u = \lim_{|\mathcal{P}|\to 0} S_{\mathcal{P}}$, where, $S_{\mathcal{P}}$ is given by $S_{\mathcal{P}} = \sum_{i=1}^{N}(\Omega_{t_i} - \Omega_{t_{i-1}})I^n_{t_{i-1}}$ for a finite partition $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ of $[s,t]$. It is easy to see that

$$|S_{\{s,t\}}| = |(\Omega_t - \Omega_s)I^n_s| \le K^{n-1}\omega(0,s)^{1/q}\omega(s,t)^{1/q}. \tag{2.6}$$

Let $t_j \in \mathcal{P}$, $(j = 1, \ldots, N-1)$. Then, we have

$$\begin{aligned} |S_{\mathcal{P}} - S_{\mathcal{P}\setminus\{t_j\}}| &= \bigl|(\Omega_{t_j} - \Omega_{t_{j-1}})I^n_{t_{j-1}} + (\Omega_{t_{j+1}} - \Omega_{t_j})I^n_{t_j} - (\Omega_{t_{j+1}} - \Omega_{t_{j-1}})I^n_{t_{j-1}}\bigr|\\ &= \bigl|(\Omega_{t_{j+1}} - \Omega_{t_j})(I^n_{t_j} - I^n_{t_{j-1}})\bigr| \le K^{n-1}\omega(t_{j-1},t_{j+1})^{2/q}.\end{aligned}$$

From this and a routine argument,

$$|S_{\mathcal{P}} - S_{\{s,t\}}| \leq K^{n-1} 2^{2/q} \zeta(2/q) \omega(s,t)^{2/q}. \tag{2.7}$$

From (2.6) and (2.7), we see that $|S_{\mathcal{P}}| \leq K^n \omega(s,t)^{1/q}$. This implies (2.5) for $n+1$ (and, hence, for all n by induction).

If T is chosen so that $K < 1$, then $t \mapsto m_t$ is convergent in q-variation topology on the restricted interval and is a solution for (2.1) which satisfies that $|m_t - m_s| \leq (1-K)^{-1} \omega(s,t)^{1/q}$ for $(s,t) \in \triangle_{[0,T]}$.

Take $0 = T_0 < T_1 < \cdots < T_k$ such that $\omega(T_{i-1}, T_i)^{1/q}\big(1 + 2^{2/q}\zeta(2/q)\big) = 1/2$ for $i = 1, \ldots, k-1$ and $\omega(T_{k-1}, T_k)^{1/q}\big(1 + 2^{2/q}\zeta(2/q)\big) \leq 1/2$. By the superadditivity of ω, $k - 1 \leq 2^q \big(1 + 2^{2/q}\zeta(2/q)\big)^q \omega(0,1)$. Hence, k is dominated by a constant which depends only q and $\omega(0,1)$. On each time interval $[T_{i-1}, T_i]$, construct M by $M_t = m_{t - T_{i-1}} M_{T_{i-1}}$. By the facts we stated above this is a (global) solution of (2.1) with desired estimate (2.3). It is easy to verify the uniqueness.

Finally, we will prove the local Lipschitz continuity (2.4). In a similar way as above, we will show by induction that

$$|(I^n_t - I^n_s) - (\hat{I}^n_t - \hat{I}^n_s)| \leq \varepsilon n K^{n-1} \omega(s,t)^{1/q}, \qquad (s,t) \in \triangle_{[0,T]},\ n \in \mathbb{N}. \tag{2.8}$$

Obviously, (2.8) holds for $n = 1$.

In the same way as above, we see that

$$\begin{aligned} |S_{\{s,t\}} - \hat{S}_{\{s,t\}}| &\leq |(\Omega_t - \Omega_s)(I^n_s - \hat{I}^n_s)| + |[(\Omega_t - \Omega_s) - (\hat{\Omega}_t - \hat{\Omega}_s)]\hat{I}^n_s| \\ &\leq \varepsilon n K^{n-1} \omega(0,s)^{1/q} \omega(s,t)^{1/q} + K^{n-1}\omega(0,s)^{1/q} \varepsilon \omega(s,t)^{1/q} \\ &\leq \varepsilon (n+1) K^{n-1} \omega(0,1)^{1/q} \omega(s,t)^{1/q}. \end{aligned} \tag{2.9}$$

Let $t_j \in \mathcal{P}, (j = 1, \ldots, N-1)$. Then,

$$\begin{aligned} &\big|(S_{\mathcal{P}} - S_{\mathcal{P}\setminus\{t_j\}}) - (\hat{S}_{\mathcal{P}} - \hat{S}_{\mathcal{P}\setminus\{t_j\}})\big| \\ &= \big|(\Omega_{t_j} - \Omega_{t_{j-1}}) I^n_{t_{j-1}} + (\Omega_{t_{j+1}} - \Omega_{t_j}) I^n_{t_j} - (\Omega_{t_{j+1}} - \Omega_{t_{j-1}}) I^n_{t_{j-1}} \\ &\qquad - (\hat{\Omega}_{t_j} - \hat{\Omega}_{t_{j-1}}) \hat{I}^n_{t_{j-1}} - (\hat{\Omega}_{t_{j+1}} - \hat{\Omega}_{t_j}) \hat{I}^n_{t_j} + (\hat{\Omega}_{t_{j+1}} - \hat{\Omega}_{t_{j-1}}) \hat{I}^n_{t_{j-1}}\big| \\ &\leq \big|\Omega_{t_{j+1}} - \Omega_{t_j}\big| \, \big|(I^n_{t_j} - I^n_{t_{j-1}}) - (\hat{I}^n_{t_j} - \hat{I}^n_{t_{j-1}})\big| \\ &\qquad + \big|(\Omega_{t_{j+1}} - \Omega_{t_j}) - (\hat{\Omega}_{t_{j+1}} - \hat{\Omega}_{t_j})\big| \, \big|(I^n_{t_j} - I^n_{t_{j-1}})\big| \\ &\leq \varepsilon (n+1) K^{n-1} \omega(t_{j-1}, t_{j+1})^{2/q}. \end{aligned}$$

From this and a routine argument,

$$\big|(S_{\mathcal{P}} - S_{\{s,t\}}) - (\hat{S}_{\mathcal{P}} - \hat{S}_{\{s,t\}})\big| \leq \varepsilon (n+1) K^{n-1} 2^{2/q} \zeta(2/q) \omega(s,t)^{2/q}. \tag{2.10}$$

From (2.9) and (2.10), we see that $|S_{\mathcal{P}} - \hat{S}_{\mathcal{P}}| \leq \varepsilon(n+1) K^n \omega(s,t)^{1/q}$. This implies (2.8) for $n+1$ (and, hence, for all n by induction).

In the same way as above, we can prolong the solutions and obtain (2.4). The proof for N is essentially the same. So we omit it. Take $q' \in (q, 1)$ and apply Corollary 2.2. Then, because of the continuity we have just shown, we see that $M_t N_t = N_t M_t = \mathrm{Id}_{\mathcal{V}}$ for all t. ∎

From now on we will prove a lemma for Duhamel's principle in the context of the rough path theory. When the operator-valued path M below is of finite variation and $[p] = 2$, the principle was checked in [11]. Here, we will consider the case where $p \geq 2$, M is of finite q-variation $(1 \leq q < 2)$ with $1/p + 1/q > 1$.

We set

$$\mathcal{C}_q(L(\mathcal{V}, \mathcal{V})) := \Big\{(M, N) \mid M, N \in C_{0,q}\big(L(\mathcal{V}, \mathcal{V})\big) + \mathrm{Id}_{\mathcal{V}}, \\ M_t N_t = N_t M_t = \mathrm{Id}_{\mathcal{V}} \text{ for } t \in [0, 1]\Big\}.$$

We say $M \in \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ if $(M, M^{-1}) \in \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ for simplicity.

We define a map $\Gamma : C_{0,q}(\mathcal{V}) \times \mathcal{C}_q(L(\mathcal{V}, \mathcal{V})) \to C_{0,q}(\mathcal{V})$ by

$$\Gamma(X, M)_t = \Gamma\big(X, (M, M^{-1})\big)_t := M_t \int_0^t M_s^{-1} dX_s, \qquad t \in [0, 1] \tag{2.11}$$

for $X \in C_{0,q}(\mathcal{V})$ and $M \in \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$. Here, the right hand side is the Young integral.

Lemma 2.6 *Let $\mathcal{V}$ be a real Banach space, $p \geq 2$, $1 \leq q < 2$ with $1/p + 1/q > 1$. Let Γ be as above. Then, we have the following assertions:*

(1). *Assume that there exists a control function ω such that*

$$|X^j_{s,t}| \leq \omega(s, t)^{j/p}, \qquad j = 1, \ldots, [p], \tag{2.12}$$

$$|M_t - M_s|_{L(\mathcal{V},\mathcal{V})} + |M_t^{-1} - M_s^{-1}|_{L(\mathcal{V},\mathcal{V})} \leq \omega(s, t)^{1/q} \tag{2.13}$$

hold for all $(s, t) \in \triangle$. Then,

$$\big|\Gamma(X, M)^j_{s,t}\big| \leq C\omega(s, t)^{j/p}, \quad j = 1, \ldots, [p], \quad (s, t) \in \triangle, \tag{2.14}$$

where C is a positive constant depending only on p, q, and $\omega(0, 1)$.

(2). *Γ extends to a continuous map from $G\Omega_p(\mathcal{V}) \times \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ to $G\Omega_p(\mathcal{V})$. (We denote it again by Γ.) Clearly, $\Gamma(\varepsilon X, M) = \varepsilon\Gamma(X, M)$ holds for any $X \in G\Omega_p(\mathcal{V})$, $M \in \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ and $\varepsilon \in \mathbb{R}$.*

Proof. The proof is not very difficult. So we give a sketch of proof. From (2.11), we have $\Gamma(X, M)_t = X_t - M_t \int_0^t (dM_s^{-1}) X_s$. Note that the map that associates (X, M) with the second term above is continuous from $G\Omega_p(\mathcal{V}) \times \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ to $C_{0,q}(\mathcal{V})$. Using Corollary 2.4, we see that $(X, M) \mapsto \Gamma(X, M)$ is continuous from $G\Omega_p(\mathcal{V}) \times \mathcal{C}_q(L(\mathcal{V}, \mathcal{V}))$ to $G\Omega_p(\mathcal{V})$. ∎

The following corollary is called (the rough path version of) Duhamel's principle and will be used frequently below.

Corollary 2.7 *Let $\mathcal{V}$ be a real Banach space, $p \geq 2$, $1 \leq q < 2$ with $1/p + 1/q > 1$. For $\Omega \in C_{0,q}(L(\mathcal{V}, \mathcal{V}))$, define $M = M_\Omega$ and $N = M_\Omega^{-1}$ as in (2.1) and (2.2). For this $(M_\Omega, M_\Omega^{-1})$ and $X \in C_{0,q}(\mathcal{V})$ define $Y := \Gamma(X, M_\Omega)$ as in (2.11). Then, the following (1)–(3) hold:*
(1). *Y is clearly the unique solution of the following ODE (in the q-variational sense):*

$$dY_t - (d\Omega_t) \cdot Y_t = dX_t, \qquad Y_0 = 0.$$

(2). *Assume that there exists a control function ω such that*

$$|X^j_{s,t}| \leq \omega(s,t)^{j/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle$$
$$|\Omega_t - \Omega_s|_{L(\mathcal{V},\mathcal{V})} \leq \omega(s,t)^{1/q}, \qquad (s,t) \in \triangle.$$

Then, $\left|Y^j_{s,t}\right| \leq C\omega(s,t)^{j/p}$, $\quad j = 1, \ldots, [p], \quad (s,t) \in \triangle$, where C is a positive constant depending only on p, q, and $\omega(0,1)$.
(3). *$(X, \Omega) \mapsto Y = \Gamma(X, M_\Omega)$ extends to a continuous map from $G\Omega_p(\mathcal{V}) \times C_{0,q}(L(\mathcal{V}, \mathcal{V}))$ to $G\Omega_p(\mathcal{V})$.*

2.3 preliminary lemmas

In this subsection we will prove several simple lemmas for later use. Proofs are easy. Let $p \geq 2$ and let $\mathcal{V}$ and $\mathcal{W}$ be real Banach spaces.

Lemma 2.8 *For $\alpha \in L(\mathcal{V}, \mathcal{W})$ and $X \in G\Omega_p(\mathcal{V})$, set $\bar{\alpha}(X) = (1, \bar{\alpha}(X)^1, \ldots, \bar{\alpha}(X)^{[p]})$ by $\bar{\alpha}(X)^j_{s,t} = \alpha^{\otimes j}(X^j_{s,t})$, $(j = 1, \ldots, [p])$.*
(1). *Then, $\bar{\alpha}(X) \in G\Omega_p(\mathcal{W})$ and*

$$|\bar{\alpha}(X)^j_{s,t} - \bar{\alpha}(\hat{X})^j_{s,t}| \leq |\alpha|^j_{L(\mathcal{V},\mathcal{W})} |X^j_{s,t} - \hat{X}^j_{s,t}|$$

for any $X \in G\Omega_p(\mathcal{V})$. In particular, $\bar{\alpha} : G\Omega_p(\mathcal{V}) \to G\Omega_p(\mathcal{W})$ is Lipschitz continuous.
(2). *If $\alpha, \beta \in L(\mathcal{V}, \mathcal{W})$, then*

$$|\bar{\alpha}(X)^j_{s,t} - \bar{\beta}(X)^j_{s,t}| \leq j|\alpha - \beta|_{L(\mathcal{V},\mathcal{W})} (|\alpha|_{L(\mathcal{V},\mathcal{W})} \vee |\beta|_{L(\mathcal{V},\mathcal{W})})^{j-1} |X^j_{s,t}|$$

for any $X \in G\Omega_p(\mathcal{V})$.

Proof. By the basic property of the projective norm, we can see that $|\alpha^{\otimes j}|_{L(\mathcal{V}^{\otimes j}, \mathcal{W}^{\otimes j})} = |\alpha|^j_{L(\mathcal{V},\mathcal{W})}$. The rest is easy ∎

The following is a slight modification of Lemma 6.3.5 in p.171, [16]. The proof is easy. Note that the choice $\delta > 0$ is independent of ω. In the following, $\Gamma_{a,b,c} : G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2}) \to G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ is defined by $\Gamma_{a,b,c} = \overline{a\mathrm{Id}_{\mathcal{V}} \oplus b\mathrm{Id}_{\mathcal{W}} \oplus c\mathrm{Id}_{\mathcal{W}}}$ $(a, b, c \in \mathbb{R})$.

Lemma 2.9 *If $K, \hat{K} \in G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ with $\pi_{\mathcal{V}}(K) = X$, $\pi_{\mathcal{V}}(\hat{K}) = \hat{X}$ and if ω is a control such that*

$$\begin{aligned} |X^j_{s,t}|, |\hat{X}^j_{s,t}| &\leq \frac{1}{2}\omega(s,t)^{j/p}, \quad |K^j_{s,t}|, |\hat{K}^j_{s,t}| \leq (C_2\omega(s,t))^{j/p} \\ |X^j_{s,t} - \hat{X}^j_{s,t}| &\leq \frac{\varepsilon}{2}\omega(s,t)^{j/p}, \quad |K^j_{s,t} - \hat{K}^j_{s,t}| \leq \varepsilon(C_2\omega(s,t))^{j/p}, \end{aligned}$$

for all $j = 1, \ldots, [p]$ and $(s,t) \in \triangle$, then there exists a constant $\delta \in (0,1]$ depending only on C_2 and $[p]$ such that, for any $\delta_1, \delta_2 \in (0, \delta]$, we have

$$\left|(\Gamma_{1,\delta_1,\delta_2}K)^j_{s,t}\right| \leq \omega(s,t)^{j/p}, \quad \left|(\Gamma_{1,\delta_1,\delta_2}K)^j_{s,t} - (\Gamma_{1,\delta_1,\delta_2}\hat{K})^j_{s,t}\right| \leq \varepsilon\omega(s,t)^{j/p},$$

for all $j = 1, \ldots, [p]$ and $(s,t) \in \triangle$.

Lemma 2.10 (1) *Let $\mathcal{V}$ be a real Banach space and $\{0 = T_0 < T_1 < \cdots < T_N = 1\}$ be a partition of $[0,1]$. For each $i = 1, 2, \ldots, N$, $A(i) : \triangle_{[T_{i-1},T_i]} \to T^{([p])}(\mathcal{V})$ is a geometric rough path which satisfies that*

$$\left|A(i)^j_{s,t}\right| \leq \omega(s,t)^{j/p}, \qquad j = 1, \ldots, [p],\ (s,t) \in \triangle_{[T_{i-1},T_i]}.$$

We define $A : \triangle \to T^{([p])}(\mathcal{V})$ by

$$A_{s,t} = A_{s,T_k}(k) \otimes A_{T_k,T_{k+1}}(k+1) \otimes \cdots \otimes A_{T_{l-1},t}(l) \qquad \text{in } T^{([p])}(\mathcal{V})$$

for $(s,t) \in \triangle$ such that $s \in [T_{k-1}, T_k]$ and $t \in [T_{l-1}, T_l]$. Then, A is a geometric rough path such that

$$\left|A^j_{s,t}\right| \leq (C\omega(s,t))^{j/p}, \qquad j = 1, \ldots, [p] \text{ and } (s,t) \in \triangle.$$

Here, $C > 0$ is a constant which depends only on p and N.

(2) *Let $A(i)$ and $\hat{A}(i)$ be two such rough paths on restricted intervals as above ($i = 1, 2, \ldots, N$). In addition to the assumption of (1) for both $A(i)$ and $\hat{A}(i)$, we also assume that*

$$\left|A(i)^j_{s,t} - \hat{A}(i)^j_{s,t}\right| \leq \varepsilon\omega(s,t)^{j/p}, \qquad \text{for } j = 1, \ldots, [p] \text{ and } (s,t) \in \triangle_{[T_{i-1},T_i]}.$$

Then, A and $\hat{A}$ defined as above satisfy that

$$\left|A^j_{s,t} - \hat{A}^j_{s,t}\right| \leq \varepsilon(C'\omega(s,t))^{j/p}, \qquad \text{for } j = 1, \ldots, [p] \text{ and } (s,t) \in \triangle.$$

Here, $C' > 0$ is a constant which depends only on p and N.

Proof. This can be done by straight forward computation. ▮

The first assertion of the following lemma is Corollary 3.2.1, [16]. The second one is a modification of Theorem 3.2.2, [16].

Lemma 2.11 *Let $\mathcal{V}$ be a real Banach space and let $A, B : \triangle \to T^{([p])}(\mathcal{V})$ be almost rough paths.*
(1). *If there exist $\theta > 1$ and a control function ω such that*

$$\begin{aligned} |A^j_{s,t}| &\le \omega(s,t)^{j/p}, \qquad j = 1, \dots, [p], \quad (s,t) \in \triangle, \\ |A^j_{s,t} - (A_{s,u} \otimes A_{u,t})^j| &\le \omega(s,t)^{\theta}, \qquad j = 1, \dots, [p], \quad (s,u),(u,t) \in \triangle, \end{aligned}$$

then, there is a unique rough path $\hat{A}$ associated to $A \in Omega_p(\mathcal{V})$ such that

$$|\hat{A}^j_{s,t}| \le C\omega(s,t)^{j/p}, \qquad j = 1, \dots, [p], \quad (s,t) \in \triangle.$$

Here, $C > 0$ is a constant which depends only on $p, \theta, \omega(0,1)$.
(2). *Assume that there exists $\theta > 1$ and a control function ω such that, for all $j = 1, \dots, [p]$, $\varepsilon > 0$ and $(s,u),(u,t) \in \triangle$,*

$$\begin{aligned} |A^j_{s,t}|,\ |B^j_{s,t}| \le \omega(s,t)^{j/p}, \qquad |A^j_{s,t} - B^j_{s,t}| &\le \varepsilon\omega(s,t)^{j/p}, \\ |A^j_{s,t} - (A_{s,u} \otimes A_{u,t})^j|,\ |B^j_{s,t} - (B_{s,u} \otimes B_{u,t})^j| &\le \omega(s,t)^{\theta}, \\ \Big|\big(A^j_{s,t} - (A_{s,u} \otimes A_{u,t})^j\big) - \big(B^j_{s,t} - (B_{s,u} \otimes B_{u,t})^j\big)\Big| &\le \varepsilon\omega(s,t)^{\theta} \end{aligned}$$

hold. Then, the associated rough paths, $\hat{A}$ and $\hat{B}$, satisfy that

$$|\hat{A}^j_{s,t} - \hat{B}^j_{s,t}| \le \varepsilon C\omega(s,t)^{j/p}, \qquad j = 1, \dots, [p], \quad (s,t) \in \triangle.$$

Here, $C > 0$ is a constant which depends only on $p, \theta, \omega(0,1)$.

Proof. For the first assertion, see Corollary 3.2.1, [16]. We can show the second assertion by modifying the proof of Theorem 3.2.2, [16]. ▮

3 A generalization of Lyons' continuity theorem

The aim of this section is to generalize Lyons' continuity theorem (also known as "the universal limit theorem") for Itô maps in the rough path theory. This section is based on Sections 5.5 and 6.3 in Lyons and Qian [16]. Notations and results in these sections will be referred frequently. The following points seem new:

1. We let the coefficient of an Itô map also vary.

2. We give an explicit estimate for the "local Lipschitz continuity" of Itô maps (Theorem 6.3.1, [16]).

3.1 A review of integration along a geometric rough path

For a real Banach space $\mathcal{V}$, we denote by $G\Omega_p(\mathcal{V})$ the space of geometric rough paths over $\mathcal{V}$, where $p \geq 2$ is the roughness and the tensor norm is the projective norm. Let $\mathcal{W}$ be another real Banach space and let $f : \mathcal{V} \to L(\mathcal{V}, \mathcal{W})$ be $C^{[p]+1}$ in the sense of Fréchet differentiation. We say $f \in C^{[p]+1}_{b,loc}(\mathcal{V}, L(\mathcal{V}, \mathcal{W}))$ if f is $C^{[p]+1}$ from $\mathcal{V}$ to $L(\mathcal{V}, \mathcal{W})$ such that $|D^j f|$ $(j = 0, 1, \ldots, [p]+1)$ are bounded on any bounded set. In this subsection we will consider $\int f(X)dX$ (for $X \in G\Omega_p(X)$).

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, let Π_n be the set of all permutations of $\{1, 2, \ldots, n\}$. We define the left action of $\pi \in \Pi_n$ on $\mathcal{V}^{\otimes n}$ by $\pi(v_1 \otimes \cdots \otimes v_n) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}$. (Note that this is different from the definition in [16], where the right action is adopted. However, this does not matter so much since no composition of permutations will appear below.) Given $\mathbf{l} = \{l_1, \ldots, l_i\}$ $(l_1, \ldots, l_i \in \mathbb{N})$, let $|\mathbf{l}| = l_1 + \cdots + l_i$. We say $\pi \in \Pi_{\mathbf{l}}$ if $\pi \in \Pi_{|\mathbf{l}|}$ satisfies the following conditions:

$$
\begin{aligned}
\pi(1) < & \cdots < \pi(l_1), \\
\pi(l_1+1) < & \cdots < \pi(l_1+l_2), \\
& \cdots \\
\pi(l_1+\cdots+l_{i-1}+1) < & \cdots < \pi(|\mathbf{l}|), \\
\pi(l_1) < \pi(l_1+l_2) < & \cdots < \pi(|\mathbf{l}|).
\end{aligned} \tag{3.1}
$$

(Note: The last condition in (3.1) is missing in p.138, [16].)

Let $X \in G\Omega_p(\mathcal{V})$ be a smooth rough path. Then, for all $(s,t) \in \triangle$,

$$
\int_{s<u_1<\cdots<u_i<t} dX^{l_1}_{s,u_1} \otimes \cdots \otimes dX^{l_i}_{s,u_i} = \sum_{\pi \in \Pi_{\mathbf{l}}} \pi X^{|\mathbf{l}|}_{s,t}. \tag{3.2}
$$

(See Lemma 5.5.1, [16]. By Corollary 2.2 and the Young integration theory, (3.2) also holds for X lying above an element of $C_{0,q}(\mathcal{V})$.)

For $f \in C^{[p]+1}_{b,loc}(\mathcal{V}, L(\mathcal{V}, \mathcal{W}))$ in Fréchet sense, we denote $f^j(x)$ for $D^j f(x)$ for simplicity $(j = 0, 1, \ldots, [p]+1)$. For $X \in G\Omega_p(\mathcal{V})$, we define $Y \in C_{0,p}(\triangle, T^{([p])}(\mathcal{W}))$ by

$$
Y^i_{s,t} = \sum_{\mathbf{l}=(l_1,\ldots,l_i), 1 \leq l_j, |\mathbf{l}| \leq [p]} f^{l_1-1}(X_s) \otimes \cdots \otimes f^{l_i-1}(X_s) \Big\langle \sum_{\pi \in \Pi_{\mathbf{l}}} \pi X^{|\mathbf{l}|}_{s,t} \Big\rangle \tag{3.3}
$$

for $(s,t) \in \triangle$ and $i = 1, \ldots, [p]$.

For a smooth rough path X, it is easy to see that, for any $(s,u), (u,t) \in \triangle$,

$$
\sum_{l=1}^{[p]} f^{l-1}(X_s)\langle dX^l_{s,t} \rangle = \sum_{l=1}^{[p]} \big(f^{l-1}(X_u) - R_l(X_s, X_u) \big) \langle dX^l_{u,t} \rangle \tag{3.4}
$$

holds. Here, for $x, y \in \mathcal{V}$,

$$
\begin{aligned}
R_l(x,y) &= f^{l-1}(y) - \sum_{k=j-1}^{[p]-1} f^k(x) \langle (y-x)^{\otimes(k-l+1)} \rangle \\
&= \int_0^1 d\theta \frac{(1-\theta)^{[p]-l}}{([p]-l)!} f^{[p]}(x + \theta(y-x)) \langle (y-x)^{\otimes([p]-l+1)} \rangle
\end{aligned} \tag{3.5}
$$

(See Lemma 5.5.2 in [16]. A key fact is that the symmetric part of $X^l_{s,t}$ is $[X^1_{s,t}]^{\otimes l}/l!$.)

Using (3.2) and (3.4), we see that, for a smooth rough path X (and for a geometric rough path $X \in G\Omega_p(\mathcal{V})$ by continuity),

$$Y_{s,t} = Y_{s,u} \otimes M_{u,t}, \qquad \text{in } T^{([p])}(\mathcal{W}) \text{ for all } (s,u),(u,t) \in \triangle. \tag{3.6}$$

Here, $M_{u,t}$ is given by $M^0_{u,t} = 1$ and, for $i = 1, \ldots, [p]$,

$$\begin{aligned} M^i_{u,t} \quad = \sum_{\mathbf{l}=(l_1,\ldots,l_i),1\le l_j,|\mathbf{l}|\le[p]} & \bigl(f^{l_1-1}(X_u) - R_{l_1}(X_s,X_u)\bigr) \otimes \\ & \cdots \otimes \bigl(f^{l_i-1}(X_u) - R_{l_i}(X_s,X_u)\bigr)\Bigl\langle \sum_{\pi\in\Pi_{\mathbf{l}}} \pi X^{|\mathbf{l}|}_{u,t}\Bigr\rangle. \end{aligned} \tag{3.7}$$

(Actually, $M_{u,t}$ depends on s, too. Eq. (3.6) is Lemma 5.5.3 in [16].)

Set $N^j_{t,u} = Y^j_{t,u} - M^j_{t,u}$. Then, for a control function ω satisfying that $|X^j_{s,t}| \le \omega(s,t)^{j/p}$ for $j = 1, \ldots, [p]$, it holds that

$$|N^j_{u,t}| \le CM(f;[p],\omega(0,1))^j \omega(s,t)^{([p]+1)/p}, \qquad j = 1,\ldots,[p], \quad (s,t) \in \triangle. \tag{3.8}$$

for some constant $C > 0$ which depends only on $p, \omega(0,1)$. Here, we set

$$M(f;k,R) := \max_{0\le j\le k} \sup\{|f^j(x)| \,:\, |x| \le R\} \qquad \text{for } R > 0 \text{ and } k \in \mathbb{N} \tag{3.9}$$

and $M(f;k) := M(f;k,\infty)$. Then, Y defined by (3.3) is an almost rough path. Indeed, noting that $(Y_{s,u} \otimes N_{u,t})^k = \sum_{i+j=k,i\ge0,j\ge1} Y^i_{s,u} \otimes N^j_{u,t}$, we can easily see from above that there exists a constant $C > 0$ which depends only on $p, \omega(0,1)$ such that

$$|Y^j_{s,t} - (Y_{s,u} \otimes Y_{u,t})^j| \le CM(f;[p],\omega(0,1))^j \omega(s,t)^{([p]+1)/p} \tag{3.10}$$

for all $j = 1, \ldots, [p]$ and $(s,u),(u,t) \in \triangle$.

We denote by $\int f(X)dX$ the unique rough path which is associated to Y. It is well-known that, if X is a smooth rough path lying above $t \mapsto X_t$, then $\int f(X)dX$ is a smooth rough path lying above $t \mapsto \int_0^t f(X_u)dX_u$. By the next proposition, $X \mapsto \int f(X)dX$ is continuous, which implies that $\int f(X)dX \in G\Omega_p(\mathcal{W})$.

The following is essentially Theorem 5.5.2 in [16]. Varying the coefficient f and giving an explicit estimate for the local Lipschitz continuity are newly added. We say $f_n \to f$ as $n \to \infty$ in $C^k_{b,loc}(\mathcal{V}, L(\mathcal{V},\mathcal{W}))$ if $M(f - f_n; k, R) \to 0$ as $n \to \infty$ for any $R > 0$.

Proposition 3.1 *We assume* $f, g \in C^{[p]+1}_{b,loc}(\mathcal{V}, L(\mathcal{V},\mathcal{W}))$.
(1). *Let f be as above. If $X \in G\Omega_p(\mathcal{V})$ and a control function ω satisfy that*

$$|X^j_{s,t}| \le \omega(s,t)^{j/p}, \qquad j = 1,\ldots,[p], \quad (s,t) \in \triangle,$$

then, for a constant $C > 0$ which depends only on p and $\omega(0,1)$, it holds that

$$\Bigl|\int_s^t f(X)dX^j\Bigr| \le CM(f;[p],\omega(0,1))^j \omega(s,t)^{j/p}, \qquad j = 1,\ldots,[p], \quad (s,t) \in \triangle.$$

(2). *Let f, g be as above and let $X, \hat{X} \in G\Omega_p(\mathcal{V})$ such that, for a control function ω,*

$$\begin{aligned} |X^j_{s,t}|,\ |\hat{X}^j_{s,t}| &\leq \omega(s,t)^{j/p}, \\ |X^j_{s,t} - \hat{X}^j_{s,t}| &\leq \varepsilon\omega(s,t)^{j/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle. \end{aligned}$$

Then,

$$\begin{aligned} \Big|\int_s^t f(X)dX^j - \int_s^t g(\hat{X})d\hat{X}^j\Big| &\leq CM(f-g; [p], \omega(0,1))\tilde{M}^{j-1}_{[p]}\omega(s,t)^{j/p} \\ &\quad + \varepsilon C\tilde{M}^j_{[p]+1}\omega(s,t)^{j/p} \end{aligned}$$

for $j = 1, \ldots, [p]$ and $(s,t) \in \triangle$. Here, $\tilde{M}_k = M(f; k, \omega(0,1)) \vee M(g; k, \omega(0,1))$ and $C > 0$ is a constant which depends only on p and $\omega(0,1)$.
(3). *In particular, the following map is continuous:*

$$(f, X) \in C^{[p]+1}_{b,loc}(\mathcal{V}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \mapsto \int f(X)dX \in G\Omega_p(\mathcal{W}).$$

Proof. In this proof the constant $C > 0$ may vary from line to line. To show the first assertion, note that

$$|Y^j_{s,t}| \leq CM(f; [p], \omega(0,1))^j\omega(s,t)^{j/p} \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle.$$

From this and (3.10) we may apply Lemma 2.11 to $M(f; [p], \omega(0,1))^{-1} \cdot Y$ to obtain the first assertion.

Now we prove the second assertion. First we consider $\int_s^t g(X)dX^j - \int_s^t g(\hat{X})d\hat{X}^j$. It is easy to see from (3.3) that

$$|Y^j_{s,t} - \hat{Y}^j_{s,t}| \leq \varepsilon CM(g; [p]+1, \omega(0,1))^j\omega(s,t)^{j/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle.$$

Similarly, we have from (3.8) that

$$|N^j_{u,t} - \hat{N}^j_{u,t}| \leq \varepsilon CM(g; [p]+1, \omega(0,1))^j\omega(s,t)^{([p]+1)/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle.$$

This implies that

$$\begin{aligned} &\Big|[Y^j_{s,t} - (Y_{s,u} \otimes Y_{u,t})^j] - [\hat{Y}^j_{s,t} - (\hat{Y}_{s,u} \otimes \hat{Y}_{u,t})^j]\Big| \\ &\leq \ \varepsilon CM(g; [p]+1, \omega(0,1))^j\omega(s,t)^{([p]+1)/p} \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle. \end{aligned}$$

Now, setting $A = M(g; [p]+1, \omega(0,1))^{-1} \cdot Y$ and $B = M(g; [p]+1, \omega(0,1))^{-1} \cdot \hat{Y}$, we may use Lemma 2.11 to obtain that, for all $j = 1, \ldots, [p], \quad (s,t) \in \triangle$,

$$\Big|\int_s^t g(X)dX^j - \int_s^t g(\hat{X})d\hat{X}^j\Big| \leq \varepsilon C\tilde{M}^j_{[p]+1}\omega(s,t)^{j/p}.$$

Next we consider $\int_s^t f(X)dX^j - \int_s^t g(X)dX^j$. It is easy to see from (3.3) that

$$|Y(f)^j_{s,t} - Y(g)^j_{s,t}| \le CM(f-g;[p],\omega(0,1))\tilde{M}^{j-1}_{[p]}\omega(s,t)^{j/p}, \quad j=1,\dots,[p], \quad (s,t)\in\triangle.$$

Similarly, we have from (3.8) that, for $j=1,\dots,[p]$ and $(s,t)\in\triangle$,

$$|N(f)^j_{u,t} - N(g)^j_{u,t}| \le CM(f-g;[p],\omega(0,1))\tilde{M}^{j-1}_{[p]}\omega(s,t)^{([p]+1)/p}.$$

This implies that

$$\begin{aligned}&\Big|[Y(f)^j_{s,t} - (Y(f)_{s,u}\otimes Y(f)_{u,t})^j] - [Y(g)^j_{s,t} - (Y(g)_{s,u}\otimes Y(g)_{u,t})^j]\Big| \\ &\quad\le\ CM(f-g;[p],\omega(0,1))\tilde{M}^{j-1}_{[p]}\omega(s,t)^{([p]+1)/p} \qquad j=1,\dots,[p], \quad (s,t)\in\triangle.\end{aligned}$$

Now, setting $A=\tilde{M}^{-1}_{[p]}\cdot Y$, $B=\tilde{M}^{-1}_{[p]}\cdot\hat{Y}$ and $\varepsilon = M(f-g;[p],\omega(0,1))\tilde{M}^{-1}_{[p]}$, we may use Lemma 2.11 below to obtain that, for all $j=1,\dots,[p]$, $\quad (s,t)\in\triangle$,

$$\Big|\int_s^t f(X)dX^j - \int_s^t g(X)dX^j\Big| \le CM(f-g;[p],\omega(0,1))\tilde{M}^{j-1}_{[p]}\omega(s,t)^{j/p}.$$

This proves the second assertion.

The third assertion is trivial from the second. ▮

3.2 Existence of solutions of differential equations

In this subsection we check existence and uniqueness of the differential equation in the rough path sense. Essentially, everything in this subsection is taken from Section 6.3, Lyons and Qian [16].

Let $\mathcal{V},\mathcal{W}$ be real Banach spaces and let $f\in C_b^{[p]+1}(\mathcal{W},L(\mathcal{V},\mathcal{W}))$. We define $F\in C_b^{[p]+1}(\mathcal{V}\oplus\mathcal{W},L(\mathcal{V}\oplus\mathcal{W},\mathcal{V}\oplus\mathcal{W}))$ by

$$F(x,y)\langle(\xi,\eta)\rangle = (\xi, f(y)\xi), \qquad (x,y),(\xi,\eta)\in\mathcal{V}\oplus\mathcal{W}.$$

For given f as above and $\beta>0$, we set

$$\Psi_\beta(y,k) = \beta\big(f(y) - f(y-\beta^{-1}k)\big), \qquad y,k\in\mathcal{W}. \tag{3.11}$$

Clearly, Ψ_β is a map from $\mathcal{W}\oplus\mathcal{W}$ to $L(\mathcal{V},\mathcal{W})$.

In this section we consider the following $\mathcal{W}$-valued differential equation for given X in the rough path sense:

$$dY_t = f(Y_t)dX_t, \qquad Y_0 = 0\in\mathcal{W}.$$

Note that, by replacing f with $f(\,\cdot\,+y_0)$, we can treat the same differential equation with an arbitrary initial condition $Y_0=y_0\in\mathcal{W}$. By a solution of the above differential equation, we mean a solution of the following integral equation:

$$Z^j_{s,t} = \int_s^t F(Z)dZ^j, \qquad j=1,\dots,[p],(s,t)\in\triangle, \text{ and } \pi_V(Z)=X. \tag{3.12}$$

Note that $Z \in G\Omega_p(V \oplus W)$ and we also say $\pi_W(Z) = Y$ is a solution for given X. (Here, $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ are the projections from $\mathcal{V} \oplus \mathcal{W}$ onto $\mathcal{V}$ and $\mathcal{W}$, respectively.)

As usual we use the Picard iteration:

$$Z(n+1) = \int F(Z(n))dZ(n), \qquad \text{with } Z(0) = (X, 0).$$

For a smooth rough path X, this is equivalent to

$$\begin{aligned} dX &= dX, \\ dY(n+1) &= f(Y(n))dX, \qquad Y(n+1)_0 = 0. \end{aligned}$$

We may include the difference $D(n) = Y(n) - Y(n-1)$ in the equations: for $n \in \mathbb{N}$,

$$\begin{aligned} dX &= dX, \\ dY(n+1) &= f(Y(n))dX, \\ dD(n+1) &= \Psi_1(Y(n), D(n))dX. \end{aligned} \tag{3.13}$$

By scaling by $\beta > 0$, we see that (3.13) is equivalent to the following: for $n \in \mathbb{N}$,

$$\begin{aligned} dX &= dX, \\ dY(n+1) &= f(Y(n))dX, \\ d\beta D(n+1) &= \Psi_\beta(Y(n), \beta D(n))dX. \end{aligned} \tag{3.14}$$

Set $\Phi_\beta : \mathcal{V} \oplus \mathcal{W}^{\oplus 2} \to L(\mathcal{V} \oplus \mathcal{W}^{\oplus 2}, \mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ by

$$\Phi_\beta(x, y, z)\langle(\xi, \eta, \zeta)\rangle = \big(\xi, f(y)\xi, \Psi_\beta(y, z)\xi\big), \qquad (x, y, z), (\xi, \eta, \zeta) \in \mathcal{V} \oplus \mathcal{W}^{\oplus 2}.$$

Clearly, Φ_β is the coefficient for (3.14).

Lemma 3.2 *Let $\beta \geq 1$ and let f and $\hat{f}$ be as above and define Φ_β and $\hat{\Phi}_\beta$, respectively. Then, for any n, there exist positive constants c_n, c'_n independent of $\beta \geq 1$ and $R > 0$ such that*

$$\begin{aligned} M(\Phi_\beta; n, R) &\leq c_n(1 + R)M(f; n+1, R), \qquad n \in \mathbb{N}, R > 0. \\ M(\Phi_\beta; 0, R) &\leq 1 + (1 + R)M(f; 1, R), \qquad R > 0, \\ M(\Phi_\beta - \hat{\Phi}_\beta; n, R) &\leq c'_n(1 + R)M(f - \hat{f}; n+1, R), \qquad n \in \mathbb{N} \cup \{0\}, R > 0. \end{aligned}$$

Proof. First note that if $|(x, y, z)| = |x| + |y| + |z| \leq R$, then $|y - \theta\beta^{-1}z| \leq R$ for any $\theta \in [0, 1]$. Clearly, $|\Phi_\beta(y, z)| = 1 + |f(y)| + |\Psi_\beta(y, z)|$. By the mean value theorem, $\Psi_\beta(y, z) = \int_0^1 d\theta Df(y - \theta\beta^{-1}z)\langle z\rangle$. Hence, we have the second inequality.

Denoting by π_2, π_3 the projection from $\mathcal{V} \oplus \mathcal{W}^{\oplus 2}$ onto the second and the third component respectively, we have

$$D\Psi_\beta(y, z) = \beta\big(Df(y) - Df(y - \beta^{-1}z)\big) \circ \pi_2 + Df(y - \beta^{-1}z) \circ \pi_3.$$

We can deal with the first term on the right hand side in the same way to prove the first inequality of the lemma for $n = 1$. By continuing straight forward computation like this, we can prove the rest. ∎

For $\varepsilon, \delta, \beta \in \mathbb{R}$ and a smooth rough path K lying above $(k, l, m) \in \mathrm{BV}(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$, we define $\Gamma_{\varepsilon,\delta,\beta} K$ by a smooth rough path lying above $(\varepsilon k, \delta l, \beta m)$. Then, $K \mapsto \Gamma_{\varepsilon,\delta,\beta} K$ extends to a continuous map from $G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ to itself. (Note that $\Gamma_{\varepsilon,\delta,\beta} = \overline{\varepsilon \mathrm{Id}_{\mathcal{V}} \oplus \delta \mathrm{Id}_{\mathcal{W}} \oplus \beta \mathrm{Id}_{\mathcal{W}}}$.) The following is essentially Lemma 6.3.4, [16].

Lemma 3.3 *For any $r, \rho, \beta \in \mathbb{R} \setminus \{0\}$ and a geometric rough path $K \in G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$, we have*

$$\Gamma_{\rho,\rho,\beta\rho} \int \Phi_r(K) dK \;\; = \;\; \int \Phi_{\beta r}(\Gamma_{\rho,1,\beta} K) d\Gamma_{\rho,1,\beta} K.$$

Now we consider the following iteration procedure for given $X \in G\Omega_p(\mathcal{V})$:

$$K(n+1) = \int \Phi_1(K(n)) dK(n) \qquad \text{for } n \in \mathbb{N} \tag{3.15}$$

with $K(0) = (X, 0, 0)$ and $K(1) = (X, f(0)X, f(0)X)$. Note that $K(0)$ and $K(1)$ are well-defined not only for a smooth rough path X, but also for any geometric rough path X. Since Φ_1 is the coefficient for (3.13), this corresponds to (3.13) at least if X is a smooth rough path.

We also set, for $n \in \mathbb{N}$,

$$Z(n+1) = \int F(Z(n)) dZ(n) \qquad \text{with } Z(0) = (X, 0). \tag{3.16}$$

Then, we have $\pi_{\mathcal{V}}(K(n)) = X$ and $\pi_{\mathcal{V}\oplus\mathcal{W}}(K(n)) = Z(n)$ for all $n \in \mathbb{N}$. These relations are trivial when X is a smooth rough path and can be shown by continuity for general $X \in G\Omega_p(\mathcal{V})$.

Now we set $H(n) = \Gamma_{1,1,\beta^{n-1}} K(n)$ for $\beta \neq 0$ and $n \in \mathbb{N}$. If X is a smooth rough path, then $H(n)$ is lying above $(X, Y(n), \beta^{n-1} D(n))$. From Lemma 3.3, we easily see that

$$H(n+1) = \Gamma_{1,1,\beta} \int \Phi_{\beta^{n-1}}(H(n)) dH(n), \qquad \text{for } \beta \neq 0 \text{ and } n \in \mathbb{N}. \tag{3.17}$$

From Proposition 3.1 and Lemma 3.2 we see the following: If $K \in G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ satisfies that, for some control ω with $\omega(0,1) \leq 1$,

$$|K^j_{s,t}| \leq \omega(s,t)^{j/p} \qquad \text{for } j = 1, \ldots, [p] \text{ and } (s,t) \in \triangle,$$

then, for any $\beta > 1$,

$$\Big| \int_s^t \Phi_\beta(K) dK^j \Big| \leq (C_1 \omega(s,t))^{j/p} \qquad \text{for } j = 1, \ldots, [p] \text{ and } (s,t) \in \triangle. \tag{3.18}$$

Here, the constant $C_1 > 0$ can be chosen so that it depends only on $p, M(f; [p]+1)$. (Note that $M(f; [p]+1, 1) \leq M(f; [p]+1)$. See Lemma 3.2. Note that (i) C_1 is independent of $\beta > 1$, (ii) we can take the same C_1 even if we replace f with $f(\,\cdot\, + y_0)$ for any $y_0 \in \mathcal{W}$.)

Proposition 3.4 *Let C_1 as in (3.18) and for this C_1 define δ as in Lemma 2.9. Choose $\beta > 1$ arbitrarily and set $\rho = \beta/\delta$. Let $X \in G\Omega_p(\mathcal{V})$ such that $|X^j_{s,t}| \leq \hat{\omega}(s,t)^{j/p}$ for $j = 1, \dots, [p]$ and $(s,t) \in \triangle$ for some control function $\hat{\omega}$.*
(1). *Set*

$$\omega(s,t) = \big(2 + \frac{\rho + 2|f|_\infty}{\rho}\big)^p \hat{\omega}(s,t). \tag{3.19}$$

Then, for all $j = 1, \dots, [p]$ and $(s,t) \in \triangle$,

$$|X^j_{s,t}| \leq \frac{1}{2}\omega(s,t)^{j/p}, \qquad |\Gamma_{\rho,1,1}K(1)^j_{s,t}| \leq \big(\rho^p \omega(s,t)\big)^{j/p}. \tag{3.20}$$

(2). *Take $T_1 > 0$ so that $\rho^p\omega(0,T_1) \leq 1$. Then, on the restricted time interval $[0,T_1]$, we have the following estimate:*

$$\big|\Gamma_{\rho,1,1}H(n)^j_{s,t}\big| \leq \big(\rho^p\omega(s,t)\big)^{j/p}, \qquad j = 1, \dots, [p], \quad (s,t) \in \triangle_{[0,T_1]}. \tag{3.21}$$

(3). *It holds that*

$$\big|H(n)^j_{s,t}\big| \leq \big(\rho^p\omega(s,t)\big)^{j/p}, \qquad j = 1, \dots, [p], \quad (s,t) \in \triangle_{[0,T_1]}. \tag{3.22}$$

Proof. We prove the first assertion for the first and the second level paths. Proofs for higher level paths are essentially the same. It is obvious that $|X^j_{s,t}| \leq 2^{-j}\omega(s,t)^{j/p}$ for $j = 1, 2$. Since $\rho \mathrm{Id}_V \oplus f(0) \oplus f(0) : \mathcal{V} \to \mathcal{V} \oplus \mathcal{W}^{\oplus 2}$ is a bounded linear map, it naturally extends to $\overline{\rho \mathrm{Id}_{\mathcal{V}} \oplus f(0) \oplus f(0)} : G\Omega_p(V) \to G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ and $\Gamma_{\rho,1,1}K(1) = \overline{\rho \mathrm{Id}_{\mathcal{V}} \oplus f(0) \oplus f(0)}\, X$. Therefore,

$$|\Gamma_{\rho,1,1}K(1)^1_{s,t}| \leq (\rho + 2|f|_\infty)|X^1_{s,t}| \leq \big(\rho^p\omega(s,t)\big)^{1/p}$$

and, in a similar way,

$$|\Gamma_{\rho,1,1}K(1)^2_{s,t}| \leq (\rho^2 + 4\rho|f|_\infty + 4|f|^2_\infty)|X^2_{s,t}| \leq \big(\rho^p\omega(s,t)\big)^{2/p}.$$

Now we prove the second assertion by induction. Assume the inequality is true for n. Then, using (3.18) with a new control function $\rho^p\omega$,

$$\Big|\int_s^t \Phi_{\beta^{n-1}}(\Gamma_{\rho,1,1}H(n))d\Gamma_{\rho,1,1}H(n)^j\Big| \leq (C_1\rho^p\omega(s,t))^{j/p}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. By Lemma 3.3,

$$\Big|\Gamma_{\rho,\rho,\rho}\int_s^t \Phi_{\beta^{n-1}}(H(n))dH(n)^j\Big| \leq (C_1\rho^p\omega(s,t))^{j/p} \tag{3.23}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Note that

$$\left|\pi_{\mathcal{V}}\Gamma_{\rho,\rho,\rho}\int_s^t \Phi_{\beta^{n-1}}(H(n))dH(n)^j\right| = |\rho^j X^j_{s,t}| \le \frac{1}{2}(\rho^p\omega(s,t))^{j/p} \tag{3.24}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Remembering that δ (in Lemma 2.9) is independent of the control function, we may use Lemma 2.9 for (3.23) and (3.24) to obtain

$$\left|\Gamma_{1,\delta\beta^{-1},\delta}\Gamma_{\rho,\rho,\rho}\int_s^t \Phi_{\beta^{n-1}}(H(n))dH(n)^j\right| \le (\rho^p\omega(s,t))^{j/p}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Note that $\Gamma_{1,\delta\beta^{-1},\delta}\Gamma_{\rho,\rho,\rho} = \Gamma_{\rho,1,1}\Gamma_{1,1,\beta}$. Then, by (3.17), we have

$$\left|\Gamma_{\rho,1,1}H(n+1)^j_{s,t}\right| \le (\rho^p\omega(s,t))^{j/p}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Thus, the induction was completed.

The third assertion is easily verified from the second and Lemma 2.8, since $\rho^{-1} \le 1$. ∎

Set $Z'(n) = (Z(n), 0)$ when X is (hence, $Z(n)$ is) a smooth rough path. Clearly, this naturally extends to the case of geometric rough paths and we use the same notation for simplicity. Remember that $K(n) = \Gamma_{1,1,\beta^{-(n-1)}}H(n)$ and $Z'(n) = \Gamma_{1,1,0}H(n)$. Therefore, it is easy to see from Lemma 2.8 that

$$\left|K(n)^j_{s,t} - Z'(n)^j_{s,t}\right| \le j\beta^{-(n-1)}(\rho^p\omega(s,t))^{j/p} \tag{3.25}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$.

Let $\alpha : \mathcal{V}\oplus\mathcal{W}^{\oplus 2} \to \mathcal{V}\oplus\mathcal{W}$ be a bounded linear map defined by $\alpha\langle(x,y,d)\rangle = (x, y-d)$. Then, $|\alpha|_{L(\mathcal{V}\oplus\mathcal{W}^{\oplus 2},\mathcal{V}\oplus\mathcal{W})} = 1$. It is obvious that $\overline{\alpha}K(n) = Z(n-1)$ and $\overline{\alpha}Z'(n) = Z(n)$. Combined with (3.25) and Lemma 2.8, these imply that

$$\left|Z(n)^j_{s,t} - Z(n-1)^j_{s,t}\right| \le j\beta^{-(n-1)}(\rho^p\omega(s,t))^{j/p} \tag{3.26}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$.

Since $\beta > 1$, the inequality above implies that there exists $Z \in G\Omega_p(\mathcal{V}\oplus\mathcal{W})$ such that $\lim_{n\to\infty} Z(n) = Z$ in $G\Omega_p(\mathcal{V}\oplus\mathcal{W})$. In particular,

$$\left|Z(n)^j_{s,t} - Z^j_{s,t}\right| \le j\sum_{m=n}^{\infty}\beta^{-(m-1)}(\rho^p\omega(s,t))^{j/p} \tag{3.27}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. This Z is the desired solution of (3.12) on the restricted time interval $[0, T_1]$.

Remark 3.5 *Let us recall how the constants are defined. First, C_1 depends only on p and $M(f;[p]+1)$. δ depends only on C_1 and p and so does $\rho := \beta/\delta$, where $\beta > 1$ is arbitrary chosen. Therefore, the constants on the right hand side of (3.26) and (3.27) depends only on p and $M(f;[p]+1)$ (and the choice of $\beta > 1$). In particular, the constants C_1, δ, C_3 (below) etc. can be chosen independent of y_0 even if we replace f with $f(\,\cdot\, + y_0)$.*

From (3.27) and Remark 3.5,

$$\left|Z^j_{s,t}\right| \le (C_3\omega(s,t))^{j/p}, \qquad j=1,\dots,[p], \quad (s,t)\in \triangle_{[0,T_1]}. \tag{3.28}$$

for some constant $C_3>0$ which depends only on p and $M(f;[p]+1)$ (and the choice of $\beta>1$).

Now we will consider prolongation of solutions. Take $0=T_0<T_1<\cdots<T_N=1$ so that $\rho^p\omega(T_{i-1},T_i)=1$ for $i=1,\dots,N-1$ and $\rho^p\omega(T_{N-1},T_N)\le 1$. By the superadditivity of ω, $N-1\le \rho^p\omega(0,1)=(3\rho+2|f|_\infty)^p\hat\omega(0,1)$. Hence, N is dominated by a constant which depends only on $\hat\omega(0,1)$, p, and $M(f;[p]+1)$ (and the choice of $\beta>1$).

On $[T_{i-1},T_i]$, we solve the differential equation (3.12) for a initial condition Y_{Ti-1} instead of $Y_0=y_0$. By Remark 3.5 and Remark 3.6, we see that (3.28) holds on each time interval $[T_{i-1},T_i]$ with the same $C_3>0$. Then, we prolong them by using Lemma 2.10. Thus, we obtain a solution on the whole interval $[0,1]$.

Remark 3.6 *By the definition of ω in (3.19), we can take the same T_i's even if we replace f by $f(\,\cdot\,+y_0)$. This is the reason why we assume the boundedness of $|f|$. This fact enables us to use a prolongation method as above. If $|f|$ is of linear growth, then the prolongation of solution may fail. The author does not know whether Lyons' continuity theorem still holds or not in such a case.*

Summing up the above arguments, we have the following existence theorem.

Theorem 3.7 *Consider the differential equation (3.12). Then, for any $X\in G\Omega_p(\mathcal{V})$ there exists a unique solution $Z\in G\Omega_p(\mathcal{V}\oplus\mathcal{W})$ of (3.12). Moreover, if X satisfies that*

$$\left|X^j_{s,t}\right| \le \hat\omega(s,t)^{j/p}, \qquad j=1,\dots,[p], \quad (s,t)\in\triangle$$

for some control function $\hat\omega$, then Z satisfies that

$$\left|Z^j_{s,t}\right| \le (L\hat\omega(s,t))^{j/p}, \qquad j=1,\dots,[p], \quad (s,t)\in\triangle.$$

Here, $L>0$ is a constant which depends only on $\hat\omega(0,1)$, p, and $M(f;[p]+1)$.

Proof. All but uniqueness have already been shown. Since we are mainly interested in estimates of solutions, we omit a proof of uniqueness. See pp. 177–178 in [16]. ∎

3.3 Local Lipschitz continuity of Itô maps

In this section we will prove the Lipschitz continuity of Itô maps. In [16] the coefficient of Itô maps is fixed. Here, we will let the coefficient vary.

Let $X,\hat X\in G\Omega_p(\mathcal{V})$ and $\hat\omega$ be a control function such that

$$|X^j_{s,t}|,|\hat X^j_{s,t}|\le\hat\omega(s,t)^{j/p}, \quad |X^j_{s,t}-\hat X^j_{s,t}| \;\le\; \varepsilon\hat\omega(s,t)^{j/p} \tag{3.29}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle$. Let $y_0, \hat{y}_0 \in \mathcal{W}$ be initial points such that

$$|y_0|, |\hat{y}_0| \le r_0, \qquad |y_0 - \hat{y}_0| \le \varepsilon' \tag{3.30}$$

Let $f, \hat{f} \in C_b^{[p]+2}(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$. For this f and $\hat{f}$, we assume that, for any $R > 0$,

$$M(f; [p]+2), M(\hat{f}; [p]+2) \le M, \qquad M(f - \hat{f}; [p]+2, R) \le \varepsilon''_R. \tag{3.31}$$

Essentially, ε' and ε''_R vary only on $0 \le \varepsilon' \le 2r_0$ and $0 \le \varepsilon''_R \le 2M$, respectively.

As in Theorem 3.7, we can solve the differential equation f and $\hat{f}$ with the initial point y_0 and $\hat{y}_0$, respectively, as in the previous subsection. Set

$$R_0 := 1 + r_0 + L\hat{\omega}(0,1), \tag{3.32}$$

where $L > 0$ is the constant in Theorem 3.7. Then, the first level paths of the solutions satisfy

$$|y_0 + Y^1_{0,t}|, |\hat{y}_0 + \hat{Y}^1_{0,t}| \le R_0 - 1, \qquad t \in [0,1]. \tag{3.33}$$

Instead of (3.30) and (3.31), we assume that

$$|y_0|, |\hat{y}_0| \le R_0 - 1, \qquad |y_0 - \hat{y}_0| \le \varepsilon' \tag{3.34}$$

and that

$$M(f; [p]+2), M(\hat{f}; [p]+2) \le M, \qquad M(f - \hat{f}; [p]+2, R_0) \le \varepsilon''(:= \varepsilon''_{R_0}). \tag{3.35}$$

Clearly, if (3.29)–(3.31) are satisfied, then so are (3.29), (3.34), and (3.35). Now we will start with (3.29), (3.34), and (3.35).

We set $f_{y_0}(y) = f(y + y_0)$ and $\hat{f}_{\hat{y}_0}(y) = \hat{f}(y + \hat{y}_0)$ for simplicity. In a similar way, by replacing y with $y + y_0$ (or with $y + \hat{y}_0$, respectively), we define F_{y_0}, $\Psi_{y_0,\beta}$, $\Phi_{y_0,\beta}$, etc. ($\hat{F}_{\hat{y}_0}$, $\hat{\Psi}_{\hat{y}_0,\beta}$, $\hat{\Phi}_{\hat{y}_0,\beta}$, etc, respectively). By straight forward computation, we see from Lemma 3.2 that

$$\begin{aligned} M(\Phi_{y_0,\beta}; [p]+1, 1) &\le CM(f; [p]+2) = CM, \\ M(\Phi_{y_0,\beta} - \hat{\Phi}_{\hat{y}_0,\beta}; [p], 1) &\le M(\Phi_{y_0,\beta} - \Phi_{\hat{y}_0,\beta}; [p], 1) + M(\Phi_{\hat{y}_0,\beta} - \hat{\Phi}_{\hat{y}_0,\beta}; [p], 1) \\ &\le CM(f_{y_0} - f_{\hat{y}_0}; [p]+1, 1) + CM(f_{\hat{y}_0} - \hat{f}_{\hat{y}_0}; [p]+1, 1) \\ &\le C(M\varepsilon' + \varepsilon''). \end{aligned} \tag{3.36}$$

Here, C is a positive constant depending only on p, which may vary from line to line. For these Φ_{β,y_0} and $\hat{\Phi}_{\beta,\hat{y}_0}$, we define $H(n), K(n), \hat{H}(n), \hat{K}(n)$ as in the previous subsection.

Assume that $K, \hat{K} \in G\Omega_p(\mathcal{V} \oplus \mathcal{W}^{\oplus 2})$ satisfy that, for some control ω with $\omega(0,1) \le 1$,

$$|K^j_{s,t}|, |\hat{K}^j_{s,t}| \le \omega(s,t)^{j/p}, \quad |K^j_{s,t} - \hat{K}^j_{s,t}| \le \varepsilon\omega(s,t)^{j/p} \tag{3.37}$$

for $j = 1, \dots, [p]$ and $(s,t) \in \triangle$. Then, from Proposition 3.1 we obtain (3.18) for $K, \hat{K}$ for the same C_1. Also from Proposition 3.1 and (3.36) we obtain

$$\Big| \int_s^t \Phi_{\beta,y_0}(K) dK^j - \int_s^t \hat{\Phi}_{\beta,\hat{y}_0}(\hat{K}) d\hat{K}^j \Big| \le \varepsilon_1 (C_1'\omega(s,t))^{j/p} \tag{3.38}$$

for $j=1,\dots,[p]$ and $(s,t)\in\triangle$. Here, $\varepsilon_1:=\varepsilon+\varepsilon'+\varepsilon''$ and the constant $C_1'>0$ depends only on M and p. (In particular, C_1' is independent of $\beta\geq 1$.)

Let $C_2:=C_1\vee(2^pC_1')$ and define $\delta\in(0,1]$ as in Lemma 2.9. Take $\beta>1$ arbitrarily and set $\rho=\beta/\delta$. (Here, ρ is defined.)

For $\hat{\omega}$ as in (3.29), we set ω essentially in the same way as in (3.19), namely,

$$\omega(s,t)=\Big(2+\frac{\rho+2|f|_\infty\vee|\hat{f}|_\infty}{\rho}\Big)^p\hat{\omega}(s,t).$$

Then we have (3.20) for $X,\hat{X},\Gamma_{\rho,1,1}K(1),\Gamma_{\rho,1,1}\hat{K}(1)$ and we also have by straight forward computation that

$$|X^j_{s,t}-\hat{X}^j_{s,t}|\leq\frac{\varepsilon_1}{2}\omega(s,t)^{j/p},\qquad|\Gamma_{\rho,1,1}K(1)^j_{s,t}-\Gamma_{\rho,1,1}\hat{K}(1)^j_{s,t}|\leq\varepsilon_1\big(\rho^p\omega(s,t)\big)^{j/p}\quad(3.39)$$

for $j=1,\dots,[p]$ and $(s,t)\in\triangle$.

Take $T_1>0$ so that $\rho^p\omega(0,T_1)\leq 1$ as in Proposition 3.4. We will prove that not only (3.21), but also the following holds for $j=1,\dots,[p]$ and $(s,t)\in\triangle_{[0,T_1]}$:

$$\big|\Gamma_{\rho,1,1}H(n)^j_{s,t}-\Gamma_{\rho,1,1}\hat{H}(n)^j_{s,t}\big|\leq\varepsilon_1\big(\rho^p\omega(s,t)\big)^{j/p}.\qquad(3.40)$$

From (3.40) and Lemma 2.8 we see immediately that

$$\big|H(n)^j_{s,t}-\hat{H}(n)^j_{s,t}\big|\leq\varepsilon_1\big(\rho^p\omega(s,t)\big)^{j/p},\qquad j=1,\dots,[p],\quad(s,t)\in\triangle_{[0,T_1]}.\qquad(3.41)$$

together with (3.22) for $H(n)$ and $\hat{H}(n)$.

Assume that (3.40) holds for n. From (3.38) we have, together with (3.23) for $H(n)$ and $\hat{H}(n)$,

$$\begin{aligned}&\Big|\int_s^t\Phi_{\beta^{n-1},y_0}(\Gamma_{\rho,1,1}H(n))d\Gamma_{\rho,1,1}H(n)^j-\int_s^t\hat{\Phi}_{\beta^{n-1},\hat{y}_0}(\Gamma_{\rho,1,1}\hat{H}(n))d\Gamma_{\rho,1,1}\hat{H}(n)^j\Big|\\&=\Big|\Gamma_{\rho,\rho,\rho}\Big[\int_s^t\Phi_{\beta^{n-1},y_0}(H(n))dH(n)^j-\int_s^t\hat{\Phi}_{\beta^{n-1},\hat{y}_0}(\hat{H}(n))d\hat{H}(n)^j\Big]\Big|\\&\leq(\varepsilon_1+M\varepsilon'+\varepsilon'')(C_1'\rho^p\omega(s,t))^{j/p}\leq\varepsilon_1(2^pC_1'\rho^p\omega(s,t))^{j/p}\end{aligned}$$

for $j=1,\dots,[p]$ and $(s,t)\in\triangle_{[0,T_1]}$. Note that

$$\begin{aligned}&\Big|\pi_{\mathcal{V}}\Gamma_{\rho,\rho,\rho}\Big[\int_s^t\Phi_{\beta^{n-1},y_0}(H(n))dH(n)^j-\int_s^t\hat{\Phi}_{\beta^{n-1},\hat{y}_0}(\hat{H}(n))d\hat{H}(n)^j\Big]\Big|\\&\qquad=|\rho^j(X^j_{s,t}-\hat{X}^j_{s,t})|\leq\frac{\varepsilon_1}{2}(\rho^p\omega(s,t))^{j/p}\end{aligned}$$

for $j=1,\dots,[p]$ and $(s,t)\in\triangle_{[0,T_1]}$. Applying Lemma 2.9 with $C_2:=C_1\vee(2^pC_1')$ and $\varepsilon=\varepsilon_1$ to these two inequalities above (and noting that $\Gamma_{1,\delta\beta^{-1},\delta}\Gamma_{\rho,\rho,\rho}=\Gamma_{\rho,1,1}\Gamma_{1,1,\beta}$), we obtain that

$$\big|\Gamma_{\rho,1,1}H(n+1)^j_{s,t}-\Gamma_{\rho,1,1}\hat{H}(n+1)^j_{s,t}\big|\leq\varepsilon_1(\rho^p\omega(s,t))^{j/p}\qquad(3.42)$$

for $j = 1, \ldots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Thus, we have shown (3.40).

From (3.41) and Lemma 2.8, we can prove the following in the same way as in the proof of (3.28):

$$\left|Z^j_{s,t}\right|, \left|\hat{Z}^j_{s,t}\right| \le (C_3 \rho^p \omega(s,t))^{j/p}, \qquad \left|Z^j_{s,t} - \hat{Z}^j_{s,t}\right| \le \varepsilon_1 (C'_3 \rho^p \omega(s,t))^{j/p} \tag{3.43}$$

for $j = 1, \ldots, [p]$ and $(s,t) \in \triangle_{[0,T_1]}$. Here, C_3, C'_3 depend only on M, p (and the choice of β).

Note that since the second component of $H(n)$ is $Y(n)$, we can easily see from (3.41) that $|Y^1_{0,T_1} - \hat{Y}^1_{0,T_1}| \le \varepsilon_1$. Therefore, the difference of the initial values on the second interval $[T_1, T_2]$ is dominated by $|(y_0 + Y^1_{0,T_1}) - (\hat{y}_0 + \hat{Y}^1_{0,T_1})| \le \varepsilon' + \varepsilon_1 \le 2\varepsilon_1$.

Therefore, from the above computation and (3.33), on the second time interval $[T_1, T_2]$, (3.29), (3.34), and (3.35) are again satisfied, with ε' in (3.34) being replaced with $2\varepsilon_1$. (Note that $\hat{\omega}$, R_0, and M are not changed.)

Thus, we can do the same argument on $[T_1, T_2]$ to obtain (3.43), with ε_1 in (3.43) being replaced with $2\varepsilon_1$. Similarly, we obtain that $|(y_0 + Y^1_{0,T_2}) - (\hat{y}_0 + \hat{Y}^1_{0,T_2})| \le 5\varepsilon_1$. Repeating this argument finitely many times and use Lemma 2.10, we can prove the following theorem on the local Lipschitz continuity of Itô maps.

Theorem 3.8 *Let $\hat{\omega}$, r_0, R_0,M, $\varepsilon, \varepsilon', \varepsilon''_R$, $X, \hat{X} \in G\Omega_p(\mathcal{V})$, $y_0, \hat{y}_0 \in \mathcal{W}$, and $f, \hat{f} \in C_b^{[p]+2}(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ satisfy (3.29)–(3.31). Set R_0 and $\varepsilon'' = \varepsilon''_{R_0}$ as in (3.32). We denote by $Z, \hat{Z}$ be the solutions of Itô maps corresponding to $f, \hat{f}$ with initial condition $y_0, \hat{y}_0$, respectively. Then, in addition to Theorem 3.7, we have the following; there is a positive constant L' such that*

$$|Z^j_{s,t} - \hat{Z}^j_{s,t}| \le (\varepsilon + \varepsilon' + \varepsilon'')(L' \hat{\omega}(s,t))^{j/p} \tag{3.44}$$

for $j = 1, \ldots, [p]$ and $(s,t) \in \triangle$. Here, L' depends only on $\hat{\omega}(0,1)$, p, r_0 and M.

For $k \in \mathbb{N}$ and $M > 0$, set $\mathcal{C}^k_M(\mathcal{W}, L(\mathcal{V}, \mathcal{W})) = \{f \in C^k_b(\mathcal{W}, L(\mathcal{V}, \mathcal{W})) \mid M(f;k) \le M\}$. We say $f_n \to f$ in $\mathcal{C}^k_M(\mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ as $n \to \infty$ if $M(f - f_n; k, R) \to 0$ as $n \to \infty$ for any $R > 0$.

Corollary 3.9 *Let Z be the solution corresponding to X, f, y_0 given as above. Then, the map*

$$(f, X, y_0) \in \mathcal{C}^{[p]+2}_M(\mathcal{W}, L(\mathcal{V}, \mathcal{W})) \times G\Omega_p(\mathcal{V}) \times \mathcal{W} \mapsto Z \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$$

is continuous for any $M > 0$.

Remark 3.10 *The above argument is based on Section 6.3, [16]. Therefore, when $[p] = 2$, the regularity condition is C^4_b in Theorem 3.8 and Corollary 3.9. The author guesses that this can be improved to C^3_b, if we mimic the argument in Sections 6.1 and 6.2, [16]. (However, it has not been done yet.)*

3.4 Estimate of difference of higher level paths of two solutions

Let $p \geq 2$ and $1/p + 1/q > 1$ and let f be as in the previous subsection. In this subsection we only consider the case $f = \hat{f}$. For given X, the solution $Z = (X, Y)$ of (3.12) is denoted by $Z_X = (X, Y_X)$. In Theorem 3.8 we estimated the "difference" of Z_X^j and $Z_{\hat{X}}^j$. If $\hat{X} \in C_{0,q}(\mathcal{V}) \subset G\Omega_p(\mathcal{V})$, Then, $Z_{\hat{X}}$ is an element of $C_{0,q}(\mathcal{W})$ and, therefore, the "difference" $Z_X - Z_{\hat{X}}$ is a $\mathcal{W}$-valued geometric rough path. The purpose of this section is to give an estimate for $(Z_X - Z_{\hat{X}})^j$ in such a case. Note that $(Z_X - Z_{\hat{X}})^j$ and $Z_X^j - Z_{\hat{X}}^j$ is not the same if $j \neq 1$.

Roughly speaking, we will show that $\|(Z_{X+\Lambda} - Z_\Lambda)^j\|_{p/j} \leq C(c_1, \kappa_0, f)\xi(X)^j$ for $\Lambda \in C_{0,q}(\mathcal{V})$ and $X \in G\Omega_p(\mathcal{V})$ with $\|\Lambda\|_q \leq c_1$ and $\xi(X) \leq \kappa_0$. Since this is a continuous function of (X, Λ), we may only think of X lying above an element of $C_{0,q}(\mathcal{V})$. Note also that if we set

$$\omega(s,t) := \|\Lambda\|_{q,[s,t]}^q + \sum_{j=1}^{[p]} \kappa^{-p} \|X^j\|_{p/j,[s,t]}^{p/j}, \qquad \text{(here, we set } \kappa := \xi(X)\text{)},$$

then this control function satisfies that $\omega(0,1) \leq c_1^q + [p]$, $|\Lambda_{s,t}^1| \leq \omega(s,t)^{1/q}$, and $|X_{s,t}^j| \leq \kappa^j \omega(s,t)^{j/p}$.

First we prove the following lemma. Heuristically, $\kappa > 0$ is a small constant.

Lemma 3.11 *Let $\kappa_0 > 0$. Assume that a control function $\hat{\omega}$, $\Lambda \in C_{0,q}(\mathcal{V})$ and $X \in G\Omega_p(\mathcal{V})$, and $\kappa \in [0, \kappa_0]$ satisfy that*

$$|\Lambda_{s,t}^1| \leq \hat{\omega}(s,t)^{1/q}, \qquad |X_{s,t}^j| \leq (\kappa\hat{\omega}(s,t))^{j/p} \qquad j = 1, \ldots, [p], (s,t) \in \triangle.$$

Then, there is a positive constant C which depends only on $\kappa_0, \hat{\omega}(0,1), p, M(f; [p]+1)$ such that

$$\left| \left(\kappa^{-1}X, \Lambda, Y_{X+\Lambda}\right)_{s,t}^j \right| \leq C\hat{\omega}(s,t)^{p/j}, \qquad s < t.$$

Here, the left hand side denotes the jth level path of a $\mathcal{V}^{\oplus 2} \oplus \mathcal{W}$-valued geometric rough path and $Y_{X+\Lambda}$ denotes (the $\mathcal{W}$-component of) the solution of (3.12) for $X + \Lambda$.

Proof. We will proceed in a similar way as in the previous subsection. We define $\tilde{F} \in C_b^{[p]+1}(\mathcal{V}^{\oplus 2} \oplus \mathcal{W}, L(\mathcal{V}^{\oplus 2} \oplus \mathcal{W}, \mathcal{V}^{\oplus 2} \oplus \mathcal{W}))$ by

$$\tilde{F}(x, x', y)\langle(\xi, \xi', \eta)\rangle = (\xi, \xi', f(y)(\xi + \xi')), \quad \text{for } (x, x', y), (\xi, \xi', \eta) \in \mathcal{V}^{\oplus 2} \oplus \mathcal{W}$$

and define $\tilde{\Phi}_\beta : \mathcal{V}^{\oplus 2} \oplus \mathcal{W}^{\oplus 2} \to L(\mathcal{V}^{\oplus 2} \oplus \mathcal{W}^{\oplus 2}, \mathcal{V}^{\oplus 2} \oplus \mathcal{W}^{\oplus 2})$ by

$$\tilde{\Phi}_\beta(x, x', y, z)\langle(\xi, \xi', \eta, \zeta)\rangle = \big(\xi, \xi', f(y)(\xi + \xi'), \Psi_\beta(y, z)(\xi + \xi')\big),$$

for $\beta > 0$ and $(x, x', y, z), (\xi, \xi', \eta, \zeta) \in \mathcal{V}^{\oplus 2} \oplus \mathcal{W}^{\oplus 2}$. Note that $\tilde{\Phi}_\beta$ satisfies a similar estimates as in Lemma 3.2.

Instead of (3.12), we now consider

$$\tilde{Z}^j_{s,t} = \int_s^t \tilde{F}(\tilde{Z})d\tilde{Z}^j, \qquad j = 1, 2, \ldots, [p], (s,t) \in \triangle, \text{ and } \pi_{\mathcal{V}^{\oplus 2}}(\tilde{Z}) = (X, \Lambda). \quad (3.45)$$

It is easy to check that the solution of this equation is $(X, \Lambda, Y_{X+\Lambda})$.

In order to solve (3.45), we use the iteration method as in (3.13) or (3.14). More explicitly,

$$\begin{aligned} dX &= dX, \qquad d\Lambda = d\Lambda \\ d\tilde{Y}(n+1) &= f(\tilde{Y}(n))d(X+\Lambda), \\ d\beta\tilde{D}(n+1) &= \Psi_\beta(\tilde{Y}(n), \beta\tilde{D}(n))d(X+\Lambda). \end{aligned} \quad (3.46)$$

Now consider the iteration procedure for given $(X, \Lambda) \in G\Omega_p(\mathcal{V}) \times C_{0,q}(\mathcal{V})$ and for $\tilde{F}$ and $\tilde{\Phi}_1$ as in (3.15) and (3.16). Also define $\tilde{K}(n)$ and $\tilde{Z}(n)$ as in (3.15) and (3.16) with $\tilde{K}(0) = (X, \Lambda, 0, 0)$ and $\tilde{K}(1) = (X, \Lambda, f(0)(X+\Lambda), f(0)(X+\Lambda))$.

Set $\tilde{H}(n) = \Gamma_{1,1,1,\beta^{n-1}}\tilde{K}(n) = (X, \Lambda, \tilde{Y}(n), \beta^{n-1}\tilde{D}(n))$. Then, in the same way as in (3.17), we have

$$\tilde{H}(n+1) = \Gamma_{1,1,1,\beta} \int \tilde{\Phi}_{\beta^{n-1}}(\tilde{H}(n))d\tilde{H}(n), \qquad \text{for } \beta \neq 0 \text{ and } n \in \mathbb{N}. \quad (3.47)$$

Slightly modifying Proposition 3.1 and Lemma 3.2, we see from the estimates for $\tilde{\Phi}_\beta$ the following: If $\tilde{K} \in G\Omega_p(\mathcal{V}^{\oplus 2} \oplus \mathcal{W}^{\oplus 2})$ satisfies that, for some control ω_0 with $\omega_0(0,1) \leq 1$,

$$|\Gamma_{1/\kappa,1,1,1}\tilde{K}^j_{s,t}| \leq \omega_0(s,t)^{j/p} \qquad \text{for } j = 1, \ldots, [p], \quad (s,t) \in \triangle$$

then, for any $\beta > 1$,

$$\Big|\Gamma_{1/\kappa,1,1,1} \int_s^t \tilde{\Phi}_\beta(\tilde{K})d\tilde{K}^j\Big| \leq (\tilde{C}_1\omega_0(s,t))^{j/p} \qquad \text{for } j = 1, \ldots, [p], (s,t) \in \triangle \quad (3.48)$$

Here, $\tilde{C}_1 > 0$ is a constant which depends only on $\kappa_0, p, M(f; [p]+1)$. (See Lemma 3.2. Note that (i) $\tilde{C}_1$ is independent of $\beta > 1$, (ii) we may take $\tilde{C}_1$ independent of y_0 even if we replace f with $f(\,\cdot\, + y_0)$.)

Let $\tilde{C}_1$ as in (3.48) and for this $\tilde{C}_1$ define δ as in Lemma 2.9. Choose $\beta > 1$ arbitrarily and set $\rho = \beta/\delta$, where δ is given in Lemma 2.9. As in (3.4), there exists a constant $c = c(\kappa_0, \rho, p, |f|_\infty)$ such that $\omega(s,t) = c\hat{\omega}(s,t)$ satisfies that , for all $j = 1, \ldots, [p]$ and $(s,t) \in \triangle$,

$$|(\kappa^{-1}X, \Lambda)^j_{s,t}| \leq \frac{1}{2}\omega(s,t)^{j/p}, \qquad |\Gamma_{\rho/\kappa,\rho,1,1}\tilde{K}(1)^j_{s,t}| \leq \big(\rho^p\omega(s,t)\big)^{j/p}. \quad (3.49)$$

Now we will show that, for $T_1 \in (0,1]$ such that $\omega(0,T_1) \leq 1$, it holds on the restricted time interval $[0, T_1]$ that

$$\big|\Gamma_{\rho/\kappa,\rho,1,1}\tilde{H}(n)^j_{s,t}\big| \leq \big(\rho^p\omega(s,t)\big)^{j/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle_{[0,T_1]}. \quad (3.50)$$

We use induction. The case $n = 1$ was already shown since $\tilde{H}(1) = \tilde{K}(1)$. Using (3.48) for $\rho^p\omega$, we have

$$\begin{aligned}\Big|\Gamma_{1/\kappa,1,1,1}\int_s^t \tilde{\Phi}_{\beta^{n-1}}(\Gamma_{\rho,\rho,1,1}\tilde{H}(n))d\Gamma_{\rho,\rho,1,1}\tilde{H}(n)^j\Big| &= \Big|\Gamma_{\rho/\kappa,\rho,\rho,\rho}\int_s^t \tilde{\Phi}_{\beta^{n-1}}(\tilde{H}(n))d\tilde{H}(n)^j\Big| \\ &\leq (\tilde{C}_1\rho^p\omega(s,t))^{j/p}. \end{aligned} \tag{3.51}$$

By projection onto the $\mathcal{V}^{\oplus 2}$-component,

$$\Big|\pi_{\mathcal{V}^{\oplus 2}}\Gamma_{\rho/\kappa,\rho,\rho,\rho}\int_s^t \tilde{\Phi}_{\beta^{n-1}}(\tilde{H}(n))d\tilde{H}(n)^j\Big| = |\rho^j(\kappa^{-1}X,\Lambda)^j_{s,t}| \leq \frac{1}{2}(\rho^p\omega(s,t))^{j/p} \tag{3.52}$$

We may use Lemma 2.9 for (3.23) and (3.24) to obtain

$$\Big|\Gamma_{1,1,\delta\beta^{-1},\delta}\Gamma_{\rho/\kappa,\rho,\rho,\rho}\int_s^t \tilde{\Phi}_{\beta^{n-1}}(\tilde{H}(n))d\tilde{H}(n)^j\Big| \leq (\rho^p\omega(s,t))^{j/p}$$

From this, we see that (3.50) for $n+1$. Hence we have shown (3.50) for any n. From (3.50), it is easy to see that

$$\big|\Gamma_{1/\kappa,1,1,1}\tilde{H}(n)^j_{s,t}\big| \leq \big(\rho^p\omega(s,t)\big)^{j/p}, \qquad j = 1,\ldots,[p], \quad (s,t)\in\triangle_{[0,T_1]}. \tag{3.53}$$

In the same way as in (3.25)–(3.28), we obtain from (3.53) that

$$\big|\Gamma_{1/\kappa,1,1}\tilde{Z}^j_{s,t}\big| \leq (\tilde{C}_3\omega(s,t))^{j/p}, \qquad j = 1,\ldots,[p], \quad (s,t)\in\triangle_{[0,T_1]}. \tag{3.54}$$

for some constant $\tilde{C}_3 > 0$ which depends only on κ_0, p, and $M(f;[p]+1)$ (and the choice of $\beta > 1$).

Note that (3.54) is the desired inequality (on the restricted interval). By prolongation of solution we can prove the lemma. ▮

For X and Λ as above, set $Q = Q(X,\Lambda) := Y_{X+\Lambda} - Y_\Lambda$. Clearly,

$$dQ = f(Y_{X+\Lambda})dX + [f(Y_{X+\Lambda}) - f(Y_\Lambda)]d\Lambda.$$

Lemma 3.12 *Let $\kappa_0 > 0$. Assume that a control function $\hat{\omega}$, $\Lambda\in C_{0,q}(\mathcal{V})$ and $X\in G\Omega_p(\mathcal{V})$, and $\kappa\in[0,\kappa_0]$ satisfy that*

$$|\Lambda^1_{s,t}| \leq \hat{\omega}(s,t)^{1/q}, \qquad |X^j_{s,t}| \leq (\kappa\hat{\omega}(s,t))^{j/p} \qquad j = 1,\ldots,[p], \quad (s,t)\in\triangle.$$

Then, there is a positive constant C which depends only on $\kappa_0, \hat{\omega}(0,1), p, M(f;[p]+2)$ such that

$$\Big|\big(\kappa^{-1}X,\Lambda,Y_{X+\Lambda},Y_\Lambda,\kappa^{-1}Q\big)^j_{s,t}\Big| \leq (C\hat{\omega}(s,t))^{j/p}, \qquad j = 1,\ldots,[p], \quad (s,t)\in\triangle.$$

Here, the left hand side denotes the jth level path of a $\mathcal{V}^{\oplus 2}\oplus\mathcal{W}^{\oplus 3}$-valued geometric rough path.

Proof. In this proof, the positive constant C may change from line to line. As before we may assume that $X \in C_{0,q}(\mathcal{V})$. Recall that $\Lambda \in C_{0,q}(\mathcal{V}) \mapsto Y_\Lambda \in C_{0,q}(\mathcal{W})$ is continuous and there exists a constant $C > 0$ such that $|(Y_\Lambda)^1_{s,t}| \le C\hat{\omega}(s,t)^{1/q}$. Combining this with Lemma 3.11, we have

$$\left|\left(\kappa^{-1}X, \Lambda, Y_{X+\Lambda}, Y_\Lambda\right)^j_{s,t}\right| \le (C\hat{\omega}(s,t))^{j/p}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle.$$

From Lemma 3.11, we easily see that, for some constant $C > 0$,

$$\left|\left(\kappa^{-1}X, \Lambda, Y_{X+\Lambda}, Y_\Lambda, \kappa^{-1}\int f(Y_{X+\Lambda})dX\right)^j_{s,t}\right| \le (C\hat{\omega}(s,t))^{j/p}, \quad j = 1, \ldots, [p], (s,t) \in \triangle.$$

Note that from the local Lipschitz continuity of (the first level path of) the Itô map (Theorem 3.8), we see that, for some constant $C > 0$,

$$\left|\kappa^{-1}\int_s^t [f(Y_{X+\Lambda}) - f(Y_\Lambda)]d\Lambda\right| \le C\hat{\omega}(s,t)^{1/q}, \qquad j = 1, \ldots, [p], \quad (s,t) \in \triangle.$$

Here, the left hand side is the Young integral. From these, we can easily obtain the theorem. ▌

4 A stochastic Taylor-like expansion

4.1 Estimates for ordinary terms in the expansion

In this section we will estimate ordinary terms in the stochastic Taylor-like expansion for Itô maps. Let $p \ge 2$ and $1 \le q < 2$ with $1/p + 1/q > 1$ and let $\mathcal{V}, \hat{\mathcal{V}}, \mathcal{W}$ be real Banach spaces. Let $\sigma \in C_b^\infty([0,1] \times \mathcal{W}, L(\mathcal{V}, \mathcal{W}))$ and $b \in C_b^\infty([0,1] \times \mathcal{W}, L(\hat{\mathcal{V}}, \mathcal{W}))$. Here, $[0,1] \times \mathcal{W}$ is considered as a subset of the direct sum $\mathbf{R} \oplus \mathcal{W}$. We will consider the following ODE: for $\varepsilon > 0$, $X \in G\Omega_p(\mathcal{V})$, and $\Lambda \in C_{0,q}(\hat{\mathcal{V}})$,

$$dY^{(\varepsilon)}_t = \sigma(\varepsilon, Y^{(\varepsilon)}_t)\varepsilon dX_t + b(\varepsilon, Y^{(\varepsilon)}_t)d\Lambda_t, \qquad Y^{(\varepsilon)}_0 = 0 \tag{4.1}$$

Note that if X is lying above an element of $C_{0,q}(\mathcal{V})$, then (4.1) makes sense in the q-variational setting.

More precisely, the above equation (4.1) can be formulated as follows. Define $\tilde{\sigma}$ by $\tilde{\sigma} = \sigma \circ p_1 + b \circ p_2$, where p_1 and p_2 are canonical projection from $\mathcal{V} \oplus \hat{\mathcal{V}}$ onto the first and the second component, respectively. Then, $\tilde{\sigma} \in C_b^\infty([0,1] \times \mathcal{W}, L(\mathcal{V} \oplus \hat{\mathcal{V}}, \mathcal{W}))$. We consider the Itô map $\Phi^\varepsilon : G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}}) \to G\Omega_p(\mathcal{W})$ which corresponds to the coefficient $\tilde{\sigma}(\varepsilon, \cdot)$ with the initial condition 0. If $X \in C_{0,q}(\mathcal{V})$ and $\Lambda \in C_{0,q}(\hat{\mathcal{V}})$, then $(X, \Lambda) \in C_{0,q}(\mathcal{V} \oplus \hat{\mathcal{V}})$. This map naturally extends to a continuous map from $G\Omega_p(\mathcal{V}) \times C_{0,q}(\hat{\mathcal{V}})$ to $G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}})$ (see Corollary 3.9). The precise meaning of (4.1) is that $Y^{(\varepsilon)} = \Phi^\varepsilon((\varepsilon X, \Lambda))_1$.

Remark 4.1 *In Azencott [3], he treated differential equations with the coefficients of the form $\sigma(\varepsilon, t, y)$ and $b(\varepsilon, t, y)$. ODE (4.1), however, includes such cases. In order to see this, set $\mathcal{W}' = \mathcal{W} \oplus \mathbb{R}$, $\hat{\mathcal{V}}' = \hat{\mathcal{V}} \oplus \mathbb{R}$, $\Lambda'_t = (\Lambda_t, t)$, and add to (4.1) the following trivial equation; $dY'_t = dt$.*

We set $Y^0 = \Phi^0((\mathbf{1}, \Lambda))_1$, where $\mathbf{1} = (1, 0, \ldots, 0)$ is the unit element in the truncated tensor algebra (which is regarded as a constant rough path). Note that $\Lambda \in C_{0,q}(\hat{\mathcal{V}}) \mapsto Y^0 \in C_{0,q}(\mathcal{W})$ is locally Lipschitz continuous (see [16]).

We will expand $Y^{(\varepsilon)}$ in the following form:

$$Y^{(\varepsilon)} \sim Y^0 + \varepsilon Y^1 + \varepsilon^2 Y^2 + \varepsilon^3 Y^3 + \cdots \qquad \text{as } \varepsilon \searrow 0.$$

(Note that Y^k does NOT denote the kth level path of Y. The kth level path of Y^j will be denoted by $(Y^j)^k$. Similar notations will be used for I^k and J^k below. This may be a little confusing. Sorry.) By considering a (formal) Taylor expansion for $\sigma(\varepsilon, Y_t^{(\varepsilon)})$ and $b(\varepsilon, Y_t^{(\varepsilon)})$, we will find explicit forms of Y^n ($n \in \mathbb{N}$) as follows. (Or equivalently, we may formally operate $(n!)^{-1}(d/d\varepsilon)^n$ at $\varepsilon = 0$ on the both sides of (4.1)).

$$dY_t^n - \partial_y b(0, Y_t^0)\langle Y_t^n, d\Lambda_t\rangle = dI_t^n + dJ_t^n, \qquad (n \in \mathbb{N}) \tag{4.2}$$

where $I^n = I^n(X, \Lambda)$ and $J^n = J^n(X, \Lambda)$ are given by

$$dI_t^1 = \sigma(0, Y_t^0) dX_t, \qquad dJ_t^1 = \partial_\varepsilon b(0, Y_t^0) d\Lambda_t, \tag{4.3}$$

with $I_0^1 = J_0^1 = 0$, and, for $n = 2, 3, \ldots$,

$$\begin{aligned} dI_t^n &= \sum_{j=0}^{n-2} \sum_{k=1}^{n-1-j} \sum_{(i_1,\ldots,i_k)\in S_k^{n-1-j}} \frac{1}{j!k!} \partial_\varepsilon^j \partial_y^k \sigma(0, Y_t^0)\langle Y_t^{i_1}, \ldots, Y_t^{i_k}, dX_t\rangle \\ &\quad + \frac{1}{(n-1)!} \partial_\varepsilon^{n-1} \sigma(0, Y_t^0) dX_t \\ dJ_t^n &= \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \sum_{(i_1,\ldots,i_k)\in S_k^{n-j}} \frac{1}{j!k!} \partial_\varepsilon^j \partial_y^k b(0, Y_t^0)\langle Y_t^{i_1}, \ldots, Y_t^{i_k}, d\Lambda_t\rangle \\ &\quad + \sum_{k=2}^{n} \frac{1}{k!} \partial_y^k b(0, Y_t^0)\langle Y_t^{i_1}, \ldots, Y_t^{i_k}, d\Lambda_t\rangle + \frac{1}{n!} \partial_\varepsilon^n b(0, Y_t^0) d\Lambda_t \end{aligned} \tag{4.4}$$

with $I_0^n = J_0^n = 0$. Here, ∂_ε and ∂_y denote the partial Fréchet derivatives in ε and in y, respectively, and

$$S_k^m = \{(i_1, \ldots, i_k) \in \mathbb{N}^k = \{1, 2, \ldots\}^k \mid i_1 + \cdots + i_k = m\}.$$

Now we define functions which appear on the right hand sides of (4.3) and (4.4). Let $\mathcal{X}_{n-1} = \mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus n}$. An element in $\mathcal{X}_{n-1}$ is denoted by $v = (x, \hat{x}; y^0, y^1, \ldots, y^{n-1})$. Partial Fréchet derivatives are denoted by ∂_x, ∂_{y^1}, etc. and the projection from $\mathcal{X}_{n-1}$ onto each components are denoted by p_x, p_{y^1}, etc. Set $f_1, g_1 \in C_{b,loc}^\infty(\mathcal{X}_0, L(\mathcal{X}_0, \mathcal{W}))$ by

$$f_1(y^0) = \sigma(0, y^0) \circ p_x, \qquad g_1(y^0) = \partial_\varepsilon b(0, y^0) \circ p_{\hat{x}}. \tag{4.5}$$

For $n = 2, 3, \dots,$ set $f_n, g_n \in C^\infty_{b,loc}(\mathcal{X}_{n-1}, L(\mathcal{X}_{n-1}, \mathcal{W}))$ by

$$\begin{aligned} f_n(y^0, \dots, y^{n-1}) &= \Big[\sum_{j=0}^{n-2}\sum_{k=1}^{n-1-j}\sum_{(i_1,\dots,i_k)\in S_k^{n-1-j}} \frac{1}{j!k!}\partial_\varepsilon^j \partial_y^k \sigma(0, y^0)\langle y^{i_1}, \dots, y^{i_k}, \cdot\rangle \\ &\quad + \frac{1}{(n-1)!}\partial_\varepsilon^{n-1}\sigma(0, y^0)\Big] \circ p_x, \\ g_n(y^0, \dots, y^{n-1}) &= \Big[\sum_{j=1}^{n-1}\sum_{k=1}^{n-j}\sum_{(i_1,\dots,i_k)\in S_k^{n-j}} \frac{1}{j!k!}\partial_\varepsilon^j \partial_y^k b(0, y^0)\langle y^{i_1}, \dots, y^{i_k}, \cdot\rangle \\ &\quad + \sum_{k=2}^{n}\frac{1}{k!}\partial_y^k b(0, y^0)\langle y^{i_1}, \dots, y^{i_k}, \cdot\rangle + \frac{1}{n!}\partial_\varepsilon^n b(0, y^0)\Big] \circ p_{\hat{x}}. \quad (4.6) \end{aligned}$$

Clearly, the functions f_n and g_n are actually independent of x and $\hat{x}$.

Lemma 4.2 *Let f_n and g_n be as above and $\delta > 0, C > 0, r \in \mathbb{N}$. For $\xi = (\xi_1, \dots, \xi_r) \in \{0, \dots, n-1\}^r$, we set $|\xi| = \sum_{k=1}^r \xi_k$. Then, on the following set*

$$\big\{(y^0, \dots, y^{n-1}) \mid \ |y^i| \le C(1+\delta)^i \text{ for } 0 \le i \le n-1\big\},$$

it holds that, for any ξ such that $|\xi| \le n-1$,

$$\big|\partial_\xi^r f_n(x, \hat{x}; y^0, \dots, y^{n-1})\big| \le C'(1+\delta)^{n-1-|\xi|},$$

Here, $\partial_\xi^r = \partial_{y^{\xi_1}} \cdots \partial_{y^{\xi_r}}$ and C' is a positive constant independent of δ. If $|\xi| > n-1$, then $\partial_\xi^r f_n = 0$.

Similarly, it holds on the same set that, for any ξ such that $|\xi| \le n$,

$$\big|\partial_\xi^r g_n(x, \hat{x}; y^0, \dots, y^{n-1})\big| \le C'(1+\delta)^{n-|\xi|}.$$

If $|\xi| > n$, then $\partial_\xi^r g_n = 0$.

Proof. This lemma can be shown by straight forward computation since f_n and g_n are (i) C_b^∞ in y^0-variable and (ii) "polynomials" in $(y^1, \dots, y^{n-1})$-variables. ▮

In fact, we can compute $\partial_\xi^r f_n = \partial_{y^{\xi_1}} \cdots \partial_{y^{\xi_r}} f_n$ explicitly as follows.

Lemma 4.3 *Let f_n $(n = 1, 2, \dots)$ be as above and $r = 1, 2, \dots$. For $\xi = (\xi_1, \dots, \xi_r) \in \{0, \dots, n-1\}^r$, set $\mu = \sharp\{k \mid 1 \le k \le r, \ \xi_k \neq 0\}$. Then, if $|\xi| \le n-1$,*

$$\begin{aligned} &(\partial_\xi^r f_n)(y^0, \dots, y^{n-1}) \\ &= \Big[\sum_{j=0}^{n-2-|\xi|}\sum_{k=\mu+1}^{n-1-j-|\xi|}\sum_{(i_1,\dots,i_{k-\mu})\in S_k^{n-1-j-|\xi|}} \frac{1}{j!(k-\mu)!}\partial_\varepsilon^j \partial_y^{k+r-\mu}\sigma(0, y^0)\langle y^{i_1}, \dots, y^{i_{k-\mu}}, \cdot\rangle \\ &\quad + \frac{1}{(n-1-|\xi|)!}\partial_\varepsilon^{n-1-|\xi|}\partial_y^r \sigma(0, y^0)\Big] \circ p_{(\xi_1,\dots,\xi_r,x)}. \end{aligned}$$

Here, the right hand side is regarded as in $L^{r+1}(\mathcal{X}_{n-1}, \dots, \mathcal{X}_{n-1}; \mathcal{W})$. Note that if $|\xi| = n-1$ the first term on the right hand is regarded as zero. If $|\xi| > n-1$, $\partial_\xi^r f_n = 0$.

Proof. We give here a slightly heuristic proof. However, since the difficulty of this lemma lies only in algebraic part, it does not cause a serious trouble.

From the Taylor expansion for σ

$$\begin{aligned} \varepsilon\sigma(\varepsilon, y^0 + \Delta y) &\sim \varepsilon \sum_{j,k} \frac{\varepsilon^j}{j!k!} \partial_\varepsilon^j \partial_y^k \sigma(0, y^0)\langle \overbrace{\Delta y, \ldots, \Delta y}^{k}, \cdot \rangle, \\ \Delta y &\sim \varepsilon^1 y^1 + \varepsilon^2 y^2 + \varepsilon^3 y^3 + \cdots, \qquad \text{as } \varepsilon \searrow 0. \end{aligned} \tag{4.7}$$

Then, we get a linear combination of the terms of the form

$$\varepsilon^{j+1+i_1+\cdots+i_k} \partial_\varepsilon^j \partial_y^k \sigma(0, y^0)\langle y^{i_1}, \ldots, y^{i_k}, \cdot \rangle.$$

(In this proof we say the above term is of order $j + 1 + i_1 + \cdots + i_k$.) Recall that the definition of f_n is the sum of terms of order n in the right hand side of (4.7).

Let us first consider $\partial_{y^s} f_n$ ($s \neq 0$). Then, if $n - s \geq 0$, $\partial_{y^s} f_n(y^0, \ldots, y^{n-1})$ is the sum of terms of order $n - s$ of the ∂_{y^s}-derivative of (4.7), which is given by

$$\sum_{j,k} \frac{\varepsilon^{j+1+s}}{j!(k-1)!} \partial_\varepsilon^j \partial_y^k \sigma(0, y^0)\langle \overbrace{\Delta y, \ldots, \Delta y}^{k-1}, \cdot \rangle, \qquad \Delta y \sim \varepsilon^1 y^1 + \varepsilon^2 y^2 + \cdots.$$

Picking up terms of order n, we easily see $\partial_{y^s} f_n(y^0, \ldots, y^{n-1})$ is given as in the statement of this lemma. The case for $\partial_{y^0} f_n$ is easier.

Thus, we have shown the lemma for $r = 1$. Repeating this argument, we can show the general case ($r \geq 2$). ∎

We set some notations for iterated integrals. Let $\mathcal{A}^i$ be real Banach spaces and let ϕ^i be $\mathcal{A}^i$-valued paths ($1 \leq i \leq n$). Define

$$\mathcal{I}^n[\phi^1, \ldots, \phi^n]_{s,t} = \int_{s<u_1<\cdots<u_n<t} d\phi^1_{u_1} \otimes \cdots \otimes d\phi^n_{u_n} \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$$

whenever possible. (For example, when $\phi^i \in C_{0,q}(\mathcal{A}_i)$ ($1 \leq i \leq n$) for some $1 \leq q < 2$.)

Let $\mathcal{B}^i$ be real Banach spaces ($1 \leq i \leq n$) and $\mathcal{B} = \otimes_{i=1}^n \mathcal{B}_i$. For $\pi \in \Pi_n$ and $b_1 \otimes \cdots \otimes b_n \in \mathcal{B}$, we write

$$\pi(b_1 \otimes \cdots \otimes b_n) = (b_{\pi^{-1}(1)}, \ldots, b_{\pi^{-1}(n)}) \in \mathcal{B}_{\pi^{-1}(1)} \otimes \cdots \otimes \mathcal{B}_{\pi^{-1}(n)}$$

Let $\mathcal{C} = \oplus_{i=1}^m \mathcal{A}_i$ and consider $\mathcal{C}^{\otimes n}$. The $(i_1, \ldots, i_n)$-component of $\eta \in \mathcal{C}^{\otimes n}$ is denoted by $\eta^{(i_1,\ldots,i_n)} \in \mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_n}$. Clearly, $\sum_{1\leq i_1,\ldots,i_n\leq m} \eta^{(i_1,\ldots,i_n)} = \eta$. Let $\psi = (\psi^1, \ldots, \psi^m)$ be a nice path in $\mathcal{C}$. The nth level path of the rough path lying above ψ is $\mathcal{I}^n[\psi, \ldots, \psi]$. The action of $\pi \in \Pi_n$ in component form is given by

$$\left(\pi\mathcal{I}^n[\psi, \ldots, \psi]_{s,t}\right)^{(i_1,\ldots,i_n)} = \pi\mathcal{I}^n[\psi^{i_{\pi(1)}}, \ldots, \psi^{i_{\pi(n)}}]_{s,t}. \tag{4.8}$$

This equality can be verified by straightforward computation.

Now we state our main theorem in this subsection. In the following we set

$$\xi(X) := \sum_{j=1}^{[p]} \|X^j\|_{p/j}^{1/j} \qquad \text{for } X \in C_{0,q}(\mathcal{V}) \text{ (or } X \in G\Omega_p(\mathcal{V})).$$

Here, X^j denotes the jth level path of (the rough path lying above) X. Clearly, $\xi(rX) = |r|\xi(X)$ for $r \in \mathbb{R}$. In the following we set $\nu(-2) = 1$, $\nu(-1) = 0$, and $\nu(i) = i$ for $i \geq 0$.

Theorem 4.4 *The map* $(X, \Lambda) \mapsto (X, \Lambda, Y^0, \ldots, Y^n)$ *extends to a continuous map from* $G\Omega_p(\mathcal{V}) \times C_{0,q}(\hat{\mathcal{V}})$ *to* $G\Omega_p(\mathcal{X}_n)$*. Moreover, for any* $X \in C_{0,q}(\mathcal{V})$ *and* $\Lambda \in C_{0,q}(\hat{\mathcal{V}})$*, there exists a control function* $\omega = \omega_{X,\Lambda}$ *such that the following* (i) *and* (ii) *hold:*
(i) *For any* $(s,t) \in \triangle$, $X \in C_{0,q}(\mathcal{V})$, $\Lambda \in C_{0,q}(\hat{\mathcal{V}})$, $j = 1, \ldots, [p]$, *and* $i_1, \ldots, i_j \in \{-2, -1, \ldots, n\}$*, it holds that*

$$\left|\mathcal{I}^j[Y^{i_1}, \ldots, Y^{i_j}]_{s,t}\right| \leq (1 + \xi(X))^{\nu(i_1)+\cdots+\nu(i_j)} \omega(s,t)^{j/p} \tag{4.9}$$

Here, for notational simplicity, we set $Y^{-2} = X$, $Y^{-1} = \Lambda$.
(ii) *For any* $r > 0$*, there exists a constant* $c = c(r) > 0$ *such that*

$$\sup\{\omega(0,1) \mid \ X \in C_{0,q}(\mathcal{V}),\, \Lambda \in C_{0,q}(\hat{\mathcal{V}}) \text{ with } \|\Lambda\|_q \leq r\} \leq c.$$

Proof. In this proof, c and ω may change from line to line and we will denote $\delta = \xi(X)$. We will use induction. The case $n = 0$ is easy, since the map $\Lambda \in C_{0,q}(\hat{\mathcal{V}}) \mapsto Y^0 \in C_{0,q}(\mathcal{W})$ is locally Lipschitz continuous (see [16]). Now we assume the statement of the theorem holds for $n-1$ and will prove the case for n.

Set $Z^{n-1} = (X, \Lambda, Y^0, \ldots, Y^{n-1})$ for $n \in \mathbb{N}$. Then, it is obvious that

$$(X, \Lambda, Y^0, \ldots, Y^{n-1}, I^n) = \int (\mathrm{Id}_{\mathcal{X}^{n-1}} \oplus f_n)(Z^{n-1}) dZ^{n-1}.$$

By using (3.3)–(3.7), we will estimate the almost rough path $\Xi \in A\Omega_p(\mathcal{X}^n)$, which defines the integral on the right hand side.

Let $1 \leq k \leq [p]$. We consider the $\mathbf{i} = (i_1, \ldots, i_k)$-component of Ξ^k, where $-2 \leq i_j \leq n$ for all $j = 1, \ldots, k$. Set $\mathcal{N}(\mathbf{i}) = \{j \mid i_j = n\}$. Note that, if $j \notin \mathcal{N}(\mathbf{i})$ (equivalently, if $i_j \neq n$), then $\mathbf{l} = (l_1, \ldots, l_k)$ in the sum of 3.3 must satisfy $l_j = 1$.

We will fix such an $\mathbf{l}$. For $j \notin \mathcal{N}(\mathbf{i})$, set $L_j := p_{i_j}$ (the projection onto the i_j-component) and $\mathbf{m}_j = \mathbf{m}'_j = i_j$. For $j \in \mathcal{N}(\mathbf{i})$, set $\mathbf{m}_j = (m_j^1, \ldots, m_j^{l_j-1})$, $\mathbf{m}'_j = (m_j^1, \ldots, m_j^{l_j-1}, -2)$, and

$$L_j = \sum_{\mathbf{m}_j \in \{0,\ldots,n-1\}^{l_j-1}} \partial_{\mathbf{m}_j}^{l_j-1} f_n(Z_s^{n-1}) =: \sum_{\mathbf{m}_j \in \{0,\ldots,n-1\}^{l_j-1}} L_j^{\mathbf{m}_j}. \tag{4.10}$$

Then, from (3.3),

$$[\Xi_{s,t}^k]^{(\mathbf{i})} \;=\; \sum_{\mathbf{l}} \Big[\sum_{\mathbf{m}_j \in \{0,\ldots,n-1\}^{l_j-1}} (L_1 \otimes \cdots \otimes L_k) \Big\langle \sum_{\pi \in \Pi_{\mathbf{l}}} [\pi(Z^{n-1})_{s,t}^{|\mathbf{l}|}]^{(\mathbf{m}'_1,\ldots,\mathbf{m}'_k)} \Big\rangle \Big]. \tag{4.11}$$

The sum is over such $\mathbf{l}$'s. From (4.8) and the assumption of induction, we easily see that

$$\big|[\pi(Z^{n-1})^{|\mathbf{l}|}_{s,t}]^{(\mathbf{m}'_1,\dots,\mathbf{m}'_k)}\big| \le c(1+\delta)^{\nu(\mathbf{m}'_1,\dots,\mathbf{m}'_k)}\omega(s,t)^{k/p}, \tag{4.12}$$

where $\nu(\mathbf{m}'_1,\dots,\mathbf{m}'_k) := \sum_{j=1}^{k}\sum_{r=1}^{l_j}\nu(m_j^r)$. Combining this with Lemma 4.2, we have that

$$\big|[\Xi^k_{s,t}]^{(\mathbf{i})}\big| \le c(1+\delta)^{\nu(\mathbf{i})}\omega(s,t)^{k/p}, \qquad k=1,\dots,[p],\ (s,t)\in\triangle, \tag{4.13}$$

Next we estimate $\mathbf{i}$-component of $\Xi^k_{s,t}-(\Xi_{s,u}\otimes\Xi_{u,t})^k$ for $s<u<t$. For that purpose, it is sufficient to estimate $R_l(x,y)$ in (3.5) for $f=\mathrm{Id}_{\mathcal{X}^{n-1}}\oplus f_n$. If $i\neq n$, ith component of $R_l(x,y)$ (that is equal to $R_l(x,y)$ for the projection p_i) vanishes. From Lemma 4.2, the nth component $R(f_n)_l(Z^{n-1}_s, Z^{n-1}_u)$ satisfies the following: for $\mathbf{m}_j$ as above,

$$\begin{aligned}\Big|\int_0^1 d\theta\frac{(1-\theta)^{[p]-l}}{([p]-l)!}D^{[p]-l+1}\partial^{l-1}_{\mathbf{m}_j}f_n\big(Z^{n-1}_s+\theta(Z^{n-1})^1_{s,u}\big)\langle(Z^{n-1})^1_{s,u})^{\otimes([p]-l+1)}\rangle\Big|\\ \le c(1+\xi(X))^{n-1-(m_j^1+\cdots+m_j^{l-1})}, \qquad \text{if } m_j^1+\cdots+m_j^{l-1}\le n-1.\end{aligned}$$

Here, the left hand side is regarded as a multilinear map from $Y^{m_j^1}\times\cdots\times Y^{m_j^{l-1}}\times X$. If $m_j^1+\cdots+m_j^{l-1}>n-1$, then the left hand side vanish.

Denoting by $\hat{L}_j$ the left hand side of the above inequality, we can do the same argument as in (4.10)–(4.13) to obtain that

$$\Big|\big[\Xi^k_{s,t}-(\Xi_{s,u}\otimes\Xi_{u,t})^k\big]^{(\mathbf{i})}\Big| \le c(1+\delta)^{\nu(\mathbf{i})}\omega(s,t)^{([p]+1)/p} \tag{4.14}$$

From (4.13)–(4.14), $(X,\Lambda,Y^0,\dots,Y^{n-1},I^n)$ satisfies a similar inequality to (4.9) (with Y^n in (4.9) being replaced with I^n).

Now, by the Young integration theory, we see that $|J^n_t-J^n_s|\le c(1+\delta)^n\omega(s,t)^{1/q}$, which implies that $(X,\Lambda,Y^0,\dots,Y^{n-1},I^n+J^n)$ satisfies a similar inequality to (4.9) (with Y^n in (4.9) being replaced with I^n+J^n).

Set $d\Omega_t=\partial_y b(0,Y^0_t)\langle\,\cdot\,,d\Lambda_t\rangle$ and set $M=M(\Lambda)$ by $dM_t=d\Omega_t\cdot M_t$ with $M_0=\mathrm{Id}_{\mathcal{W}}$. It is easy to see that $\Lambda\in C_{0,q}(\mathcal{V})\mapsto M\in\mathcal{C}_q(L(\mathcal{W},\mathcal{W}))$ is continuous (See Proposition 2.5). Also set $\hat{M}=\mathrm{Id}_{\mathcal{X}_{n-1}}\oplus M\in\mathcal{C}_q(L(\mathcal{X}_{n-1},\mathcal{X}_{n-1}))$. Then, applying Duhamel's principle (Corollary 2.7) for $\hat{M}$ and $(rX,Y^0,rY^1,\dots,r^{n-1}Y^{n-1},r^n(I^n+J^n))$ with $1/r=(1+\delta)$, we obtain $(rX,Y^0,rY^1,\dots,r^{n-1}Y^{n-1},r^nY^n)$. This completes the proof. ▌

4.2 Estimates for remainder terms in the expansion

In this subsection we give estimates for remainder terms in the stochastic Taylor-like expansion. We keep the same notations as in the previous subsection. If $X\in C_{0,q}(\mathcal{V})$ and $\Lambda\in C_{0,q}(\hat{\mathcal{V}})$, then

$$Q^{n+1,(\varepsilon)}=Q^{n+1,(\varepsilon)}(X,\Lambda):=Y^{(\varepsilon)}-(Y^0+\varepsilon Y^1+\cdots+\varepsilon^nY^n), \qquad \varepsilon\in[0,1]$$

is clearly well-defined. We prove that the correspondence $(X,\Lambda)\mapsto Q^{n+1,(\varepsilon)}(X,\Lambda)$ extends to a continuous map from $G\Omega_p(\mathcal{V})\times C_{0,q}(\hat{\mathcal{V}})$ to $G\Omega_p(\mathcal{W})$ and that $Q^{n+1,(\varepsilon)}(X,\Lambda)$ is a term of "order $n+1$".

For simplicity we assume that $X\in C_{0,q}(\mathcal{V})$ and $\Lambda\in C_{0,q}(\hat{\mathcal{V}})$. From (4.1) and (4.2), we see that

$$
\begin{aligned}
dQ_t^{n+1,(\varepsilon)} - \partial_y b(0,Y_t^0)\langle Q_t^{n+1,(\varepsilon)}, d\Lambda_t\rangle &= \sigma(\varepsilon, Y_t^{(\varepsilon)})\varepsilon dX_t - \sum_{k=1}^{n}\varepsilon^k dI_t^k \\
+b(\varepsilon,Y_t^{(\varepsilon)})d\Lambda_t - b(0,Y_t^0)d\Lambda_t &- \partial_y b(0,Y_t^0)\langle Y_t^{(\varepsilon)}-Y_t^0, d\Lambda_t\rangle - \sum_{k=1}^{n}\varepsilon^k dJ_t^k. \quad (4.15)
\end{aligned}
$$

Note that I^k and J^k $(k=1,\dots,n)$ depends only on $Y^0,Y^1,\dots,Y^{n-1}$, but not on Y^n.

Set $\hat{\mathcal{X}}_n=\mathcal{V}\oplus\hat{\mathcal{V}}\oplus\mathcal{W}^{\oplus n+2}$. An element in $\mathcal{X}_n$ is denoted by $(x,\hat{x};y^{-1},y^0,\dots,y^n)$. Then, in a natural way, $f_n,g_n\in C_{b,loc}^{\infty}(\hat{\mathcal{X}}_n, L(\hat{\mathcal{X}}_n,\mathcal{W}))$ and Lemmas 4.2 and 4.3 hold with trivial modification.

In the following theorem, we set for simplicity

$$
Z^{n,(\varepsilon)}=Z^{n,(\varepsilon)}(X,\Lambda):=(\varepsilon X,\Lambda,Y^{(\varepsilon)},Y^0,\varepsilon Y^1,\dots,\varepsilon^{n-1}Y^{n-1},Q^{n,(\varepsilon)}).
$$

This is a $\hat{\mathcal{X}}_n$-valued path. We also set $\hat{Y}^{i,(\varepsilon)}=\varepsilon^i Y^i$ for $0\le i\le n-1$, $\hat{Y}^{n,(\varepsilon)}=Q^{n,(\varepsilon)}$, $\hat{Y}^{-1,(\varepsilon)}=Y^{(\varepsilon)}$, $\hat{Y}^{-2,(\varepsilon)}=\Lambda$, and $\hat{Y}^{-3,(\varepsilon)}=\varepsilon X$. Then, $Z^{n,(\varepsilon)}=(\hat{Y}^{-3},\dots,\hat{Y}^n)$. We define $\nu(i)=i$ for $i\ge 0$, $\nu(-1)=\nu(-2)=0$, and $\nu(-3)=1$.

Theorem 4.5 *For each $n\in\mathbb{N}$ and $\varepsilon\in[0,1]$, the map $(X,\Lambda)\mapsto Z^{n,(\varepsilon)}(X,\Lambda)$ extends to a continuous map from $G\Omega_p(\mathcal{V})\times C_{0,q}(\hat{\mathcal{V}})$ to $G\Omega_p(\hat{\mathcal{X}}_n)$. Moreover, for any $X\in C_{0,q}(\mathcal{V})$ and $\Lambda\in C_{0,q}(\hat{\mathcal{V}})$, there exists a control function $\omega=\omega_{X,\Lambda}$ such that the following* (i) *and* (ii) *hold:*
(i) *For any $(s,t)\in\triangle$, $X\in C_{0,q}(\mathcal{V})$, $\Lambda\in C_{0,q}(\hat{\mathcal{V}})$, $j=1,\dots,[p]$, and $i_s\in\{-3,-2,\dots,n\}$ $(1\le s\le j)$, it holds that*

$$
\left|\mathcal{I}^j[\hat{Y}^{i_1,(\varepsilon)},\dots,\hat{Y}^{i_j,(\varepsilon)}]_{s,t}\right|\le(\varepsilon+\xi(\varepsilon X))^{\nu(i_1)+\cdots+\nu(i_j)}\omega(s,t)^{j/p} \quad (4.16)
$$

(ii) *For any $r_1,r_2>0$, there exists a constant $c=c(r_1,r_2)>0$ depending only on r_1,r_2,n such that*

$$
\sup\{\omega(0,1)\mid\ X\in C_{0,q}(\mathcal{V})\ \text{with}\ \xi(\varepsilon X)\le r_1,\ \Lambda\in C_{0,q}(\hat{\mathcal{V}})\ \text{with}\ \|\Lambda\|_q\le r_2\}\le c.
$$

Proof. In this proof the constant c and the control function ω may change from line to line. As before we write $\delta=\xi(X)$. We use induction. First we consider the case $n=1$. The estimate of the difference of $\Phi^{\varepsilon}(X,\Lambda)$ and $\Phi^{\varepsilon}(\mathbf{1},\Lambda)$ is essentially shown in Lemma 3.12. The estimate of the difference of $\Phi^{\varepsilon}(\mathbf{1},\Lambda)$ and $\Phi^0(\mathbf{1},\Lambda)$ is a simple exercise for ODEs in q-variational sense. Thus, combining these, we can easily show the case $n=1$.

Now we assume the statement of the theorem is true for n, and will prove the case for $n+1$. First we give estimates for the first term on the right hand side of (4.15). Slightly modifying the definition of $Z^{n,(\varepsilon)}$, we set $\tilde{Z}^{n,(\varepsilon)} = (X, \Lambda, Y^{(\varepsilon)}, Y^0, \dots, Y^{n-1}, Q^{n,(\varepsilon)})$. We also set

$$F_n^{(\varepsilon)}(y^{-1}, y^0, \dots, y^{n-1}) = \varepsilon\sigma(\varepsilon, y^{-1}) - \sum_{k=1}^{n} \varepsilon^k f_k(y^0, \dots, y^{k-1}). \tag{4.17}$$

Then, $F_n^{(\varepsilon)} \in C_{b,loc}^{\infty}(\hat{\mathcal{X}}_n, L(\hat{\mathcal{X}}_n, \mathcal{W}))$ and $\tilde{F}_n^{(\varepsilon)} := h_n^{(\varepsilon)} \oplus F_n^{(\varepsilon)} \in C_{b,loc}^{\infty}(\hat{\mathcal{X}}_n, L(\hat{\mathcal{X}}_n, \hat{\mathcal{X}}_{n+1})$. Here, $h_n^{(\varepsilon)} \in L(\hat{\mathcal{X}}_n, \hat{\mathcal{X}}_n)$ is defined by

$$h_n^{(\varepsilon)}(y^{-3}, \dots, y^n) = (\varepsilon y^{-3}, y^{-2}, y^{-1}, y^0, \varepsilon y^1, \dots, \varepsilon^{n-1} y^{n-1}, y^n).$$

We now consider $\int \tilde{F}_n^{(\varepsilon)}(\tilde{Z}^{n,(\varepsilon)}) d\tilde{Z}^{n,(\varepsilon)}$.

Since it is too complicated to give at once estimates like (3.3) for all the components of all the level paths of the above integral, we first consider (3.3) for the first level path of the last component of the above integral, i.e., $[\int F_n^{(\varepsilon)}(\tilde{Z}^{n,(\varepsilon)}) d\tilde{Z}^{n,(\varepsilon)}]^1$. Substitute $i = 1$, $f = F_n^{(\varepsilon)}$, and $X = \tilde{Z}^{n,(\varepsilon)}$ in (3.3). Then, (the first level of) the almost rough path Θ which approximates $[\int F_n^{(\varepsilon)}(\tilde{Z}^{n,(\varepsilon)}) d\tilde{Z}^{n,(\varepsilon)}]^1$ satisfies that

$$[\Theta^1]^{n+1} := \sum_{l=1}^{[p]} D^{l-1} F_n^{(\varepsilon)}(\tilde{Z}_s^{n,(\varepsilon)}) \langle [\tilde{Z}^{n,(\varepsilon)}]_{s,t}^l \rangle. \tag{4.18}$$

Here, D denotes the Fréchet derivative on $\hat{\mathcal{X}}_n$.

Now we estimate the right hand side of (4.18). Choose l and fix it. The contribution from the first term on the right hand side of (4.17) (i.e., $\varepsilon\sigma(\varepsilon, y^{-1})$) is given as follows;

$$\partial_y^{l-1} \sigma(\varepsilon, Y_s^{(\varepsilon)}) \langle \mathcal{I}^l [Y^{(\varepsilon)}, \dots, Y^{(\varepsilon)}, \varepsilon X]_{s,t} \rangle. \tag{4.19}$$

By the Taylor expansion for σ,

$$\begin{aligned} &\varepsilon \partial_y^{l-1} \sigma(\varepsilon, Y_s^{(\varepsilon)}) \\ &= \sum_{j,k; j+k \le n-1} \frac{\varepsilon^{j+1}}{j!k!} \partial_y^{l-1+k} \partial_\varepsilon^j \sigma(\varepsilon, Y_s^0) \langle \overbrace{Y_s^{(\varepsilon)} - Y_s^0, \dots, Y_s^{(\varepsilon)} - Y_s^0}^{k}, \cdot \rangle + \varepsilon S_n, \\ Y^{(\varepsilon)} &= Y^0 + \varepsilon Y^1 + \dots + \varepsilon^{n-1} Y^{n-1} + Q^{n,(\varepsilon)}. \end{aligned}$$

with $|\varepsilon S_n| \le c(\varepsilon + \varepsilon\delta)^{n+1}$. Therefore, (4.19) is equal to a sum of terms of the following form;

$$\frac{\varepsilon^j}{j!k!} \partial_y^{l-1+k} \partial_\varepsilon^j \sigma(\varepsilon, Y_s^0) \langle \hat{Y}_s^{i_1,(\varepsilon)}, \dots, \hat{Y}_s^{i_k,(\varepsilon)}; \mathcal{I}^l [\hat{Y}^{i_{k+1},(\varepsilon)}, \dots, \hat{Y}^{i_{l-1+k},(\varepsilon)}, \varepsilon X]_{s,t} \rangle. \tag{4.20}$$

Here, $1 \le i_1, \dots, i_k \le n$ and $0 \le i_{k+1}, \dots, i_{l-1+k} \le n$. We say this term is of order $1 + j + \sum_{a=1}^{l-1+k} \nu(i_a)$, since this is dominated by $c(\varepsilon + \varepsilon\delta)^{1+j+\sum_{a=1}^{l-1+k} \nu(i_a)} \omega(s,t)^{l/p}$. Note that, if a term of this form involves $\hat{Y}^{n,(\varepsilon)} = Q^{n,(\varepsilon)}$, then its order is larger than n.

Let $m \le n$. It is sufficient to show that the terms of order m in (4.20) cancel off with

$$\begin{aligned} &\varepsilon^m D^{l-1} f_m^{(\varepsilon)}(\tilde{Z}_s^{n,(\varepsilon)})\langle [\tilde{Z}^{n,(\varepsilon)}]_{s,t}^l \rangle \\ &= \varepsilon^m \sum_{\xi \in \{0,\dots,m-1\}^{l-1}} \partial_\xi^{l-1} f_m^{(\varepsilon)}(Y_s^0,\dots,Y_s^{m-1})\langle [\mathcal{I}^l[Y^{\xi_1},\dots,Y^{\xi_{m-1}},X]_{s,t}\rangle. \end{aligned}$$

By Lemma 4.3 and its proof, this cancels off with all the terms of order m in (4.20). (Note that $l-1, m, k$, and $(i_{k+1},\dots,i_{l-1+k})$ in (4.20) correspond to $r, n, k-\mu$, and $(\xi_1,\dots,\xi_r)$ in Lemma 4.3, respectively.) Hence, $D^{l-1}F_n^{(\varepsilon)}(\tilde{Z}_s^{n,(\varepsilon)})\langle [\tilde{Z}^{n,(\varepsilon)}]_{s,t}^l\rangle$ is dominated by $c(\varepsilon+\varepsilon\delta)^{n+1}\omega(s,t)^{l/p}$. Thus, we have obtained an estimate for (4.18).

Let $1 \le k \le [p]$. We consider the $\mathbf{i} = (i_1,\dots,i_k)$-component of Θ^k, where $-3 \le i_j \le n+1$ for all $j = 1,\dots,k$. Set $\mathcal{N}(\mathbf{i}) = \{j \mid i_j = n+1\}$. Note that, if $j \notin \mathcal{N}(\mathbf{i})$ (equivalently, if $i_j \neq n+1$), then $\mathbf{l} = (l_1,\dots,l_k)$ in the sum in 3.3 must satisfy $l_j = 1$.

We will fix such an $\mathbf{l}$. For $j \notin \mathcal{N}(\mathbf{i})$, set $L_j := \varepsilon^{\nu(i_j)} p_{i_j}$ if $i_j \neq n$ and $L_j := p_{i_j}$ if $i_j = n$ and also set $\mathbf{m}_j = \mathbf{m}'_j = i_j$. For $j \in \mathcal{N}(\mathbf{i})$, set $\mathbf{m}_j = (m_j^1,\dots,m_j^{l_j-1})$, $\mathbf{m}'_j = (m_j^1,\dots,m_j^{l_j-1},-3)$, and

$$L_j = D^{l_j-1}F_n^{(\varepsilon)}(\tilde{Z}_s^{n,(\varepsilon)}) = \sum_{\mathbf{m}_j \in \{0,\dots,n-1\}^{l_j-1}} \partial_{\mathbf{m}_j}^{l_j-1} F_n^{(\varepsilon)}(\tilde{Z}_s^{n,(\varepsilon)}) \circ p_{\mathbf{m}'_j}.$$

Then, from (3.3),

$$\begin{aligned} [\Theta_{s,t}^k]^{(\mathbf{i})} &= \sum_{\mathbf{l}} (L_1 \otimes \cdots \otimes L_k)\Big\langle \sum_{\pi \in \Pi_{\mathbf{l}}} \pi(\tilde{Z}^{n,(\varepsilon)})_{s,t}^{|\mathbf{l}|}\Big\rangle \\ &= \sum_{\mathbf{l}} (L_1 \otimes \cdots \otimes L_k)\Big\langle \mathcal{I}^k[(\tilde{Z}^{n,(\varepsilon)})_{s,\cdot}^{l_1},\dots,(\tilde{Z}^{n,(\varepsilon)})_{s,\cdot}^{l_k}]\Big\rangle. \end{aligned} \tag{4.21}$$

The sum is over such $\mathbf{l}$'s as in (3.3).

Now we estimate the right hand side of (4.21). Fix $\mathbf{l}$ for a while. For $j \notin \mathcal{N}(\mathbf{i})$, Pairing of L_j and $(\tilde{Z}^{n,(\varepsilon)})^{l_j}$ is clearly of order $\nu(i_j)$. For $j \in \mathcal{N}(\mathbf{i})$, consider the pairing of L_j and $(\tilde{Z}^{n,(\varepsilon)})^{l_j}$. Then, we can see that the same cancellation takes place as in (4.17)–(4.20) and that this is of order $n+1$. If we notice that, for nice paths $\psi^1,\dots,\psi^{|\mathbf{l}|}$,

$$\begin{aligned} &\mathcal{I}^k\big[\mathcal{I}^{l_1}[\psi^1,\dots,\psi^{l_1}]_{s,\cdot},\mathcal{I}^{l_2}[\psi^{l_1+1},\dots,\psi^{l_1+l_2}]_{s,\cdot},\dots,\mathcal{I}^{l_k}[\psi^{l_1+\cdots+l_{k-1}+1},\dots,\psi^{|\mathbf{l}|}]_{s,\cdot}\big]_{s,t} \\ &\qquad = \sum_{\pi \in \Pi_{\mathbf{l}}} \pi \mathcal{I}^{|\mathbf{l}|}[\psi^{\pi(1)},\dots,\psi^{\pi(|\mathbf{l}|)}]_{s,t}, \end{aligned}$$

then, by using (4.16) for n, we obtain from the above observation for (4.21) that

$$\big|[\Theta_{s,t}^k]^{(\mathbf{i})}\big| \le c(\varepsilon+\varepsilon\delta)^{\nu(i_1)+\cdots+\nu(i_k)}\omega(s,t)^{k/p}. \tag{4.22}$$

Next we estimate $R_l(X_s,X_u)$, $(s<u<t)$ in (3.5) with $f = F_n^{(\varepsilon)}$ and $X = \tilde{Z}^{n,(\varepsilon)}$ (since R_l of other components in $\tilde{F}_n^{(\varepsilon)}$ clearly vanish). Fix $1 \le l \le [p]$. From (4.17), the first term in $R_l(F_n^{(\varepsilon)})(\tilde{Z}_s^{n,(\varepsilon)},\tilde{Z}_u^{n,(\varepsilon)})\langle (\tilde{Z}^{n,(\varepsilon)})_{u,t}^l\rangle$ is given by

$$\int_0^1 d\theta \frac{(1-\theta)^{[p]-l}}{([p]-l)!}\partial_y^{[p]}\sigma(\varepsilon,Y_{s,u;\theta}^\varepsilon)\langle [(Y^\varepsilon)_{s,u}^1]^{\otimes [p]-l+1},\mathcal{I}^l[Y^{(\varepsilon)},\dots,Y^{(\varepsilon)},\varepsilon X]_{u,t}\rangle, \tag{4.23}$$

where $Y^{\varepsilon}_{s,u;\theta} := Y^{\varepsilon}_s + \theta(Y^{\varepsilon})^1_{s,u}$. Then, by expanding this as in the previous section, we can see that the same cancellation takes place as in (4.17)–(4.20) and that

$$|R_l(F_n^{(\varepsilon)})(\tilde{Z}_s^{n,(\varepsilon)}, \tilde{Z}_u^{n,(\varepsilon)})\langle(\tilde{Z}^{n,(\varepsilon)})^l_{u,t}\rangle| \le c(\varepsilon + \varepsilon\delta)^{n+1}\omega(s,t)^{([p]+1)/p}.$$

This implies that

$$\begin{aligned} |[\Theta^1_{s,t} - \Theta^1_{s,u} - \Theta^1_{u,t}]^{n+1}| &\le \sum_{l=1}^{[p]} |R_l(F_n^{(\varepsilon)})(\tilde{Z}_s^{n,(\varepsilon)}, \tilde{Z}_u^{n,(\varepsilon)})\langle(\tilde{Z}^{n,(\varepsilon)})^l_{u,t}\rangle| \\ &\le c(\varepsilon + \varepsilon\delta)^{n+1}\omega(s,t)^{([p]+1)/p}, \quad s < u < t. \end{aligned}$$

From (3.5)–(3.7) and the above estimate for R_l, we may compute in the same way as in (4.21)–(4.22) to obtain that

$$\big|[\Theta^k_{s,t} - (\Theta_{s,u} \otimes \Theta_{u,t})^k]^{(\mathbf{i})}\big| \le c(\varepsilon + \varepsilon\delta)^{\nu(i_1)+\cdots+\nu(i_k)}\omega(s,t)^{([p]+1)/p}. \tag{4.24}$$

From (4.22) and (4.24) we see that the map

$$(X,\Lambda) \mapsto \big(\varepsilon X, \Lambda, Y^{(\varepsilon)}, Y^0, \varepsilon Y^1, \ldots, \varepsilon^{n-1}Y^{n-1}, Q^{n,(\varepsilon)}, \int \sigma(\varepsilon, Y^{(\varepsilon)})\varepsilon dX - \sum_{k=1}^n \varepsilon^k I^k\big)$$

is continuous and satisfies the inequality (4.16) for $n+1$ (with $\hat{Y}^{n,(\varepsilon)}$ and $\hat{Y}^{n+1,(\varepsilon)}$ being replaced with $Q^{n,(\varepsilon)}$ and $\int \sigma(\varepsilon, Y^{(\varepsilon)})\varepsilon dX - \sum_{k=1}^n \varepsilon^k I^k$, respectively).

By expanding $\int b(\varepsilon, Y^{(\varepsilon)})d\Lambda$ in (4.15) with $Y^{(\varepsilon)} = Y^0 + \cdots + \varepsilon^{n-1}Y^{n-1} + Q^{n,(\varepsilon)}$, we see that

$$\begin{aligned} \Big|\int_s^t \big(b(\varepsilon, Y_u^{(\varepsilon)})d\Lambda_u - b(0, Y_u^0)d\Lambda_u - \partial_y b(0, Y_u^0)\langle Y_u^{(\varepsilon)} - Y_u^0, d\Lambda_u\rangle - \sum_{k=1}^n \varepsilon^k dJ_u^k\big)\Big| \\ \le c(\varepsilon + \varepsilon\delta)^{n+1}\omega(s,t)^{1/q}. \end{aligned}$$

(Thanks to the third term $\partial_y b(0, Y_u^0)\langle Y_u^{(\varepsilon)} - Y_u^0, d\Lambda_u\rangle$, all the terms that involve $Q^{n,(\varepsilon)}$ in the expansion are of order $n+1$ or larger.)

Let $K^{n+1,(\varepsilon)}$ be the right hand side of (4.15). From these we see that the map

$$(X,\Lambda) \mapsto \big(\varepsilon X, \Lambda, Y^{(\varepsilon)}, Y^0, \varepsilon Y^1, \ldots, \varepsilon^{n-1}Y^{n-1}, Q^{n,(\varepsilon)}, K^{n+1,(\varepsilon)}\big)$$

is continuous and satisfies the inequality (4.16) for $n+1$ (with $\hat{Y}^{n,(\varepsilon)}$ and $\hat{Y}^{n+1,(\varepsilon)}$ being replaced with $Q^{n,(\varepsilon)}$ and $K^{n+1,(\varepsilon)}$, respectively).

By applying Duhamel's principle (Lemma 2.7) in the same way as in the proof of Theorem 4.4 in the previous subsection, we see that the map

$$(X,\Lambda) \mapsto \big(\varepsilon X, \Lambda, Y^{(\varepsilon)}, Y^0, \varepsilon Y^1, \ldots, \varepsilon^{n-1}Y^{n-1}, Q^{n,(\varepsilon)}, Q^{n+1,(\varepsilon)}\big) \tag{4.25}$$

is continuous and satisfies the inequality (4.16) for $n+1$ (with $\hat{Y}^{n,(\varepsilon)}$ and $\hat{Y}^{n+1,(\varepsilon)}$ being replaced with $Q^{n,(\varepsilon)}$ and $Q^{n+1,(\varepsilon)}$, respectively).

Finally define a bounded linear map $\alpha \in L(\hat{\mathcal{V}}_{n+1}, \hat{\mathcal{V}}_{n+1})$ by

$$\alpha(y^{-3}, \ldots, y^n, y^{n+1}) = (y^{-3}, \ldots, -y^n + y^{n+1}, y^{n+1})$$

and apply $\overline{\alpha}$ to (4.25), which completes the proof of Theorem 4.5. ▮

5 Laplace approximation for Itô functionals of Brownian rough paths

In this section, by using the expansion for Itô maps in the rough path sense in the previous sections, we generalize the Laplace approximation for Itô functionals of Brownian rough paths, which was shown in Aida [2] or Inahama and Kawabi [11] (Theorem 3.2). In this section we always assume $2 < p < 3$.

5.1 Setting of the Laplace approximation

Let $(\mathcal{V}, \mathcal{H}, \mu)$ be an abstract Wiener space, that is, $\mathcal{V}$ is a real separable Banach space, $\mathcal{H}$ is a real separable Hilbert space embedded continuously and densely in $\mathcal{V}$, and μ is a Gaussian measure on $\mathcal{V}$ such that

$$\int_{\mathcal{V}} \exp(\sqrt{-1}\langle \phi, x\rangle)\mu(dx) = \exp(-\|\phi\|^2_{\mathcal{H}^*}/2), \qquad \text{for any } \phi \in \mathcal{V}^*.$$

By the general theory of abstract Wiener spaces, there exists a $\mathcal{V}$-valued Brownian motion $w = (w_t)_{t\geq 0}$ associated with μ. The law of the scaled Brownian motion εw on $P(\mathcal{V}) = \{y : [0,1] \to \mathcal{V} | \text{continuous and } y_0 = 0\}$ is denoted by $\hat{\mathbb{P}}_\varepsilon$ $(\varepsilon > 0)$.

We assume the exactness condition **(EX)** below for the projective norm on $\mathcal{V}\otimes\mathcal{V}$ and μ. This condition implies the existence of the Brownian rough paths W. (See Ledoux, Lyons, and Qian [14].) The law of the scaled Brownian rough paths εW is a probability measure on $G\Omega_p(\mathcal{V})$ and is denoted by $\mathbb{P}_\varepsilon$ $(\varepsilon > 0)$.

(EX): We say that the Gaussian measure μ and the projective norm on $X\otimes X$ satisfies the *exactness condition* if there exist $C > 0$ and $1/2 \leq \alpha < 1$ such that, for all $n = 1, 2, \ldots,$

$$\mathbb{E}\Big[\Big|\sum_{i=1}^{n} \eta_{2i-1} \otimes \eta_{2i}\Big|\Big] \leq Cn^\alpha.$$

Here, $\{\eta_i\}_{i=1}^\infty$ are an independent and identically distributed random variables on X such that the law of η_i is μ.

We consider an ODE in the rough path sense in the following form. Let $\mathcal{W}$ be another real Banach space. For $\sigma \in C_b^\infty([0,1]\times\mathcal{W}, L(\mathcal{V},\mathcal{W}))$ and $b \in C_b^\infty([0,1]\times\mathcal{W}, L(\mathbf{R},\mathcal{W})) = C_b^\infty([0,1]\times\mathcal{W},\mathcal{W})$,

$$dY_t^{(\varepsilon)} = \sigma(\varepsilon, Y_t^{(\varepsilon)})\varepsilon dW + b(\varepsilon, Y_t^{(\varepsilon)})dt, \qquad Y_0^{(\varepsilon)} = 0.$$

Using the notation of the previous section, we may write $Y^{(\varepsilon)} = \Phi^\varepsilon(\varepsilon W, \Upsilon)$. Here Φ^ε is the Itô map corresponding to (σ, b) and Υ is the $\mathbf{R}$-valued path defined by $\Upsilon_t = t$.

We will study the asymptotics of $(Y^{(\varepsilon)})_1 = \Phi^\varepsilon(\varepsilon W, \Upsilon)_1$ as $\varepsilon \searrow 0$. We impose the following conditions on the functions F and G. In what follows, we especially denote by D the Fréchet derivatives on $L_2^{0,1}(\mathcal{H})$ and $P(\mathcal{W})$. The Cameron-Martin space for $\hat{\mathbb{P}}_1$ is denoted by $L_2^{0,1}(\mathcal{H})$, which is a linear subspace of $P(\mathcal{W})$. Note that $L_2^{0,1}(\mathcal{H}) \subset \mathrm{BV}(\mathcal{V}) \subset G\Omega_p(\mathcal{V})$. Set $\Psi^0 : L_2^{0,1}(\mathcal{H}) \to P(\mathcal{W})$ by $\Psi^0(\Lambda) = \Phi^0(\Lambda, \Upsilon)_1$.

(H1): F and G are real-valued bounded continuous functions defined on $P(\mathcal{W})$.

(H2): The function $\tilde{F} := F \circ \Psi^0 + \| \cdot \|^2_{L^{0,1}_2(\mathcal{H})}/2$ defined on $L^{0,1}_2(\mathcal{H})$ attains its minimum at a unique point $\Lambda \in L^{0,1}_2(\mathcal{H})$. For this Λ, we write $\phi := \Psi^0(\Lambda)$.

(H3): The functions F and G are $n+3$ and $n+1$ times Fréchet differentiable on a neighbourhood $B(\phi)$ of $\phi \in P(\mathcal{W})$, respectively. Moreover there exist positive constants $M_1, \ldots, M_{n+3}$ such that

$$
\begin{aligned}
\big|D^k F(\eta)\big[y, \ldots, y\big]\big| &\le M_k \|y\|^k_{P(\mathcal{W})}, \quad k = 1, \ldots, n+3, \\
\big|D^k G(\eta)\big[y, \ldots, y\big]\big| &\le M_k \|y\|^k_{P(\mathcal{W})}, \quad k = 1, \ldots, n+1,
\end{aligned}
$$

hold for any $\eta \in B(\phi)$ and $y \in P(\mathcal{W})$.

(H4): At the point $\Lambda \in L^{0,1}_2(\mathcal{H})$, consider the Hessian $A := D^2(F \circ \Psi_0)(\Lambda)|_{L^{0,1}_2(\mathcal{H}) \times L^{0,1}_2(\mathcal{H})}$. As a bounded self-adjoint operator on $L^{0,1}_2(\mathcal{H})$, the operator A is strictly larger than $-\mathrm{Id}_{L^{0,1}_2(\mathcal{H})}$ in the form sense.

Now we are in a position to state our main theorem. This can be considered as a rough path version of Azencott [3] or Ben Arous [5]. The key of the proof is the stochastic Taylor-like expansion of the Itô map around the minimal point Λ, which will be explained in the next subsection. (There are many other nice results on this topic in the conventional SDE theory. See Section 5.2 of Pitarbarg and Fatalov [18]. For results in the Malliavin calculus, see Kusuoka and Stroock [12, 13], and Takanobu and Watanabe [19].)

Theorem 5.1 *Under conditions* **(EX)**, **(H1)**–**(H4)**, *we have the following asymptotic expansion: ($\mathbb{E}$ is the integration with respect to $\mathbb{P}_1$ or $\hat{\mathbb{P}}_1$.)*

$$
\begin{aligned}
&\mathbb{E}\Big[G(Y^{(\varepsilon)}) \exp\big(-F(Y^{(\varepsilon)})/\varepsilon^2\big)\Big] \\
&= \exp\big(-\tilde{F}(\Lambda)/\varepsilon^2\big) \exp\big(-c(\Lambda)/\varepsilon\big) \cdot \big(\alpha_0 + \alpha_1 \varepsilon + \cdots + \alpha_n \varepsilon^n + O(\varepsilon^{n+1})\big),
\end{aligned}
\tag{5.1}
$$

where the constant $c(\Lambda)$ is given by $c(\Lambda) := DF(\phi)[\Xi(\Lambda)]$. Here $\Xi(\Lambda) \in P(\mathcal{W})$ is the unique solution of the differential equation

$$
d\Xi_t - \partial_y \sigma(0, \phi_t)[\Xi_t, d\Lambda_t] - \partial_y b(0, \phi_t)[\Xi_t] dt = \partial_\varepsilon b(0, \phi_t) dt \quad \text{with} \quad \Xi_0 = 0. \tag{5.2}
$$

Remark 5.2 *In [11], only equations of the following form were discussed.*

$$
dY^{(\varepsilon)}_t = \sigma(Y^{(\varepsilon)}_t)\varepsilon dW_t + \sum_{i=1}^n a_i(\varepsilon) b_i(Y^{(\varepsilon)}_t) dt, \qquad Y^{(\varepsilon)}_0 = 0.
$$

Here, $a_i : [0,1] \to \mathbf{R}$ are "nice" functions. This may be somewhat unnatural. However, since we extended the stochastic Taylor-like expansion to the "ε-dependent case" in the previous section, we are able to slightly generalize the Laplace asymptotics (and the large deviation) as in the above theorem.

5.2 Sketch of proof for Theorem 5.1

The proof for Theorem 5.1 is essentially the same as the one for Theorem 3.2, [11], once the stochastic Taylor-like expansion is obtained. Therefore, we only give a sketch of proof in this subsection.

Roughly speaking, there are three steps in the proof:
Step 1: A large deviation principal for the laws of $Y^{(\varepsilon)}$ as $\varepsilon \searrow 0$.
Step 2: The stochastic Taylor expansion around the maximal point. We expand $\hat{Y}^{(\varepsilon)}$ as $\varepsilon \searrow 0$ as in the previous sections, where $\hat{Y}^{(\varepsilon)}$ is given by the following differential equation:

$$d\hat{Y}^{(\varepsilon)} = \sigma(\varepsilon, \hat{Y}_t^{(\varepsilon)})(\varepsilon dW_t + d\Lambda_t) + b(\varepsilon, \hat{Y}_t^{(\varepsilon)})dt, \qquad \hat{Y}_0^{(\varepsilon)} = 0. \tag{5.3}$$

Here, $\Lambda \in L_2^{0,1}(\mathcal{H})$ is given in Assumption **(H2)**. Note that $\phi = \hat{Y}^0$ if we use the notation in the previous section.
Step 3: Combine the expansion for $\hat{Y}^{(\varepsilon)}$ with the Taylor expansion for F, G, and exp.

Firstly, we explain Step 1. We use the large deviation for Brownian rough paths (Theorem 1 in Ledoux, Qian, and Zhang [15]. The infinite dimensional case is in [9]) and then use the contraction principle of Itô map, which is continuous. This strategy was established in [15].

Proposition 5.3 *The law of $Y^{(\varepsilon)}$ on $P(\mathcal{W})$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with the following rate function I:*

$$I(y) = \begin{cases} \inf\{\|X\|^2_{L_2^{0,1}(\mathcal{H})}/2 \mid y = \Psi^0(X)\} & \text{if } y = \Phi^0(X) \text{ for some } X \in L_2^{0,1}(\mathcal{H}), \\ \infty & \text{otherwise.} \end{cases}$$

Proof. First recall that $\mathbb{P}_\varepsilon$ on $G\Omega_p(\mathcal{V})$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with the following rate function J (see Theorem 1, [15] or Theorem 3.2, [9]):

$$J(X) = \begin{cases} \|X\|^2_{L_2^{0,1}(\mathcal{H})}/2 & \text{if } X \in L_2^{0,1}(\mathcal{H}), \\ \infty & \text{otherwise.} \end{cases}$$

Then, from the slight extension of Lyons' continuity theorem (Theorem 3.9) and the slight extension of the contraction principle (Lemma 3.9, [9], for instance), we can prove the proposition. ∎

Secondly, we explain Step 2. We can use Theorems 4.4 and 4.5, if we set $\mathcal{V} = \mathcal{V}$, $\hat{\mathcal{V}} = \mathcal{V} \oplus \mathbf{R}$ and regard the equation (5.3) as follows:

$$d\hat{Y}^{(\varepsilon)} = \sigma(\varepsilon, \hat{Y}_t^{(\varepsilon)})\varepsilon dW_t + \big[\sigma(\varepsilon, \hat{Y}_t^{(\varepsilon)})d\Lambda_t + b(\varepsilon, \hat{Y}_t^{(\varepsilon)})dt\big], \qquad \hat{Y}_0^{(\varepsilon)} = 0.$$

Finally, we explain Step 3. This step is essentially the same as in Section 6, [9]. In this step, a Fernique type theorem and a Cameron-Martin type theorem for the Brownian rough paths are used. (See, for instance, Theorem 2.2 and Lemma 2.3, [8].)

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