Higher regulators of Fermat curves and values of L-functions

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1. Introduction

Let X be a smooth projective curve over $\mathbb{Q}$ with the genus g and $L(H^1(X), s)$ be the Hasse-Weil L-function. In [B1], Beilinson defines a regulator map

$$\mathrm{reg}_{\mathcal{D}} : H^2_{\mathcal{M}}(X, \mathbb{Q}(m+2)) \to H^2_{\mathcal{D}}(X, \mathbb{R}(m+2))$$

for an integer $m \geqq 0$, where $H^2_{\mathcal{M}}(X, \mathbb{Q}(r))$ is the motivic cohomology (or the absolute chohomology) and $H^2_{\mathcal{D}}(X, \mathbb{R}(r))$ is the Deligne cohomology. Let $N \geqq 3$ be integers and denote by X_N the Fermat curve of exponent N, which is the smooth projective curve given by the affine equation : $x^N + y^N = 1$. In this paper, we will construct non-zero element of the image of the regulator map $\mathrm{reg}_{\mathcal{D}}$ for Fermat curves X_N. This element is connected to the Beilinson's conjecture which relates the special values of L-function $L(H^1(X), s)$ in [B1]. We have the canonical isomorphism $H^2_{\mathcal{D}}(X, \mathbb{R}(m+2)) \cong \left[H^1_B(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}(2\pi\sqrt{-1})^{m+1}\right]^{DR}$ from the Betti cohomology. Here, DR represents the invariant subspace under the action of complex conjugation on $X(\mathbb{C})$ and $2\pi\sqrt{-1}$. The Beilinson's conjecture predicts that

1. The regulator map induces the $\mathbb{Q}$-lattice structure of $H^2_{\mathcal{D}}(X, \mathbb{R}(m+2))$.
2. Define $c \in \mathbb{R}^\times/\mathbb{Q}^\times$ by

$$\bigwedge^g \left[\mathrm{reg}_{\mathcal{D}} H^2_{\mathcal{M}}(X, \mathbb{Q}(m+2))\right] = c \bigwedge^g \left[(H^1_B(X(\mathbb{C}), \mathbb{Q}(2\pi\sqrt{-1})^{m+1})^{DR}\right]$$

Then, $c \equiv L^{(g)}(H^1(X), -m) \mod \mathbb{Q}^\times$.

This conjecture is formulated for a projective smooth variety and has shown when X is an cyclotomic field and an elliptic curve in [B], [B1]. In the case of the cyclotomic field, the element of the motivic cohomology or its image of the regulator map is called by cyclotomic element. In view point of l-adic story and Hodge story, the cyclotomic element was studies by Deligne and Beilinson in [B2] and [BD]. We will try to construct the analog of cyclotomic element on the Fermat curve.

2. Premieres

2.1 Let F be a number field and $S = \mathrm{Spec}F$. We denote by Sm/S the category of smooth schemes of finite type over S. We will consider the following three cohomology theories on Sm/S with coefficients $\mathcal{F} = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_l$. Let X be a scheme X of Sm/S.

(i)When $\mathcal{F} = \mathbb{Q}$, we set

$$H^q(X, \mathbb{Q}(r)) := H^q_{\mathcal{M}}(X, \mathbb{Q}(r)) = Gr^r_\gamma K_{2r-q}(X) \otimes \mathbb{Q}$$

where Gr^r_γ is the subquotient of γ-filtration on the K-group.
(ii)When $\mathcal{F} = \mathbb{R}$, we set

$$H^q(X, \mathbb{R}(r)) := H^q_{\mathcal{D}}(X, \mathbb{R}(r)) = \mathbb{H}^q(X(\mathbb{C}), \mathbb{R}(q)_{\mathcal{D}})^{DR}$$

which is the hyper cohomology of the complex

$$\mathbb{R}(r)_{\mathcal{D}} := 0 \to \overset{\text{degree } 0}{\mathbb{R}(2\pi\sqrt{-1})^r} \to \overset{\text{degree } 1}{O_X} \to \overset{\text{degree } 2}{\Omega^1_X} \to \cdots \to \overset{\text{degree } r}{\Omega^{r-1}_X} \to 0.$$

(iii)When $\mathcal{F} = \mathbb{Q}_l$ for a prime l, we set

$$H^q(X, \mathbb{Q}_l(r)) := H^q_{et}(X, \mathbb{Q}_l(r)).$$

These cohomology theories satisfy the nice cohomological properties which is formulated in [Gi]. We have the regulator maps between these cohomologies

$$\begin{aligned} \mathrm{reg}_{\mathcal{D}} :& H^q(X, \mathbb{Q}(r)) \to H^q(X, \mathbb{R}(r)), \\ \mathrm{reg}_{\mathfrak{et}} :& H^q(X, \mathbb{Q}(r)) \to H^q(X, \mathbb{Q}_l(r)). \end{aligned}$$

2.2 We need the notion of homotopical algebra by Quillen in [Q1]. We will give the interpretation of three cohomology theories from a viewpoint of homotopical algebra. Let $\mathcal{S}_*$ be the category of pointed simplicial sets. For a scheme X of Sm/S, we denote by $\mathcal{S}_*(X_{zar})$ the category of sheaves of pointed simplicial sets on X_{zar}. In Brown[Br], Brown and Gersten[BG], they have shown that the category $\mathcal{S}_*(X_{zar})$ has a structure as a closed model category. We denote by $\mathcal{H}o(\mathcal{S}_*(X_{zar}))$(resp. $\mathcal{H}o(\mathcal{S}_*)$) the homotopical algebra of $\mathcal{S}_*(X_{zar})$(resp. $\mathcal{S}_*$). Then, we have a functor $R\Gamma(X, \ ) : \mathcal{H}o(\mathcal{S}_*(X_{zar})) \to \mathcal{H}o(\mathcal{S}_*)$. We denote by $\mathcal{K}_X = \mathbb{Z} \times \mathbb{Z}_\infty BGL(O_X)$ in $\mathcal{H}o(\mathcal{S}_*(X_{zar}))$, where $\mathbb{Z}_\infty$ is the completion functor defined by Bousfield-Kan in [BoK]. Since X is regular, $\mathcal{K}_X$ coincides with the pointed simplicial sheaf $\mathcal{G}_X = QCoh_S$ induced from the presheaf of pointed simplicial sets $U \mapsto QCoh(U)$ on X_{zar}. Here, $Coh(U)$ is the abelian category of coherent shaef on open set $U \subset X$ and Q is the Quillen's Q-construction in [Q]. Note that $K_q(X) = \pi_q R\Gamma(X, \mathcal{K}_X)$, where π_q denote q-th homotopy group of pointed simplicial set. By Gillet's methods in [Gi], we have Chern class maps

$$\begin{aligned} c^{\mathcal{D}}_r &: \mathcal{K}_X \to \mathbb{K}(2r, R\pi_{an}\mathbb{R}(r)_{\mathcal{D}}) \\ c^{\mathfrak{et}}_r &: \mathcal{K}_X \to \mathbb{K}(2r, R\pi_{et}\mathbb{Q}_l(r)) \end{aligned}$$

in the homotopical algebra, where $R\alpha_{et} : D(X_{et}) \to D(X_{zar})$ and $R\alpha_{an} : D(X(\mathbb{C})_{an}) \to D(X_{zar})$ are the canonical derived functors and $\mathbb{K}$ is the Dold-Puppe's construction $D(X_{zar}) \to \mathcal{H}o(\mathcal{S}_*(X_{zar}))$. Note that $H^{2r-q}_{\mathcal{D}}(X, \mathbb{R}(r)) = \pi_q R\Gamma(X, \mathbb{K}(2r, R\pi_{an}\mathbb{R}(r)_{\mathcal{D}})))$ and $H^{2r-q}_{\mathfrak{et}}(X, \mathbb{Q}_l) = \pi_q R\Gamma(X, \mathbb{K}(2r, R\pi_{et}\mathbb{Q}_l(r)))$. The regulator maps $\mathrm{reg}_{\mathcal{D}}$ and $\mathrm{reg}_{\mathfrak{et}}$ are induced from the Chern class map $c^{\mathcal{D}}_r$ and $c^{\mathfrak{et}}_r$.
2.3 For integer $n \geqq 0$, we set the finite ordered set $[n] := \{0 < 1 < 2 < \cdots < n\}$. Let I be the category whose objects are $[n]$ for $n \geqq 0$ and whose morphisms are all maps with preserving the order of elements of objects. A simplicial scheme

of Sm/S is a contravariant functor from I to Sm/S. We define an augmented simplicial scheme of Sm/S by a pair of scheme X and a simplicial scheme over X. When $X_\bullet$ is an augmented simplicial scheme of Sm/S, we denote by X_{-1} the augmented scheme of $X_\bullet$ and by $X_{\geqq 0}$ the simplicial scheme of $X_\bullet$ and by X_n the scheme of n-simplices of $X_{\geqq 0}$ for $n \geqq 0$. Let $asSm/S$ be the category of augmented simplicial schemes of Sm/S. For $X_\bullet$ of $asSm/S$, we define the cohomology theories by the following way. We set the objects of $\mathcal{H}o(\mathcal{S}_*)$

$$
\begin{aligned}
K(X_{\geqq 0}) &:= \underset{n\in I}{coholim} R\Gamma(X_n, \mathcal{K}_{X_n}),\\
R\Gamma(X_{\geqq 0}, \mathbb{R}(r)) &:= \underset{n\in I}{coholim} R\Gamma(X_n, \mathbb{K}(2r, R\pi_{an}\mathbb{R}(r)_{\mathcal{D}}))),\\
R\Gamma(X_{\geqq 0}, \mathbb{Q}_l(r)) &:= \underset{n\in I}{coholim} R\Gamma(X_n, \mathbb{K}(2r, R\pi_{et}\mathbb{Q}_l(r))),\\
K(X_\bullet) &:= coholim \left(R\Gamma(X_{-1}, \mathcal{K}_{X_{-1}}) \to K(X_{\geqq 0}\right)),\\
R\Gamma(X_\bullet, \mathbb{R}(r)) &:= coholim \left(R\Gamma(X_{-1}, \mathbb{K}(2r, R\pi_{an}\mathbb{R}(r)_{\mathcal{D}}))) \to R\Gamma(X_{\geqq 0}, \mathbb{R}(r))\right),\\
R\Gamma(X_\bullet, \mathbb{Q}_l(r)) &:= coholim \left(R\Gamma(X_{-1}, \mathbb{K}(2r, R\pi_{et}\mathbb{Q}_l(r))) \to R\Gamma(X_{\geqq 0}, \mathbb{Q}_l(r)).\right).
\end{aligned}
$$

Here, *coholim* is th homotopy inverse limit functor constructed by Bousfield-Kan in [BoK]. Then, the cohomologies of $X_\bullet$ are defined by

$$
\begin{aligned}
H^{2r-q}_{\mathcal{M}}(X_\bullet, \mathbb{Q}(r)) &:= Gr^r_\gamma \pi_q K(X_\bullet),\\
H^{2r-q}_{\mathcal{D}}(X_\bullet, \mathbb{R}(r)) &:= \pi_q R\Gamma(X_\bullet, \mathbb{R}(r)),\\
H^{2r-q}_{\mathfrak{et}}(X_\bullet, \mathbb{Q}_l) &:= \pi_q R\Gamma(X_\bullet, \mathbb{Q}_l(r)).
\end{aligned}
$$

We have the following exact sequence.

$$\cdots \to H^n(X_\bullet) \to H^n(X_{-1}) \to H^n(X_{\geqq 0}) \to H^{n+1}(X_\bullet) \to \cdots$$

There exists the regulator maps $\mathrm{reg}_{\mathcal{D}}$ and $\mathrm{reg}_{\mathfrak{et}}$ between these cohomologies.

3. Constructions

3.1 Let S be a base scheme. We set $T = {\mathbb{G}_m}_S^2$ the 2-dimensional torus over S with coordinate functions x, y. We define the augmented simplicial scheme $T_\bullet$ over S by the following. We denote $T_{i(nd)}$ the non-degenarate i-simplices of $T_\bullet$.

$$
\begin{aligned}
T_{-1} &= T,\\
T_{0(nd)} &= T_x \sqcup T_y,\\
T_{1(nd)} &= T_x \cap T_y = V(x = y = 1),\\
T_{i(nd)} &= \emptyset \qquad\qquad \text{if } i \neq 0, 1
\end{aligned}
$$

The boundary maps of $T_\bullet$ are induced from the canonical imbeddings.

Let $n \in \mathbb{Z}_{\geqq 1}$. We define $T(-n)_\bullet$ the augmented simplicial scheme over S by the following inductive way.

$$
\begin{aligned}
T(-1)_\bullet &= T_\bullet,\\
T(-n)_\bullet &= tot(T(-n+1)_\bullet \times_S T_\bullet) \qquad \text{for } n \geqq 2
\end{aligned}
$$

Here, we take $tot(T(-n+1)_\bullet \times_S T_\bullet$ the total simlicial scheme from the bi-simplicial scheme $T(-n+1)_\bullet \times_S T_\bullet$. The -1-part of $T(-n)$ is the $2n$-dimensional torus $T^n = {\mathbb{G}_m}^{2n}$.

Proposition 3.1.1 *Let $X_\bullet$ be an augmented simplicial scheme over S. Then there exists a following canonical isomorphism of cohomologies.*

$$H^{q-2n}(X_\bullet, \mathcal{F}(r-2n)) \cong H^q(tot(X_\bullet \times_S T(-n)_\bullet), \mathcal{F}(r))$$

If $x_1, y_1, \cdots, x_n, y_n$ are the coordinate functions of $T(-n)_{-1} = T^n$, then the above isomorphism coincide with the cup product of $x_1 \cup y_1 \cup \cdots x_n \cup y_n$.

Remark 3.1.2 In [B2], Beilinson has described the same augmented simplicial scheme $S(-n)$ which is constructed from $T = \mathbb{G}_m$ instead of $T = {\mathbb{G}_m}^2$. As the above notation, we have $H^{q-n}(X_\bullet, \mathcal{F}(r-n)) \cong H^q(tot(X_\bullet \times_S S(-n)_\bullet), \mathcal{F}(r))$.

3.2 Let $n \in \mathbb{Z}_{\geqq 1}$ and $T^{n+1} = \overbrace{T \times_S \cdots \times_S T}^{(n+1)\text{-times}}$ be the $2n+2$-dimensional torus with coordinate $x_0, y_0, x_1, y_1, \cdots, x_n, y_n$. From subschemes of T^{n+1}, we want to define an augmented bi-simlicial scheme $Y^{(n)}_{\bullet\bullet}$ over S whose -1-part is $Y^{(n)}_{-1\bullet} = T^{n+1}_\bullet$. For $0 \leqq i \leqq n$, we set the S-scheme

$$Y^{(n)}_{\{i\}\bullet} = \overbrace{T_\bullet \times_S \cdots \times_S T_\bullet}^{n\text{-times}}$$

and we take the immersion

$$\iota_{\{i\}} : Y^{(n)}_{\{i\}\bullet} \hookrightarrow Y^{(n)}_{-1\bullet},$$

which is defined by

$$\iota_{\{i\}} = \begin{cases} id_T \times_S \cdots \times_S id_T \times_S \overset{\text{i-th}}{diag} \times_S id_T \times_S \cdots \times_S id_T & \text{for } 0 \leqq i \leqq n-1 \\ id_T \times_S \cdots \times_S id_T \times_S \{(1,1)\} & \text{for } i = n \end{cases},$$

where the i-th map is the diagonal map: $diag : T_\bullet \to T_\bullet \times_S T_\bullet; \alpha \mapsto (\alpha, \alpha)$. We regard $Y^{(n)}_{\{i\}\bullet}$ as a simplicial subscheme of $Y^{(n)}_{-1\bullet} = T^{n+1}_\bullet$ by the immersions $\iota_{\{i\}}$. For any subset $A \subseteq \{0, 1, \cdots n\}$, we set

$$Y^{(n)}_{A\bullet} = \bigcap_{i \in A} Y^{(n)}_{\{i\}\bullet} \subseteq Y^{(n)}_{-1\bullet}.$$

There exists the canonical isomorphism $Y^{(n)}_{A\bullet} \cong T^{n+1-\#A}_\bullet$ over S. For $k \in \mathbb{Z}_{\geqq 0}$, the non-degenarate k-simplices part $Y^{(n)}_{k(nd)\bullet}$ of $Y^{(n)}_{\bullet\bullet}$ is defined by

$$Y^{(n)}_{k(nd)\bullet} = \begin{cases} \bigsqcup_{\#A=k+1} T^{(r)}_{A\bullet} & \text{if } 0 \leqq k \leqq n \\ \emptyset & \text{otherwise.} \end{cases}$$

and the boundary maps are induced form canonical injections. Then $Y^{(n)}_{\bullet\bullet}$ is a bi-simplicial scheme over S.

3.3 We take a divisor D of T defined by equation $xy = 1$ and we set an open subscheme $U = T \setminus D$. Let $t : U \to T$ be the canonical immersion.

Lemma 3.3.1 *there exists a weak homotopy equivalence of cohomology theories induced from the following canonical morphism of S-simplicial schemes*

$$Y^{(n-1)}_{U,t} \times_S S(-1)_\bullet \xrightarrow{\cong} mapping\!-\!fiber \left[U \times_S Y^{(n)}_\bullet \xrightarrow{j_n} Y^{(n)}_{U,t} \right].$$

Lemma 3.3.2 *There exists an isomorphism*

$$H^{p-2n}(U, \mathcal{F}(q-2n)) \cong H^p(U \times_S Y_\bullet^{(n)}, \mathcal{F}(q))$$

From (3.3.1) and (3.3.2), we know that there exists the following long exact sequence

$$\begin{array}{ccccc} \cdots & \longrightarrow & H^q(Y_{U,t}^{(l)}, \mathcal{F}(r)) & \longrightarrow & H^{q-2l}(U, \mathcal{F}(r-2l)) \longrightarrow \\ H^{q-1}(Y_{U,t}^{(l-1)}, \mathcal{F}(r-1)) & \longrightarrow & H^{q+1}(Y_{U,t}^{(l)}, \mathcal{F}(r)) & \longrightarrow & \cdots \end{array}$$

by the assumptions for each cohomology theory. Using the theory of exact couples, we obtain a spectral sequence

$$E_1^{l,q} = H^{2(q+l)+\alpha}(U, \mathcal{F}(2l+\beta)) \Rightarrow H^{2(q+l+n)+\alpha}(Y_{U,t}^{(n)}, \mathcal{F}(2n+\beta)) \qquad \textbf{(3.3.3)}$$

Remark 3.3.4 The above spectral sequence is an analog of the Beilinson's work in [B2]. When $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, he has constructed the following spectral sequence concerned with the cyclotomic element.

$$E_1^{l,q} = H^{q+l}(U, \mathcal{F}(l+\beta)) \Rightarrow H^{q+l+n}(\tilde{Y}_{U,t}^{(n)}, \mathcal{F}(n+\beta))$$

Here, $\tilde{Y}_{U,t}^{(n)}$ is the same augmented simplicial scheme which is constructed by the same method.

Proposition 3.3.5 *For $n \geqq 1$, there exists the following commutative diagram of exact sequences.*

$$\begin{array}{ccccc} 0 \to H^1_{\mathcal{M}}(S, \mathbb{Q}(2n+1)) & \longrightarrow & H^{2n+1}_{\mathcal{M}}(Y_{U,t}^{(n)}, \mathbb{Q}(2n+1)) & \xrightarrow{\alpha_{\mathcal{M}}} & H^1_{\mathcal{M}}(U, \mathbb{Q}(1)) \\ \downarrow \mathrm{reg} & & \downarrow \mathrm{reg} & & \downarrow \mathrm{reg} \\ 0 \to H^1(S, \mathcal{F}(n+1)) & \longrightarrow & H^{2n+1}(Y_{U,t}^{(n)}, \mathcal{F}(2r+1) & \xrightarrow{\alpha} & H^1(U, \mathcal{F}(1)) \end{array}$$

The image of $\alpha_{\mathcal{M}}$ is generated by the elements $1-xy, x, y \in O(U)^\times = H^1_{\mathcal{M}}(U, \mathbb{Q}(1))$.

Remark 3.3.6 The same statement for $H^{2n+2}_{\mathcal{M}}(Y_{U,t}^{(n)}, \mathbb{Q}(2n+2))$ is not evident. But, there exist the canonical map $\alpha_{\mathcal{M}} : H^{2n+2}_{\mathcal{M}}(Y_{U,t}^{(n)}, \mathbb{Q}(2n+2)) \to H^2_{\mathcal{M}}(U, \mathbb{Q}(2))$ which induced from the spectral sequence.

3.4 We will construct elements which is an analog of the motivic polylogarithm on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Definition 3.4.1 *Let N be an integer $N \geqq 1$. We will construct a polylogarithm element Π_N by the following way.*
(i) When $N = 2n+1$ is odd, we define an element Π_{2n+1} by

$$\Pi_{2n+1} \in H^{2n+1}(Y_{U,t}^{(n)}, \mathbb{Q}(2n+1))$$

such that $\alpha_{\mathcal{M}}(\Pi_{2n+1}) = 1 - xy \in \mathrm{Im}(\alpha_{\mathcal{M}}) \subset H^1_{\mathcal{M}}(U, \mathbb{Q}(1)) = O(U)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ from (3.3.5).
(ii) When $N = 2n+2$ is even, we define an element Π_{2n+2} by

$$\Pi_{2n+2} := \Pi_{2n+1} \cup \pi^*(x) \in H^{2n+2}(Y_{U,t}^{(n)}, \mathbb{Q}(2n+2)).$$

Here, when $\pi : Y^{(n)}_{U,t} \to U$ is the canonical projection, $\pi^*(x)$ is the image of $x \in O(U)^\times \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(U, \mathbb{Q}(1))$ under the map $\pi^* : H^1_{\mathcal{M}}(U, \mathbb{Q}(1)) \to H^1(Y^{(n)}_{U,t}, \mathbb{Q}(1))$.

By the projection formula, we know that $\alpha_{\mathcal{M}}(\Pi_{2n+2}) = \{1 - xy, x\} \in K_2(U) \otimes_{\mathbb{Z}} \mathbb{Q} = H^2_{\mathcal{M}}(U, \mathbb{Q}(2))$.

4. Results

4.1 Let $N \geqq 3$ be integers and denote by $\bar{X}_N$ the Fermat curve of exponent N, which is the smooth projective curve given by the affine equation : $x^N + y^N = 1$. We denote by $X_N := \bar{X}_N \cap U$ an affine Fermat curve. We take the canonical injection $t_N : X_N \to U$. We obtain the augmented bi-simplicial scheme $Y^{(n)}_{X_N,t_N}$ and the canonical morphism $t_N : Y^{(n)}_{X_N,t_N} \to Y^{(n)}_{U,t}$. The following morphism of the spectral sequences is induced form t_N.

$$\begin{array}{ccc} E_1^{l,q} = H^{2(q+l)+\alpha}(U, \mathcal{F}(2l+\beta)) & \Rightarrow & H^{2(q+l+n)+\alpha}(Y^{(n)}_{U,t}, \mathcal{F}(2n+\beta)) \\ \downarrow & & \downarrow \\ E_1^{l,q} = H^{2(q+l)+\alpha}(X_N, \mathcal{F}(2l+\beta)) & \Rightarrow & H^{2(q+l+n)+\alpha}(Y^{(n)}_{X_N,t_N}, \mathcal{F}(2n+\beta)) \end{array} \qquad \mathbf{(4.1.1)}$$

We get the following proposition on the affine Fermat curve.

Proposition 4.1.2 *The above bottom spectral sequence on the Fermat curve degenerates at E_1.*

Proof: The boundary map $E_1^{l,q} \to E_1^{l+1,q}$ is the cup product of the element $\{x, y\} \in K_2(X_N) \otimes_{\mathbb{Z}} \mathbb{Q}$. In Milnor K-group $K_2(X_N)$, we know that $N^2\{x, y\} = \{x^N, y^N\} = \{x^N, 1 - x^N\} = 0$. So, the proof is complete. □

From the above proposition, we get the filtration F_q of $H^{2n+2}(Y^{(n)}_{X_N,t_N}, \mathcal{F}(2n+2))$ such that $Gr^q_F = H^{2q}(X_N, \mathcal{F}(2q)$. So, this implies that there exists the following element.

$$\alpha_n = \Pi_{2n+2} - \Pi_{2n} \cup x_n \cup y_n \in H^2_{\mathcal{M}}(X_N, \mathbb{Q}(2n+2))$$

If $\mathrm{reg}_{\mathcal{D}}(\alpha_n)$ is not zero, then $\mathrm{reg}_{\mathcal{D}}(\alpha_n)$ sholud present the value of L-function $L(H^1(\bar{X}_N), s)$ at $s = -2n$. We will compute the value of $\mathrm{reg}_{\mathcal{D}}(\alpha_n)$. When $n = 0$, we have $\alpha_0 = \{1 - xy, x\}$. In [R], Ross compute the value of $\mathrm{reg}_{\mathcal{D}}\{1 - xy, x\}$.

4.2 Let $\zeta = \exp(2\imath\sqrt{-1})$, and let $A_{i,j}$ denote the automorphism of $X_N(\mathbb{C})$ given by $(x, y) \mapsto (\zeta^i x, \zeta^j y)$. Let $t^{1/N}$ denote the principal branch of the N-th root function, and let $\gamma : [0, 1] \to X_N(\mathbb{C})$ denote the path from $(1, 0)$ to $(0, 1)$ given by $\gamma(t) = (t^{1/N}, (1-t)^{1/N})$. For integers m and n, let $\gamma_{m,n}$ denote the following closed path on $X_N(\mathbb{C})$.

$$\gamma_{m,n} := \gamma - A_{m,0}\gamma + A_{m,n} - A_{0,n}\gamma$$

From the computation of the double complex of the differential modules on $X_N(\mathbb{C})$, we obtain the following formula.

$$\mathrm{reg}(\alpha_1) = \log(1 - xy)(\log x)^2 \frac{dx}{x} \in H^2_{\mathcal{D}}(U, \mathbb{R}(4))$$

For $\alpha > 0$, we define by

$$B_m(\alpha) := \int_0^1 \frac{dt_m}{t_m} \cdots \int_0^{t_3} \frac{dt_2}{t_2} \int_0^{t_2} t_1^\alpha (1-t)^\alpha \frac{dt_1}{t_1}.$$

When $m = 1$, This function coincides with the beta function $B_1(\alpha) = \frac{1}{2}B(\alpha, \alpha)$.

Theorem 4.2.1 *The element* $\alpha_1 \in H^2_{\mathcal{M}}(X_N \cap U, \mathbb{Q}(4))$ *has non-zero image under the regulator map* $\mathrm{reg}_{\mathcal{D}} : H^2_{\mathcal{M}}(X_N, \mathbb{Q}(4)) \to H^2_{\mathcal{D}}(X_N, \mathbb{R}(4))$.

Proof: We describe the pairing of $\mathrm{reg}(\alpha_1)$ and some close path $\gamma = \gamma_{1,1} + \gamma_{1,-1}$ by using the above functions. We computes that

$$\langle \mathrm{reg}(\alpha), \gamma \rangle = \frac{3}{8\pi^3} \sum_{k=1}^{\infty} \sin \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi k}{N}\right) B_3 \left(\frac{k}{N}\right)$$

For $0 \leqq M \leqq N - 1$, we set $\beta_M = \displaystyle\sum_{k \equiv M \mod N} \frac{1}{k} B_3 \left(\frac{k}{N}\right)$. Then, we get the following.

$$\langle \mathrm{reg}(\alpha), \gamma \rangle = \sum_{M=1}^{[\frac{N-1}{2}]} \sin \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi k}{N}\right) (\beta_M - \beta_{N-M})$$

Since $\beta_1 > \cdots > \beta_{N-1}$, the above is not zero. So, the proof is complete. □

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