Asymptotic Expansions for the Laplace Approximations for Itô Functionals of Brownian Rough Paths

Yuzuru Inahama

Department of Mathematics, Graduate School of Science and Engineering
Tokyo Institute of Technology
2-12-1, Oh-okayama, Meguro-ku, Tokyo 152-8551, JAPAN

Hiroshi Kawabi

Faculty of Mathematics, Kyushu University
6-10-1, Hakozaki, Higashi-ku, Fukuoka 812-8581, JAPAN

Dedicated to Professor Shinzo Watanabe on the occasion of his 70th birthday

Abstract

In this paper, we establish asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths under the condition that the phase function has finitely many non-degenerate minima. Our main tool is the Banach space-valued rough path theory of T. Lyons. We use a large deviation principle and the stochastic Taylor expansion with respect to the topology of the space of geometric rough paths. This is a continuation of a series of papers by Inahama [7] and by Inahama and Kawabi [8], [9].

Key words: Asymptotic expansions, Laplace approximation, Itô functional, Large deviation principle, Rough path theory, Stochastic Taylor expansion

1 Introduction

Let (X, H, μ) be an abstract Wiener space, i.e., X is a real separable Banach space, H is the Cameron-Martin space and μ is the Wiener measure on X. Let

Email addresses: inahama@math.titech.ac.jp (Yuzuru Inahama), kawabi@math.kyushu-u.ac.jp (Hiroshi Kawabi).

Y be another real separable Banach space and $w := (w_t)_{0 \le t \le 1}$ be the X-valued Brownian motion on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ associated with μ. We consider a class of Y-valued Wiener functionals $X^\varepsilon := (X_t^\varepsilon)_{0 \le t \le 1}$ defined through the following formal stochastic differential equation (SDE) on Y:

$$dX_t^\varepsilon = \sigma(X_t^\varepsilon) \circ \varepsilon dw_t + b(\varepsilon, X_t^\varepsilon)dt, \qquad X_0^\varepsilon = 0, \tag{1.1}$$

where the coefficients σ and b take values in $L(X, Y)$ and Y, respectively, with a suitable regularity condition. In this paper, $L(B, B')$ denotes the space of bounded linear maps from B to B' for real separable Banach spaces B and B'. We note that, since the diffusion coefficient σ takes values in $L(X, Y)$, we cannot interpret the equation (1.1) through the usual theory of SDEs generally when X and Y are infinite dimensional Banach spaces. See Section 3 for the precise formulation of these Wiener functionals X^ε. As examples of (1.1), we can give a class of heat processes on loop spaces and the solutions of SDEs on M-type 2 Banach spaces. See Inahama and Kawabi [9] for details.

The main objective of this paper is to discuss the precise asymptotic behavior of the Laplace type functional integral $\mathbb{E}\Big[G(X^\varepsilon)\exp(-F(X^\varepsilon)/\varepsilon^2)\Big]$ as $\varepsilon \searrow 0$. For heat processes on loop spaces, Laplace's method was studied in the earlier paper Inahama [7]. In this paper, as a continuation of [7] and [9], we establish the asymptotic expansion formulas for wider classes of (infinite dimensional) Banach space-valued Wiener functionals by using the fact that the rough path theory of T. Lyons works on any Banach space.

To establish the Freidlin-Wentzell type large deviation principle for X^ε, due to the lack of the continuity of the Itô map $w \mapsto X^\varepsilon$, Schilder's theorem and the contraction principle may not be used directly. To overcome this difficulty, Freidlin and Wentzell developed refined techniques involving the exponential continuity (see Deuschel and Stroock [6]). On the other hand, recently, Ledoux, Qian and Zhang [14] gave a new proof for the large deviation principle by using the rough path theory. The basic idea in [14] is summarized as follows: First, they show that the laws of Brownian rough paths satisfy the large deviation principle. Next, they use the contraction principle since the Itô map is continuous in the framework of the rough path theory. Hence their approach seems straightforward and much simpler than conventional proofs. In [8], it is shown that their approach is also applicable to a class of stochastic processes on infinite dimensional spaces.

As an application of the large deviation principle, Laplace's method is investigated in many research fields of probability theory and mathematical physics. In finite dimensional settings, Schilder [16] initiated the study in the case of $X^\varepsilon = \varepsilon w$ and Azencott [4] and Ben Arous [5] continued this study for (1.1). (For results concerning with more general Wiener functionals, see Kusuoka and Stroock [11], [12] and Takanobu and Watanabe [17].) In these papers, the stochastic Taylor expansion for X^ε plays an essential role. The problem of [16] is rather easier because each term of the expansion is continuous, which comes from the fact that X^ε is nothing but the scaled Brownian motion. So, there is no ambiguity in the formulation. However, in general, it is very complicated to give a precise interpretation on each term of this expansion through conventional stochastic analysis because the Itô map is not a continuous Wiener functional. On the other hand, Aida [2] proposed a new proof with the rough path theory for this problem recently. In [2], he obtained the stochastic Taylor expansion with respect to the topology of the space of geometric rough paths for finite dimensional cases. Since the Itô map is continuous in the rough path sense, each term of the expansion is continuous. Hence we do not need to face the difficulty mentioned above. By these reasons, the authors guess that, on the space of geometric rough paths, there could be many other probability measures to which this method is applicable. Based on the idea of [2], the first author [7] has already showed the stochastic Taylor expansion up to the order 2 in an infinite dimensional setting.

Our method of the stochastic Taylor expansion is slightly different from Aida's method in [2]. He uses the derivative equation, whose coefficient is of course of linear growth. Since it is not known whether Lyons' continuity theorem holds or not for unbounded coefficients, he extends the continuity theorem for the case of the derivative equation in [1]. On the other hand, we use the method in Azencott [4] and we only need the continuity theorem for the given equation, whose coefficient is bounded. The price we have to pay is that notations and proofs may seem slightly long. However, the strategy of this method is quite simple and straight forward.

The organization of this paper is as follows: In Section 2, we give a simple review of the rough path theory and introduce the Cameron-Martin theorem and Fernique's theorem in the framework of Brownian rough paths. In Section 3, we give the framework and state the asymptotic expansion formula in the case where the phase function has a unique non-degenerate minimum point (Theorem 3.2). In Section 4, we establish the Taylor expansion for our Wiener functionals X^ε in the sense of rough paths. This expansion plays an essential role in this paper. It is deterministic in this case and, hence, the term "stochastic Taylor expansion" may not be appropriate anymore. In Section 5, we estimate the remainder terms of the Taylor expansion. In Section 6, we prove Theorem 3.2 and give the explicit representation for the coefficients in the asymptotic expansion (Theorem 6.5). Finally, we also establish the asymptotic expansion formula in the case where the phase function has finitely many non-degenerate minima (Theorem 6.7).

Throughout this paper, we denote by c unimportant positive constants which may vary from line to line. When their dependence on some parameters are significant, we specify as $c(\|\gamma\|_1)$, $c(r_0, r_1)$, etc.

2 Preliminaries from the rough path theory

In this section we set notations and review some basic results of the rough path theory.

2.1 A review of the rough path theory

First, we recall the definition of spaces of geometric rough paths. Let B be a real separable Banach space. The algebraic tensor product is denoted by $B \otimes_{\mathrm{a}} B$. We consider a norm $|\cdot|$ on $B \otimes_{\mathrm{a}} B$ such that $|x \otimes y| \le |x|_B \cdot |y|_B$ holds for all $x, y \in B$. We denote by $B \otimes B$ the completion of $B \otimes_{\mathrm{a}} B$ by this norm. We often suppress the subscripts of Banach norms when there is no fear of confusion. We also use the notation $B^n := \underbrace{B \oplus \cdots \oplus B}_{n-\text{times}}$.

Let $2 < p < 3$ be the roughness and fix it throughout this paper. A continuous map $\overline{x} = (1, \overline{x}_1, \overline{x}_2)$ from the simplex $\Delta := \{(s,t) |\ 0 \le s \le t \le 1\}$ to the truncated tensor algebra $T^{(2)}(B) := \mathbb{R} \oplus B \oplus (B \otimes B)$ is said to be a B-valued rough path of roughness p if it satisfies that, for every $s \le u \le t$,

$$\begin{aligned} \overline{x}_1(s,t) &= \overline{x}_1(s,u) + \overline{x}_1(u,t), \\ \overline{x}_2(s,t) &= \overline{x}_2(s,u) + \overline{x}_2(u,t) + \overline{x}_1(s,u) \otimes \overline{x}_1(u,t) \end{aligned} \tag{2.1}$$

and

$$\|\overline{x}_j\|_{p/j} := \Big(\sup_D \sum_{l=1}^n |\overline{x}_j(t_{l-1}, t_l)|^{p/j} \Big)^{j/p} < \infty \qquad \text{for } j = 1, 2,$$

where $D = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ runs over all finite partition of the interval $[0,1]$. Equation (2.1) is called Chen's identity and the norm $\|\cdot\|_p$ is called p-variation norm. $\|\cdot\|_{p,[s,t]}$ denotes the p-variation norm on the time interval $[s,t]$. We define by $\Omega_p(B)$ the set of B-valued rough paths of finite p-variation. The distance between $\overline{x}$ and $\overline{y}$ in $\Omega_p(B)$ is defined by

$$d_p(\overline{x}, \overline{y}) = \|\overline{x}_1 - \overline{y}_1\|_p + \|\overline{x}_2 - \overline{y}_2\|_{p/2}.$$

We also set $\xi_B(\overline{x}) := \|\overline{x}_1\|_p + \|\overline{x}_2\|_{p/2}^{1/2}$. In the sequel, we will suppress the subscript in the case where B is the abstract Wiener space X.

Let $P(B) := \{x \in C([0,1], B) \mid x_0 = 0\}$. For $x \in P(B)$, we define the norm by $\|x\|_{P(B)} := \sup_{0 \le t \le 1} |x_t|_B$ and sometimes write $x(t)$ for x_t. We often write $\overline{x}_1(\cdot)$ for $\overline{x}_1(0, \cdot) \in P(B)$ for simplicity. We define by $\mathrm{BV}(B) := \{\gamma \in P(B) \mid \|\gamma\|_1 <$

$\infty\}$, where $\|\gamma\|_1$ denotes the total variation norm of γ. For $\gamma \in \mathrm{BV}(B)$, we set $\overline{\gamma} = (1, \overline{\gamma}_1, \overline{\gamma}_2)$ by

$$\overline{\gamma}_1(s,t) := \gamma_t - \gamma_s, \quad \overline{\gamma}_2(s,t) := \int_s^t (\gamma_u - \gamma_s) \otimes d\gamma_u, \quad 0 \le s \le t \le 1,$$

where the right-hand side of $\overline{\gamma}_2$ is the Riemann-Stieltjes integral. A rough path obtained in this way is called the smooth rough path lying above γ. A rough path obtained as the d_p-limit of a sequence of smooth rough paths is called a geometric rough path and the set of all the geometric rough paths is denoted by $G\Omega_p(B)$. It is well-known that $G\Omega_p(B)$ is a complete separable metric space.

We set

$$\mathcal{H}(B) = L_0^{2,1}(B) := \left\{ y \in P(B) \,\Big|\, y_t = \int_0^t y'_s ds \text{ with } \|y\|^2_{\mathcal{H}(B)} := \int_0^1 |y'_t|_B^2 dt < \infty \right\}.$$

Clearly, there are natural continuous injections $\mathcal{H}(B) \hookrightarrow \mathrm{BV}(B) \hookrightarrow G\Omega_p(B)$. Note that $\mathcal{H}(B)$ is dense in $G\Omega_p(B)$ and, when B is a Hilbert space, it has a natural Hilbert structure.

Now we present a simple lemma which will be used in Section 4. We set

$$\begin{aligned}\mathcal{BV}(L(B,B)) := \Big\{ (M,N) \,\Big|\, & (M - \mathrm{Id}_B, N - \mathrm{Id}_B) \in \mathrm{BV}\Big(L(B,B)^{\oplus 2}\Big), \\ & M_t N_t = N_t M_t = \mathrm{Id}_B \text{ for } t \in [0,1] \Big\}.\end{aligned}$$

We say $M \in \mathcal{BV}(L(B,B))$ if $(M, M^{-1}) \in \mathcal{BV}(L(B,B))$ for simplicity.

We define a map $\Gamma : \mathrm{BV}(B) \times \mathcal{BV}(L(B,B)) \to \mathrm{BV}(B)$ by

$$\Gamma(h, M)_t = \Gamma\Big(h, (M, M^{-1})\Big)_t := M_t \int_0^t M_s^{-1} dh_s, \qquad 0 \le t \le 1$$

for $h \in \mathrm{BV}(B)$ and $M \in \mathcal{BV}(L(B,B))$. The next lemma shows that it has a continuous extension from $G\Omega_p(B) \times \mathcal{BV}(L(B,B))$ to $G\Omega_p(B)$, which will be denoted by Γ again.

Lemma 2.1 *Let $\Gamma : \mathrm{BV}(B) \times \mathcal{BV}(L(B,B)) \to \mathrm{BV}(B)$ be as above. Assume that there exists a control function ω such that*

$$|\overline{h}_i(s,t)| \le \omega(s,t)^{i/p}, \qquad i = 1, 2, \tag{2.2}$$

$$|M_t - M_s|_{L(B,B)} + |M_t^{-1} - M_s^{-1}|_{L(B,B)} \le \omega(s,t), \tag{2.3}$$

hold for $0 \le s \le t \le 1$. Then we have the following assertions:

(1) *For* $h \in \mathrm{BV}(B)$ *and* $M \in \mathcal{BV}(L(B,B))$,

$$\left|\overline{\Gamma(h,M)}_i(s,t)\right| \le c_i\,\omega(s,t)^{i/p}, \quad i=1,2, \quad 0 \le s \le t \le 1 \tag{2.4}$$

where c_1 *and* c_2 *are positive constants depending only on* $\omega(0,1)$.
(2) Γ *extends to a continuous map from* $G\Omega_p(B) \times \mathcal{BV}(L(B,B))$ *to* $G\Omega_p(B)$. *(We denote it again by* $(\overline{h},M) \in G\Omega_p(B) \times \mathcal{BV}(L(B,B)) \mapsto \Gamma(\overline{h},M) \in G\Omega_p(B)$*.) Clearly,* $\Gamma(\overline{\varepsilon h},M) = \varepsilon\Gamma(\overline{h},M)$ *holds for any* $\overline{h} \in G\Omega_p(B), M \in \mathcal{BV}(L(B,B))$ *and* $\varepsilon \in \mathbb{R}$.

Proof. At the beginning, we define $\lambda(h,M) \in \mathrm{BV}(B)$ by

$$\lambda(h,M)_t := \int_0^t dM_s\Big(M_s^{-1}h_s - \int_0^s dM_u^{-1}h_u\Big), \quad t \ge 0.$$

We note $\Gamma(h,M)_0 = 0$ and

$$d\Gamma(h,M)_t = dM_t\int_0^t M_s^{-1}dh_s + dh_t, \quad t > 0.$$

Then by the integration by parts formula, it is easy to see

$$\begin{aligned}\Gamma(h,M)_t &= \int_0^t dh_s + \int_0^t dM_s\Big(\int_0^s M_u^{-1}dh_u\Big)\\ &= h_t + \int_0^t dM_s\Big(M_s^{-1}h_s - \int_0^s dM_u^{-1}h_u\Big) = h_t + \lambda(h,M)_t \quad t \ge 0.\end{aligned} \tag{2.5}$$

On the other hand, we have an estimate

$$\begin{aligned}&\Big|\lambda(h,M)_t - \lambda(h,M)_s\Big|\\ &\le \Big|\int_s^t dM_u\Big(M_u^{-1}h_u\Big)\Big| + \Big|\int_s^t dM_u\Big(\int_0^u dM_u^{-1}h_u\Big)\Big|\\ &\le \|h\|_{P(B)}(1+\|M^{-1}\|_1)\|M\|_{1,[s,t]} + \|h\|_{P(B)}\cdot\|M^{-1}\|_1\cdot\|M\|_{1,[s,t]}\\ &\le 2\|h\|_{P(B)}(1+\|M^{-1}\|_1)\|M\|_{1,[s,t]}.\end{aligned} \tag{2.6}$$

Then by noting $\|h\|_{P(B)} \le \omega(0,1)^{1/p}$, (2.5) and (2.6), we obtain an estimate on the first level path of $\Gamma(h,M)$ as follows:

$$
\begin{aligned}
\left|\overline{\Gamma(h,M)}_1(s,t)\right| &= |\Gamma(h,M)_t - \Gamma(h,M)_s| \\
&\le |\overline{h}_1(s,t)| + |\lambda(h,M)_t - \lambda(h,M)_s| \\
&\le \omega(s,t)^{1/p} + 2\|h\|_{P(B)}(1+\|M^{-1}\|_1)\|M\|_{1,[s,t]} \\
&\le \omega(s,t)^{1/p} + 2\omega(0,1)^{1/p}\Big(1+\omega(0,1)\Big)\omega(s,t) \\
&\le \Big(1+2\omega(0,1)+2\omega(0,1)^2\Big)\omega(s,t)^{1/p}. \qquad (2.7)
\end{aligned}
$$

For the second level path of $\Gamma(h,M)$, we also have

$$
\begin{aligned}
\left|\overline{\Gamma(h,M)}_2(s,t)\right| &= \left|\int_s^t \Big\{(h_u-h_s)+\Big(\lambda(h,M)_u-\lambda(h,M)_s\Big)\Big\}\otimes d\Big(h+\lambda(h,M)\Big)_u\right| \\
&\le \left|\overline{h}_2(s,t)\right| + \left|\overline{\lambda(h,M)}_2(s,t)\right| + \left|\int_s^t (h_u-h_s)\otimes d\lambda(h,M)_u\right| \\
&\quad + \left|\Big(\lambda(h,M)_t-\lambda(h,M)_s\Big)\otimes(h_t-h_s) - \int_s^t d\lambda(h,M)_u\otimes(h_u-h_s)\right| \\
&\le \omega(s,t)^{2/p} + \|\lambda(h,M)\|_{1,[s,t]}^2 + \|h\|_{p,[s,t]}\cdot\|\lambda(h,M)\|_{1,[s,t]} \\
&\quad +2\|\lambda(h,M)\|_{1,[s,t]}\cdot\|h\|_{p,[s,t]} \\
&\le \omega(s,t)^{2/p} + 4\|h\|_{P(B)}^2(1+\|M^{-1}\|_1)^2\|M\|_{1,[s,t]}^2 \\
&\quad +3\Big\{2\|h\|_{P(B)}(1+\|M^{-1}\|_1)\|M\|_{1,[s,t]}\Big\}\cdot\|h\|_{p,[s,t]} \\
&\le \omega(s,t)^{2/p} + 4\omega(0,1)^{2/p}(1+\omega(0,1))^2\omega(s,t)^2 \\
&\quad +3\Big\{2\omega(0,1)^{1/p}(1+\omega(0,1))\omega(s,t)\Big\}\omega(s,t)^{1/p} \\
&\le \Big(1+6\omega(0,1)+10\omega(0,1)^2+8\omega(0,1)^3+4\omega(0,1)^4\Big)\omega(s,t)^{2/p}. \quad (2.8)
\end{aligned}
$$

This completes the proof of (1).

Next, we aim to show (2). For $h,k\in \mathrm{BV}(B)$ and $M,N\in\mathcal{BV}(L(B,B))$, we assume that (2.2), (2.3) and

$$
\begin{aligned}
&|\overline{k}_i(s,t)| \le \omega(s,t)^{i/p}, \quad |\overline{h}_i(s,t)-\overline{k}_i(s,t)| \le \varepsilon\omega(s,t)^{i/p}, \quad i=1,2, \\
&|N_t-N_s|_{L(B,B)} + |N_t^{-1}-N_s^{-1}|_{L(B,B)} \le \omega(s,t), \\
&\Big|(M-N)_t-(M-N)_s\Big|_{L(B,B)} \le \varepsilon\omega(s,t), \\
&\Big|(M^{-1}-N^{-1})_t-(M^{-1}-N^{-1})_s\Big|_{L(B,B)} \le \varepsilon\omega(s,t),
\end{aligned}
$$

hold for $0\le s\le t\le 1$. Under these assumptions, we have

$$
\begin{aligned}
&\left|\overline{\lambda(h,M)}_1(s,t)-\overline{\lambda(h,N)}_1(s,t)\right|\\
&\le\Big|\int_s^t dM_u(M_u^{-1}h_u)-\int_s^t dN_u(N_u^{-1}h_u)\Big|\\
&\quad+\Big|\int_s^t dM_u\Big(\int_0^u dM_\tau^{-1}h_\tau\Big)-\int_s^t dN_u\Big(\int_0^u dN_\tau^{-1}h_\tau\Big)\Big|\\
&\le\Big|\int_s^t d(M-N)_u(M_u^{-1}h_u)\Big|+\Big|\int_s^t dN_u(M_u^{-1}-N_u^{-1})h_u\Big|\\
&\quad+\Big|\int_s^t d(M-N)_u\Big(\int_0^u dM_\tau^{-1}h_\tau\Big)\Big|+\Big|\int_s^t dN_u\Big\{\int_0^u d(M^{-1}-N^{-1})_\tau h_\tau\Big\}\Big|\\
&\le\|h\|_{P(B)}\Big(1+\|M^{-1}\|_1\Big)\cdot\|M-N\|_{1,[s,t]}+\|h\|_{P(B)}\|M^{-1}-N^{-1}\|_1\cdot\|N\|_{1,[s,t]}\\
&\quad+\|h\|_{P(B)}\|M^{-1}\|_1\cdot\|M-N\|_{1,[s,t]}+\|h\|_{P(B)}\|M^{-1}-N^{-1}\|_1\cdot\|N\|_{1,[s,t]}\\
&\le\varepsilon\omega(0,1)^{1/p}\Big(1+4\omega(0,1)\Big)\omega(s,t).
\end{aligned}
\tag{2.9}
$$

Then by noting (2.6), (2.9) and the equality $\lambda(h,M)-\lambda(k,M)=\lambda(h-k,M)$, we have the following estimate on the first level path:

$$
\begin{aligned}
&\left|\overline{\Gamma(h,M)}_1(s,t)-\overline{\Gamma(k,N)}_1(s,t)\right|\\
&\le|\overline{h}_1(s,t)-\overline{k}_1(s,t)|+\left|\overline{\lambda(h-k,M)}_1(s,t)\right|+\left|\overline{\lambda(k,M)}_1(s,t)-\overline{\lambda(k,N)}_1(s,t)\right|\\
&\le\varepsilon\omega(s,t)^{1/p}+\Big\{2\|h-k\|_{P(B)}(1+\|M^{-1}\|_1)\cdot\|M\|_{1,[s,t]}\Big\}\\
&\quad+\varepsilon\omega(0,1)^{1/p}\Big(1+4\omega(0,1)\Big)\omega(s,t)\\
&\le\varepsilon\omega(s,t)^{1/p}+2\varepsilon\omega(0,1)^{1/p}\Big(1+\omega(0,1)\Big)\omega(s,t)+\varepsilon\Big(\omega(0,1)+4\omega(0,1)^2\Big)\omega(s,t)^{1/p}\\
&\le\varepsilon\Big(1+3\omega(0,1)+6\omega(0,1)^2\Big)\omega(s,t)^{1/p}.
\end{aligned}
\tag{2.10}
$$

For the second level path, we can proceed as

$$
\begin{aligned}
&\left|\overline{\Gamma(h,M)}_2(s,t)-\overline{\Gamma(k,N)}_2(s,t)\right|\\
&\le\Big\{\ \Big|\overline{h}_2(s,t)-\overline{k}_2(s,t)\Big|+\Big|\overline{\lambda(h,M)}_2(s,t)-\overline{\lambda(k,M)}_2(s,t)\Big|\\
&\qquad\qquad+\Big|\int_s^t\overline{h}_1(s,u)\otimes d\lambda(h,M)_u-\int_s^t\overline{k}_1(s,u)\otimes d\lambda(k,M)_u\Big|\\
&\qquad\qquad\qquad+\Big|\int_s^t\overline{\lambda(h,M)}_1(s,u)\otimes dh_u-\int_s^t\overline{\lambda(k,M)}_1(s,u)\otimes dk_u\Big|\ \Big\}\\
&\quad+\Big\{\ \Big|\overline{\lambda(k,M)}_2(s,t)-\overline{\lambda(k,N)}_2(s,t)\Big|+\Big|\int_s^t\overline{k}_1(s,u)\otimes d\Big(\lambda(k,M)-\lambda(k,N)\Big)_u\Big|\\
&\qquad\qquad+\Big|\int_s^t\Big(\overline{\lambda(k,M)}_1(s,u)-\overline{\lambda(k,N)}_1(s,u)\Big)\otimes dk_u\Big|\ \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big\{ \ \varepsilon\omega(s,t)^{2/p} + \Big| \int_s^t \overline{\lambda(h-k,M)}_1(s,u)\otimes d\lambda(h,M)_u \\
&\qquad\qquad\qquad\qquad + \int_s^t \overline{\lambda(k,M)}_1(s,u)\otimes d\lambda(h-k,M)_u \Big| \\
&\qquad\quad + \Big| \int_s^t \overline{(h-k)}_1(s,u)\otimes d\lambda(h,M)_u + \int_s^t \overline{k}_1(s,u)\otimes d\lambda(h-k,M)_u \Big| \\
&\qquad\quad + \Big| \int_s^t \overline{\lambda(h-k,M)}_1(s,u)\otimes dh_u + \int_s^t \overline{\lambda(k,M)}_1(s,u)\otimes d(h-k)_u \Big| \ \Big\} \\
&\quad + \Big\{ \ \Big| \int_s^t \Big(\overline{\lambda(k,M)}_1(s,u) - \overline{\lambda(k,N)}_1(s,u) \Big) \otimes d\lambda(k,M)_u \Big| \\
&\qquad\qquad + \Big| \int_s^t \overline{\lambda}(k,N)_1(s,u)\otimes d\Big(\lambda(k,M)-\lambda(k,N)\Big)_u \Big| \\
&\qquad\qquad\quad + \Big| \int_s^t \overline{k}_1(s,u)\otimes d\Big(\lambda(k,M)-\lambda(k,N)\Big)_u \Big| \\
&\qquad\qquad\quad + \Big| \int_s^t \Big(\overline{\lambda(k,M)}_1(s,u) - \overline{\lambda(k,N)}_1(s,u) \Big) \otimes dk_u \Big| \ \Big\} \\
&\le \Big\{ \varepsilon\omega(s,t)^{2/p} + \|\lambda(h-k,M)\|_{1,[s,t]} \cdot \Big(\|\lambda(h,M)\|_{1,[s,t]} + \|\lambda(k,M)\|_{1,[s,t]} \Big) \\
&\qquad + \|h-k\|_{p,[s,t]} \cdot \Big(\|\lambda(h,M)\|_{1,[s,t]} + 2\|\lambda(k,M)\|_{1,[s,t]} \Big) \\
&\qquad + \|\lambda(h-k,M)\|_{1,[s,t]} \cdot \Big(2\|h\|_{p,[s,t]} + \|k\|_{p,[s,t]} \Big) \Big\} \\
&\quad + \Big\{ \|\lambda(k,M)-\lambda(k,N)\|_{1,[s,t]} \cdot \Big(\|\lambda(k,M)\|_{1,[s,t]} + \|\lambda(k,N)\|_{1,[s,t]} \Big) \\
&\qquad + 3\|\lambda(k,M)-\lambda(k,N)\|_{1,[s,t]} \cdot \|k\|_{p,[s,t]} \Big\} \\
&\le \Big\{ \varepsilon\omega(s,t)^{2/p} + 4\|h-k\|_{P(B)} \cdot \Big(\|h\|_{P(B)} + \|k\|_{P(B)} \Big) \cdot \Big(1 + \|M^{-1}\|_1 \Big)^2 \|M\|_{1,[s,t]}^2 \\
&\qquad + 2\Big(\|h\|_{P(B)} + 2\|k\|_{P(B)} \Big) \cdot \Big(1 + \|M^{-1}\|_1 \Big) \cdot \|M\|_{1,[s,t]} \cdot \|h-k\|_{p,[s,t]} \\
&\qquad + 2\|h-k\|_{P(B)} \cdot \Big(1 + \|M^{-1}\|_1 \Big) \cdot \|M\|_{1,[s,t]} \cdot \Big(2\|h\|_{p,[s,t]} + \|k\|_{p.[s,t]} \Big) \Big\} \\
&\quad + \|\lambda(k,M)-\lambda(k,N)\|_{1,[s,t]} \cdot \Big(\|\lambda(k,M)\|_{1,[s,t]} + \|\lambda(k,N)\|_{1,[s,t]} + 3\|k\|_{p,[s,t]} \Big) \\
&\le \Big\{ \varepsilon\omega(s,t)^{2/p} + 8\varepsilon\omega(0,1)^{2/p}(1+\omega(0,1))^2\omega(s,t)^2 \\
&\qquad\qquad\qquad + 12\varepsilon\omega(0,1)^{1/p}(1+\omega(0,1))\omega(s,t)^{1+1/p} \Big\} \\
&\quad + \varepsilon\omega(0,1)^{1/p}(1+4\omega(0,1))\omega(s,t) \cdot \Big\{ 4\omega(0,1)^{1/p}(1+\omega(0,1))\omega(s,t) + 3\omega(s,t)^{1/p} \Big\} \\
&\le \varepsilon \Big\{ 1 + 8(1+\omega(0,1))^2\omega(0,1)^2 + 12(1+\omega(0,1))\omega(0,1) \Big\} \omega(s,t)^{2/p} \\
&\quad + \varepsilon \Big\{ 4\omega(0,1)^2 \Big(1 + 5\omega(0,1) + 4\omega(0,1)^2 \Big) + 3\omega(0,1)(1+4\omega(0,1)) \Big\} \omega(s,t)^{2/p} \\
&\le \varepsilon \Big(1 + 15\omega(0,1) + 36\omega(0,1)^2 + 36\omega(0,1)^3 + 24\omega(0,1)^4 \Big) \omega(s,t)^{2/p}. \qquad (2.11)
\end{aligned}
$$

Then (2.10) and (2.11) lead us that Γ is locally Lipschitz continuous on $\mathrm{BV}(B) \times \mathcal{BV}(L(B,B))$ with respect to the product topology induced by the distance d_p on $G\Omega_p(B)$ and $\|\cdot\|_1$ on $\mathrm{BV}(L(B,B))$. Hence by remembering that $\mathrm{BV}(B)$ is dense in $G\Omega_p(B)$, the map Γ extends to a continuous map from

$G\Omega_p(B) \times \mathcal{BV}(L(B,B))$ to $G\Omega_p(B)$. The final assertion is almost trivial. This completes the proof. ∎

Before closing this section, we review integrals along rough paths. Let B' be another separable Banach space and $f \in C^3_{b,loc}(B, L(B,B'))$, i.e., $\nabla^i f$, $i = 0,1,2,3$, exist and are bounded on every bounded set of B. Here, ∇ denotes the Fréchet derivative on B. For $k \in \mathbb{N}$, $\nabla^k f$ is a map from B to $L^k(B,\dots,B; L(B,B'))$. Here $L^k(B_1,\dots,B_k; B_{k+1})$ denotes the space of bounded k-linear maps from the direct sum of Banach spaces $\bigoplus_{i=1}^k B_i$ to another Banach space B_{k+1}.

For B-valued rough path $\overline{x} \in G\Omega_p(B)$, we consider

$$J_{s,t} := f(x_s)\overline{x}_1(s,t) + \nabla f(x_s)\Big[\overline{x}_2(s,t)\Big], \quad 0 \le s \le t \le 1.$$

We see that, for $s < u < t$,

$$\begin{aligned} J_{s,t} - J_{s,u} - J_{u,t} = &- \int_0^1 d\eta' \int_0^{\eta'} d\eta \nabla^2 f\Big(x_s + \eta \overline{x}_1(s,u)\Big)\Big[\overline{x}_1(s,u) \otimes \overline{x}_1(s,u) \otimes \overline{x}_1(u,t)\Big] \\ &- \int_0^1 d\eta \nabla^2 f\Big(x_s + \eta \overline{x}_1(s,u)\Big)\Big[\overline{x}_1(s,u) \otimes \overline{x}_2(u,t)\Big], \end{aligned} \qquad (2.12)$$

where we used the Taylor expansion for the function f and Chen's identity (2.1) for $\overline{x}$. Equation (2.12) will be used in Section 5.

For f denoted above and $\overline{x} \in G\Omega_p(B)$, we define $\int f(x)d\overline{x} \in G\Omega_p(B')$ by

$$\begin{aligned} \Big(\int f(x)d\overline{x}\Big)_1(s,t) &:= \lim_{m(D)\to 0} \sum_{i=0}^{N-1} J_{t_i,t_{i+1}}, \\ \Big(\int f(x)d\overline{x}\Big)_2(s,t) &:= \lim_{m(D)\to 0} \sum_{i=0}^{N-1} \Big\{\Big(f(x_{t_i}) \otimes f(x_{t_i})\Big)\Big[\overline{x}_2(t_i,t_{i+1})\Big] \\ &\qquad + \Big(\int f(x)d\overline{x}\Big)_1(s,t_i) \otimes \Big(\int f(x)d\overline{x}\Big)_1(t_i,t_{i+1})\Big\}, \end{aligned}$$

where $D := \{s = t_0 < t_1 < t_2 < \cdots < t_N = t\}$. When $\overline{x}$ is in a bounded set of $G\Omega_p(B)$, the sup-norm of the first level path $\overline{x}_1(\cdot)$ is also bounded. Hence, we easily have the following continuity theorem by Theorems 5.2.3 and 5.3.1 in Lyons and Qian [15]. In the sequel, we often take integrands of the form $\mathrm{Id}_B \oplus f \in C^3_{b,loc}(B, L(B, B \oplus B'))$, where $f \in C^3_{b,loc}(B, L(B,B'))$.

Theorem 2.2 *Let $\overline{x}, \overline{y} \in G\Omega_p(B)$. We assume that there exists a control function ω such that*

$$|\overline{x}_i(s,t)| \vee |\overline{y}_i(s,t)| \le \omega(s,t)^{i/p}, \ |\overline{x}_i(s,t) - \overline{y}_i(s,t)| \le \varepsilon\omega(s,t)^{i/p}, \ i = 1,2.$$

Then there exist positive constants C_1 and C_2 depending only on p, $\omega(0,1)$ and $\sup\{|\nabla^j f(x)| : |x| \le \omega(0,1)^{1/p}\}, j = 0,1,2,3$, *such that*

$$\Big|\Big(\int f(x)d\overline{x}\Big)_i(s,t)\Big| \le C_1\,\omega(s,t)^{i/p},$$
$$\Big|\Big(\int f(x)d\overline{x}\Big)_i(s,t) - \Big(\int f(y)d\overline{y}\Big)_i(s,t)\Big| \le \varepsilon\, C_2\,\omega(s,t)^{i/p},$$

hold for $i = 1, 2$.

2.2 Some fundamental results for Brownian rough paths

In this subsection, we introduce Brownian rough paths on an abstract Wiener space (X, H, μ). Let $w = (w_t)_{t\ge 0}$ be the X-valued Brownian motion introduced in the previous section. For $\varepsilon > 0$, the law of εw on $P(X)$ is denoted by $\mathbb{P}'_\varepsilon$. Then $(P(X), \mathcal{H}(H), \mathbb{P}'_1)$ is also an abstract Wiener space. We write $\mathcal{H} := \mathcal{H}(H)$ for simplicity. When $|\cdot|_{X\otimes X}$ and μ satisfy the following *exactness condition*

(EX): There exist constants $C > 0$ and $\alpha \in [1/2, 1)$ such that, for any $N \in \mathbb{N}$ and for any sequence $\{G_l\}_{l=1}^{2N}$ of independent X-valued random variables with common distribution μ, it holds

$$\mathbb{E}\Big[\Big|\sum_{l=1}^{N} G_{2l-1} \otimes G_{2l}\Big|_{X\otimes X}\Big] \le CN^{\alpha}. \tag{2.13}$$

(cf. Definition 1 in Ledoux, Lyons and Qian [13]), the Brownian rough path exists (see Theorem 3 in [13]). Let $\overline{w} = (1, \overline{w}_1, \overline{w}_2)$ be the Brownian rough path. It is the $\mathbb{P}$-almost sure limit of the $\overline{w(m)}$ as $m \to \infty$ in $G\Omega_p(X)$ with respect to d_p-topology, where $w(m)$ is the m-th dyadic polygonal approximation of w. Note that $\overline{w}_1(s,t) = w_t - w_s$ for $\mathbb{P}$-almost surely. We denote by $\mathbb{P}_\varepsilon, \varepsilon > 0$, the law of the scaled Brownian rough path $\overline{\varepsilon w} = (1, \varepsilon\overline{w}_1, \varepsilon^2\overline{w}_2)$.

Now we present a theorem of Fernique type for Brownian rough paths. The following proposition is taken from Theorem 2.2 in [7]

Proposition 2.3 *There exists a positive constant β such that*

$$\mathbb{E}\Big[\exp\Big(\beta\,\xi^2\Big)\Big] = \int_{G\Omega_p(X)} \exp\Big(\beta\,\xi(\overline{w})^2\Big)\mathbb{P}_1(d\overline{w}) < \infty$$

Finally, we give a theorem for absolute continuity of the laws of shifted Brownian rough paths. It is similar to the well-known Cameron-Martin theorem. For $\overline{x} \in G\Omega_p(X)$ and $\gamma \in \mathrm{BV}(X)$, we define the shifted rough path $\overline{x+\gamma} \in G\Omega_p(X)$ by

$$\overline{(x+\gamma)}_1(s,t) = \overline{x}_1(s,t) + \overline{\gamma}_1(s,t),$$
$$\overline{(x+\gamma)}_2(s,t) = \overline{x}_2(s,t) + \int_s^t \overline{x}_1(s,u) \otimes d\gamma_u + \int_s^t (\gamma_u - \gamma_s) \otimes \overline{x}_1(s,du) + \overline{\gamma}_2(s,t).$$

Here the second and the third terms on the right-hand side are Young integrals. It is well-known that the map $(\overline{x}, \gamma) \mapsto \overline{x \pm \gamma}$ is continuous from $G\Omega_p(X) \times \mathrm{BV}(X)$ to $G\Omega_p(X)$. (See Theorem 3.3.2 in [15].) The following proposition is taken from Lemma 2.3 in [7].

Proposition 2.4 *Let $\varepsilon > 0$ and $h \in \mathcal{H}$. Then for every bounded measurable function F on $G\Omega_p(X)$, it holds that*

$$\int_{G\Omega_p(X)} F(\overline{w+h}) \mathbb{P}_\varepsilon(d\overline{w}) = \int_{G\Omega_p(X)} F(\overline{w}) \exp\left(\frac{1}{\varepsilon^2}\int_0^1 h'(t) d\overline{w}_1(t) - \frac{1}{2\varepsilon^2}\|h\|^2_{\mathcal{H}}\right) \mathbb{P}_\varepsilon(d\overline{w}),$$

where $\int_0^1 h'(t) d\overline{w}_1(t)$ is the stochastic integral with respect to the scaled Brownian motion $(\overline{w}_1(0,t))_{0\le t\le 1}$ defined on the probability space $(G\Omega_p(X), \mathbb{P}_\varepsilon)$. (Hereafter we sometimes denote it by $[h](\overline{w})$ for simplicity.)

3 Framework and the main result

In this section, we set notations, introduce our Wiener functionals through the Itô map in the rough path sense and state our results. From now on, we only consider the projective norm on the tensor product of any pair of Banach spaces, and we assume condition **(EX)** for $|\cdot|_{X\otimes X}$ and μ to treat Brownian rough paths. Note that **(EX)** holds with $\alpha = 1/2$ if $\dim(X) < \infty$.

First, we set notations for coefficients. Let $\sigma \in C_b^\infty(Y, L(X,Y))$ and $b_1, \dots, b_N \in C_b^\infty(Y,Y)$, $N \in \mathbb{N}$. ∇ denotes the Fréchet derivative on Y. For $k \in \mathbb{N}$, $\nabla^k \sigma$ and $\nabla^k b$ are maps from Y to $L^k(Y,\dots,Y; L(X,Y))$ and $L^k(Y,\dots,Y;Y)$, respectively. We set $\tilde{X} := X \oplus \mathbb{R}^N$ and define $\tilde{\sigma} \in C_b^\infty(Y, L(\tilde{X}, Y))$ by

$$\tilde{\sigma}(y)\big[(x,u)\big]_{\tilde{X}} := \sigma(y)x + \sum_{i=1}^N b_i(y)u_i, \quad y \in Y, x \in X, u = (u_1, \dots, u_N) \in \mathbb{R}^N.$$

Next, we consider the following differential equation in the rough path sense:

$$dy_t = \tilde{\sigma}(y_t) d\tilde{x}_t \quad \text{with } y_0 = 0. \tag{3.1}$$

Then for any $\overline{\tilde{x}} \in G\Omega_p(\tilde{X})$, there exists a unique solution $\overline{z} \in G\Omega_p(\tilde{X} \oplus Y)$ in the rough path sense. Note that the natural projection of $\overline{z}$ onto the first component is $\overline{\tilde{x}}$. Projection of $\overline{z}$ onto the second component is denoted by $\overline{y} \in G\Omega_p(Y)$ and we write $\overline{y} = \Phi(\overline{\tilde{x}})$ and call it a solution of (3.1). The map $\Phi : G\Omega_p(\tilde{X}) \to G\Omega_p(Y)$ is called the Itô map and is locally Lipschitz continuous in the sense of Lyons and Qian. See Theorem 6.2.2 in [15] for details. If $\tilde{x}_t = (\gamma_t, \lambda_t^{(1)}, \ldots, \lambda_t^{(N)})$ is a $\tilde{X}$-valued continuous path of finite variation, the map $t \mapsto \Phi(\overline{\tilde{x}})_1(0,t)$ is the solution of

$$dy_t = \sigma(y_t)d\gamma_t + \sum_{i=1}^{N} b_i(y_t)d\lambda_t^{(i)} \quad \text{ with } y_0 = 0$$

in the usual sense and $\overline{z}$ is the smooth rough path lying above $(\tilde{x}_t, \Phi(\overline{\tilde{x}})_1(0,t))_{0\le t\le 1}$.

For $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \mathrm{BV}(\mathbb{R}^N)$ and $\overline{x} \in G\Omega_p(X)$, we set $\iota(\overline{x}, \lambda) \in G\Omega_p(\tilde{X})$ by $\iota(\overline{x}, \lambda)_1(s,t) = (\overline{x}_1(s,t), \lambda_t - \lambda_s)$ and

$$\iota(\overline{x}, \lambda)_2(s,t) = \Big(\overline{x}_2(s,t), \int_s^t \overline{x}_1(s,u)\otimes d\lambda_u, \int_s^t (\lambda_u - \lambda_s)\otimes \overline{x}_1(s,du), \int_s^t (\lambda_u - \lambda_s)\otimes d\lambda_u\Big).$$

Here the second and the third component are Young integrals. If $\overline{h}$ is a smooth rough path lying above $h \in \mathrm{BV}(X)$, then $\iota(\overline{h}, \lambda)$ is a smooth rough path lying above $(h, \lambda) \in \mathrm{BV}(\tilde{X})$. Note that the map $\iota : G\Omega_p(X) \times \mathrm{BV}(\mathbb{R}^N) \to G\Omega_p(\tilde{X})$ is continuous.

For $\varepsilon \in [0,1]$, we define $\lambda^\varepsilon \in \mathrm{BV}(\mathbb{R}^N)$ by $\lambda^\varepsilon(t) := (a_1(\varepsilon)t, \ldots, a_N(\varepsilon)t)$, where $a = (a_1, \ldots, a_N) : [0,1] \to \mathbb{R}^N$ is a $\mathbb{R}^N$-valued smooth curve. In what follows, we usually use the notation

$$a^j(\varepsilon)\cdot \nabla^k b(y) := \sum_{i=1}^{N} \frac{d^j}{d\varepsilon^j} a_i(\varepsilon)\nabla^k b_i(y), \qquad j,k \in \mathbb{N}\cup\{0\}.$$

Next, we regard the Itô map defined above as a map from $\mathrm{BV}(X)$ to $\mathrm{BV}(Y)$. We define $\Psi_\varepsilon : \mathrm{BV}(X) \to \mathrm{BV}(Y)$ by $\Psi_\varepsilon(h)_t := \Phi\Big(\iota(\overline{h}, \lambda^\varepsilon)\Big)_1(0,t)$ for $0 \le t \le 1$. That is, $y := \Psi_\varepsilon(h)$ is the unique solution of the ordinary differential equation

$$dy_t = \sigma(y_t)dh_t + a(\varepsilon)\cdot b(y_t)dt \quad \text{ with } y_0 = 0. \tag{3.2}$$

We note that Ψ_ε also maps $\mathcal{H}(X)$ to $\mathcal{H}(Y)$.

For the X-valued Brownian motion w, let $\overline{w}$ be the Brownian rough path over X. For $\varepsilon \in [0,1]$, we define a Wiener functional $X^\varepsilon \in P(Y)$ by

$$X_t^\varepsilon := \Phi\Big(\iota(\overline{\varepsilon w}, \lambda^\varepsilon)\Big)_1(0,t), \qquad 0 \le t \le 1.$$

We investigate the asymptotic behavior of the law of X^ε as $\varepsilon \searrow 0$. First, we recall a large deviation principle which was essentially shown in Theorem 4.9 of Inahama and Kawabi [8].

Theorem 3.1 *For $\varepsilon > 0$, we denote by $\mathcal{V}_\varepsilon$ the law of the process X^ε. Then, $\{\mathcal{V}_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with the good rate function Λ, where*

$$\Lambda(\phi) = \begin{cases} \frac{1}{2} \inf \Big\{ \|\gamma\|_{\mathcal{H}}^2 \big| \ \phi = \Psi_0(\gamma) \Big\}, & \text{if } \phi = \Psi_0(\gamma) \text{ for some } \gamma \in \mathcal{H}, \\ \infty, & \text{otherwise.} \end{cases}$$

More precisely, for any measurable set $K \subset P(Y)$, it holds that

$$- \inf_{\phi \in K^\circ} \Lambda(\phi) \leq \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \mathcal{V}_\varepsilon(K) \leq \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \mathcal{V}_\varepsilon(K) \leq - \inf_{\phi \in \overline{K}} \Lambda(\phi).$$

As a consequence of Theorem 3.1, we have the following asymptotics for every bounded continuous function F on $P(Y)$:

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{E}\Big[\exp\Big(- F(X^\varepsilon)/\varepsilon^2 \Big) \Big] = -\inf \Big\{ F(\phi) + \Lambda(\phi) \mid \phi \in P(Y) \Big\}.$$

This is Varadhan's integral lemma. See [6] for example. Our next concern is to investigate the more precise asymptotics of a generalization of the integral on the left-hand side of above equality. That is, we aim to establish the asymptotic expansions of the integral $\mathbb{E}\Big[G(X^\varepsilon) \exp\Big(- F(X^\varepsilon)/\varepsilon^2 \Big)\Big]$ as $\varepsilon \searrow 0$.

In this paper, we impose the following conditions on the functions F and G. In what follows, we especially denote by D the Fréchet derivatives on $\mathrm{BV}(X)$ and $P(Y)$.

(H1): F and G are real-valued bounded continuous functions defined on $P(Y)$.

(H2): The function $F_\Lambda := F \circ \Psi_0 + \| \cdot \|_{\mathcal{H}}^2/2$ defined on $\mathcal{H}$ attains its minimum at a unique point $\gamma \in \mathcal{H}$. For this γ, we write $\phi := \Psi_0(\gamma)$.

(H3): The functions F and G are $n+3$ and $n+1$ times Fréchet differentiable on a neighborhood $B(\phi)$ of $\phi \in P(Y)$, respectively. Moreover there exist positive constants $M_1, \ldots, M_{n+3}$ such that

$$\begin{aligned} \Big| D^k F(\eta) \Big[y, \ldots, y \Big] \Big| &\leq M_k \|y\|_{P(Y)}^k, \quad k = 1, \ldots, n+3, \\ \Big| D^k G(\eta) \Big[y, \ldots, y \Big] \Big| &\leq M_k \|y\|_{P(Y)}^k, \quad k = 1, \ldots, n+1, \end{aligned}$$

hold for any $\eta \in B(\phi)$ and $y \in P(Y)$.

(H4): At the point $\gamma \in \mathcal{H}$, we consider the Hessian $A := D^2(F \circ \Psi_0)(\gamma)|_{\mathcal{H} \times \mathcal{H}}$. As a bounded self-adjoint operator on $\mathcal{H}$, the operator A is strictly larger than $-\mathrm{Id}_{\mathcal{H}}$ in the form sense. (By the min-max principle, it is equivalent to assume that all eigenvalues of A are strictly larger than -1.)

Now we are in a position to state our main theorem. The explicit values of $\{\alpha_m\}_{m=0}^n$ will be given later since we need to introduce a few more notations which we cannot introduce briefly. See Theorem 6.5 for the detail.

Theorem 3.2 *Under conditions* **(EX)**, **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *we have the following asymptotic expansion:*

$$\begin{aligned}&\mathbb{E}\Big[G(X^\varepsilon)\exp\big(-F(X^\varepsilon)/\varepsilon^2\big)\Big]\\&=\exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big)\exp\big(-c(\gamma)/\varepsilon\big)\cdot\big(\alpha_0+\alpha_1\varepsilon+\cdots+\alpha_n\varepsilon^n+O(\varepsilon^{n+1})\big),\end{aligned} \tag{3.3}$$

where the constant $c(\gamma)$ *in* (3.3) *is given by* $c(\gamma) := DF(\phi)[\Xi_1(\gamma)]$. *Here* $\Xi_j(\gamma)\in\mathcal{H}(Y)\subset P(Y), j\in\mathbb{N}$, *is the unique solution of the differential equation*

$$d\Xi_t-\nabla\sigma(\phi_t)[\Xi_t,d\gamma_t]-a(0)\cdot\nabla b(\phi_t)[\Xi_t]dt=a^{(j)}(0)\cdot b(\phi_t)dt \quad \textit{with} \quad \Xi_t=0. \tag{3.4}$$

4 Taylor expansion in the sense of rough paths

In this section, we establish the Taylor expansion for the differential equation (3.2) in the sense of rough paths. In Sections 4 and 5, we discuss without conditions **(EX)**, **(H1)**, **(H2)**, **(H3)** and **(H4)**. In particular, $\gamma\in\mathrm{BV}(X)$ and $\phi=\Psi_0(\gamma)$ are not the special elements as in **(H2)**. Notice also that we do not need the imbedded Hilbert space $H\subset X$, neither.

At the beginning, we discuss in a heuristic way in order to find out what the terms in the expansion are like. Fix $\gamma\in\mathrm{BV}(X)$ and $\phi=\Psi_0(\gamma)\in\mathrm{BV}(Y)$. Suppose that we have an expansion around ϕ as

$$\Delta\phi:=\Phi(\iota(\gamma+\varepsilon h,\lambda^\varepsilon))_1-\phi\sim\varepsilon\phi^1+\cdots+\varepsilon^n\phi^n+\cdots, \qquad \text{as } \varepsilon\searrow 0.$$

Of course, we also have

$$a(\varepsilon)\sim a(0)+\varepsilon a'(0)+\cdots+\varepsilon^n\frac{a^{(n)}(0)}{n!}+\cdots, \qquad \text{as } \varepsilon\searrow 0.$$

From the equation (3.2),

$$\begin{aligned}
d(\phi+\Delta\phi) &\sim \sigma(\phi+\Delta\phi)d(\gamma+\varepsilon h) + \Big(\sum_{n=0}^{\infty}\varepsilon^n \frac{a^{(n)}(0)}{n!}\Big)\cdot b(\phi+\Delta\phi)dt, \\
&\sim \Big(\sum_{n=0}^{\infty}\frac{1}{n!}\nabla^n\sigma(\phi)\Big[\Delta\phi,\ldots,\Delta\phi, d(\gamma+\varepsilon h)\Big]\Big) \\
&\quad +\Big(\sum_{n=0}^{\infty}\varepsilon^n \frac{a^{(n)}(0)}{n!}\Big)\cdot\Big(\sum_{n=0}^{\infty}\frac{1}{n!}\nabla^n b(\phi)\Big[\Delta\phi,\ldots,\Delta\phi\Big]dt\Big). \qquad (4.1)
\end{aligned}$$

Picking up terms of order $n \in \mathbb{N}$, we see the following definition is quite natural.

Definition 4.1 *For fixed $\gamma \in \mathrm{BV}(X)$, We set $\phi^0 = \phi$ by*

$$d\phi_t = \sigma(\phi_t)d\gamma_t + a(0)\cdot b(\phi_t)dt \qquad \text{with } \phi_0 = 0 \qquad (4.2)$$

and set ϕ^1 by

$$d\phi^1_t - \nabla\sigma(\phi_t)[\phi^1_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\phi^1_t]dt = \sigma(\phi_t)dh_t + a'(0)\cdot b(\phi_t)dt \qquad \text{with } \phi^1_0 = 0.$$

For $n = 2, 3, \ldots,$ we set $\phi^n = \phi^n(h,\gamma)$ by

$$\begin{aligned}
&d\phi^n_t - \nabla\sigma(\phi_t)[\phi^n_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\phi^n_t]dt \\
&= dk(\phi,\phi^1,\ldots,\phi^{n-1};h)_t + d\tilde{k}(\phi,\phi^1,\ldots,\phi^{n-1};\gamma)_t \qquad \text{with } \phi^n_0 = 0. \quad (4.3)
\end{aligned}$$

Here $k(\phi,\phi^1,\ldots,\phi^{n-1};h)_t$ and $\tilde{k}(\phi,\phi^1,\ldots,\phi^{n-1};\gamma)_t$ are defined by

$$\begin{aligned}
k(\phi,\phi^1,\ldots,\phi^{n-1};h)_t &:= \int_0^t \sum_{k=1}^{n-1}\sum_{(i_1,\ldots,i_k)\in S_k^{n-1}} \frac{1}{k!}\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_k}, dh_s\Big], \qquad (4.4) \\
\tilde{k}(\phi,\phi^1,\ldots,\phi^{n-1};\gamma)_t &:= \int_0^t \sum_{k=2}^{n}\sum_{(i_1,\ldots,i_k)\in S_k^{n}} \frac{1}{k!}\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_k}, d\gamma_s\Big] \\
&\quad + \int_0^t \sum_{k=2}^{n}\sum_{(i_1,\ldots,i_k)\in S_k^{n}} \frac{a(0)}{k!}\cdot\nabla^k b(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_k}\Big]ds \\
&\quad + \int_0^t \sum_{j=1}^{n-1}\sum_{k=1}^{n-j}\sum_{(i_1,\ldots,i_k)\in S_k^{n-j}} \frac{a^{(j)}(0)}{j!k!}\cdot\nabla^k b(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_k}\Big]ds \\
&\quad + \int_0^t \frac{1}{n!}a^{(n)}(0)\cdot b(\phi_s)ds, \qquad (4.5)
\end{aligned}$$

where the sum on the right-hand side runs over

$$S_k^n := \{(i_1,\ldots,i_k)\in\mathbb{N}^k \mid i_j \geq 1 \text{ for all } 1\leq j\leq k \text{ and } i_1+\cdots+i_k = n\}.$$

Before providing the main theorem of this section, we recall a lemma which will play a key-role in the sequel.

Lemma 4.2 *Let $\gamma \in \mathrm{BV}(X)$. We define by $M(\gamma) : [0,1] \to L(Y,Y)$ the solution of*

$$dM_t = d\Omega(\gamma)_t M_t \qquad \textit{with} \quad M_0 = \mathrm{Id}_Y, \tag{4.6}$$

where

$$d\Omega(\gamma)_t := \nabla\sigma(\phi_t)\Big[\,\cdot\,, d\gamma_t\Big] + a(0)\cdot\nabla b(\phi_t)dt, \quad t \geq 0. \tag{4.7}$$

Then we have the following assertions:
(1) *For all $t \in [0,1]$, the inverse $M(\gamma)_t^{-1}$ exists and it is the solution of*

$$dM_t^{-1} = -M_t^{-1} d\Omega(\gamma)_t \qquad \textit{with} \quad M_0^{-1} = \mathrm{Id}_Y. \tag{4.8}$$

(2) *For $k \in \mathrm{BV}(Y)$, we define by $\Gamma(k,\gamma) := \Gamma(k, M(\gamma)) \in \mathrm{BV}(Y)$, i.e.,*

$$\Gamma(k,\gamma)_t := M(\gamma)_t \int_0^t M(\gamma)_s^{-1} dk_s, \qquad t \geq 0. \tag{4.9}$$

Then it is the unique solution of

$$d\Gamma_t - \nabla\sigma(\phi_t)\Big[\Gamma_t, d\gamma_t\Big] - a(0)\cdot\nabla b(\phi_t)[\Gamma_t]dt = dk_t \qquad \textit{with} \quad \Gamma_0 = 0.$$

(3) $\Gamma : \mathrm{BV}(Y)\times\mathrm{BV}(X) \to \mathrm{BV}(Y)$ *extends to a continuous map from $G\Omega_p(Y)\times \mathrm{BV}(X)$ to $G\Omega_p(Y)$. (In the sequel, we denote it again by $(\overline{k},\gamma) \in G\Omega_p(Y)\times \mathrm{BV}(X) \mapsto \Gamma(\overline{k},\gamma) \in G\Omega_p(Y)$, and usually abbreviate as $\Gamma(\overline{k})$ if the dependence of γ is not significant.)*

Proof. Since (1) and (2) are shown in Lemma 3.1 of Inahama [7], we only give the proof of (3). Let $\gamma, \hat{\gamma} \in \mathrm{BV}(X)$. We set a control function ω_1 by

$$\omega_1(s,t) := \|\gamma\|_{1,[s,t]} + \|\hat{\gamma}\|_{1,[s,t]} + \varepsilon^{-1}\|\gamma - \hat{\gamma}\|_{1,[s,t]} + \sum_{i=1}^N |a_i(0)|\cdot(t-s), \quad 0 \leq s \leq t \leq 1.$$

We note that ω_1 satisfies

$$|\gamma_t - \gamma_s| \vee |\hat{\gamma}_t - \hat{\gamma}_s| \leq \omega_1(s,t), \quad \Big|(\gamma-\hat{\gamma})_t - (\gamma-\hat{\gamma})_s\Big| \leq \varepsilon\omega_1(s,t), \quad 0 \leq s \leq t \leq 1.$$

First, we set $\tilde{\gamma}_t := (\gamma_t, a_1(0)t, \ldots, a_N(0)t)$ and $\tilde{\hat{\gamma}}_t := (\hat{\gamma}_t, a_1(0)t, \ldots, a_N(0)t)$. These are of $\mathrm{BV}(\tilde{X})$. We consider $\phi = \Psi_0(\gamma)$ and $\hat{\phi} = \Psi_0(\hat{\gamma})$. Since ϕ is the solution of the differential equation

$$d\phi_t = \tilde{\sigma}(\phi_t)d\tilde{\gamma}_t \qquad \text{with} \quad \phi_0 = 0,$$

we may apply Theorems 2.3.1 and 2.3.2 in Lyons and Qian [15]. Then there exists a positive constant K_1 depending only on $\|\nabla\sigma\|_\infty, \|\nabla b_1\|_\infty, \ldots, \|\nabla b_N\|_\infty$ and $\omega_1(0,1)$ such that

$$|\phi_t - \phi_s| \le K_1 \omega_1(s,t), \quad \Big|(\phi - \hat{\phi})_t - (\phi - \hat{\phi})_s\Big| \le \varepsilon K_1 \omega_1(s,t), \tag{4.10}$$

hold for $0 \le s \le t \le 1$.

Next, we consider $\Omega(\gamma)$. By (4.7) and (4.10), we have the following estimates:

$$\begin{aligned}\Big|\Omega(\gamma)_t - \Omega(\gamma)_s\Big| &\le \|\nabla\sigma\|_\infty \|\gamma\|_{1,[s,t]} + \sum_{i=1}^N |a_i(0)| \cdot \|\nabla b_i\|_\infty \cdot (t-s) \\ &\le \Big(\|\nabla\sigma\|_\infty + \max_{1\le i\le N} \|\nabla b_i\|_\infty\Big) \cdot \omega_1(s,t), \end{aligned} \tag{4.11}$$

$$\begin{aligned}&\Big|\Big(\Omega(\gamma) - \Omega(\hat{\gamma})\Big)_t - \Big(\Omega(\gamma) - \Omega(\hat{\gamma})\Big)_s\Big| \\ &\le \|\nabla\sigma\|_\infty \|\gamma - \hat{\gamma}\|_{1,[s,t]} + \|\nabla^2\sigma\|_\infty \|\phi - \hat{\phi}\|_{1,[s,t]} \|\hat{\gamma}\|_{1,[s,t]} \\ &\quad + \sum_{i=1}^N |a_i(0)| \cdot \|\nabla b_i\|_\infty \|\phi - \hat{\phi}\|_{1,[s,t]} \cdot (t-s) \\ &\le \varepsilon \Big\{\|\nabla\sigma\|_\infty + K_1\omega_1(0,1) \cdot \Big(\|\nabla^2\sigma\|_\infty + \max_{1\le i\le N} \|\nabla b_i\|_\infty\Big)\Big\} \cdot \omega_1(s,t). \end{aligned} \tag{4.12}$$

Here we set a control function ω_2 by

$$\omega_2(s,t) := \Big\{\|\nabla\sigma\|_\infty + \Big(1 + K_1\omega_1(0,1)\Big) \cdot \Big(\|\nabla^2\sigma\|_\infty + \max_{1\le i\le N} \|\nabla b_i\|_\infty\Big)\Big\}\omega_1(s,t).$$

Then (4.11) and (4.12) imply

$$\Big|\Omega(\gamma)_t - \Omega(\gamma)_s\Big| \vee \Big|\Omega(\hat{\gamma})_t - \Omega(\hat{\gamma})_s\Big| \le \omega_2(s,t), \tag{4.13}$$

$$\Big|\Big(\Omega(\gamma) - \Omega(\hat{\gamma})\Big)_t - \Big(\Omega(\gamma) - \Omega(\hat{\gamma})\Big)_s\Big| \le \varepsilon\omega_2(s,t), \tag{4.14}$$

for $0 \le s \le t \le 1$. Hence, we may apply Theorems 2.3.1 and 2.3.2 in [15] again for differential equations (4.6) and (4.8), and we also have that

$$\begin{aligned}&\Big|M(\gamma)_t - M(\gamma)_s\Big| \vee \Big|M(\gamma)_t^{-1} - M(\gamma)_s^{-1}\Big| \le K_2\omega_2(s,t), \\ &\Big|\Big(M(\gamma) - M(\hat{\gamma})\Big)_t - \Big(M(\gamma) - M(\hat{\gamma})\Big)_s\Big| \le \varepsilon K_2\omega_2(s,t), \\ &\Big|\Big(M(\gamma)^{-1} - M(\hat{\gamma})^{-1}\Big)_t - \Big(M(\gamma)^{-1} - M(\hat{\gamma})^{-1}\Big)_s\Big| \le \varepsilon K_2\omega_2(s,t), \end{aligned}$$

hold for $0 \le s \le t \le 1$, where K_2 is a positive constant depending only on $\omega_2(0,1)$. We note that these estimates means that the maps $\gamma \in \mathrm{BV}(X) \mapsto M(\gamma) \in \mathcal{BV}(L(Y,Y))$ and $\gamma \in \mathrm{BV}(X) \mapsto M(\gamma)^{-1} \in \mathcal{BV}(L(Y,Y))$ are locally Lipschitz continuous. Hence by recalling Lemma 2.1, we can see our desired continuity. This completes the proof. ∎

Next we introduce another maps which are similar to ϕ^1 and ϕ^2. For given $\gamma \in \mathrm{BV}(X)$ and each $h, \hat{h} \in \mathrm{BV}(X)$, we define $\chi = \chi(h) := \chi(h;\gamma)$ and $\psi = \psi(h,\hat{h}) := \psi(h,\hat{h};\gamma)$ through Y-valued differential equations

$$d\chi_t - \nabla\sigma(\phi_t)[\chi_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\chi_t]dt = \sigma(\phi_t)dh_t \quad \text{with } \chi_0 = 0, \tag{4.15}$$

and

$$\begin{aligned} &d\psi_t - \nabla\sigma(\phi_t)[\psi_t, d\gamma_t] - a(0)\cdot\nabla b(\phi_t)[\psi_t]dt \\ &= \nabla\sigma(\phi_t)[\chi(\hat{h};\gamma)_t, dh_t] + \nabla\sigma(\phi_t)[\chi(h;\gamma)_t, d\hat{h}_t] \\ &\quad + \nabla^2\sigma(\phi_t)[\chi(h;\gamma)_t, \chi(\hat{h};\gamma)_t, d\gamma_t] + a(0)\cdot\nabla^2 b(\phi_t)[\chi(h;\gamma)_t, \chi(\hat{h};\gamma)_t]dt \quad \text{with } \psi_0 = 0. \end{aligned} \tag{4.16}$$

We should note that

$$\chi(h;\gamma) = D\Psi_0(\gamma)[h], \quad \psi(h;\gamma) = D^2\Psi_0(\gamma)[h,\hat{h}]. \tag{4.17}$$

By recalling Definition 4.1 and Lemma 4.2, we can easily see the representation of $\Xi_j(\gamma), j \in \mathbb{N}$, and the following relationship between ϕ^1, ϕ^2 and χ, ψ.

Lemma 4.3 *Let $\Psi_0 : \mathrm{BV}(X) \to \mathrm{BV}(Y)$ and $\Xi_j(\gamma) \in \mathrm{BV}(Y), j \in \mathbb{N}$, be defined as in Section 3. For $h \in \mathrm{BV}(X)$, we consider $\chi(h), \psi(h,h) \in \mathrm{BV}(Y)$ as above. Then we have*

$$\begin{aligned} \Xi_j(\gamma)_t &= \Gamma\Big(\int_0^\cdot a^{(j)}(0)\cdot b(\phi_s)ds\Big)_t, \\ \phi^1(h,\gamma)_t &= \chi(h)_t + \Xi_1(\gamma)_t, \\ \phi^2(h,\gamma)_t &= \frac{1}{2}\Big(\psi(h,h)_t + Y(h;\gamma)_t + \Xi_2(\gamma)_t\Big), \end{aligned}$$

hold for $t \geq 0$. Here $Y(h;\gamma) \in \mathrm{BV}(Y)$ is defined by

$$\begin{aligned} &Y(h;\gamma)_t \\ &:= \Gamma\Big(\int_0^\cdot 2(\nabla\sigma)(\phi_s)\big[\Xi_1(\gamma)_s, dh_s\big] + (\nabla^2\sigma)(\phi_s)\big[2\chi(h)_s + \Xi_1(\gamma)_s, \Xi_1(\gamma)_s, d\gamma_s\big]\Big)_t \\ &\quad + \Gamma\Big(\int_0^\cdot \Big(2a'(0)\cdot b(\phi_s)\big[\chi(h)_s + \Xi_1(\gamma)_s\big] \\ &\qquad\qquad + a(0)\cdot\nabla^2 b(\phi_s)\big[2\chi(h)_s + \Xi_1(\gamma)_s, \Xi_1(\gamma)_s\big]\Big)ds\Big)_t, \quad t \geq 0. \end{aligned}$$

Now we are in a position to state the main theorem in this section.

Theorem 4.4 *Let $\gamma \in \mathrm{BV}(X)$ be given and $\phi^n = \phi^n(h,\gamma), n \in \mathbb{N} \cup \{0\}$, be as in Definition* 4.1*. Then, the map $\phi^n : \mathrm{BV}(X) \times \mathrm{BV}(X) \to \mathrm{BV}(Y)$ extends to a continuous map $\phi^n : G\Omega_p(X) \times \mathrm{BV}(X) \to G\Omega_p(Y)$. Moreover, there exists a positive constant $c = c(\|\gamma\|_1)$ independent of $\overline{h} \in G\Omega_p(X)$ and $\gamma \in \mathrm{BV}(X)$ such that the following estimate holds:*

$$\left\|\phi^n(\overline{h},\gamma)_1\right\|_p \le c\Big(1+\xi(\overline{h})\Big)^n. \tag{4.18}$$

Proof. As we will see the continuity is almost obvious from the integration theory in the sense of rough paths. Since we prove the estimate (4.18) by mathematical induction, we divide the proof into two steps. Throughout the proof, we set $\kappa := \xi(\overline{h})$. By recalling (4.10), the boundedness of $\|\gamma\|_1$ leads us to the boundedness of $\|\phi\|_1$.

Step 1: We consider the case $n = 1$. We set a control function ω_1 by

$$\omega_1(s,t) := \|\overline{\gamma}_1\|_{1,[s,t]} + \kappa^{-p}\|\overline{h}_1\|^p_{p,[s,t]} + \kappa^{-p}\|\overline{h}_2\|^{p/2}_{p/2,[s,t]}, \quad 0 \le s \le t \le 1.$$

Then we can see that $\omega_1(0,1) \le 1 + \|\overline{\gamma}_1\|_1$. It is also easy to see

$$\|\overline{h}_1\|_{p,[s,t]} \le \kappa\omega_1(s,t)^{1/p}, \quad \|\overline{h}_2\|_{p/2,[s,t]} \le \kappa^2\omega_1(s,t)^{2/p} \quad \text{and} \quad \|\overline{\gamma}_1\|_{1,[s,t]} \le \omega_1(s,t). \tag{4.19}$$

We denote by $\overline{(h,\phi)} := \iota(\overline{h},\phi)$ for $\overline{h} \in G\Omega_p(X)$ and $\phi \in \mathrm{BV}(Y)$. We note that $\overline{(h,\phi)}_1(s,t) = \Big(\overline{h}_1(s,t), \phi_t - \phi_s\Big)$. Then by (4.19), we have

$$|\overline{(h,\phi)}_1(s,t)| \le c(1+\kappa)\omega_1(s,t)^{1/p}, \quad |\overline{(h,\phi)}_2(s,t)| \le c(1+\kappa)^2\omega_1(s,t)^{2/p} \tag{4.20}$$

Let $V_1 := X \oplus Y$ and $V_2 := X \oplus Y \oplus Y$. We set $f : V_1 \to L(V_1, V_2)$ by

$$f(x_1,x_2)[\eta_1,\eta_2] := \Big(\eta_1, \eta_2, \sigma(x_2)\eta_1\Big) \quad \text{for } x_1,\eta_1 \in X, x_2,\eta_2 \in Y.$$

Clearly, $f \in C_b^\infty(V_1, L(V_1,V_2))$. Hence, the integration with respect to f defines a continuous map from $G\Omega_p(V_1)$ to $G\Omega_p(V_2)$. It is also bounded on every bounded set in the following sense: if $\overline{v} \in G\Omega_p(V_1)$ satisfies $\xi_{V_1}(\overline{v}) \le c$ for a constant $c > 0$, then $\xi_{V_2}(\int f(v)d\overline{v}) \le c'$ holds for some constant $c' > 0$ which depends only on c, p and f, but not on v.

Therefore we have the following composition of continuous maps:

$$(\overline{h},\gamma) \in G\Omega_p(X) \times \mathrm{BV}(X) \xmapsto{\tilde{\iota}} \overline{(h,\phi)} \in G\Omega_p(V_1) \xmapsto{\Phi} \int f(h,\phi)d\overline{(h,\phi)} \in G\Omega_p(V_2),$$

where $\tilde{\iota} = \iota_{G\Omega_p(V_1)} \circ (\mathrm{Id}_{G\Omega_p(X)} \times \Psi_0)$. Hence by (4.20), we have

$$\left|\left(\int f(h,\phi)d\overline{(h,\phi)}\right)_1(s,t)\right| \le c\,(1+\kappa)\omega_1(s,t)^{1/p}, \tag{4.21}$$

$$\left|\left(\int f(h,\phi)d\overline{(h,\phi)}\right)_2(s,t)\right| \le c\,(1+\kappa)^2\omega_1(s,t)^{2/p} \tag{4.22}$$

holds for some constant $c>0$ depending only on p, r_0 and f, but independent of $h,\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \le r_0$.

Here we note that if $\overline{h}$ is a smooth rough path lying above $h \in \mathrm{BV}(X)$,

$$\left(\int f(h,\phi)d\overline{(h,\phi)}\right)_1(s,t) = \left(\overline{h}_1(s,t), \phi_t - \phi_s, \int_s^t \sigma(\phi_u)dh_u\right),$$

where the third component on the right-hand side is the usual Riemann-Stieltjes integral.

Then by (4.21) and (4.22), we can conclude that

$$\left|\int_s^t \sigma(\phi_u)dh_u\right| \le c\left(1+\xi(\overline{h})\right)\omega_1(s,t)^{1/p},$$

$$\left|\int_s^t \left(\int_s^u \sigma(\phi_v)dh_v\right)\otimes dh_u\right| \le c\left(1+\xi(\overline{h})\right)^2\omega_1(s,t)^{2/p}.$$

Next, let $M(\gamma)_t$ be as in Lemma 4.2 and set $\hat{M}(\gamma)_t = \mathrm{Id}_X \oplus \mathrm{Id}_Y \oplus M(\gamma)_t$. Note that, if $\|\gamma\|_1$ is bounded, then $\|\hat{M(\gamma)}\|_1 + \|\hat{M}(\gamma)^{-1}\|_1$ is bounded, too. Clearly, $\hat{M}(\gamma)$ satisfies the assumption of Lemma 2.1 with $B = V_2 = X \oplus Y \oplus Y$. By using $\hat{M}(\gamma)$, we can define $\hat{\Gamma}$ in the same manner as (4.9). Furthermore we set $g(h,\phi) \in \mathrm{BV}(V_2)$ by

$$g(h,\phi)_t := \left(0, 0, \int_0^t a'(0)\cdot b(\phi_s)ds\right), \quad t \ge 0.$$

Then we can see that

$$\hat{\Gamma}\left(\int f(h,\phi)d\overline{(h,\phi)}\right)_1(s,t) + \hat{\Gamma}\left(\overline{g(h,\phi)}\right)_1(s,t) = \left(h_t - h_s, \phi_t - \phi_s, \phi^1_t - \phi^1_s\right)$$

holds at least if $h \in \mathrm{BV}(X)$. Clearly, the map $(h,\gamma) \longmapsto (h,\phi,\phi^1)$ extends to a continuous map from $G\Omega_p(X)\times \mathrm{BV}(X)$ to $G\Omega_p(V_2)$. By Lemma 2.1, the third component of the first level path satisfies that

$$\left|\phi^1_t - \phi^1_s\right| \le c\left(1+\xi(\overline{h})\right)\omega_2(s,t)^{1/p},$$

$$\left|\int_s^t (\phi^1_u - \phi^1_s)\otimes dh_u\right| \le c\left(1+\xi(\overline{h})\right)^2\omega_2(s,t)^{2/p},$$

for a control function ω_2 defined by

$$\omega_2(s,t) := \omega_1(s,t) + (t-s) + \|\hat{M}(\gamma)\|_{1,[s,t]} + \|\hat{M}^{-1}(\gamma)\|_{1,[s,t]}, \quad 0 \le s \le t \le 1.$$

Note that $\omega_2(0,1)$ is dominated by a positive constant independent of $h, \gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \le r_0$. Hence we have (4.18) for $n = 1$.

Step 2: We set $V_n := X \oplus Y^{n+1}$ for $n \in \mathbb{N}$. In order to use mathematical induction, we aim to show Proposition $\mathbf{P(n)}$ below. Note that $\mathbf{P(1)}$ has already been shown in Step 1. As before, $\gamma \in \mathrm{BV}(X)$ is arbitrarily given.

Proposition P(n): *The map* $(h,\gamma) \mapsto \big(h, \phi, \phi^1, \ldots, \phi^n\big)$ *extends to a continuous map from* $G\Omega_p(X) \times \mathrm{BV}(X)$ *to* $G\Omega_p(V_{n+1})$. *There exists a control function* ω *such that*

$$\begin{aligned} |\phi^j_t - \phi^j_s| &\le \Big(1 + \xi(\overline{h})\Big)^j \omega(s,t)^{1/p}, \\ |(\phi^j \cdot dh)(s,t)| &\le \Big(1 + \xi(\overline{h})\Big)^{j+1} \omega(s,t)^{2/p}, \quad j = 1, \ldots, n, \end{aligned}$$

holds for all $h \in \mathrm{BV}(X)$ *and* $\gamma \in \mathrm{BV}(X)$ *with* $\|\gamma\|_1 \le r_0$. *Here*

$$(\phi^j \cdot dh)(s,t) := \int_s^t (\phi^j_u - \phi^j_s) \otimes dh_u$$

and $\omega(0,1)$ *is dominated by a positive constant* $c = c(r_0)$ *which may depend on* n, *but not on* $h, \gamma \in \mathrm{BV}(X)$ *with* $\|\gamma\|_1 \le r_0$.

From now, we will prove Proposition $\mathbf{P(n)}$ under Proposition $\mathbf{P(n\text{-}1)}$. First we treat the first term on the right-hand side of (4.3). For simplicity, we set $k_t := k(\phi, \phi^1, \ldots, \phi^{n-1}; h)_t$ and $\tilde{k}_t := \tilde{k}(\phi, \phi^1, \ldots, \phi^{n-1}; \gamma)_t$. Then we easily see that

$$k_t - k_s = \int_s^t \sum_{k=1}^{n-1} \sum_{(i_1,\ldots,i_k) \in S^{n-1}_k} \frac{1}{k!} \nabla^k \sigma(\phi_u)\Big[\phi^{i_1}_u, \ldots, \phi^{i_k}_u, dh_u\Big].$$

For each $(i_1, \ldots, i_k)$ in the above sum and $s < t$, we set

$$\begin{aligned} J^{i_1,\ldots,i_k}_{s,t} :=& \nabla^k \sigma(\phi_s)\Big[\phi^{i_1}_s, \ldots, \phi^{i_k}_s, \overline{h}_1(s,t)\Big] \\ &+ \nabla^{k+1}\sigma(\phi_s)\Big[\bullet, \phi^{i_1}_s, \ldots, \phi^{i_k}_s, \bullet\Big]\Big[(\phi \cdot dh)(s,t)\Big] \\ &+ \sum_{j=1}^k \nabla^k \sigma(\phi_s)\Big[\phi^{i_1}_s, \ldots, \phi^{i_{j-1}}_s, \bullet, \phi^{i_{j+1}}_s, \ldots, \phi^{i_k}_s, \bullet\Big]\Big[(\phi^{i_j} \cdot dh)(s,t)\Big], \\ J_{s,t} :=& \sum_{k=1}^{n-1} \sum_{(i_1,\ldots,i_k) \in S^{n-1}_k} \frac{1}{k!} J^{i_1,\ldots,i_k}_{s,t}. \end{aligned}$$

For a partition $D = \{s = t_0 < t_1 < \cdots < t_N = t\}$ of the interval $[s,t]$, we set $J_{s,t}(D) := \sum_{i=1}^{N} J_{t_{i-1},t_i}$. It is well-known that $\lim_{|D|\to 0} J_{s,t}(D) = k_t - k_s$, where $|D|$ denotes the mesh of the partition D.

Here we set a control function ω_3 by

$$\begin{aligned}\omega_3(s,t) :=\, &(t-s) + \|\overline{\gamma}_1\|_{1,[s,t]} + \|\overline{\phi}_1\|_{1,[s,t]} + \kappa^{-p}\|\overline{h}_1\|_{p,[s,t]}^{p} + \kappa^{-p}\|\overline{h}_2\|_{p/2,[s,t]}^{p/2} \\ &+ \sum_{j=1}^{n-1}(1+\kappa)^{-pj}\|\overline{\phi^j}_1\|_{p,[s,t]}^{p} + \sum_{j=0}^{n-1}(1+\kappa)^{-p(j+1)/2}\|\phi^j \cdot dh\|_{p/2,[s,t]}^{p/2},\end{aligned}$$

for $0 \leq s \leq t \leq 1$. Since we assume **P(n-1)**, $\omega_3(0,1)$ is dominated by a constant independent of $\overline{h} \in G\Omega_p(X)$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$. It is easy to see that

$$|J_{s,t}| \leq \sum_{k=1}^{n-1} \sum_{(i_1,\dots,i_k)\in S_k^{n-1}} \frac{1}{k!}|J_{s,t}^{i_1,\dots,i_k}| \leq c\,(1+\kappa)^n\Big(\omega_3(s,t)^{1/p} + \omega_3(s,t)^{2/p}\Big) \tag{4.23}$$

holds for some constant $c > 0$ independent of $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$.

Now we estimate $|J_{s,t} - J_{s,u} - J_{u,t}|$ for $s < u < t$. By Chen's identity, we see that

$$\begin{aligned}&J_{s,t}^{i_1,\dots,i_k} - J_{s,u}^{i_1,\dots,i_k} - J_{u,t}^{i_1,\dots,i_k} \\ &= \Big(\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\dots,\phi_s^{i_k},\overline{h}_1(u,t)\Big] - \nabla^k\sigma(\phi_u)\Big[\phi_u^{i_1},\dots,\phi_u^{i_k},\overline{h}_1(u,t)\Big]\Big) \\ &\quad +\nabla^{k+1}\sigma(\phi_s)\Big[\bullet,\phi_s^{i_1},\dots,\phi_s^{i_k},\bullet\Big]\Big[\overline{\phi}_1(s,u)\otimes\overline{h}_1(u,t)\Big] \\ &\quad +\Big(\nabla^{k+1}\sigma(\phi_s)\Big[\bullet,\phi_s^{i_1},\dots,\phi_s^{i_k},\bullet\Big] \\ &\qquad\qquad -\nabla^{k+1}\sigma(\phi_u)\Big[\bullet,\phi_u^{i_1},\dots,\phi_u^{i_k},\bullet\Big]\Big)\Big[(\phi\cdot dh)(u,t)\Big] \\ &\quad +\sum_{j=1}^{k}\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\dots,\phi_s^{i_{j-1}},\bullet,\phi_s^{i_{j+1}},\dots,\phi_s^{i_k},\bullet\Big]\Big[(\overline{\phi^{i_j}}_1(s,u)\otimes\overline{h}_1(u,t)\Big] \\ &\quad +\sum_{j=1}^{k}\Big(\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\dots,\phi_s^{i_{j-1}},\bullet,\phi_s^{i_{j+1}},\dots,\phi_s^{i_k},\bullet\Big] \\ &\qquad\qquad -\nabla^k\sigma(\phi_u)\Big[\phi_u^{i_1},\dots,\phi_u^{i_{j-1}},\bullet,\phi_u^{i_{j+1}},\dots,\phi_u^{i_k},\bullet\Big]\Big)\Big[(\phi^{i_j}\cdot dh)(u,t)\Big] \\ &=: I_1 + \cdots + I_5.\end{aligned} \tag{4.24}$$

By using **P(n-1)**, it is easy to see that $|I_3| + |I_5| \leq c\,(1+\kappa)^n\omega_3(s,t)^{3/p}$ for some constant $c > 0$ independent of $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$.

For $\mathbf{y} = (y_0, y_1, \ldots, y_k) \in Y^{k+1}$, we set $g : Y^{k+1} \to L(X, Y)$ by

$$g(\mathbf{y}) := \nabla^k \sigma(y_0)\Big[y_1, \ldots, y_k, \bullet\Big]_{Y^k \times X}.$$

Then, by straight forward computation,

$$\begin{aligned}\nabla^2 g(\mathbf{y})\Big[\Delta\mathbf{y}, \Delta\mathbf{y}\Big] = &\nabla^{k+2}\sigma(y_0)\Big[\Delta y_0, \Delta y_0, y_1, \ldots, y_k, \bullet\Big] \\ &+2\sum_{j=1}^{k} \nabla^{k+1}\sigma(y_0)\Big[\Delta y_0, y_1, \ldots, \Delta y_j, \ldots, y_k, \bullet\Big] \\ &+2\sum_{1\le i<j\le k} \nabla^{k}\sigma(y_0)\Big[y_1, \ldots, \Delta y_i, \ldots, \Delta y_j, \ldots, y_k, \bullet\Big].\end{aligned}$$

It is well-known that

$$g(\mathbf{y} + \Delta\mathbf{y}) - g(\mathbf{y}) - \nabla g(\mathbf{y})\Big[\Delta\mathbf{y}\Big] = \int_0^1 d\theta' \int_0^{\theta'} d\theta \nabla^2 g(\mathbf{y} + \theta\Delta\mathbf{y})\Big[\Delta\mathbf{y}, \Delta\mathbf{y}\Big].$$

By letting $y_j = \phi_s^{i_j}$ and $\Delta y_j = \phi_u^{i_j} - \phi_s^{i_j} = \overline{\phi^{i_j}}_1(s,u)$ for $j = 0, 1, \ldots, k$, we see from (4.24) that

$$I_1 + I_2 + I_4 = -\int_0^1 d\theta' \int_0^{\theta'} d\theta \nabla^2 g(\mathbf{y} + \theta\Delta\mathbf{y})\Big[\Delta\mathbf{y}, \Delta\mathbf{y}\Big]\Big[\overline{h}_1(u,t)\Big].$$

Then we have

$$|I_1 + I_2 + I_4| \le c\,(1+\kappa)^n \omega_3(s,t)^{3/p}$$

for some constant $c > 0$ independent of $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \le r_0$. Therefore, we see that

$$\begin{aligned}|J_{s,t} - J_{s,u} - J_{u,t}| \le &\sum_{k=1}^{n-1} \sum_{(i_1,\ldots,i_k)\in S_k^{n-1}} \frac{1}{k!}\,|J_{s,t}^{i_1,\ldots,i_k} - J_{s,u}^{i_1,\ldots,i_k} - J_{u,t}^{i_1,\ldots,i_k}| \\ \le &\, c\,(1+\kappa)^n \omega_3(s,t)^{3/p},\end{aligned} \tag{4.25}$$

where c is a positive constant independent of $s < u < t$, $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \le r_0$.

Then it is a routine to see from (4.25) that

$$|J_{s,t}(D) - J_{s,t}| \le c\,2^{3/p}\zeta(p/3)(1+\kappa)^n \omega_3(s,t)^{3/p} \tag{4.26}$$

for any partition D of the interval $[s,t]$, where ζ denotes the ζ-function. Combining this with (4.23), we have

$$
\begin{aligned}
|k_t - k_s| &= \Big|\lim_{|D|\to 0} J_{s,t}(D)\Big| \\
&\le c\,(1+\kappa)^n\Big(\omega_3(s,t)^{1/p} + \omega_3(s,t)^{2/p} + \omega_3(s,t)^{3/p}\Big) \\
&\le c'\,(1+\kappa)^n \omega_3(s,t)^{1/p}.
\end{aligned} \tag{4.27}
$$

Next we estimate the $p/2$-variation norm of $(s,t) \mapsto \int_s^t (k_u - k_s) \otimes dh_u$. For each $(i_1, \ldots, i_k) \in S_k^{n-1}$ and $s < t$, we set

$$
\begin{aligned}
K_{s,t}^{i_1,\ldots,i_k} &:= \Big(\nabla^k \sigma(\phi_s)\Big[\phi_s^{i_1}, \ldots, \phi_s^{i_k}, \bullet\Big] \otimes \mathrm{Id}_X\Big)\Big[\overline{h}_2(s,t)\Big] \in Y \otimes X, \\
K_{s,t} &:= \sum_{k=1}^{n-1} \sum_{(i_1,\ldots,i_k)\in S_k^{n-1}} \frac{1}{k!}\, K_{s,t}^{i_1,\ldots,i_k}.
\end{aligned}
$$

Clearly, it holds that

$$
|K_{s,t}| \le c\,(1+\kappa)^{n+1}\omega_3(s,t)^{2/p}
$$

for some $c > 0$ independent of $s < t$, $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \le r_0$.

Let D be a partition of the interval $[s,t]$ as above. We set $K_{s,t}(D)$ by

$$
K_{s,t}(D) := \sum_{i=1}^{N}\Big(K_{t_{i-1},t_i} + \overline{k}_1(t_0, t_{i-1}) \otimes \overline{h}_1(t_{i-1}, t_i)\Big). \tag{4.28}
$$

It is well-known that $\lim_{|D|\to 0} K_{s,t}(D) = \int_s^t (k_u - k_s) \otimes dh_u$ holds.

For each $(i_1, \ldots, i_k)$ in the above sum and $s < u < t$, we see from **P(1)**,...,**P(n-1)** and Chen's identity that

$$
\begin{aligned}
&\Big|K_{s,t}^{i_1,\ldots,i_k} - K_{s,u}^{i_1,\ldots,i_k} - K_{u,t}^{i_1,\ldots,i_k} - J_{s,u}^{i_1,\ldots,i_k} \otimes \overline{h}_1(u,t)\Big| \\
&\le \Big|\Big(\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_k},\bullet\Big] - \nabla^k\sigma(\phi_u)\Big[\phi_u^{i_1},\ldots,\phi_u^{i_k},\bullet\Big]\Big) \otimes \mathrm{Id}_X\Big[\overline{h}_2(u,t)\Big]\Big| \\
&\quad + \Big|\Big(\nabla^{k+1}\sigma(\phi_s)\Big[\bullet, \phi_s^{i_1},\ldots,\phi_s^{i_k},\bullet\Big]\Big[(\phi\cdot dh)(s,t)\Big]\Big) \otimes \overline{h}_1(u,t)\Big| \\
&\quad + \sum_{j=1}^{k}\Big|\Big(\nabla^k\sigma(\phi_s)\Big[\phi_s^{i_1},\ldots,\phi_s^{i_{j-1}},\bullet,\phi_s^{i_{j+1}},\ldots,\phi_s^{i_k},\bullet\Big]\Big[(\phi^{i_j}\cdot dh)(s,t)\Big]\Big) \otimes \overline{h}_1(u,t)\Big| \\
&\le c\,(1+\kappa)^{n+1}\omega_3(s,t)^{3/p}.
\end{aligned} \tag{4.29}
$$

Summing up with respect to $(i_1, \ldots, i_k)$,

$$
\Big|K_{s,t} - K_{s,u} - K_{u,t} - J_{s,u} \otimes \overline{h}_1(u,t)\Big| \le c\,(1+\kappa)^{n+1}\omega_3(s,t)^{3/p}, \tag{4.30}
$$

where $c > 0$ is independent of $s < u < t$, $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$.

Then it is a routine to see from (4.26), (4.28) and (4.30) that

$$|K_{s,t}(D) - K_{s,t}| \leq c\, 2^{3/p}\zeta(p/3)(1+\kappa)^{n+1}\omega_3(s,t)^{3/p}.$$

Therefore, we have

$$\Big|\int_s^t (k_u - k_s) \otimes dh_u\Big| = \Big|\lim_{|D|\to 0} K_{s,t}(D)\Big| \leq c\,(1+\kappa)^{n+1}\omega_3(s,t)^{2/p}, \qquad (4.31)$$

where $c > 0$ is independent of $s < t$, $\overline{h}$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$.

From (4.3) we see that

$$d\phi_t^n - \nabla\sigma(\phi_t)\Big[\phi_t^n, d\gamma_t\Big] - a(0)\cdot\nabla b(\phi_t)\Big[\phi_t^n\Big]dt = dk_t + d\tilde{k}_t,$$

where $\tilde{k}$ satisfies that

$$\|\tilde{k}\|_{1,[s,t]} \leq c\,(1+\kappa)^n\Big\{\|\gamma\|_{1,[s,t]} + (t-s)\Big\}.$$

Therefore, we see that

$$\begin{aligned} |(k+\tilde{k})_t - (k+\tilde{k})_s| &\leq c\,(1+\kappa)^n\omega_3(s,t)^{1/p}, \\ \Big|\int_s^t \Big((k+\tilde{k})_t - (k+\tilde{k})_s\Big)\otimes dh_u\Big| &\leq c\,(1+\kappa)^{n+1}\omega_3(s,t)^{2/p}. \end{aligned} \qquad (4.32)$$

Here, $c > 0$ is independent of $s < t$, $\overline{h} \in G\Omega_p(X)$ and $\gamma \in \mathrm{BV}(X)$ with $\|\gamma\|_1 \leq r_0$. Note that we have seen that $(h,\gamma) \mapsto (h,\phi,\phi_1,\ldots,\phi_{n-1},k+\tilde{k})$ extends to a continuous map from $G\Omega_p(X)\times\mathrm{BV}(X)$ to $G\Omega_p(V_{n+1})$, which maps a bounded subset to a bounded subset in $G\Omega_p(V_{n+1})$.

Finally, let $M(\gamma)_t$ be as in Lemma 4.2 and set $\hat{M}(\gamma)_t = \mathrm{Id}_X \oplus \mathrm{Id}_{Y^{n-1}} \oplus M(\gamma)_t$ and a control function

$$\omega_4(s,t) := \omega_3(s,t) + \|\hat{M}(\gamma)\|_{1,[s,t]} + \|\hat{M}^{-1}(\gamma)\|_{1,[s,t]}.$$

Then by using Lemma 2.1 for $\hat{M}(\gamma)_t$ and $(h,\phi,\phi_1,\ldots,\phi_{n-1},k+\tilde{k})$, we can draw the same argument as in Step 1. Hence, we obtain **P(n)**. This completes the proof. ∎

By combining Lemma 4.3 and Theorem 4.4, we can easily see

Corollary 4.5 *Let $\chi_t = \chi(h;\gamma)_t, \psi_t = \psi(h,h;\gamma)_t$ and $Y = Y(h;\gamma)_t$ be as in Lemma 4.3. Then the maps $(h,\gamma) \in \mathrm{BV}(X)\times\mathrm{BV}(X) \mapsto \chi(h;\gamma), \psi(h,h;\gamma), Y(h;\gamma) \in$*

$\mathrm{BV}(Y)$ *extend to continuous maps from* $G\Omega_p(X) \times \mathrm{BV}(X)$ *to* $G\Omega_p(Y)$. *Moreover, there exists a positive constant* $c = c(\|\gamma\|_1)$ *independent of* $\overline{h} \in G\Omega_p(X)$ *and* $\gamma \in \mathrm{BV}(X)$ *such that*

$$\|\chi(\overline{h};\gamma)_1\|_p + \|Y(\overline{h};\gamma)_1\|_p \le c\Big(1+\xi(\overline{h})\Big), \qquad \|\psi(\overline{h},\overline{h};\gamma)_1\|_p \le c\Big(1+\xi(\overline{h})\Big)^2. \tag{4.33}$$

5 Estimate for remainder terms

In this section, we estimate the remainder terms of the Taylor expansion for the differential equation (3.2). Let $\gamma \in \mathrm{BV}(X)$. For $\varepsilon \in (0,1]$ and $h \in \mathrm{BV}(X)$, we define $R^n_\varepsilon = R^n_\varepsilon(h) = R^n_\varepsilon(h,\gamma), n \in \mathbb{N}$, by

$$R^n_\varepsilon(h,\gamma) := \Phi\Big(\iota(\overline{\gamma+\varepsilon h}, \lambda^\varepsilon)\Big)_1 - \phi - \sum_{j=1}^{n-1} \varepsilon^j \phi^j(h,\gamma).$$

The following theorem gives an estimate of the remainder term R^n_ε defined above.

Theorem 5.1 *For* $n \in \mathbb{N}$, $\varepsilon \in (0,1]$, *and* $h,\gamma \in \mathrm{BV}(X)$, *let* $R^n_\varepsilon = R^n_\varepsilon(h,\gamma)$ *be as above. Let* r_0 *and* r_1 *be any positive constants. Then, there is a positive constant* $c = c(r_0, r_1)$ *such that*

$$\|\overline{R^n_\varepsilon(h,\gamma)}_1\|_p \le c\Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)^n = c\varepsilon^n(1+\xi(\overline{h}))^n \tag{5.1}$$

holds for all γ *with* $\|\overline{\gamma}\|_1 \le r_0$ *and* h *with* $\xi(\overline{\varepsilon h}) \le r_1$. *Moreover, for fixed* ε, $(h,\gamma) \in \mathrm{BV}(X) \times \mathrm{BV}(X) \mapsto R^n_\varepsilon(h,\gamma) \in \mathrm{BV}(Y)$ *extends to a continuous map from* $G\Omega_p(X) \times \mathrm{BV}(X)$ *to* $G\Omega_p(Y)$. *(We denote it again by* $\overline{x} \in G\Omega_p(X) \times \mathrm{BV}(X) \mapsto R^n_\varepsilon(\overline{x}) = R^n_\varepsilon(\overline{x},\gamma) \in G\Omega_p(Y)$.*)*

Proof. We show the estimate (5.1) by mathematical induction. We divide the proof into several steps. Throughout the proof, we set $\kappa := \xi(\overline{\varepsilon h})$ for simplicity. We also note that $\overline{\varepsilon h}_i = \varepsilon^i \overline{h}_i$ holds for $i = 1,2$.

Step 1: We consider the case $n = 1$. Set a control function ω by

$$\begin{aligned}\omega(s,t) := {} & \|\overline{\gamma}_1\|_{1,[s,t]} + \kappa^{-p}\|\overline{\varepsilon h}_1\|^p_{p,[s,t]} + \kappa^{-p}\|\overline{\varepsilon h}_2\|^{p/2}_{p/2,[s,t]} \\ & + \|\overline{\lambda^\varepsilon}_1\|_{1,[s,t]} + \|\overline{\lambda^0}_1\|_{1,[s,t]} + \varepsilon^{-1}\|\overline{\lambda^\varepsilon}_1 - \overline{\lambda^0}_1\|_{1,[s,t]}, \quad 0 \le s \le t \le 1.\end{aligned}$$

Then it is easy to check that there exists a positive constant $c = c(r_0, r_1) > 0$ such that $\omega(0,1) \le c$, and that, for $j = 1,2$,

$$\left|\iota(\overline{\gamma}, \lambda^0)_j(s,t)\right| + \left|\iota(\overline{\gamma + \varepsilon h}, \lambda^\varepsilon)_j(s,t)\right| \le c\,\omega(s,t)^{j/p},$$
$$\left|\iota(\overline{\gamma}, \lambda^0)_j(s,t) - \iota(\overline{\gamma + \varepsilon h}, \lambda^\varepsilon)_j(s,t)\right| \le c\,(\varepsilon+\kappa)\omega(s,t)^{j/p}$$

hold. Then by Lyons' continuity theorem (Theorem 6.2.1 in Lyons and Qian [15]), it holds that there exists a positive constant $c' = c'(r_0, r_1)$ such that

$$\left|\overline{z}\Big(\iota(\overline{\gamma}, \lambda^0)\Big)_j(s,t) - \overline{z}\Big(\iota(\overline{\gamma + \varepsilon h}, \lambda^\varepsilon)\Big)_j(s,t)\right| \le c'\,(\varepsilon+\kappa)\omega(s,t)^{j/p} \quad \text{for } j = 1, 2. \tag{5.2}$$

Here, $\overline{z}(\overline{x}) \in G\Omega_p(\tilde{X} \oplus Y)$ is the unique solution of the differential equation (3.1). Note that the Y-component of $\overline{z}(\overline{x})$ coincides with $\Phi(\overline{x})$. The Y-component of the above inequality (5.2) immediately implies that there exists a positive constant $c = c(r_0, r_1)$ such that

$$\left|\overline{R^1_\varepsilon(h)}_1(s,t)\right| \le c\,\Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)\omega(s,t)^{1/p}$$

holds. Therefore we obtain our desired estimate $\|\overline{R^1_\varepsilon(h)}_1\|_p \le c\,\Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)$ for some positive constant $c = c(r_0, r_1)$.

Step 2: Next we prove that

$$\varepsilon\left|\Big(R^1_\varepsilon(h)\cdot dh\Big)(s,t)\right| = \left|\int_s^t \overline{R^1_\varepsilon(h)}_1(s,u)\otimes \varepsilon dh_u\right| \le (\varepsilon+\kappa)^2\omega(s,t)^{2/p} \tag{5.3}$$

for some control function ω such that $\omega(0,1) \le c = c(r_0, r_1)$. In what follows, we denote $\Phi\Big(\iota(\overline{\gamma + \varepsilon h}, \lambda^\varepsilon)\Big)_1$ by $\phi^\varepsilon = \phi^\varepsilon(h, \gamma)$ for simplicity of notation. Then by considering $Y \otimes X$-component of $(\tilde{X} \oplus Y)^{\otimes 2} = ((X \oplus \mathbb{R}^N) \oplus Y)^{\otimes 2}$, we see from (5.2) that

$$\left|\int_s^t (\phi^\varepsilon_u - \phi^\varepsilon_s)\otimes d(\gamma_u + \varepsilon h_u) - \int_s^t (\phi_u - \phi_s)\otimes d\gamma_u\right| \le c\,(\varepsilon+\kappa)\omega(s,t)^{2/p}. \tag{5.4}$$

From the estimate for $R^1_\varepsilon(h) = \phi^\varepsilon - \phi$, we see that

$$\left|\int_s^t (\phi^\varepsilon_u - \phi^\varepsilon_s)\otimes d\gamma_u - \int_s^t (\phi_u - \phi_s)\otimes d\gamma_u\right| \le c\,(\varepsilon+\kappa)\omega(s,t)^{2/p}. \tag{5.5}$$

(5.4) and (5.5) imply that

$$\left|\int_s^t (\phi^\varepsilon_u - \phi^\varepsilon_s)\otimes \varepsilon dh_u\right| \le c\,(\varepsilon+\kappa)\omega(s,t)^{2/p}. \tag{5.6}$$

From (3.2) we see that

$$\int_s^t \overline{R^1_\varepsilon(h)}_1(s,u)\otimes \varepsilon dh_u = \int_s^t \int_s^u \Big(\sigma(\phi^\varepsilon_v)-\sigma(\phi_v)\Big)d\gamma_v \otimes \varepsilon dh_u + \int_s^t \int_s^u \Big(a(\varepsilon)\cdot b(\phi^\varepsilon_v)-a(0)\cdot b(\phi_v)\Big)dv\otimes \varepsilon dh_u + \int_s^t \int_s^u \sigma(\phi^\varepsilon_v)\varepsilon dh_v \otimes \varepsilon dh_u. \tag{5.7}$$

From Step 1 and the fact that $1+1/p>2/p$, the first and second terms on the right hand side of (5.7), regarded as Young integrals, are easily dominated by $c(\varepsilon+\kappa)^2\omega(s,t)^{2/p}$, respectively.

Next we fix the interval $[s,t]$ and consider $\int_s^t\int_s^u \sigma(\phi^\varepsilon_v)\varepsilon dh_v\otimes \varepsilon dh_u$. For $s\le t$, we define $J_{s,t}=(\sigma(\phi^\varepsilon_s)\otimes \mathrm{Id}_X)[\varepsilon^2\overline{h}_2(s,t)]$. By straight forward computation, $|J_{s,t}|\le c\kappa^2\omega(s,t)^{2/p}$ and, for $s<u<t$,

$$\begin{aligned}&|J_{s,t}-J_{s,u}-J_{u,t}-\sigma(\phi^\varepsilon_s)\overline{\varepsilon h}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)|\\ &=\Big|\Big((\sigma(\phi^\varepsilon_u)-\sigma(\phi^\varepsilon_s)\Big)\otimes \mathrm{Id}_X)[\overline{\varepsilon h}_2(u,t)]\Big|\le c\kappa^2\omega(s,t)^{3/p}.\end{aligned} \tag{5.8}$$

Similarly, by using the results in Step 1, we easily see that

$$\begin{aligned}&\Big|\int_s^t \sigma(\phi^\varepsilon_v)\varepsilon dh_v-\sigma(\phi^\varepsilon_s)\overline{\varepsilon h}_1(s,t)\Big|\\ &\le\Big|\int_s^t \sigma(\phi^\varepsilon_v)\varepsilon dh_v-\sigma(\phi^\varepsilon_s)\overline{\varepsilon h}_1(s,t)-\nabla\sigma(\phi^\varepsilon_s)[(\phi^\varepsilon\cdot\varepsilon dh)(s,t)]\Big|\\ &\qquad+\Big|\nabla\sigma(\phi^\varepsilon_s)[(\phi^\varepsilon\cdot\varepsilon dh)(s,t)]\Big|\\ &\le c\kappa\omega(s,t)^{3/p}+c\kappa\omega(s,t)^{2/p}\le c\kappa\omega(s,t)^{2/p}.\end{aligned} \tag{5.9}$$

Here, we used the integration theory for first level paths.

For any partition $D=\{s=t_0<\cdots<t_N=t\}$ of $[s,t]$, we set

$$J_{s,t}(D)=\sum_{i=1}^N\Big(J_{t_{i-1},t_i}+\int_s^{t_{i-1}}\sigma(\phi^\varepsilon_v)\varepsilon dh_v\otimes\overline{\varepsilon h}_1(t_{i-1},t_i)\Big). \tag{5.10}$$

It is well-known that $\lim_{|D|\to 0}J_{s,t}(D)=\int_s^t\int_s^u\sigma(\phi^\varepsilon_v)\varepsilon dh_v\otimes\varepsilon dh_u$. Let $t_i\in D$ ($i\ne 0,N$) be as above. From (5.8)– (5.10),

$$\begin{aligned}&|J_{s,t}(D\setminus t_i)-J_{s,t}(D)|\\ &\le\Big|\int_{t_{i-1}}^{t_i}\sigma(\phi^\varepsilon_v)\varepsilon dh_v-\sigma(\phi^\varepsilon_{t_{i-1}})\overline{\varepsilon h}_1(t_{i-1},t_i)\Big|\cdot|\overline{\varepsilon h}_1(t_i,t_{i+1})|\\ &\qquad+|J_{t_{i-1},t_{i+1}}-J_{t_{i-1},t_i}-J_{t_i,t_{i+1}}-\sigma(\phi^\varepsilon_{t_{i-1}})\overline{\varepsilon h}_1(t_{i-1},t_i)\otimes\overline{\varepsilon h}_1(t_i,t_{i+1})|\\ &\le c\kappa^2\omega(t_{i-1},t_{i+1})^{3/p}\end{aligned}$$

By a routine argument, we see that

$$|J_{s,t} - J_{s,t}(D)| \leq c2^{3/p}\zeta(3/p)\kappa^2\omega(s,t)^{3/p},$$

which implies (5.3).

Step 3: We denote by $(R^n_\varepsilon \cdot dh)(s,t) := \int_s^t \Big(R^n_\varepsilon(h)_u - R^n_\varepsilon(h)_s\Big) \otimes dh_u$ and $V_n = X \oplus Y^{n+1}$ as before. Here we aim to prove the following proposition:

Proposition Q(n): *The map*

$$(h,\gamma) \mapsto \Big(\varepsilon h, \Phi\Big(\iota(\gamma+\varepsilon h, \lambda^\varepsilon)\Big)_1, \phi, \phi_1, \ldots, \phi_{n-1}, R^n_\varepsilon\Big)$$

extends to a continuous map from $G\Omega_p(X) \times \mathrm{BV}(X)$ to $G\Omega_p(V_{n+2})$. Moreover,

$$\begin{aligned} \Big|R^n_\varepsilon(h)_t - R^n_\varepsilon(h)_s\Big| &\leq \Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)^n \omega(s,t)^{1/p}, \\ \Big|\int_s^t \Big(R^n_\varepsilon(h)_u - R^n_\varepsilon(h)_s\Big) \otimes \varepsilon dh_u\Big| &\leq \Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)^{n+1} \omega(s,t)^{2/p} \end{aligned} \tag{5.11}$$

hold for all $h \in \mathrm{BV}(X)$ with $\xi(\overline{\varepsilon h}) \leq r_1$ and $\gamma \in \mathrm{BV}(X)$ with $\|\overline{\gamma}\|_1 \leq r_0$. Here, $\omega(0,1)$ is dominated by a positive constant $c = c(r_0, r_1)$ which depends only on r_0 and r_1.

We will show **Q(n+1)** under assuming Propositions **Q(1)**,..., **Q(n)**. We also set $\eta^{n+1}_\varepsilon = \eta^{n+1}_\varepsilon(h)$ by

$$\begin{aligned} d\eta^{n+1}_\varepsilon(h)_t :=& \,\sigma(\phi^\varepsilon_t)\varepsilon dh_t - \sigma(\phi_t)\varepsilon dh_t \\ &- \sum_{k=1}^{n-1} \sum_{(i_1,\ldots,i_k)\in \cup_{j=1}^{n-1} S^j_k} \frac{1}{k!}\nabla^k\sigma(\phi_t)\Big[\varepsilon^{i_1}\phi^{i_1}_t, \ldots, \varepsilon^{i_k}\phi^{i_k}_t, \varepsilon dh_t\Big]. \end{aligned}$$

Since we assume **Q(n)**, the right-hand side can be regarded as integral in the rough path sense. We set

$$
\begin{aligned}
J_{s,t} := &\Big(\sigma(\phi^\varepsilon_s) - \sigma(\phi_s)\Big)\varepsilon\overline{h}_1(s,t) \\
&-\sum_{k=1}^{n-1} \sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1} S^j_k} \frac{1}{k!}\nabla^k\sigma(\phi_s)\Big[\varepsilon^{i_1}\phi^{i_1}_s,\dots,\varepsilon^{i_k}\phi^{i_k}_s,\varepsilon\overline{h}_1(s,t)\Big] \\
&+\nabla\sigma(\phi^\varepsilon_s)\Big[(\phi^\varepsilon\cdot\varepsilon dh)(s,t)\Big] - \nabla\sigma(\phi_s)\Big[(\phi\cdot\varepsilon dh)(s,t)\Big] \\
&-\sum_{k=1}^{n-1} \sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1} S^j_k} \frac{1}{k!}\nabla^{k+1}\sigma(\phi_s)\Big[\,\bullet\,,\varepsilon^{i_1}\phi^{i_1}_s,\dots,\varepsilon^{i_k}\phi^{i_k}_s,\,\bullet\,\Big]\Big[(\phi\cdot\varepsilon dh)(s,t)\Big] \\
&-\sum_{k=1}^{n-1} \sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1} S^j_k} \sum_{j=1}^{k}\frac{1}{k!} \\
&\quad\times\nabla^k\sigma(\phi_s)\Big[\varepsilon^{i_1}\phi^{i_1}_s,\dots,\,\bullet\,,\dots,\varepsilon^{i_k}\phi^{i_k}_s,\,\bullet\,\Big]\Big[\varepsilon^{i_j}(\phi^{i_j}\cdot\varepsilon dh)(s,t)\Big], \quad s<t.
\end{aligned}
\tag{5.12}
$$

We will show that $|J_{s,t}| \le c\Big(\varepsilon + \xi(\overline{\varepsilon h})\Big)^{n+1}\Big(\omega(s,t)^{1/p} + \omega(s,t)^{2/p}\Big)$ for some control function ω such that $\omega(0,1) \le c = c(r_0, r_1)$. First we will consider the first and the second terms on the right-hand side of (5.12). By the Taylor expansion of σ at ϕ_s,

$$
\begin{aligned}
&\Big(\sigma(\phi^\varepsilon_s) - \sigma(\phi_s)\Big)\varepsilon\overline{h}_1(s,t) \\
&= \sum_{k=1}^{n-1}\frac{1}{k!}\nabla^k\sigma(\phi_s)\Big[R^\varepsilon_1(h)_s,\dots,R^\varepsilon_1(h)_s,\overline{\varepsilon h}_1(s,t)\Big] \\
&\quad+\int_0^1 d\theta_1\cdots\int_0^{\theta_{n-1}} d\theta_n \nabla^n\sigma(\phi_s + \theta_n R^\varepsilon_1(s))\Big[R^\varepsilon_1(h)_s,\dots,R^\varepsilon_1(h)_s,\overline{\varepsilon h}_1(s,t)\Big],
\end{aligned}
\tag{5.13}
$$

where

$$
R^\varepsilon_1(h) := \phi^\varepsilon - \phi = \varepsilon\phi^1 + \cdots + \varepsilon^{n-1}\phi^{n-1} + R^\varepsilon_n(h). \tag{5.14}
$$

From the estimates for $R^\varepsilon_1(h)$ and from the boundedness of $\nabla^n\sigma$, we easily see that the second term on the right-hand side of (5.13) is dominated by $c(\varepsilon+\kappa)^{n+1}\omega(s,t)^{1/p}$, where $\kappa = \xi(\overline{\varepsilon h})$. Put (5.14) in the first term on the right-hand side of (5.13) and, then, expand it. In that expansion, terms of order $1,\dots,n$ is exactly the same as the second term on the right-hand side of (5.12). (Here, we say a term is of order k if its absolute value is dominated by $c(\varepsilon+\kappa)^k\omega(s,t)^{1/p}$.) Because of the cancellation, the first and the second term on the right-hand side of (5.12) is dominated by $c(\varepsilon+\kappa)^{n+1}\omega(s,t)^{1/p}$.

Next, we will consider the last three terms on the right-hand side of (5.12). Set $Z = Y\otimes X$ and $\mathbf{f}_1(y,z) := \nabla\sigma(y)[z]$ for $y\in Y$ and $z\in Z$. Clearly, $\mathbf{f}_1 \in C^\infty_{b,loc}(Y\oplus Z, Y)$. It is easy to see that

$$\nabla^k \mathbf{f}_1(y,z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big] = \nabla^{k+1}\sigma(y)\Big[\Delta y,\dots,\Delta y, z\Big] \\ +k\nabla^k\sigma(y)\Big[\Delta y,\dots,\Delta y,\Delta z\Big]. \quad (5.15)$$

It is also easy to see that

$$\mathbf{f}_1(y+\Delta y, z+\Delta z) - \mathbf{f}_1(y,z) = \sum_{k=1}^{n-1}\frac{1}{k!}\nabla^k\mathbf{f}_1(y,z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big] \\ +\int_0^1 d\theta_1\cdots\int_0^{\theta_{n-1}} d\theta_n \nabla^n \mathbf{f}_1(y+\theta_n\Delta y, z+\theta_n\Delta z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big]. \quad (5.16)$$

Now we will use (5.15) with $y=\phi_s$, $\Delta y = \phi^\varepsilon_s - \phi_s = R^1_\varepsilon(h)$, $z = (\phi\cdot\varepsilon dh)(s,t)$, and

$$\Delta z = (\phi^\varepsilon\cdot\varepsilon dh)(s,t) - (\phi\cdot\varepsilon dh)(s,t) = (R^1_\varepsilon(h)\cdot\varepsilon dh)(s,t).$$

Note that Δz is of order 2 by (5.3) while Δy and z are of order 1. Therefore, the second term on the right hand side of (5.16) is dominated by $c(\varepsilon+\kappa)^{n+1}\omega(s,t)^{2/p}$.

We put (5.14) and

$$\Delta z = (R^\varepsilon_1(h)\cdot\varepsilon dh)(s,t) = \Big(\Big(\varepsilon\phi^1+\cdots+\varepsilon^{n-1}\phi^{n-1}+R^\varepsilon_n(h)\Big)\cdot\varepsilon dh\Big)(s,t), \quad (5.17)$$

in the first term on the right hand side of (5.16) and, then, expand it. We will use $\mathbf{P(n)}$, $\mathbf{Q(k)}$, $1\le k\le n$, and the symmetry of $\nabla^k\sigma$. In that expansion, terms of order $1,\dots,n$, is exactly the same as the fourth and the fifth terms on the right-hand side of (5.12). Therefore, the third, the fourth, and the fifth terms on the right-hand side of (5.12) are dominated by $c(\varepsilon+\kappa)^{n+1}\omega(s,t)^{2/p}$. Thus, we have obtained the desired estimates for $|J_{s,t}|$.

Now we show that, for all $s<u<t$,

$$|J_{s,t}-J_{s,u}-J_{u,t}| \le c\Big(\varepsilon+\xi(\overline{\varepsilon h})\Big)^{n+1}\omega(s,t)^{3/p}$$

holds for some control function ω such that $\omega(0,1)\le c = c(r_0,r_1)$.

Here we apply (2.12) to (5.12). In this case, (minus of) the first term on the right-hand side of (2.12) is given by the integration $\int_0^1 d\eta' \int_0^{\eta'} d\eta$ of the following quantity:

$$
\begin{aligned}
&\nabla^2\sigma(\phi^{\varepsilon}_{s,u:\eta})\Big[\overline{\phi^{\varepsilon}}_1(s,u)\otimes\overline{\phi^{\varepsilon}}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)\Big]-\nabla^2\sigma(\phi_{s,u:\eta})\Big[\overline{\phi}_1(s,u)\otimes\overline{\phi}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)\Big]\\
&\quad-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\frac{1}{k!}\nabla^{k+2}\sigma(\phi_{s,u:\eta})\Big[\bullet,\bullet,\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\qquad\qquad\Big[\overline{\phi}_1(s,u)\otimes\overline{\phi}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)\Big]\\
&\quad-2\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\sum_{j=1}^{k}\frac{1}{k!}\nabla^{k+1}\sigma(\phi_{s,u:\eta})\Big[\bullet,\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\bullet,\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\qquad\qquad\Big[\overline{\phi}_1(s,u)\otimes\varepsilon^{i_j}\overline{\phi^{i_j}}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)\Big]\\
&\quad-2\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\sum_{1\le j<l\le k}\frac{1}{k!}\nabla^{k}\sigma(\phi_{s,u:\eta})\Big[\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\bullet,\dots,\bullet,\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\qquad\qquad\Big[\varepsilon^{i_j}\overline{\phi^{i_j}}_1(s,u)\otimes\varepsilon^{i_l}\overline{\phi^{i_l}}_1(s,u)\otimes\overline{\varepsilon h}_1(u,t)\Big],
\end{aligned}
\tag{5.18}
$$

where $\phi^{i_k}_{s,u:\eta}$ is a shorthand for $\phi^{i_k}_s+\eta\overline{\phi}^{i_k}_1(s,u)$, etc. Similarly, (minus of) the second term on the right-hand side of (2.12) is given by the integration $\int_0^1 d\eta$ of the following quantity:

$$
\begin{aligned}
&\nabla^2\sigma(\phi^{\varepsilon}_{s,u:\eta})\Big[\overline{\phi^{\varepsilon}}_1(s,u)\otimes(\phi^{\varepsilon}\cdot\varepsilon dh)(u,t)\Big]-\nabla^2\sigma(\phi_{s,u:\eta})\Big[\overline{\phi}_1(s,u)\otimes(\phi\cdot\varepsilon dh)(u,t)\Big]\\
&\quad-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\frac{1}{k!}\nabla^{k+2}\sigma(\phi_{s,u:\eta})\Big[\bullet,\bullet,\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\qquad\qquad\Big[\overline{\phi}_1(s,u)\otimes(\phi\cdot\varepsilon dh)(u,t)\Big]\\
&\quad-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\sum_{j=1}^{k}\frac{1}{k!}\nabla^{k+1}\sigma(\phi_{s,u:\eta})\Big[\bullet,\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\bullet,\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\Big[\overline{\phi}_1(s,u)\otimes(\varepsilon^{i_j}\phi^{i_j}\cdot\varepsilon dh)(u,t)+\varepsilon^{i_j}\overline{\phi^{i_j}}_1(s,u)\otimes(\phi\cdot\varepsilon dh)(u,t)\Big]\\
&\quad-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\sum_{1\le j<l\le k}\frac{1}{k!}\nabla^{k}\sigma(\phi_{s,u:\eta})\Big[\varepsilon^{i_1}\phi^{i_1}_{s,u:\eta},\dots,\bullet,\dots,\bullet,\dots,\varepsilon^{i_k}\phi^{i_k}_{s,u:\eta},\bullet\Big]\\
&\qquad\Big[\varepsilon^{i_j}\overline{\phi^{i_j}}_1(s,u)\otimes(\varepsilon^{i_l}\phi^{i_l}\cdot\varepsilon dh)(u,t)+\varepsilon^{i_l}\overline{\phi^{i_l}}_1(s,u)\otimes(\varepsilon^{i_j}\phi^{i_j}\cdot\varepsilon dh)(u,t)\Big].
\end{aligned}
\tag{5.19}
$$

Now we will show that (5.18) and (5.19) are dominated by $c(\varepsilon+\kappa)^{n+1}\omega(s,t)^{3/p}$. Here, we note that neither c nor $\omega(0,1)$ must depend on η. For $y,z\in Y$, we set $\mathbf{f}_2\in C^{\infty}_{b,loc}(Y^2,L(X,Y))$ by $\mathbf{f}_2(y,z):=\nabla^2\sigma(y)\Big[z\otimes z,\bullet\Big]$.

Then for $k\in\mathbb{N}$, we have

$$
\begin{aligned}
&\nabla^k \mathbf{f}_2(y,z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big] \\
&= \nabla^{k+2}\sigma(y)\Big[\Delta y,\dots,\Delta y, z\otimes z, \bullet\Big] + 2k\nabla^{k+1}\sigma(y)\Big[\Delta y,\dots,\Delta y, z\otimes \Delta z, \bullet\Big] \\
&\qquad + k(k-1)\nabla^k\sigma(y)\Big[\Delta y,\dots,\Delta y, \Delta z\otimes \Delta z, \bullet,\Big]. \qquad (5.20)
\end{aligned}
$$

By the Taylor expansion for $\mathbf{f}_2$, we see that

$$
\begin{aligned}
&\mathbf{f}_2(y+\Delta y, z+\Delta z) - \mathbf{f}_2(y,z) \\
&= \sum_{k=1}^{n-1}\frac{1}{k!}\nabla^k\mathbf{f}_2(y,z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big] \\
&+ \int_0^1 d\theta_1\cdots\int_0^{\theta_{n-1}} d\theta_n \nabla^n\mathbf{f}_2(y+\theta_n\Delta y, z+\theta_n\Delta z)\Big[\Delta(y,z),\dots,\Delta(y,z)\Big].
\end{aligned}
\qquad (5.21)
$$

As before, we set $y=\phi_{s,u:\eta}$ and $z=\overline{\phi}_1(s,u)$. By recalling Proposition $\mathbf{Q(n)}$, we can write as

$$
\Delta y = \phi^{\varepsilon}_{s,u:\eta} - \phi_{s,u:\eta} = R^1_{\varepsilon}(h)(s,u:\eta) = \varepsilon\phi^1_{s,u:\eta} + \cdots + \varepsilon^{n-1}\phi^{n-1}_{s,u:\eta} + R^n_{\varepsilon}(h)(s,u:\eta),
$$

and

$$
\Delta z = \overline{\phi^{\varepsilon}}_1(s,u) - \overline{\phi}_1(s,u) = \overline{R^1_{\varepsilon}(h)}(s,u) = \varepsilon\overline{\phi^1}(s,u) + \cdots + \varepsilon^{n-1}\overline{\phi^{n-1}}(s,u) + \overline{R^n_{\varepsilon}(h)}(s,u).
$$

Then, from (5.20), the remainder term on the right hand side of (5.21) is dominated as follows:

$$
\begin{aligned}
&\Big|\nabla^n\mathbf{f}_2(y+\theta\Delta y, z+\theta\Delta z)\Big[\Delta(y,z),\dots,\Delta(y,z),\varepsilon\overline{h}_1(u,t)\Big]\Big|_Y \\
&\qquad\qquad\qquad \le c\,(\varepsilon+\kappa)^{n+1}\omega(s,t)^{3/p}, \qquad (5.22)
\end{aligned}
$$

where c is independent of η,θ and of $s<u<t$.

Then, we expand the first term on the right-hand side of (5.21) by using (5.20). In the same way as before, the terms of order $k, 1\le k\le n$, are exactly the same as the third, the fourth, and the fifth terms on on the right-hand side of (5.18). Hence, they cancel each other. Notice that we have repeatedly used the symmetry of $\nabla^k\sigma$.

In a similar way, we will estimate (5.19). In this case we set, for $y,z\in Y$ and $w\in Y\otimes X$, $\mathbf{f}_3(y,z,w) := \nabla^2\sigma(y)\Big[z,w\Big]$. Then, it is easy to see that $\mathbf{f}_3\in C^{\infty}_{b,loc}(Y^2\oplus(Y\otimes X),Y)$.

By using the Taylor expansion for $\mathbf{f}_3$, we obtain in the same way that (5.19) is dominated by $c\,(\varepsilon+\kappa)^{n+1}\omega(s,t)^{3/p}$. In this case, however, we also need the following identity:

$$\begin{aligned}\Delta w &= \Big((\phi^\varepsilon - \phi)\cdot \varepsilon dh\Big)(u,t)\\ &= \Big(R^1_\varepsilon(h)\cdot \varepsilon dh\Big)(u,t)\\ &= (\varepsilon\phi^1\cdot\varepsilon dh)(u,t) + \cdots + (\varepsilon^{n-1}\phi^{n-1}\cdot\varepsilon dh)(u,t) + \Big(R^n_\varepsilon(h)\cdot\varepsilon dh\Big)(u,t),\end{aligned}$$

where we used Proposition **Q(n)** for the last term above.

Then by combining these, we obtain the desired estimate for $|J_{s,t} - J_{s,u} - J_{u,t}|$. By a routine argument, we see that

$$|\eta^{n+1}_{\varepsilon,t} - \eta^{n+1}_{\varepsilon,s}| \le c\,(\varepsilon+\kappa)^{n+1}\omega(s,t)^{1/p},$$

where c and $\omega(0,1)$ are independent of s, t, ε and $h \in \mathrm{BV}(X)$.

Next, we will prove the estimate

$$\Big|(\eta^{n+1}\cdot\varepsilon dh)(s,t)\Big| = \Big|\int_s^t (\eta^{n+1}_{\varepsilon,u} - \eta^{n+1}_{\varepsilon,s})\otimes\varepsilon dh_u\Big| \le c\,(\varepsilon+\kappa)^{n+2}\omega(s,t)^{2/p}, \tag{5.23}$$

where c and $\omega(0,1)$ are independent of s, t, ε, and $h \in \mathrm{BV}(X)$. We set, for $s < t$,

$$\begin{aligned}K_{s,t} := &\Big(\big(\sigma(\phi^\varepsilon_s) - \sigma(\phi_s)\big)\otimes \mathrm{Id}_X\Big)\Big[\,\overline{\varepsilon h}_2(s,t)\Big]\\ &-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S^j_k}\frac{1}{k!}\Big(\nabla^k\sigma(\phi_s)\Big[\varepsilon^{i_1}\phi^{i_1}_s,\dots,\varepsilon^{i_k}\phi^{i_k}_s,\bullet\Big]\otimes\mathrm{Id}_X\Big)\Big[\,\overline{\varepsilon h}_2(s,t)\Big].\end{aligned} \tag{5.24}$$

Then, in the same way as above, we can see that

$$|K_{s,t}| \le c\,(\varepsilon+\kappa)^{n+2}\omega(s,t)^{2/p},$$

where c and $\omega(0,1)$ are independent of s, t, h, ε.

Now, we will estimate $|K_{s,t} - K_{s,u} - K_{u,t} - J_{s,u}\otimes\overline{\varepsilon h}_1(u,t)|$ for $s < u < t$. Here $J_{s,u}$ is defined in (5.12). By using Chen's identity, we obtain that

$$K_{s,t} - K_{s,u} - K_{u,t} - J_{s,u}\otimes\overline{\varepsilon h}_1(u,t) =: I_1 - I_2,$$

where

$$
\begin{aligned}
I_1 = &\Big(\sigma(\phi^\varepsilon_s)\otimes \mathrm{Id}_X - \sigma(\phi^\varepsilon_u)\otimes \mathrm{Id}_X\Big)\Big[\overline{\varepsilon h}_2(u,t)\Big] \\
&-\Big(\sigma(\phi_s)\otimes \mathrm{Id}_X - \sigma(\phi_u)\otimes \mathrm{Id}_X\Big)\Big[\overline{\varepsilon h}_2(u,t)\Big] \\
&-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\frac{1}{k!}\Big(\nabla^k\sigma(\phi_s)\Big[\varepsilon^{i_1}\phi_s^{i_1},\dots,\varepsilon^{i_k}\phi_s^{i_k},\bullet\Big]\otimes \mathrm{Id}_X \\
&\qquad\qquad -\nabla^k\sigma(\phi_u)\Big[\varepsilon^{i_1}\phi_u^{i_1},\dots,\varepsilon^{i_k}\phi_u^{i_k},\bullet\Big]\otimes \mathrm{Id}_X\Big)\Big[\overline{\varepsilon h}_2(u,t)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = &\Big(\nabla\sigma(\phi^\varepsilon_s)\Big[(\phi^\varepsilon\cdot\varepsilon dh)(s,u)\Big]\Big)\otimes\overline{\varepsilon h}_1(u,t) - \Big(\nabla\sigma(\phi_s)\Big[(\phi\cdot\varepsilon dh)(s,u)\Big]\Big)\otimes\overline{\varepsilon h}_1(u,t) \\
&-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\frac{1}{k!}\Big(\nabla^{k+1}\sigma(\phi_s)\Big[\bullet,\varepsilon^{i_1}\phi_s^{i_1},\dots,\varepsilon^{i_k}\phi_s^{i_k},\bullet\Big] \\
&\qquad\qquad \Big[(\phi\cdot\varepsilon dh)(s,u)\Big]\Big)\otimes\overline{\varepsilon h}_1(u,t) \\
&-\sum_{k=1}^{n-1}\sum_{(i_1,\dots,i_k)\in\cup_{j=1}^{n-1}S_k^j}\frac{1}{k!}\sum_{j=1}^{k}\Big(\nabla^k\sigma(\phi_s)\Big[\varepsilon^{i_1}\phi_s^{i_1},\dots,\bullet\dots,\varepsilon^{i_k}\phi_s^{i_k},\bullet\Big] \\
&\qquad\qquad \Big[(\varepsilon^{i_j}\phi^{i_j}\cdot\varepsilon dh)(s,u)\Big]\Big)\otimes\overline{\varepsilon h}_1(u,t).
\end{aligned}
$$

In the same way as before, we can see that the Taylor expansion leads us to

$$
\begin{aligned}
|K_{s,t}-K_{s,u}-K_{u,t}-J_{s,u}\otimes\overline{\varepsilon h}_1(u,t)| &\le |I_1|+|I_2| \\
&\le c\,(\varepsilon+\kappa)^{n+2}\omega(s,t)^{3/p},
\end{aligned}
$$

where c and $\omega(0,1)$ are independent of s,u,t,ε and $h\in \mathrm{BV}(X)$. We have already obtained

$$
|\Big(J_{s,t}-\overline{\eta^{n+1}}_1(s,u)\Big)\otimes\overline{\varepsilon h}_1(u,t)| \le c\,(\varepsilon+\kappa)^{n+2}\omega(s,t)^{4/p}.
$$

For a partition $D=\{s=t_0<\dots<t_N=t\}$ of the interval $[s,t]$, we set

$$
K_{s,t}(D) := \sum_{i=1}^{N}\Big(K_{t_{i-1},t_i}+\overline{\eta^{n+1}}_1(t_0,t_{i-1})\otimes\overline{\varepsilon h}_1(t_{i-1},t_i)\Big). \tag{5.25}
$$

It is well-known that $\lim_{|D|\to 0}K_{s,t}(D)=\int_s^t(\eta_u^{n+1}-\eta_s^{n+1})\otimes\varepsilon dh_u$. From above inequalities, we see from a routine argument that

$$
|K_{s,t}(D)-K_{s,t}| \le c\,2^{3/p}\zeta(p/3)\cdot(\varepsilon+\kappa)^{n+2}\omega(s,t)^{3/p}.
$$

Combining this with the estimate for $|K_{s,t}|$, we obtain that

$$\left|(\eta^{n+1} \cdot \varepsilon dh)(s,t)\right| = \left|\int_s^t (\eta_u^{n+1} - \eta_s^{n+1}) \otimes \varepsilon dh_u\right| \le c\,(\varepsilon+\kappa)^{n+2}\omega(s,t)^{2/p}.$$

Thus, we have shown the desired estimates for $\overline{\eta^{n+1}}_1(s,t)$ and for $(\eta^{n+1} \cdot \varepsilon dh)(s,t)$.

From (4.3) in Definition 4.1, we see that

$$dR_\varepsilon^{n+1}(h)_t - \nabla\sigma(\phi_t)\Big[R_\varepsilon^{n+1}(h)_t, d\gamma_t\Big] - a(0)\cdot\nabla b(\phi_t)\Big[R_\varepsilon^{n+1}(h)_t\Big]dt = d\eta_\varepsilon^{n+1}(t) + dC_t,$$

where C_t is a continuous path of finite total variation, which is explicitly given by

$$\begin{aligned}
dC_t = &\Big(\sigma(\phi_t^\varepsilon) - \sigma(\phi_t)\Big)d\gamma_t - \nabla\sigma(\phi_t)\Big[\phi_t^\varepsilon - \phi_t, d\gamma_t\Big] \\
&- \sum_{k=2}^{n} \sum_{(i_1,\dots,i_k)\in\cup_{l=1}^n S_k^l} \frac{1}{k!}\nabla^k\sigma(\phi_t)\Big[\varepsilon^{i_1}\phi_t^{i_1},\dots,\varepsilon^{i_k}\phi_t^{i_k}, d\gamma_t\Big] \\
&+ a(\varepsilon)\cdot b(\phi_t^\varepsilon)dt - a(0)\cdot b(\phi_t)dt - a(0)\cdot\nabla b(\phi_t)\Big[\phi_t^\varepsilon - \phi_t\Big]dt \\
&- \sum_{k=2}^{n} \sum_{(i_1,\dots,i_k)\in\cup_{l=1}^n S_k^l} \frac{a(0)}{k!}\cdot\nabla^k b(\phi_t)\Big[\varepsilon^{i_1}\phi_t^{i_1},\dots,\varepsilon^{i_k}\phi_t^{i_k}\Big]dt \\
&- \sum_{j=1}^{n-1}\sum_{k=1}^{n-j} \sum_{(i_1,\dots,i_k)\in\cup_{l=1}^{n-j} S_k^l} \frac{\varepsilon^j a^{(j)}(0)}{j!k!}\cdot\nabla^k b(\phi_t)\Big[\varepsilon^{i_1}\phi_t^{i_1},\dots,\varepsilon^{i_k}\phi_t^{i_k}\Big]dt \\
&- \sum_{l=1}^{n} \frac{\varepsilon^l a^{(l)}(0)}{l!}\cdot b(\phi_t)dt.
\end{aligned}$$

Here we note that the terms $\eta_\varepsilon^{n+1}, C$ do not involve ϕ^n. By the Taylor expansion for σ and b, we can easily see as before that

$$|C_t - C_s| \le c\,(\varepsilon+\kappa)^{n+1}\omega(s,t),$$

where c and $\omega(0,1)$ are independent of s,t,ε and $h \in \mathrm{BV}(X)$.

Therefore, $\overline{(\eta_\varepsilon^{n+1}+C)}_1(s,t)$ and $\Big[(\eta_\varepsilon^{n+1}+C)\cdot\varepsilon dh\Big](s,t)$ satisfy the desired estimates. By using Lemma 2.1, $\overline{R_\varepsilon^{n+1}}_1(s,t)$ and $(R_\varepsilon^{n+1}\cdot\varepsilon dh)(s,t)$ satisfy the desired estimates, too. Thus, we have obtained **Q(n+1)**. This completes the proof. ∎

6 Proof of the main result

In this section, we prove our main results. First, we devote ourselves to give the proof of Theorem 3.2 based on the argument of Albeverio, Röckle and Steblovskaya [3]. We consider the precise asymptotic behavior of the following integral as $\varepsilon \searrow 0$:

$$
\begin{aligned}
I(\varepsilon) := &\exp\Big(F_\Lambda(\gamma)/\varepsilon^2\Big)\exp\Big(c(\gamma)/\varepsilon\Big) \\
&\quad\times \int_{G\Omega_p(X)} G\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big)\exp\Big[-\frac{1}{\varepsilon^2}F\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big)\Big]\mathbb{P}_1(d\overline{w}) \\
&= \int_{G\Omega_p(X)} G\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big)\exp\Big[\frac{c(\gamma)}{\varepsilon}-\frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big)\Big]\mathbb{P}_1(d\overline{w}) \qquad (6.1) \\
&= \int_{G\Omega_p(X)\times\overline{\mathcal{H}(\mathbb{R}^N)}} G\Big(\Phi(\iota(\overline{w},\lambda))_1\Big) \\
&\qquad\times\exp\Big[\frac{c(\gamma)}{\varepsilon}-\frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w},\lambda))_1\Big)\Big](\mathbb{P}_\varepsilon\otimes\delta_{\lambda^\varepsilon})(d\overline{w}d\lambda). \qquad (6.2)
\end{aligned}
$$

where $\tilde{F}_\Lambda$ is a non-negative bounded continuous function on $P(Y)$ defined by $\tilde{F}_\Lambda(\cdot) := F(\cdot) - F_\Lambda(\gamma)$. In (6.2), $\overline{\mathcal{H}(\mathbb{R}^N)}$ denotes the closure of $\mathcal{H}(\mathbb{R}^N) := L_0^{2,1}(\mathbb{R}^N)$ in $\mathrm{BV}(\mathbb{R}^N)$. We work on this subspace since it is clearly separable and complete unlike $\mathrm{BV}(\mathbb{R}^N)$.

We divide the proof into several steps.

Step 1: Let $B(\phi,\rho')$ be an open ball centered at ϕ with radius ρ' in $P(Y)$ on which $D^iF,\ i=1,\dots,n+3$, and $D^iG,\ i=1,\dots,n+1$, exist and are bounded. Let $\gamma\in\mathcal{H}$ and $\phi\in\mathcal{H}(Y)$ be as in condition **(H2)**. For $\rho>0$ and $\gamma\in\mathcal{H}$, we denote by $\overline{\gamma}+U_\rho := \{\overline{x+\gamma}\in G\Omega_p(X)\mid \xi(\overline{x})<\rho\}$ and by $(\overline{\gamma}+U_\rho)^c$ the complement set of $\overline{\gamma}+U_\rho$ in $G\Omega_p(X)$. Then by the continuity of the maps $\overline{x}\mapsto\overline{x\pm\gamma}$, $\{\overline{\gamma}+U_\rho\}_{\rho>0}$ forms a fundamental system of neighborhood of $\overline{\gamma}$. Then by the continuity of the Itô map, there exists $\rho>0$ such that $\Phi(\iota(\overline{x},\lambda))_1(0,\cdot)\in B(\phi)$ if $\overline{x}\in\overline{\gamma}+U_\rho$ and $\lambda\in\{\lambda\in\overline{\mathcal{H}(\mathbb{R}^N)}\mid\|\lambda-\lambda^0\|_1<\rho\}$. Here $B(\phi)$ is the neighborhood of ϕ in $P(Y)$ which is denoted in condition **(H3)**. In the following we assume that $\rho>0$ is sufficiently small so that it satisfies the above condition. Later, we will choose $\rho>0$ sufficient small so that the integrability theorem of Fernique-type holds.

We divide the integral $I(\varepsilon)$ into as

$$I(\varepsilon) = I_0(\varepsilon)+I_1(\varepsilon),$$

where $I_0(\varepsilon)$ and $I_1(\varepsilon)$ are defined as (6.1), with $G(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1)$ replaced by $\mathbf{1}_{\overline{\gamma}+U_\rho}(\overline{\varepsilon w})\cdot G(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1)$ and $\mathbf{1}_{(\overline{\gamma}+U_\rho)^c}(\overline{\varepsilon w})\cdot G(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1)$, respectively.

Here we investigate the decay of the integral $I_1(\varepsilon)$. We recall that the family of measures $\{\mathbb{P}_\varepsilon \otimes \delta_{\lambda^\varepsilon}\}_{\varepsilon>0}$ satisfies a large deviation principle on the space $G\Omega_p(X) \times \overline{\mathcal{H}(\mathbb{R}^N)}$ with a good rate function $\tilde{\Lambda}$ given by

$$\tilde{\Lambda}(\overline{x}, \lambda) = \begin{cases} \frac{1}{2}\|h\|_{\mathcal{H}}^2, & \text{if } (\overline{x}, \lambda) = (\overline{h}, \lambda^0) \text{ for some } h \in \mathcal{H}, \\ \infty, & \text{otherwise.} \end{cases}$$

It is a corollary of the large deviation principle for Brownian rough paths. See Lemma 3.9 in [8] for the proof. As a consequence of this large deviation principle, we have the following assertion: there exist positive constants a and ε_0 such that

$$\begin{aligned}
&\exp\Big(- c(\gamma)/\varepsilon\Big) \cdot |I_1(\varepsilon)| \\
&\leq \int_{(\overline{\gamma}+U_\rho)^c} \Big|G\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big| \cdot \exp\Big[- \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big]\mathbb{P}_1(d\overline{w}) \\
&= \|G\|_\infty \int_{(\overline{\gamma}+U_\rho)^c \times \overline{\mathcal{H}(\mathbb{R}^N)}} \exp\Big[- \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{w}, \lambda))_1\Big)\Big](\mathbb{P}_\varepsilon \otimes \delta_{\lambda^\varepsilon})(d\overline{w}d\lambda) \\
&\leq \|G\|_\infty \cdot \exp(-a^2/\varepsilon^2)
\end{aligned} \tag{6.3}$$

holds for all $0 < \varepsilon \leq \varepsilon_0$. See Lemma 8.2 in [7] for the proof. Then we have

$$|I_1(\varepsilon)| \leq c\varepsilon^m, \qquad m \in \mathbb{N},\ 0 < \varepsilon \leq \varepsilon_0, \tag{6.4}$$

where c is a positive constant depending on $\|G\|_\infty, a, c(\gamma)$ and m.

Step 2: Let us now consider the integral $I_0(\varepsilon)$. By using the Cameron-Martin formula for Brownian rough paths (Proposition 2.4), we can calculate $I_0(\varepsilon)$ as

$$
\begin{aligned}
&\int_{G\Omega_p(X)} \mathbf{1}_{\overline{\gamma}+U_\rho}(\overline{\varepsilon w})\, G\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big) \exp\Big[\frac{c(\gamma)}{\varepsilon} - \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w},\lambda^\varepsilon))_1\Big)\Big]\mathbb{P}_1(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathbf{1}_{\overline{\gamma}+U_\rho}(\overline{w+\gamma})\, G\Big(\Phi(\iota(\overline{w+\gamma},\lambda^\varepsilon))_1\Big) \\
&\qquad\qquad \times \exp\Big[\frac{c(\gamma)}{\varepsilon} - \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{w+\gamma},\lambda^\varepsilon))_1\Big)\Big]\mathbb{P}_{\varepsilon,-\gamma}(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathbf{1}_{\overline{\gamma}+U_\rho}(\overline{w+\gamma})\, G\Big(\Phi(\iota(\overline{w+\gamma},\lambda^\varepsilon))_1\Big) \\
&\qquad\qquad \times \exp\Big[\frac{c(\gamma)}{\varepsilon} - \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{w+\gamma},\lambda^\varepsilon))_1\Big)\Big] \exp\Big(\frac{[-\gamma](\overline{w})}{\varepsilon^2} - \frac{\|-\gamma\|^2_{\mathcal{H}}}{2\varepsilon^2}\Big)\mathbb{P}_\varepsilon(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathbf{1}_{\overline{\gamma}+U_\rho}(\overline{\varepsilon w+\gamma})\, G\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big) \\
&\qquad\qquad \times \exp\Big[\frac{c(\gamma)}{\varepsilon} - \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big)\Big] \exp\Big(-\frac{[\gamma](\overline{w})}{\varepsilon} - \frac{\|\gamma\|^2_{\mathcal{H}}}{2\varepsilon^2}\Big)\mathbb{P}_1(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, G\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big) \\
&\qquad\qquad \times \exp\Big[\frac{c(\gamma)}{\varepsilon} - \frac{1}{\varepsilon^2}\tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big)\Big] \exp\Big(-\frac{[\gamma](\overline{w})}{\varepsilon} - \frac{\|\gamma\|^2_{\mathcal{H}}}{2\varepsilon^2}\Big)\mathbb{P}_1(d\overline{w}),
\end{aligned}
\tag{6.5}
$$

where $\mathbb{P}_{\varepsilon,-\gamma}$ is the law of $\mathbb{P}_1$ induced by the map $\overline{w} \mapsto \overline{\varepsilon w - \gamma}$. We note that $\overline{w} \mapsto \Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1$ is the continuous extension of the map $h \mapsto \Psi_\varepsilon(\varepsilon h+\gamma)$. Thus all the integrands are everywhere-defined on $G\Omega_p(X)$ and we may change variables.

Then by using the Taylor expansion for F and Theorems 4.4 and 5.1, we can develop $F\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big)$ into a Taylor expansion with respect to a parameter $\varepsilon \in (0,1]$. For $n \in \mathbb{N}\cup\{0\}$, we have

$$
F\Big(\Phi(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon))_1\Big) = \sum_{m=0}^{n+2} \varepsilon^m J_F^{(m)}(\phi)(\overline{w}) + R_F^{(n+3)}(\varepsilon,\phi)(\overline{w}). \tag{6.6}
$$

Here the functions $J_F^{(m)}(\phi)(\cdot) : G\Omega_p(X) \to \mathbb{R}, m = 0,\dots,n+2$, are given by

$$
\begin{aligned}
J_F^{(0)}(\phi)(\overline{w}) &:= F(\phi), \\
J_F^{(m)}(\phi)(\overline{w}) &:= \sum_{k=1}^{m} \frac{1}{k!}\Big(\sum_{(i_1,\dots,i_k)\in S_k^m} D^k F(\phi)\Big[\phi^{i_1}(\overline{w})_1,\dots,\phi^{i_k}(\overline{w})_1\Big]\Big),\ m = 1,\dots,n+2,
\end{aligned}
$$

where S_k^m and $\phi^i(\overline{w}) := \phi^i(\overline{w},\gamma), i = 1,\dots,n+2$, are defined in Definition 4.1 and Theorem 4.4, respectively.

Besides, the remainder term $R_F^{(n+3)}(\varepsilon,\phi) : G\Omega_p(X) \to \mathbb{R}$ is given by

$$
\begin{aligned}
R_F^{(n+3)}(\varepsilon,\phi)(\overline{w}) = & \sum_{k=1}^{n+2}\frac{1}{k!}\sum_{(i_1,\dots,i_k)\in S_k^{n+3}} D^kF(\phi)\Big[\hat{R}_\varepsilon^{n+3}(i_1;\overline{w})_1,\dots,\hat{R}_\varepsilon^{n+3}(i_k;\overline{w})_1\Big] \\
& +\frac{1}{(n+3)!}\int_0^1 D^{n+3}F\Big(\theta\phi+(1-\theta)\Phi\Big(\iota(\overline{\varepsilon w+\gamma},\lambda^\varepsilon)\Big)_1\Big) \\
& \qquad \Big[R_\varepsilon^1(\overline{w})_1,\dots,R_\varepsilon^1(\overline{w})_1\Big]d\theta,
\end{aligned} \tag{6.7}
$$

where $\hat{R}_\varepsilon^{n+3}(m;\cdot): G\Omega_p(X)\to\mathbb{R}, m=1,\dots,n+3$, are defined by

$$
\hat{R}_\varepsilon^{n+3}(m;\overline{w}) := \varepsilon^m\phi^m(\overline{w}), \quad m=1,\dots,n+2, \quad \text{and} \quad \hat{R}_\varepsilon^{n+3}(n+3;\overline{w}) := R_\varepsilon^{n+3}(\overline{w}).
$$

We note that the all functions in (6.6) are continuous on $G\Omega_p(X)$.

For our later use, we need estimates on $J_F^{(m)}(\phi)(\overline{w}),\ m=0,\dots,n+2$, and $R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})$. By Theorem 4.4, we have

$$
\begin{aligned}
\Big|J_F^{(m)}(\phi)(\overline{w})\Big| & \le \sum_{k=1}^{m}\frac{1}{k!}\Big(\sum_{(i_1,\dots,i_k)\in S_k^m} M_k\prod_{j=1}^{k}\|\phi^{i_j}(\overline{w})_1\|_p\Big) \\
& \le c\left\{\sum_{k=1}^{m}\frac{1}{k!}\cdot\begin{pmatrix} m-1 \\ k-1\end{pmatrix}\right\}\cdot\Big(1+\xi(\overline{w})\Big)^m = c\Big(1+\xi(\overline{w})\Big)^m,
\end{aligned} \tag{6.8}
$$

where c is a positive constant independent of $\overline{w}$.

On the other hand, by Theorem 5.1, we can obtain the following estimate for the remainder term $R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})$ under $\|\overline{\gamma}\|_1\le r_0$ and $\xi(\overline{\varepsilon w})<\rho$:

$$
\begin{aligned}
\Big|R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})\Big| & \le \sum_{k=1}^{n+2}\Big(\frac{1}{k!}\sum_{(i_1,\dots,i_k)\in S_k^{n+3}} M_k\prod_{j=1}^{k}\Big\|\hat{R}_\varepsilon^{n+3}(i_j;\overline{w})_1\Big\|_p\Big) \\
& \quad +\frac{M_{n+3}}{(n+3)!}\Big\|R_\varepsilon^{(1)}(\overline{w})_1\Big\|_p^{n+3} \\
& \le c\left\{\sum_{k=1}^{n+3}\frac{M_k}{k!}\cdot\begin{pmatrix} n+2 \\ k-1\end{pmatrix}\right\}\cdot\varepsilon^{n+3}\Big(1+\xi(\overline{w})\Big)^{n+3} \\
& = c\,\varepsilon^{n+3}\Big(1+\xi(\overline{w})\Big)^{n+3},
\end{aligned} \tag{6.9}
$$

where $c=c(n+3,r_0,\rho)$ is a positive constant independent of $\overline{w}\in G\Omega_p(X)$.

Next we recall condition **(H2)**. Since the function $F_\Lambda:\mathcal{H}\to\mathbb{R}$ attains its minimum at a unique point γ, it holds that

$$
\begin{aligned}
0 &= \Big(\gamma, h\Big)_{\mathcal{H}} + DF(\phi)[\chi(h)] \\
&= \Big(\gamma, h\Big)_{\mathcal{H}} + DF(\phi)[\phi^1(h)] - c(\gamma)
\end{aligned}
$$

for any $h \in \mathcal{H}$. Here we used Lemma 4.3 for the second line. By Theorem 4.4, $h \mapsto \phi^1(h)$ extends a continuous map on $G\Omega_p(X)$. Hence the extension

$$
\int_0^1 \gamma'(t) d\overline{w}_1(t) + DF(\phi)[\phi^1(\overline{w})_1] = c(\gamma) \tag{6.10}
$$

holds $\mathbb{P}_\varepsilon$-almost surely, where $\int_0^1 \gamma'(t) d\overline{w}_1(t)$ is the stochastic integral and we write $[\gamma](\overline{w})$ for it. For the more detailed derivation of (6.10), the readers are referred to see pages 190 and 191 in [7]. Then (6.10) leads us that

$$
J_F^{(1)}(\phi)(\overline{w}) = DF(\phi)\Big[\phi^1(\overline{w})_1\Big] = c(\gamma) - [\gamma](\overline{w}) \tag{6.11}
$$

holds $\mathbb{P}_\varepsilon$-almost surely.

Now we return to the integral $I_0(\varepsilon)$. By (6.5), (6.6) and (6.11), we can proceed as

$$
\begin{aligned}
I_0(\varepsilon) = &\int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, G\Big(\Phi(\iota(\overline{\varepsilon w + \gamma}, \lambda^\varepsilon))_1\Big) \cdot \exp\Big(\frac{c(\gamma)}{\varepsilon} - \frac{[\gamma](\overline{w})}{\varepsilon} - \frac{\|\gamma\|_{\mathcal{H}}^2}{2\varepsilon^2}\Big) \\
&\qquad \times \exp\Big[- \frac{1}{\varepsilon^2}\Big\{ \tilde{F}_\Lambda(\phi) + \varepsilon\Big(c(\gamma) - [\gamma](\overline{w})\Big) \\
&\qquad\qquad + \sum_{m=2}^{n+2} \varepsilon^m J_F^{(m)}(\phi)(\overline{w}) + R_F^{(n+3)}(\varepsilon, \phi)(\overline{w}) \Big\}\Big] \mathbb{P}_1(d\overline{w}), \\
= &\int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, G\Big(\Phi(\iota(\overline{\varepsilon w + \gamma}, \lambda^\varepsilon))_1\Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big) \\
&\qquad \times \exp\Big(- \sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w}) + \varepsilon^{-2} R_F^{(n+3)}(\varepsilon, \phi)(\overline{w})\Big) \mathbb{P}_1(d\overline{w}) \\
= &\int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, G\Big(\Phi(\iota(\overline{\varepsilon w + \gamma}, \lambda^\varepsilon))_1\Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big) \\
&\qquad \times \exp\Big(- \sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w})\Big) \mathbb{P}_1(d\overline{w}) \\
&+ \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, G\Big(\Phi(\iota(\overline{\varepsilon w + \gamma}, \lambda^\varepsilon))_1\Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big) \\
&\qquad \times \exp\Big(- \sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w})\Big) \cdot \Big\{ \exp\Big(- \frac{1}{\varepsilon^2} R_F^{(n+3)}(\varepsilon, \phi)(\overline{w})\Big) - 1 \Big\} \mathbb{P}_1(d\overline{w}) \\
=: &\, I_{0,1}(\varepsilon) + I_0^{(1)}(\varepsilon).
\end{aligned} \tag{6.12}
$$

Next we expand in a similar way $G\Big(\Phi(\iota(\overline{\varepsilon w + \gamma}, \lambda^\varepsilon))_1\Big)$ in a Taylor series with respect to ε, and insert this expansion into $I_{0,1}(\varepsilon)$. Then we obtain

$$
\begin{aligned}
I_{0,1}(\varepsilon) &= \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} \varepsilon^j J_G^{(j)}(\phi)(\overline{w}) \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
&\qquad\qquad \times \exp\Big(- \sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w}) \Big) \mathbb{P}_1(d\overline{w}) \\
&\quad + \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, R_G^{(n+1)}(\varepsilon,\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
&\qquad\qquad \times \exp\Big(- \sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w}) \Big) \mathbb{P}_1(d\overline{w}) \\
&=: I_{0,2}(\varepsilon) + I_0^{(2)}(\varepsilon). \qquad (6.13)
\end{aligned}
$$

Moreover by expanding the exp-function appearing in the right-hand side of $I_{0,2}(\varepsilon)$, we can rewrite as

$$
\begin{aligned}
I_{0,2}(\varepsilon) &= \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} \varepsilon^j J_G^{(j)}(\phi)(\overline{w}) \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
&\qquad\qquad \times \Big\{ 1 + \sum_{k=1}^{n} \frac{(-1)^k}{k!} \Big(\sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w}) \Big)^k \Big\} \mathbb{P}_1(d\overline{w}) \\
&\quad + \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} \varepsilon^j J_G^{(j)}(\phi)(\overline{w}) \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
&\qquad\qquad \times \Big\{ \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \Big(\sum_{m=3}^{n+2} \varepsilon^{m-2} J_F^{(m)}(\phi)(\overline{w}) \Big)^k \Big\} \mathbb{P}_1(d\overline{w}) \\
&=: I_{0,3}(\varepsilon) + I_0^{(3)}(\varepsilon). \qquad (6.14)
\end{aligned}
$$

We denote by $\mathcal{G}(k,n)$ the set of all maps $\pi : \{1,\dots,k\} \to \{3,\dots,n+2\}$ for $k = 1,\dots,n$. By using this notation, we can rewrite $I_{0,3}(\varepsilon)$ as follows:

$$
\begin{aligned}
I_{0,3}(\varepsilon) &= \sum_{j=0}^{n} \varepsilon^j \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \mathbb{P}_1(d\overline{w}) \\
&\quad + \sum_{j=0}^{n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\, J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
&\qquad \times \sum_{\pi \in \mathcal{G}(k,n)} \varepsilon^{j + \sum_{i=1}^{k} (\pi(i)-2)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w}) \mathbb{P}_1(d\overline{w}). \qquad (6.15)
\end{aligned}
$$

We now have the disjoint union $\mathcal{G}(k,n) = \mathcal{G}_{\downarrow}(j,k,n) \cup \mathcal{G}(j,k,n) \cup \mathcal{G}^{\uparrow}(j,k,n)$, with

$$
\begin{aligned}
\mathcal{G}_{\downarrow}(j,k,n) &:= \Big\{\pi \in \mathcal{G}(k,n) \;\Big|\; j + \sum_{i=1}^{k}(\pi(i)-2) < n\Big\},\\
\mathcal{G}(j,k,n) &:= \Big\{\pi \in \mathcal{G}(k,n) \;\Big|\; j + \sum_{i=1}^{k}(\pi(i)-2) = n\Big\},\\
\mathcal{G}^{\uparrow}(j,k,n) &:= \Big\{\pi \in \mathcal{G}(k,n) \;\Big|\; j + \sum_{i=1}^{k}(\pi(i)-2) > n\Big\}.
\end{aligned}
$$

For our later use, we also set

$$
\begin{aligned}
\hat{\mathcal{G}}_{\downarrow}(j,k,n) &:= \Big\{\pi \in \mathcal{G}(k,n) \;\Big|\; j + \sum_{i=1}^{k}(\pi(i)-2) \le n\Big\}\\
&= \mathcal{G}_{\downarrow}(j,k,n) \cup \mathcal{G}(j,k,n).
\end{aligned}
$$

By inserting these into the right-hand side of the second term of (6.15), we obtain

$$
\begin{aligned}
I_{0,3}(\varepsilon) = \Bigg\{ &\sum_{j=0}^{n} \varepsilon^{j} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\; J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\\
&+ \sum_{j=0}^{n}\sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\; J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\\
&\times \sum_{\pi \in \hat{\mathcal{G}}_{\downarrow}(j,k,n)} \varepsilon^{j+\sum_{i=1}^{k}(\pi(i)-2)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w})\mathbb{P}_1(d\overline{w})\Bigg\}\\
+\Bigg\{ &\sum_{j=0}^{n}\sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})\; J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\\
&\times \sum_{\pi \in \mathcal{G}^{\uparrow}(j,k,n)} \varepsilon^{j+\sum_{i=1}^{k}(\pi(i)-2)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w})\mathbb{P}_1(d\overline{w})\Bigg\}\\
=:\; &I_{0,4}(\varepsilon) + I_0^{(4)}(\varepsilon).
\end{aligned}
\tag{6.16}
$$

We observe that $I_{0,4}(\varepsilon)$ contains only terms where ε is taken to a power less than or equal to n and $I_0^{(1)}, \dots, I_0^{(4)}$ contain only terms with higher powers of ε.

Finally, we want to remove the characteristic function $\mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w})$ from $I_{0,4}(\varepsilon)$. We divide $I_{0,4}(\varepsilon)$ by $I_{0,4}(\varepsilon) = I_{0,5}(\varepsilon) + I_0^{(5)}(\varepsilon)$, where

$$
\begin{aligned}
I_{0,5}(\varepsilon) := &\sum_{j=0}^{n} \varepsilon^j \int_{G\Omega_p(X)} J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}) \\
&+ \sum_{j=0}^{n}\sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{G\Omega_p(X)} J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big) \\
&\times \sum_{\pi\in\hat{\mathcal{G}}_{\downarrow}(j,k,n)} \varepsilon^{j+\sum_{i=1}^{k}(\pi(i)-2)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w})\mathbb{P}_1(d\overline{w}), \qquad (6.17)
\end{aligned}
$$

$$
\begin{aligned}
I_0^{(5)}(\varepsilon) := &- \sum_{j=0}^{n} \varepsilon^j \int_{G\Omega_p(X)} \mathbf{1}_{(U_{\rho/\varepsilon})^c}(\overline{w})\; J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}) \\
&- \sum_{j=0}^{n}\sum_{k=1}^{n} \frac{(-1)^k}{k!} \int_{G\Omega_p(X)} \mathbf{1}_{(U_{\rho/\varepsilon})^c}(\overline{w})\; J_G^{(j)}(\phi)(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big) \\
&\times \sum_{\pi\in\hat{\mathcal{G}}_{\downarrow}(j,k,n)} \varepsilon^{j+\sum_{i=1}^{k}(\pi(i)-2)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w})\mathbb{P}_1(d\overline{w}). \qquad (6.18)
\end{aligned}
$$

Here we note that Theorem 4.4 and Lemma 6.4 denoted below, the integrand in (6.17) is integrable on $G\Omega_p(X)$ with respect to $\mathbb{P}_1$. (See Step 3 for more detailed discussions.)

We remark that $I_{0,5}(\varepsilon)$ can also be written as $I_{0,5}(\varepsilon) = \sum_{m=0}^{n} \alpha_m(\gamma)\varepsilon^m$ with

$$
\alpha_0(\gamma) := G(\phi) \int_{G\Omega_p(X)} \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})
$$

and

$$
\begin{aligned}
\alpha_m(\gamma) := \int_{G\Omega_p(X)} \Big\{ &J_G^{(m)}(\phi)(\overline{w}) + \sum_{j=0}^{m-1}\sum_{k=1}^{m} \frac{(-1)^k}{k!} J_G^{(j)}(\phi)(\overline{w}) \\
&\times \sum_{\pi\in\mathcal{G}(j,k,m)} \prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w}) \Big\} \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})
\end{aligned}
$$

for $m = 1, \ldots, n$. This proves the asymptotic expansion (3.3) claimed in Theorem 3.2, provided we give suitable estimates on $\sum_{k=1}^{5} I_0^{(k)}(\varepsilon)$ given below. Note that we have already estimated $I_1(\varepsilon)$ in Step 1.

Step 3: We start by estimating the term $I_0^{(1)}(\varepsilon)$. By the mean value theorem, there exists some constant $\theta \in (0, |R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})|/\varepsilon^2)$ such that

$$
\Big| \exp\Big(- \frac{1}{\varepsilon^2} R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})\Big) - 1 \Big| \le e^{\theta}\Big(\frac{1}{\varepsilon^2}\Big|R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})\Big|\Big)
$$

holds. Moreover by using the estimates (6.8) and (6.9), we obtain the following estimates for $\overline{w} \in U_{\rho/\varepsilon}$:

$$\varepsilon^{k-l}\Big|J_F^{(k)}(\phi)(\overline{w})\Big| \leq c_1(\varepsilon+\rho)^{k-l}(1+\xi(\overline{w}))^l, \quad k \geq l \geq 0, \tag{6.19}$$

$$\frac{1}{\varepsilon^2}\Big|R_F^{(n+3)}(\varepsilon,\phi)(\overline{w})\Big| \leq c_2\varepsilon^{n+1}(1+\xi(\overline{w}))^{n+3} \leq c_2(\varepsilon+\rho)^{n+1}(1+\xi(\overline{w}))^2, \tag{6.20}$$

where $c_1 = c_1(k)$ and $c_2 = c_2(n, r_0)$ are positive constants.

Then we have

$$\begin{aligned}\Big|I_0^{(1)}(\varepsilon)\Big| \leq \Big\|G\Big\|_\infty \int_{G\Omega_p(X)} &\exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big) \\ &\times \exp\Big[\Big(\sum_{m=3}^{n+2} c_1(m)(\varepsilon+\rho)^m\Big)(1+\xi(\overline{w}))^2\Big] \\ &\times \exp\Big[c_2(\varepsilon+\rho)^{n+1}(1+\xi(\overline{w}))^2\Big]\cdot\Big\{c_2\varepsilon^{n+1}(1+\xi(\overline{w}))^{n+3}\Big\}\mathbb{P}_1(d\overline{w}) \\ \leq c\varepsilon^{n+1}\int_{G\Omega_p(X)} &\exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big) \\ &\times \exp\Big[c_3(1+\xi(\overline{w}))^2\Big]\Big(1+\xi(\overline{w})\Big)^{n+3}\mathbb{P}_1(d\overline{w}),\end{aligned} \tag{6.21}$$

where $c_3 = c_3(n, r_0, \varepsilon, \rho)$ is a positive constant, and note that the constant c_3 converges to 0 as $\varepsilon, \rho \searrow 0$.

Now we use the fact that $\exp(-J_F^{(2)}(\phi)(\cdot))$ belongs to $L^q(G\Omega_p(X), \mathbb{P}_1)$ for some $q > 1$. (This is shown in Lemma 6.4 below.) We take $\varepsilon, \rho > 0$ sufficiently small such that $4c_3q < \beta(q-1)$ holds. Moreover by Proposition 2.3, it holds that for any $m > 0$,

$$\int_{G\Omega_p(X)} \exp\Big(m\xi(\overline{w})\Big)\mathbb{P}_1(d\overline{w}) + \int_{G\Omega_p(X)} \xi(\overline{w})^m\mathbb{P}_1(d\overline{w}) < \infty. \tag{6.22}$$

Then Hölder's inequality leads us that

$$\begin{aligned}&\int_{G\Omega_p(X)} \exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\cdot\exp\Big[c_3(1+\xi(\overline{w}))^2\Big]\Big(1+\xi(\overline{w})\Big)^{n+3}\mathbb{P}_1(d\overline{w}) \\ &\quad\leq \Big\{\int_{G\Omega_p(X)} \exp\Big(-qJ_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{1/q} \\ &\qquad\times\Big\{\int_{G\Omega_p(X)} \exp\Big(\frac{2c_3q}{q-1}(1+\xi(\overline{w}))^2\Big)\mathbb{P}_1(d\overline{w})\Big\}^{\frac{q-1}{2q}} \\ &\qquad\quad\times\Big\{\int_{G\Omega_p(X)} \Big(1+\xi(\overline{w})\Big)^{\frac{2q(n+3)}{q-1}}\mathbb{P}_1(d\overline{w})\Big\}^{\frac{q-1}{2q}} < \infty.\end{aligned} \tag{6.23}$$

By combining (6.21) and (6.23), we see that $I_0^{(1)}(\varepsilon) \leq c_1(n)\varepsilon^{n+1}$ holds.

The contribution for $I_0^{(2)}(\varepsilon)$ is treated in the same way as above and we can easily see the estimate $I_0^{(2)}(\varepsilon) \le c_2(n)\varepsilon^{n+1}$.

Next we proceed to estimate the term $I_0^{(3)}(\varepsilon)$. We use an elementary inequality

$$\Big|\sum_{k=n}^{\infty} \frac{1}{k!} x^k \Big| \le \frac{|x|^n}{n!} e^{|x|}, \qquad x \in \mathbb{R}, n \in \mathbb{N}.$$

By inserting this estimate and recalling (6.19) and (6.20), we have the following estimate:

$$\begin{aligned}
|I_0^{(3)}(\varepsilon)| \le & \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} \varepsilon^j \Big| J_G^{(j)}(\phi)(\overline{w}) \Big| \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \Big\{ \frac{1}{(n+1)!} \Big(\sum_{m=3}^{n+2} \varepsilon^{m-2} \Big| J_F^{(m)}(\phi)(\overline{w}) \Big| \Big)^{n+1} \\
& \qquad \times \exp\Big(\sum_{m=3}^{n+2} \varepsilon^{m-2} \Big| J_F^{(m)}(\phi)(\overline{w}) \Big| \Big) \Big\} \mathbb{P}_1(d\overline{w}) \\
\le & \frac{\varepsilon^{n+1}}{(n+1)!} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} \varepsilon^j \Big| J_G^{(j)}(\phi)(\overline{w}) \Big| \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \Big\{ \Big(\sum_{m=3}^{n+2} \varepsilon^{m-3} \Big| J_F^{(m)}(\phi)(\overline{w}) \Big| \Big)^{n+1} \exp\Big(\sum_{m=3}^{n+2} \varepsilon^{m-2} \Big| J_F^{(m)}(\phi)(\overline{w}) \Big| \Big) \Big\} \mathbb{P}_1(d\overline{w}) \\
\le & \frac{\varepsilon^{n+1}}{(n+1)!} \int_{G\Omega_p(X)} \mathbf{1}_{U_{\rho/\varepsilon}}(\overline{w}) \cdot \Big(\sum_{j=0}^{n} c(j) \Big(\varepsilon + \rho\Big)^j \Big) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \Big\{ \Big(\sum_{m=3}^{n+2} c(m)(\varepsilon+\rho)^{m-3} \cdot \Big(1 + \xi(\overline{w})\Big)^3 \Big)^{n+1} \\
& \qquad \times \exp\Big[\sum_{m=3}^{n+2} c(m) \Big(\varepsilon + \rho\Big)^{m-2} \cdot \Big(1 + \xi(\overline{w})\Big)^2 \Big] \Big\} \mathbb{P}_1(d\overline{w}) \\
\le & c\, \varepsilon^{n+1} \int_{G\Omega_p(X)} \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \cdot \Big(1 + \xi(\overline{w})\Big)^{3(n+1)} \\
& \qquad \times \exp\Big(c' \Big(1 + \xi(\overline{w})\Big)^2 \Big) \mathbb{P}_1(d\overline{w}), \qquad (6.24)
\end{aligned}$$

where $c' = c'(n, \varepsilon, \rho)$ is a positive constant, and note that the constant c' converges to 0 as $\varepsilon, \rho \searrow 0$. Now we take $\varepsilon, \rho > 0$ sufficiently small. Then we may apply Proposition 2.3. Therefore by using Hölder's inequality and (6.22), we get, in a similar way as for the term $I_0^{(1)}(\varepsilon)$, that $I_0^{(3)}(\varepsilon) \le c_3(n)\varepsilon^{n+1}$.

Let us now consider the term $I_0^{(4)}(\varepsilon)$. We note that the number of elements of $\mathcal{G}(k, n)$ is less than n^k. Then we have

$$
\begin{aligned}
|I_0^{(4)}(\varepsilon)| \leq & \, \varepsilon^{n+1} \sum_{j=0}^{n} \sum_{k=1}^{n} \frac{1}{k!} \int_{G\Omega_p(X)} |J_G^{(j)}(\phi)(\overline{w})| \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \sum_{\pi \in \hat{\mathcal{G}}^{\uparrow}(j,k,n)} \varepsilon^{j+\sum_{i=1}^{k}(\pi(i)-2)-(n+1)} \prod_{i=1}^{k} \Big| J_F^{(\pi(i))}(\phi)(\overline{w}) \Big| \, \mathbb{P}_1(d\overline{w}) \\
\leq & \, c\, \varepsilon^{n+1}(n+1) \sum_{k=1}^{n} \frac{n^k}{k!} \int_{G\Omega_p(X)} \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \Big(1+\xi(\overline{w})\Big)^{n+k(n+2)} \mathbb{P}_1(d\overline{w}).
\end{aligned} \tag{6.25}
$$

Hence Hölder's inequality and (6.22) lead us that $I_0^{(4)}(\varepsilon) \leq c_4(n)\varepsilon^{n+1}$.

Finally, we give an estimate on the term $I_0^{(5)}(\varepsilon)$. By using similar estimate to (6.25), we have

$$
\begin{aligned}
|I_0^{(5)}(\varepsilon)| \leq & \, c \sum_{j=0}^{n} \int_{G\Omega_p(X)} \mathbf{1}_{(U_{\rho/\varepsilon})^c}(\overline{w}) \, \Big(1+\xi(\overline{w})\Big)^{j} \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \mathbb{P}_1(d\overline{w}) \\
& + c\,(n+1) \sum_{k=1}^{n} \frac{n^k}{k!} \int_{G\Omega_p(X)} \mathbf{1}_{(U_{\rho/\varepsilon})^c}(\overline{w}) \cdot \exp\Big(- J_F^{(2)}(\phi)(\overline{w}) \Big) \\
& \times \Big(1+\xi(\overline{w})\Big)^{n+k(n+2)} \mathbb{P}_1(d\overline{w}).
\end{aligned}
$$

Here we see the following Gaussian estimate holds by remembering Proposition 2.3:

$$
\mathbb{P}_1\Big(\xi(\overline{w}) \geq \frac{\rho}{\varepsilon}\Big) \leq \exp\Big[- \beta\Big(\frac{\rho}{\varepsilon}\Big)^2 \Big] \int_{G\Omega_p(X)} \exp\Big(\beta \xi(\overline{w})^2\Big) \mathbb{P}_1(d\overline{w}). \tag{6.26}
$$

For the constant $q > 1$ denoted in Lemma 6.4 below, we take positive constants q_1, q_2 such that $1 < q_1 < q$ and $1/q_1 + 1/q_2 = 1$. Then by Hölder's inequality and (6.26), we can continue to estimate as

$$
\begin{aligned}
|I_0^{(5)}(\varepsilon)| \le & \, c\,\mathbb{P}_1\Big(\xi(\overline{w}) \ge \rho/\varepsilon\Big)^{1/q_2} \\
& \times\Big[\sum_{j=0}^{n}\Big\{\int_{G\Omega_p(X)}\Big(1+\xi(\overline{w})\Big)^{jq_1}\cdot\exp\Big(-q_1 J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{1/q_1} \\
& \quad + \sum_{k=1}^{n}\frac{n^k}{k!}\Big\{\int_{G\Omega_p(X)}\Big(1+\xi(\overline{w})\Big)^{nq_1+k(n+2)q_1} \\
& \qquad\qquad \times\exp\Big(-q_1 J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{1/q_1}\Big] \\
\le & \, c\,\exp\Big[-\frac{\beta}{q_2}\Big(\frac{\rho}{\varepsilon}\Big)^2\Big].
\end{aligned}
$$

Therefore this estimate leads us that $|I_0^{(5)}(\varepsilon)| \le c_5(m)\varepsilon^m$ holds for all $m \in \mathbb{N}$.

Hence by getting together this estimate and the previous estimates on $I_1, I_0^{(1)}, \dots, I_0^{(4)}$, we complete the proof of the asymptotic expansion formula (3.3).

Step 4: In the following, we discuss the higher integrability of $\exp\Big(-J_F^2(\phi)(\cdot)\Big)$ based on [7] and [9]. First we give an explicit representation of the Hessian A which are defined in Section 3. By recalling (4.17) and using Lemma 4.2, we have that for each $h, \hat{h} \in \mathcal{H}$,

$$
\begin{aligned}
\Big(Ah, \hat{h}\Big)_{\mathcal{H}} = & \, DF(\phi)\Big[D^2\Psi_0(\gamma)[h,\hat{h}]\Big] + D^2F(\phi)\Big[D\Psi_0(\gamma)[h], D\Psi_0(\gamma)[\hat{h}]\Big] \\
= & \, DF(\phi)\Big[\Gamma\Big(\int_0^{\cdot}(\nabla\sigma)(\phi_s)\Big[\chi(h)_s, d\hat{h}_s\Big] + (\nabla\sigma)(\phi_s)\Big[\chi(\hat{h})_s, dh_s\Big]\Big) \\
& + \Gamma\Big(\int_0^{\cdot}(\nabla^2\sigma)(\phi_s)\Big[\chi(h)_s, \chi(\hat{h})_s, d\gamma_s\Big] + a(0)\cdot\nabla^2 b(\phi_s)\Big[\chi(h)_s, \chi(\hat{h})_s\Big]ds\Big)\Big] \\
& + D^2F(\phi)\Big[\chi(h), \chi(\hat{h})\Big]
\end{aligned} \tag{6.27}
$$

holds, where $\chi(h) := \chi(h;\gamma)$ is defined through the differential equation (4.15).

We define a bilinear map $V : \mathcal{H}\times\mathcal{H} \to P(Y)$ by

$$
V(h,\hat{h})_t := \Gamma\Big(\int_0^{\cdot}(\nabla\sigma)(\phi_s)\Big[\chi(h)_s, d\hat{h}_s\Big] + (\nabla\sigma)(\phi_s)\Big[\chi(\hat{h})_s, dh_s\Big]\Big)_t, \quad t \ge 0, \tag{6.28}
$$

and define a bounded self-adjoint operator $\tilde{A}$ on $\mathcal{H}$ by

$$
\Big(\tilde{A}h, \hat{h}\Big)_{\mathcal{H}} := DF(\phi)\Big[V(h,\hat{h})\Big] \qquad \text{for } h, \hat{h} \in \mathcal{H}. \tag{6.29}
$$

Then by (6.27), (6.28) and (6.29), we obtain

$$
\begin{aligned}
&\Big((A-\tilde{A})h,\hat{h}\Big)_{\mathcal{H}}\\
&= DF(\phi)\Big[\Gamma\Big(\int_0^{\cdot}(\nabla^2\sigma)(\phi_s)\big[\chi(h)_s,\chi(\hat{h})_s,d\gamma_s\big]+a(0)\cdot\nabla^2 b(\phi_s)\big[\chi(h)_s,\chi(\hat{h})_s\big]ds\Big)\Big]\\
&\quad +D^2F(\phi)\big[\chi(h),\chi(\hat{h})\big].
\end{aligned}
\tag{6.30}
$$

By drawing the proof of Lemma 5.1 in [7], we can see the following: there exists a positive constant c such that $\|\chi(h)\|_{P(Y)} \leq c\|h\|_{P(X)}$ holds for any $h \in \mathcal{H} \subset \mathrm{BV}(X)$. Hence (6.30) leads us that

$$
\Big|\Big((A-\tilde{A})h,\hat{h}\Big)_{\mathcal{H}}\Big| \leq c\,\|\chi(h)\|_{P(Y)}\cdot\|\chi(\hat{h})\|_{P(Y)} \leq c\,\|h\|_{P(X)}\cdot\|\hat{h}\|_{P(X)}
$$

for some constant $c>0$. Now we may apply Theorem 4.6 in [10] to an abstract Wiener space $(P(X),\mathcal{H},\mathbb{P}_1')$ to obtain that $A-\tilde{A}$ is of trace class operator on $\mathcal{H}$.

The following properties on the operators A and $\tilde{A}$ are taken from Lemma 4.6 in [9].

Lemma 6.1 (1) *A and $\tilde{A}$ are self-adjoint Hilbert-Schmidt operators on $\mathcal{H}$.*
(2) *The continuous extension of the quadratic form defined by $A-\tilde{A}$ is expressed as*

$$
\begin{aligned}
\langle (A-\tilde{A})\overline{w},\overline{w}\rangle = {} & D^2F(\phi)\big[\chi(\overline{w})_1,\chi(\overline{w})_1\big]\\
& +DF(\phi)\Big[\Gamma\Big(\int_0^{\cdot}(\nabla^2\sigma)(\phi_s)\big[\chi(\overline{w})_1(s),\chi(\overline{w})_1(s),d\gamma_s\big]\\
& \qquad +\int_0^{\cdot} a(0)\cdot\nabla^2 b(\phi_s)\big[\chi(\overline{w})_1(s),\chi(\overline{w})_1(s)\big]ds\Big)\Big].
\end{aligned}
$$

Next, we consider the stochastic integration of the kernel associated with $\tilde{A}$. Since $\mathcal{H} \cong L^2([0,1],H) \cong L^2([0,1],\mathbb{R})\otimes H$, any self-adjoint Hilbert-Schmidt operator S on $\mathcal{H}$ corresponds to a kernel function $K_S \in L^2([0,1]\times[0,1], H\otimes H)$ with $K_S(u,s)=K_S(s,u)^*$ for almost all (u,s). Here $\otimes$ denotes the Hilbert-Schmidt tensor product. The correspondence $S \mapsto K_S$ is isometric. Then for the X-valued Brownian motion $w=(w_t)_{0\leq t\leq 1}$, an iterated stochastic integral

$$
\hat{K}_S(w) := 2\int_0^1\int_0^s K_S(u,s)[dw_u,dw_s]
$$

is well-defined. Clearly, this random variable is in $L^2(\mathbb{P}_1')$ with expectation 0. The correspondence $S \mapsto \hat{K}_S \in L^2(\mathbb{P}_1')$ is isometric.

For $y \in Y$, we define $Q_2(y) \in L^2(X, X; Y)$ by

$$Q_2(y)[x_1, x_2] := \frac{1}{2}(\nabla\sigma)(y)\Big[\sigma(y)x_1, x_2\Big]_{Y^2}, \qquad x_1, x_2 \in X.$$

Then by Theorem 3.1 in [10], we can define $\mathrm{Tr}(Q_2)(y) \in Y$ by

$$\mathrm{Tr}(Q_2)(y) := \int_X Q_2(y)[x, x]\mu(dx), \qquad y \in Y.$$

The following lemma is essentially shown in Corollary 7.3 and Lemma 7.4 in [7].

Lemma 6.2 (1) *For each $\alpha \in P(Y)^*$, $\alpha \circ V$ is a Hilbert-Schmidt symmetric bilinear form on $\mathcal{H}$. (We also denote by $\alpha \circ V$ the self-adjoint Hilbert-Schmidt operator on $\mathcal{H}$ associated with this bilinear form.)*
(2) *For any $\alpha \in P(Y)^*$, it holds that*

$$\alpha(\Theta(\overline{w})) = \hat{K}_{\alpha\circ V}(\overline{w}_1), \qquad \mathbb{P}_1\text{-almost surely,}$$

where

$$\begin{aligned}\Theta(\overline{w}) := &\, \psi(\overline{w}, \overline{w})_1 - \Gamma\Big(\int_0^{\cdot} \mathrm{Tr}(Q_2)(\phi_s)ds\Big) \\ &-\Gamma\Big(\int_0^{\cdot} (\nabla^2\sigma)(\phi_s)[\chi(\overline{w})_1(s), \chi(\overline{w})_1(s), d\gamma_s] \\ &\qquad\qquad +a(0)\cdot\nabla^2 b(\phi_s)\Big[\chi(\overline{w})_1(s), \chi(\overline{w})_1(s)\Big]ds\Big).\end{aligned}$$

In particular, $DF(\phi)[\Theta(\overline{w})] = \hat{K}_{\tilde{A}}(\overline{w}_1)$ holds $\mathbb{P}_1$-almost surely.

Next we present the following integration formula. See the proof of Lemma 8.3 in [7] for details. This formula plays an important role to compute the quantity α_0.

Lemma 6.3 *It holds that*

$$\begin{aligned}\int_{G\Omega_p(X)} \exp\Big[-\frac{1}{2}\Big(\hat{K}_{\tilde{A}}(\overline{w}_1) - \langle (A-\tilde{A})\overline{w}, \overline{w}\rangle\Big)\Big]\mathbb{P}_1(d\overline{w}) \\ = e^{-\frac{1}{2}\mathrm{Tr}(A-\tilde{A})}\cdot \det{}_2(\mathrm{Id}_{\mathcal{H}} + A)^{-1/2},\end{aligned}$$

where $\det_2$ *denotes the Carleman-Fredholm determinant.*

Now we discuss the integrabilities of $\exp(-J_F^{(2)}(\phi)(\cdot))$. By Lemma 4.3, we have

$$
\begin{aligned}
J_F^{(2)}(\phi)(\overline{w}) &= DF(\phi)[\phi^2(\overline{w})_1] + \frac{1}{2}D^2F(\phi)[\phi^1(\overline{w})_1, \phi^1(\overline{w})_1] \\
&= \frac{1}{2}\Big(DF(\phi)\Big[\psi(\overline{w},\overline{w})_1 + Y(\overline{w})_1 + \Xi_2(\gamma)\Big] \\
&\qquad\quad + D^2F(\phi)\Big[\chi(\overline{w})_1 + \Xi_1(\gamma), \chi(\overline{w})_1 + \Xi_1(\gamma)\Big]\Big) \\
&= \frac{1}{2}\Big(DF(\phi)\Big[\psi(\overline{w},\overline{w})_1\Big] + D^2F(\phi)\Big[\chi(\overline{w})_1, \chi(\overline{w})_1\Big]\Big) \\
&\quad + \frac{1}{2}\Big(DF(\phi)[Y(\overline{w})_1] + 2D^2F(\phi)\Big[\chi(\overline{w})_1, \Xi_1(\gamma)\Big]\Big) \\
&\quad + \frac{1}{2}\Big(DF(\phi)[\Xi_2(\gamma)] + D^2F(\phi)[\Xi_1(\gamma), \Xi_1(\gamma)]\Big) \\
&=: J_F^{(2,1)}(\phi)(\overline{w}) + J_F^{(2,2)}(\phi)(\overline{w}) + J_F^{(2,3)}(\phi).
\end{aligned}
$$

Here we note that $J_F^{(2,2)}(\phi)(\overline{w}) = 0$ and $J_F^{(2,3)}(\phi) = \frac{1}{2}DF(\phi)[\Xi_2(\gamma)]$ hold if $a'(0) = 0$. By Lemmas 6.1 and 6.2, we can proceed as follows on the term $J_F^{(2,1)}(\phi)(\overline{w})$:

$$
J_F^{(2,1)}(\phi)(\overline{w}) = \frac{1}{2}\Big(\hat{K}_{\tilde{A}}(\overline{w}_1) + DF(\phi)\Big[\Gamma\Big(\int_0^{\cdot} \mathrm{Tr}(Q_2)(\phi_s)ds\Big)\Big] + \langle (A-\tilde{A})\overline{w}, \overline{w}\rangle \Big).
$$

Then Lemma 6.3 implies

$$
\begin{aligned}
&\int_{G\Omega_p(X)} \exp\Big(- J_F^{(2,1)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}) \\
&= \exp\Big(-\frac{1}{2}\mathrm{Tr}(A-\tilde{A}) - DF(\phi)\Big[\Gamma\Big(\int_0^{\cdot} \mathrm{Tr}(Q_2)(\phi_s)ds\Big)\Big]\Big) \cdot \det{}_2(\mathrm{Id}_{\mathcal{H}} + A)^{-1/2}.
\end{aligned} \tag{6.31}
$$

On the other hand, by condition **(H4)**, there exists a constant $q_0 > 1$ such that $(\mathrm{Id}_{\mathcal{H}} + q_0 A)$ is strictly positive. Hence by (6.31) and $q_0 J_F^{(2,1)}(\phi)(\overline{w}) = J_{q_0 F}^{(2,1)}(\phi)(\overline{w})$, we can see that $\exp\Big(- J_F^{(2,1)}(\phi)(\cdot)\Big) \in L^{q_0}(G\Omega_p(X), \mathbb{P}_1)$ holds. Moreover, by Corollary 4.5, we have $|J_F^{(2,2)}(\phi)(\overline{w})| \leq c\Big(1 + \xi(\overline{w})\Big)$ for $\mathbb{P}_1$-almost surely $\overline{w} \in G\Omega_p(X)$. Therefore for any $1 < q < q_0$, by (6.22) and Hölder's inequality, we have

$$
\begin{aligned}
&\int_{G\Omega_p(X)} \exp\Big(-qJ_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}) \\
&\le \exp\Big(-qJ_F^{(2,3)}(\phi)\Big)\cdot\Big\{\int_{G\Omega_p(X)} \exp\Big(-q_0J_F^{(2,2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{q/q_0} \\
&\qquad\qquad \times\Big\{\int_{G\Omega_p(X)} \exp\Big(-\frac{q_0q}{q_0-q}J_F^{(2,2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{1-q/q_0} \\
&\le \exp\Big(-qJ_F^{(2,3)}(\phi)\Big)\cdot\Big\{\int_{G\Omega_p(X)} \exp\Big(-q_0J_F^{(2,2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w})\Big\}^{q/q_0} \\
&\qquad\qquad \times\Big\{\int_{G\Omega_p(X)} \exp\Big[c\Big(1+\xi(\overline{w})\Big)\Big]\mathbb{P}_1(d\overline{w})\Big\}^{1-q/q_0} < \infty.
\end{aligned}
$$

Hence we have shown

Lemma 6.4 *There exists a constant $q > 1$ such that*

$$
\exp\Big(-J_F^{(2)}(\phi)(\cdot)\Big) \in L^q(G\Omega_p(X),\mathbb{P}_1).
$$

Finally, by summarizing above all arguments, we can state

Theorem 6.5 *We have the asymptotic expansion* (3.3) *and the coefficients $\{\alpha_m\}_{m=0}^n$ are given by*

$$
\begin{aligned}
\alpha_0 = \alpha_0(\gamma) &:= G(\phi)\int_{G\Omega_p(X)} \exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}), \\
\alpha_m = \alpha_m(\gamma) &:= \int_{G\Omega_p(X)} \Big\{J_G^{(m)}(\phi)(\overline{w}) + \sum_{j=0}^{m-1}\sum_{k=1}^{m}\frac{(-1)^k}{k!}J_G^{(j)}(\phi)(\overline{w}) \\
&\qquad \times \sum_{\pi\in\mathcal{G}(j,k,m)}\prod_{i=1}^{k} J_F^{(\pi(i))}(\phi)(\overline{w})\Big\}\cdot\exp\Big(-J_F^{(2)}(\phi)(\overline{w})\Big)\mathbb{P}_1(d\overline{w}), \\
&\qquad\qquad \textit{for } m = 1,\dots,n,
\end{aligned}
$$

where

$$
J_F^n(\phi)(\overline{w}) := \sum_{k=1}^{n}\frac{1}{k!}\Big(\sum_{(i_1,\cdots,i_k)\in S_k^n} D^kF(\phi)\Big[\phi^{i_1}(\overline{w})_1,\cdots,\phi^{i_k}(\overline{w})_1\Big]\Big),
$$

and

$$
\mathcal{G}(j,k,m) := \Big\{\pi:\{1,\dots,k\}\to\{3,\dots,m+2\}\ \Big|\ j+\sum_{i=1}^{k}(\pi(i)-2) = m\Big\}.
$$

In particular, if $a'(0) = 0$, then $c(\gamma) = 0$ and the coefficient α_0 has the explicit representation

$$\alpha_0 = G(\phi)\cdot\exp\Big(-\frac{1}{2}\mathrm{Tr}(A-\tilde{A}) - DF(\phi)\Big[\frac{1}{2}\Xi_2(\gamma) + \Gamma\Big(\int_0^{\cdot}\mathrm{Tr}(Q_2)(\phi_s)ds\Big)\Big]\Big)$$
$$\times \det{}_2(\mathrm{Id}_{\mathcal{H}} + A)^{-1/2},$$

where $\det_2$ *denotes the Carleman-Fredholm determinant.*

Remark 6.6 *If* $a^{(j)}(0) = 0$ *for odd* j*, then by putting* $a(-\varepsilon) := a(\varepsilon)$, $\varepsilon > 0$*, we can extend* $a : [0,1] \to \mathbb{R}^N$ *to a smooth even function defined on the interval* $[-1,1]$*. Moreover, by the symmetry of the law of the Brownian motion, our functional integral* $\mathbb{E}\Big[G(X^\varepsilon)\exp(-F(X^\varepsilon)/\varepsilon^2)\Big]$ *is an even function of* ε*. Hence we see that the coefficients* α_m *are equal to zero for odd* m*.*

Finally, we generalize Theorem 6.5 to the case where the phase function has finitely many non-degenerate minima.

Theorem 6.7 *We assume that the phase function* F_Λ *has a finite set* $\mathcal{M} := \{\gamma_1, \dots, \gamma_k\}$ *of minimum points and conditions* **(H3)** *and* **(H4)** *hold separately for every* γ_i, $i = 1, \dots, k$*. We denote by* $c(\mathcal{M}) := \inf\{c(\gamma) | \gamma \in \mathcal{M}\}$ *and* $\mathcal{J} := \{j \in \{1, \dots, k\} | c(\gamma_j) = c(\mathcal{M})\}$*. Then we have the following asymptotic expansion formula:*

$$\mathbb{E}\Big[G(X^\varepsilon)\exp\Big(-F(X^\varepsilon)/\varepsilon^2\Big)\Big] = \exp\Big(-F_\Lambda(\gamma_1)/\varepsilon^2\Big)\exp\Big(-c(\mathcal{M})/\varepsilon\Big)$$
$$\times\Big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_n\varepsilon^n + O(\varepsilon^{n+1})\Big), \tag{6.32}$$

where the coefficients $\{\alpha_m\}_{m=0}^n$ *in* (6.32) *are given by* $\alpha_m = \sum_{j\in\mathcal{J}} \alpha_m(\gamma_j)$, $0 \le m \le n$*.*

Proof. We denote $\phi_i := \Psi_0(\gamma_i), i = 1, \dots, k$, and choose $\rho, \rho' > 0$ sufficiently small such that open balls $B(\phi_i, \rho') \subset B(\phi_i), i = 1, \dots, k$, are pairwise disjoint in $P(Y)$ and

$$\Big\{\Phi\Big(\iota(\overline{x}, \lambda)\Big)_1(0, \cdot) \;\Big|\; \overline{x} \in \overline{\gamma}_i + U_\rho, \|\lambda - \lambda^0\|_1 < \rho\Big\} \subset B(\phi_i, \rho'/4)$$

holds. Here $B(\phi_i)$ is the neighborhood of $\phi_i \in P(Y)$ in condition **(H3)**. We define a cut-off function $\eta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ by $\eta(x) \equiv 1$ for $|x| \le \rho/2$, $\eta(x) \equiv 0$ for $|x| \ge \rho$ and $0 \le \eta \le 1$. For each $i = 1, \dots, k$, we define a continuous function $\mathcal{G}_i : P(Y) \to \mathbb{R}$ by

$$\mathcal{G}_i(y) := \eta\Big(2\|y - \phi_i\|_{P(Y)}\Big)\cdot G(y), \quad y \in P(Y).$$

We define $\mathcal{G}_0 := G - \sum_{i=1}^k \mathcal{G}_i$, so that $\sum_{i=0}^k \mathcal{G}_i \equiv G$. For each $i = 1, \dots, k$, we also define a continuous function $\mathcal{F}_i : P(Y) \to \mathbb{R}$ by

$$\mathcal{F}_i(y) := \Big\{1 - \eta\Big(\|y - \phi_i\|_{P(Y)}\Big)\Big\} + F(y), \quad y \in P(Y).$$

Then the functions $\{\mathcal{F}_i\}_{i=1}^k$ and $\{\mathcal{G}_i\}_{i=0}^k$ satisfy conditions **(H1)**, **(H2)**, **(H3)** and **(H4)** in Section 3, where each $(\mathcal{F}_i)_\Lambda := \mathcal{F}_i \circ \Psi_0 + \|\cdot\|_{\mathcal{H}}^2/2$ achieves its minimum at the unique point $\gamma_i \in \mathcal{H}$. Moreover we can see that for $i = 1, \ldots, k$,

$$\begin{aligned}
&\int_{G\Omega_p(X)} \mathcal{G}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} F\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathcal{G}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} \mathcal{F}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}).
\end{aligned}$$

Then by noting above, we have

$$\begin{aligned}
&\int_{G\Omega_p(X)} G\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} F\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}) \\
&= \sum_{i=0}^k \int_{G\Omega_p(X)} \mathcal{G}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} F\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}) \\
&= \int_{G\Omega_p(X)} \mathcal{G}_0\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} F\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}) \\
&\quad + \sum_{i=1}^k \int_{G\Omega_p(X)} \mathcal{G}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \exp\Big[-\frac{1}{\varepsilon^2} \mathcal{F}_i\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}) \\
&=: \mathcal{I}_0(\varepsilon) + \sum_{i=1}^k \mathcal{I}_i(\varepsilon). \qquad (6.33)
\end{aligned}$$

For the term $\mathcal{I}_0(\varepsilon)$, we note that the following equality holds:

$$\begin{aligned}
&\exp\Big(F_\Lambda(\gamma_1)\Big) \cdot \mathcal{I}_0(\varepsilon) \\
&= \int_{\bigcap_{i=1}^k (\overline{\gamma}_i + U_\rho)^c} \mathcal{G}_0\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big) \cdot \exp\Big[-\frac{1}{\varepsilon^2} \tilde{F}_\Lambda\Big(\Phi(\iota(\overline{\varepsilon w}, \lambda^\varepsilon))_1\Big)\Big] \mathbb{P}_1(d\overline{w}).
\end{aligned}$$

Here we recall the estimate (6.3). Then there exists positive constants a and ε_0 such that

$$\exp\Big(F_\Lambda(\gamma_1)\Big) \cdot \Big|\mathcal{I}_0(\varepsilon)\Big| \le k\|G\|_\infty \cdot \exp(-a^2/\varepsilon^2) \qquad (6.34)$$

holds for all $0 < \varepsilon \le \varepsilon_0$.

For each term $\mathcal{I}_i(\varepsilon), i = 1, \ldots, k$, we may apply Theorem 3.2. Therefore we get together estimates (3.3) on $\mathcal{I}_1(\varepsilon), \ldots, \mathcal{I}_k(\varepsilon)$ with (6.34), and insert these into

(6.33). Then we can obtain the desired asymptotic expansion formula (6.32). ∎

Acknowledgment. The authors express their deepest gratitudes to Professor Shigeki Aida for valuable suggestions. They also thank Professors Sergio Albeverio, Bruce Driver, David Elworthy, Shizan Fang, Paul Malliavin, Zhongmin Qian and Shinzo Watanabe for their helpful comments and encouragements. The authors were supported by JSPS Research Fellowships for Young Scientists and the second author is supported by 21st century COE program "Development of Dynamic Mathematics with High Functionality" at Faculty of Mathematics, Kyushu University.

References

[1] S. Aida: *Notes on proofs of continuity theorem in rough path analysis*, preprint, 2006.

[2] S. Aida: *Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold*, preprint, 2006.

[3] S. Albeverio, H. Röckle and V. Steblovskaya: *Asymptotic expansions for Ornstein-Uhlenbeck semigroups perturbed by potentials over Banach spaces*, Stochastics and Stochastics Reports, **69**, (2000), pp. 195–238.

[4] R. Azencott: *Formule de Talyor stochastique et développement asymptotique d'intégrales de Feynman*, in "Seminar on Probability, XVI, Supplement ", pp. 237–285, Lecture Notes in Math., **921**, Springer, Berlin-New York, 1982.

[5] G. Ben Arous: *Methods de Laplace et de la phase stationnaire sur l'espace de Wiener*, Stochastics **25**, (1988), no.3, pp. 125–153.

[6] J.D. Deuschel and D.W. Stroock: Large Deviations, Academic Press, Boston, 1989.

[7] Y. Inahama: *Laplace's method for the laws of heat processes on loop spaces*, J. Funct. Anal. **232** (2006), pp. 148–194.

[8] Y. Inahama and H. Kawabi: *Large deviations for heat kernel measures on loop spaces via rough paths*, J. London Math. Society **73** (2006), no. 3, pp. 797–816.

[9] Y. Inahama and H. Kawabi: *On asymptotics of certain Banach space-valued Itô functionals of Brownian rough paths*, To appear in Proceedings of the Abel Symposium 2005, Stochastic Analysis and Applications, A Symposium in Honor of Kiyosi Itô, Springer.

[10] H.H. Kuo: Gaussian Measures on Banach Spaces, Lecture Notes in Math., **463**, Springer, Berlin, 1985.

[11] S. Kusuoka and D.W. Stroock: *Precise asymptotics of certain Wiener functionals*, J. Funct. Anal. **99** (1991), no. 1, pp. 1–74.

[12] S. Kusuoka and D.W. Stroock: *Asymptotics of certain Wiener functionals with degenerate extrema*, Comm. Pure Appl. Math. **47** (1994), no. 4, pp. 477–501.

[13] M. Ledoux, T. Lyons and Z. Qian: *Lévy area of Wiener processes in Banach spaces*, Ann. Probab. **30** (2002), no. 2, pp. 546–578.

[14] M. Ledoux, Z. Qian and T.S. Zhang: *Large deviations and support theorem for diffusion processes via rough paths*, Stochastic Process. Appl. **102** (2002), no. 2, pp.265–283.

[15] T. Lyons and Z. Qian: System control and rough paths, Oxford University Press, Oxford, 2002.

[16] M. Schilder: *Some asymptotic formulas for Wiener integrals*, Trans. Amer. Math. Soc. **125** (1966), 63–85.

[17] S. Takanobu and S. Watanabe: *Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integrations*, in “Asymptotic problems in probability theory: Wiener functionals and asymptotics ”(Sanda/Kyoto, 1990), pp. 194–241, Pitman Res. Notes Math. Ser., **284**, Longman Sci. Tech., Harlow, 1993.