Integral representations of nonnegative solutions for parabolic equations and elliptic Martin boundaries*

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Abstract

We consider nonnegative solutions of a parabolic equation in a cylinder $D \times (0,T)$, where D is a noncompact domain of a Riemannian manifold. Under the assumption [IU] (i.e., the associated heat kernel is intrinsically ultracontractive), we establish an integral representation theorem: Any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0,T))$, where $\partial_M D$ is the Martin boundary of D for the associated elliptic operator. We apply it in a unified way to several concrete examples to explicitly represent nonnegative solutions. We also show that [IU] implies the condition [SP] (i.e., the constant function 1 is a small perturbation of the elliptic operator on D).

1 Introduction

This paper is concerned with integral representations of nonnegative solutions to parabolic equations, and gives a representation theorem which is general

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and applicable to many concrete examples for establishing explicit integral representations.

We consider nonnegative solutions of a parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times (0, T), \tag{1.1}$$

where T is a positive number, D is a noncompact domain of a Riemannian manifold M, $\partial_t = \partial/\partial t$, and L is a second order elliptic operator on D. We study the following

Problem. Determine all nonnegative solutions of the parabolic equation (1.1).

This problem is closely related to the Widder type uniqueness theorem for a parabolic equation, which asserts that any nonnegative solution is determined uniquely by its initial value. (For Widder type uniqueness theorems, see [6], [8], [22], [29], [31], [40], [41], [44], [45] and references therein.) We say that **[UP]** (i.e., uniqueness for the positive Cauchy problem) holds for (1.1) when any nonnegative solution of (1.1) with zero initial value is identically zero. When [UP] holds for (1.1) the answer to our problem is extremely simple: for any nonnegative solution u of (1.1) there exists a unique Borel measure μ on D such that

$$u(x,t) = \int_D p(x,y,t)d\mu(y), \quad x \in D, \ 0 < t < T,$$

where p is the minimal fundamental solution for (1.1) (see [6], [8]). While [UP] does not hold, however, only a few explicit integral representations of nonnegative solutions to parabolic equations are given (see [20], [26], [35], [48], [54]). (For related representation theorems, see [31] and [50].) On the other hand, for elliptic equations, there has been a significant progress in determining explicitly Martin boundaries in many important cases (see [1], [2], [3], [4], [7], [16], [23], [37], [38], [43], [45], [47] and references therein). Recall that any nonnegative solution of a subcritical elliptic equation is represented by an integral of Martin kernels with respect to a Borel measure on the Martin boundary.

The aim of this paper is to give explicit integral representations of nonnegative solutions to parabolic equations for which [UP] does not hold. We give a general and sharp condition under which any nonnegative solution of (1.1) with zero initial value is represented by an integral on the product of the Martin boundary of D for an elliptic operator associated with L and the time interval $[0, T)$.

Now, in order to state our main results, we fix notations and recall several notions and facts. Let M be a connected separable n-dimensional smooth manifold with Riemannian metric of class C^0. Denote by ν the Riemannian measure on M. T_xM and TM denote the tangent space to M at $x \in M$ and the tangent bundle, respectively. We denote by $\mathrm{End}(T_xM)$ and $\mathrm{End}(TM)$ the set of endmorphisms in T_xM and the corresponding bundle, respectively. The inner product on TM is denoted by $\langle X, Y \rangle$, where $X, Y \in TM$; and $|X| = \langle X, X \rangle^{1/2}$. The divergence and gradient with respect to the metric on M are denoted by div and ∇, respectively. Let D be a noncompact domain of M. Let L be an elliptic differential operator on D of the form

$$Lu = -m^{-1}\mathrm{div}(mA\nabla u) + Vu, \tag{1.2}$$

where m is a positive measurable function on D such that m and m^{-1} are bounded on any compact subset of D, A is a symmetric measurable section on D of $\mathrm{End}(TM)$, and V is a real-valued measurable function on D such that

$$V \in L^p_{\mathrm{loc}}(D, md\nu), \qquad \text{for some } p > \max(\frac{n}{2}, 1).$$

Here $L^p_{\mathrm{loc}}(D, md\nu)$ is the set of real-valued functions on D locally p-th integrable with respect to $md\nu$. We assume that L is locally uniformly elliptic on D, i.e., for any compact set K in D there exists a positive constant λ such that

$$\lambda|\xi|^2 \le \langle A_x\xi, \xi \rangle \le \lambda^{-1}|\xi|^2, \quad x \in K,\ (x, \xi) \in TM.$$

We assume that the quadratic form Q on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A\nabla u, \nabla u \rangle + Vu^2) md\nu$$

is bounded from below, and put

$$\lambda_0 = \inf\{Q[u]; u \in C_0^\infty(D),\ \int_D u^2 md\nu = 1\}.$$

Then, for any $a < \lambda_0$, $(L - a, D)$ is subcritical, i.e., there exists the (minimal positive) Green function of $L - a$ on D. Denote by L_D the selfadjoint operator in $L^2(D; md\nu)$ associated with the closure of Q. We assume that λ_0 is an eigenvalue of L_D. Let ϕ_0 be the normalized positive eigenfunction for λ_0. Let $p(x, y, t)$ be the minimal fundamental solution for (1.1), which is equal to the integral kernel of the semigroup e^{-tL_D} on $L^2(D, md\nu)$.

Our main assumption is the following condition [IU] (i.e., intrinsic ultracontractivity).

[IU] For any $t > 0$, there exists a constant $C_t > 0$ such that

$$p(x, y, t) \le C_t \ \phi_0(x)\phi_0(y), \quad x, y \in D.$$

(For results related to [IU], see [9], [10], [11], [12], [14], [18], [19], [27], [36], [39], [40], [42], [43] and references therein.) This condition [IU] implies that L_D admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^{\infty}$ with eigenvalues $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ repeated according to multiplicity. Furthermore,

$$\frac{p(x, y, t)}{\phi_0(x)\phi_0(y)} = \sum_{j=0}^{\infty} e^{-\lambda_j t} \frac{\phi_j(x)\phi_j(y)}{\phi_0(x)\phi_0(y)}, \qquad x, y \in D, \ t > 0, \tag{1.3}$$

where the series converges uniformly on $D \times D \times [\delta, \infty)$ for any $\delta > 0$ (see [18]). Recall that if [IU] holds, then [UP] does not hold for (1.1) and the equation admits a positive solution with zero initial value (see [39]); and for a class of parabolic equations, [IU] is equivalent to the existence of such a solution (see [40]).

We show in this paper that [IU] also implies the following condition [SP] (i.e., small perturbation) for any $a < \lambda_0$:

[SP] The constant function 1 is a small perturbation of $L - a$ on D, i.e., for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D \setminus K} G(x, z)G(z, y)m(z)d\nu(z) \le \varepsilon G(x, y), \qquad x, y \in D \setminus K,$$

where G is the Green function of $L - a$ on D (see [51]).

Namely, we shall show in Section 3 the following theorem which is of independent interest.

Theorem 1.1 *Suppose that [IU] holds. Then, for any $a < \lambda_0$, 1 is a small perturbation of $L - a$ on D.*

This theorem is completely new when the dimension of M is higher than 1; while it is known in the one dimensional case (see Proposition 3.5 of [42], Theorems 6.1 and 6.3 of [43]). Note that the converse to Theorem 1.1 does not hold (see Remark 3.2 in Section 3). Recall that [SP] implies the following condition [SSP] (i.e., semismall perturbation) (see [42]):

[SSP] 1 is a semismall perturbation of $L-a$ on D, i.e., for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D\setminus K} G(x^0,z)G(z,y)m(z)d\nu(z) \le \varepsilon G(x^0,y), \qquad y \in D\setminus K,$$

where x^0 is a fixed reference point in D.

Fix $a<\lambda_0$, and suppose that [SSP] holds. Then, for any $j=1,2,\cdots$, the function ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of D for $L-a$ (see Theorem 6.3 of [53]). (For semismall perturbations, see also [43].) Let $D^* = D\cup\partial_M D$ be the Martin compactification of D for $L-a$, which is a compact metric space. We denote by $\partial_m D$ the minimal Martin boundary of D for $L-a$. This is a Borel subset of $\partial_M D$. Here, we note that $\partial_M D$ and $\partial_m D$ are independent of a in the following sense: if [SSP] holds, then for any $b<\lambda_0$ there is a homeomorphism Φ from the Martin compactification of D for $L-a$ onto that for $L-b$ such that $\Phi|_D = identity$, and Φ maps the Martin boundary and minimal Martin boundary of D for $L-a$ onto those for $L-b$, respectively (see Theorem 1.4 of [42]).

Now, we are ready to state our main theorem.

Theorem 1.2 Assume [IU]. Then, for any nonnegative solution u of (1.1) there exists a unique pair of Borel measures μ on D and λ on $\partial_M D\times[0,T)$ such that λ is supported by the set $\partial_m D\times[0,T)$, and

$$u(x,t) = \int_D p(x,y,t)d\mu(y) + \int_{\partial_M D\times[0,t)} q(x,\xi,t-s)d\lambda(\xi,s) \tag{1.4}$$

for any $(x,t)\in D\times(0,T)$. Here $q(x,\xi,\tau)$ is a continuous function on $D\times\partial_M D\times(-\infty,\infty)$ defined by

$$\begin{aligned} q(x,\xi,\tau) &= \sum_{j=0}^{\infty} e^{-\lambda_j\tau}\phi_j(x)[\phi_j/\phi_0](\xi), \quad \tau>0, \qquad (1.5)\\ q(x,\xi,\tau) &= 0, \qquad \tau\le 0, \end{aligned}$$

where the series in (1.5) converges uniformly on $K \times \partial_M D \times [\delta, \infty)$ for any compact subset K of D and $\delta > 0$. Furthermore,

$$q > 0 \quad \text{on} \quad D \times \partial_M D \times (0, \infty), \tag{1.6}$$

$$(\partial_t + L) q(\cdot, \xi, \cdot) = 0 \quad \text{on} \quad D \times (-\infty, \infty). \tag{1.7}$$

Conversely, for any Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that λ is supported by $\partial_m D \times [0, T)$ and

$$\int_D p(x^0, y, t) d\mu(y) < \infty, \quad 0 < t < T, \tag{1.8}$$

$$\int_{\partial_M D \times [0,t)} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad 0 < t < T, \tag{1.9}$$

where x^0 is a fixed point in D, the right hand side of (1.4) is a nonnegative solution of (1.1).

A preliminary version of Theorem 1.2 was announced at the International Workshop on Potential Theory 2004 in Matsue [46].

The proof of this theorem will be given in Section 4. It is based upon the abstract parabolic Martin representation theorem and Choquet's theorem (see [30], [34], [49]), and its key step is to identify the parabolic Martin boundary.

Remark 1.3 We can also establish an integral representation theorem for nonnegative solutions of

$$(\partial_t + L) u = 0 \quad \text{in} \quad D \times (0, \infty), \tag{1.10}$$

which is completely analogous to Theorem 1.2: Assume [IU]. Then the conclusions of Theorem 1.2 hold with T replaced by ∞.

Here, in order to illustrate a scope of Theorem 1.2, we give simple examples. Further examples will be given in Section 5.

Example 1.4 Let $\alpha \in \mathbf{R}$ and

$$L = -\Delta + (1 + |x|^2)^{\alpha/2} \quad \text{on} \quad D = \mathbf{R}^n.$$

Then [UP] holds for (1.1) if and only if $\alpha \le 2$; while [IU] is satisfied if and only if $\alpha > 2$ (see [40]). (Concerning [IU] for more general Schrödinger operators, see [9], [18], [19], [40], [43].)

(i) Suppose that $\alpha \le 2$. Then for any nonnegative solution u of (1.1) there exists a unique Borel measure μ on D such that

$$u(x,t) = \int_D p(x,y,t)d\mu(y), \quad x \in D,\ 0 < t < T. \tag{1.11}$$

Conversely, for any Borel measure μ on D satisfying (1.8), the right hand side of (1.11) is a nonnegative solution of (1.1).

(ii) Suppose that $\alpha > 2$. Then the conclusions of Theorem 1.2 hold with

$$\partial_M D = \partial_m D = \infty S^{n-1}, \tag{1.12}$$

where ∞S^{n-1} is the sphere at infinity of $\mathbf{R}^n$, and the Martin compactification D^* of $D = \mathbf{R}^n$ with respect to L is obtained by attaching a sphere S^{n-1} at infinity: $D^* = \mathbf{R}^n \sqcup \infty S^{n-1}$. (As for (1.12), see [37].)

Note that the Martin boundary $\partial_M D$ in the case $-2 < \alpha \le 2$ is also equal to that for $\alpha > 2$. Nevertheless, when [UP] holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters when [UP] does not hold.

Example 1.5 Let $A(x) = (a_{ij}(x))_{i,j=1}^n$, $n \ge 2$, be a symmetric matrix-valued measurable function on $\mathbf{R}^n$ satisfying $\lambda \le A(x) \le \lambda^{-1}$, $x \in \mathbf{R}^n$, for some positive constant λ. Let

$$L = -\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j), \tag{1.13}$$

where $\partial_j = \partial/\partial x_j$. Let D be a bounded John domain in $\mathbf{R}^n$, i.e., D is a bounded domain, and there exist a point $z^0 \in D$ and a positive constant c_J such that each $z \in D$ can be joined to z^0 by a rectifiable curve $\gamma(t), 0 \le t \le 1$, with $\gamma(0) = z,\ \gamma(1) = z^0,\ \gamma \subset D$, and

$$\mathrm{dist}(\gamma(t), \partial D) \ge c_J \ell(\gamma[0,t]), \quad 0 \le t \le 1,$$

where $\ell(\gamma[0,t])$ is the length of the curve $\gamma(s), 0 \le s \le t$. Then the condition [IU] is satisfied (see Theorem 1 and Corollary 2.6 of [9]). Thus the conclusions of Theorem 1.2 hold.

Note that the Martin boundary $\partial_M D$ of D with respect to L may be different from the topological boundary ∂D in $\mathbf{R}^n$, although they coincide if ∂D is not bad, for example, if D is a Lipschitz domain (see [1], [3], [4], [16] and references therein).

This example is a generalization of the integral representation theorem for a Lipschitz cylinder by Fabes-Garofalo-Salsa [20], although they use kernel functions instead of $q(x, \xi, t - s)$. The integral representation (1.4) via $q(x, \xi, t-s)$ is more explicit, and seems to be new even for a Lipschitz cylinder.

Example 1.6 Let L be the elliptic operator (1.13). For $\beta \in \mathbf{R}$, put

$$D = \{(x_1, x') \in \mathbf{R}^n ; x_1 > 1, |x'| < x_1^\beta\}.$$

Then [IU] is satisfied if and only if $\beta < -1$. Indeed, if $\beta < -1$, then D is a uniformly Hölder domain of order $1 - 1/\beta < 2$ (see [9] and (5.28) of [41]); thus [IU] is satisfied by Theorem 1.(a) of [9] (see also Remark 1 after Theorem 9.6 of [19]). If $\beta \geq -1$, then any nonnegative solution of the initial and boundary value problem

$$\begin{aligned} (\partial_t + L)u &= 0 \quad \text{in } D \times (0, T), \\ u(x, 0) &= 0 \quad \text{on } D, \\ u(x, t) &= 0 \quad \text{on } \partial D \times (0, T) \end{aligned}$$

must be identically zero (see Theorem 1.1 of [41]); thus [IU] is not satisfied by Theorem 3.1.D of [39]. For a more direct proof, see Theorem 6 of [11].

Now, suppose that $\beta < -1$. Then the conclusions of Theorem 1.2 hold with

$$\partial_M D = \partial_m D = \partial D \cup \{\infty\}, \tag{1.14}$$

where ∞ is the point at infinity outside of $\overline{D}$. Here (1.14) can be shown by the boundary Harnack principle and the scaling argument as in Appendix of [38].

As for the case $\beta \geq -1$, the structure of nonnegative solution of (1.1) will be determined elsewhere.

The remainder of this paper is organized as follows. Sections 2, 3 and 4 are devoted to proofs of Theorems 1.1 and 1.2. In Section 2 we describe the abstract parabolic Martin boundary $\partial_M^\beta Q$ of $Q = D \times (0, T)$ with respect to

$\partial_t + L$ and a measure β on Q, and establish without assuming [IU] an abstract integral representation theorem for nonnegative solutions to the parabolic equation (1.1). It asserts that for any nonnegative solution u of (1.1) there exists a unique Borel measure μ on D such that

$$v(x,t) \equiv u(x,t) - \int_D p(x,y,t) d\mu(y)$$

is a nonnegative solution of (1.1) with zero initial value, and v is represented uniquely by an integral on $\partial_M^\beta Q \setminus D \times \{0\}$ (see Theorem 2.1 in Section 2). In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 on the basis of the abstract integral representation theorem in Section 2. In Section 5 we give several concrete examples as applications of Theorem 1.2. Finally, in Section 6 we give an integral representation theorem for nonnegative solutions of the equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times (-\infty, 0)$$

(see Theorem 6.1 in Section 6); since it is of independent interest and can be shown in the same way as Theorem 1.2.

2 Parabolic Martin boundary

In this section we describe the parabolic Martin boundary, and give an abstract integral representation theorem for nonnegative solutions to the parabolic equation (1.1). Throughout this section we do not assume the condition [IU].

For $x \in D$ and $r > 0$, we denote by $B(x,r)$ the geodesic ball in the Riemannian manifold M with center x and radius r. Let x^0 be a reference point in D. Choose a nonnegative continuous function a on D such that $a(x) = 1$ on $B(x^0, r^0)$ and $a(x) = 0$ outside $B(x^0, 2r^0)$ for some $r^0 > 0$ with $B(x^0, 3r^0) \Subset D$. Choose a nonnegative continuous function b on $\mathbf{R}$ such that $b(t) > 0$ on $(T/2, T)$ and $b(t) = 0$ on $\mathbf{R} \setminus (T/2, T)$. Denote by β the measure defined by $d\beta(x,t) = a(x)b(t)m(x)\, d\nu(x)dt$. For any nonnegative measurable function u on $Q = D \times (0,T)$, we write

$$\beta(u) = \iint_Q u(x,t)\, d\beta(x,t).$$

Denote by $P(Q)$ the set of all nonnegative solutions of (1.1), and put

$$P_\beta(Q) = \{u \in P(Q); \beta(u) < \infty\}.$$

Note that for any $u \in P(Q)$ there exists a function b as above such that $\beta(u) < \infty$; thus $P(Q) = \bigcup_\beta P_\beta(Q)$. Furthermore, the parabolic Harnack inequality shows that if $\beta(u) = 0$, then $u = 0$.

Throughout this section we fix a measure β. Let us define the β-Martin boundary $\partial_M^\beta Q$ of Q with respect to $\partial_t + L$ along the line given in [34] and [30]. Put

$$\begin{aligned} p(x,t;y,s) &= p(x,y,t-s), \quad t > s,\ x,y \in D, \\ p(x,t;y,s) &= 0, \quad\quad\quad\quad\quad\ \ t \le s,\ x,y \in D. \end{aligned}$$

Define the β-Martin kernel K_β by

$$K_\beta(x,t;y,s) = \frac{p(x,t;y,s)}{\beta\left(p(\,\cdot\,;y,s)\right)}, \quad (x,t),\ (y,s) \in Q,$$

where $\beta\left(p(\,\cdot\,;y,s)\right) = \iint_Q p(z,r;y,s)\,d\beta(z,r)$. Let $\{D_j\}_{j=1}^\infty$ be an exhaustion of D such that each D_j is a domain with smooth boundary, $D_j \Subset D_{j+1} \Subset D$, $\bigcup_{j=1}^\infty D_j = D$, and $B(x^0, 3r^0) \Subset D_1$. Put $Q_j = D_j \times (T/4j, T(1-1/4j))$. For $Y = (y,s),\ Z = (z,r) \in Q$, let

$$\delta_\beta(Y,Z) = \sum_{j=1}^\infty 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X;Y) - K_\beta(X;Z)|}{1 + |K_\beta(X;Y) - K_\beta(X;Z)|}.$$

Then we see that δ_β is a metric on Q, and the topology on Q induced by δ_β is equivalent to the original topology of Q. Denote by $Q^{\beta *}$ the completion of Q with respect to the metric δ_β. Put $\partial_M^\beta = Q^{\beta *} \setminus Q$. A sequence $\{Y^k\}_{k=1}^\infty$ in Q is called a fundamental sequence if $\{Y^k\}_{k=1}^\infty$ has no point of accumulation in Q and $\left\{K_\beta(\,\cdot\,;Y^k)\right\}_{k=1}^\infty$ converges uniformly on any compact subset of Q to a nonnegative solution of (1.1). By the local a priori estimates for solutions of (1.1), for any $\Xi \in \partial_M^\beta Q$ there exist a unique nonnegative solution $K_\beta(\,\cdot\,;\Xi)$ of (1.1) and a fundamental sequence $\{Y^k\}_{k=1}^\infty$ in Q such that

$$\lim_{k\to\infty} \sum_{j=1}^\infty 2^{-j} \sup_{X \in Q_j} \frac{\left|K_\beta(X;Y^k) - K_\beta(X;\Xi)\right|}{1 + |K_\beta(X;Y^k) - K_\beta(X;\Xi)|} = 0.$$

Thus the metric δ_β is canonically extended to $Q^{\beta *}$. Furthermore, $Q^{\beta *}$ becomes a compact metric space, since by the parabolic Harnack inequality, any sequence $\{Y^k\}_{k=1}^\infty$ with no point of accumulation in Q has a fundamental subsequence. We call $K_\beta(\,\cdot\,;\Xi)$, $\partial_M^\beta Q$ and $Q^{\beta *}$ the β-Martin kernel, β-Martin boundary and β-Martin compactification for $(Q,\ \partial_t + L)$, respectively. Note that $\beta\left(K_\beta(\,\cdot\,;\Xi)\right) \le 1$ by Fatou's lemma; and so $K_\beta(\,\cdot\,;\Xi) \in P_\beta(Q)$. A nonnegative solution $u \in P_\beta(Q)$ is said to be minimal if for any nonnegative solution $v \le u$ there exists a nonnegative constant C such that $v = Cu$. Put

$$\partial_m^\beta Q = \left\{ \Xi \in \partial_M^\beta Q; K_\beta(\,\cdot\,;\Xi) \text{ is minimal and } \beta\left(K_\beta(\,\cdot\,;\Xi)\right) = 1 \right\},$$

which we call the minimal β-Martin boundary for $(Q,\ \partial_t + L)$.

Observe that $D \times [0,T)$ is embedded into $Q^{\beta *}$, and $D \times \{0\} \subset \partial_M^\beta Q$. Indeed, with $y \in D$ fixed, for any sequence $\{Y^k\}_{k=1}^\infty$ in Q with $\lim_{k\to\infty} Y^k = (y,0)$ we have $\lim_{k\to\infty} K_\beta(x,t;Y^k) = p(x,t;y,0)/\beta\left(p(\,\cdot\,;y,0)\right)$; furthermore, $K_\beta(\,\cdot\,;y,0) \ne K_\beta(\,\cdot\,;z,0)$ if $y \ne z$. We also note that any sequence $\left\{Y^k = (y^k, s^k)\right\}_{k=1}^\infty$ in Q with $\lim_{k\to\infty} s^k = T$ is a fundamental sequence, since $\lim_{k\to\infty} K_\beta(\,\cdot\,;Y^k) = 0$. We denote by ϖ the point in $\partial_M^\beta Q$ corresponding to the Martin kernel which is identically zero : $K_\beta(\,\cdot\,;\varpi) = 0$. Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus \left(D \times \{0\} \cup \{\varpi\}\right).$$

We are now ready to state a main theorem of this section.

Theorem 2.1 For any $u \in P_\beta(Q)$, there exists a unique pair of finite Borel measures κ on D and λ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that λ is supported by the set $\mathcal{L}_m^\beta Q$,

$$u(x,t) = \int_D \frac{p(x,t;y,0)}{\beta\left(p(\,\cdot\,;y,0)\right)}\, d\kappa(y) + \int_{\mathcal{L}_m^\beta Q} K_\beta(x,t;\Xi)\, d\lambda(\Xi), \tag{2.1}$$

for any $(x,t) \in Q$, and

$$\beta(u) = \kappa(D) + \lambda(\mathcal{L}_m^\beta Q). \tag{2.2}$$

Conversely, for any finite Borel measures κ on D and λ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that λ is supported by the set $\mathcal{L}_m^\beta Q$, the right hand side of (2.1) belongs to $P_\beta(Q)$.

This theorem may be shown via the theory of Martin boundaries of general harmonic spaces in the axiomatic potential theory (see Remark 2.7 at the end of this section), but it can not be found in the literature. We rather show Theorem 2.1 directly by making use of Choquet's theorem. We show only the first half of it, since the second half can be shown easily. The proof is decomposed into several steps. We denote by $C_0^0(D)$ the set of all continuous functions with compact support in D, and put $C_0^{0,+}(D) = \{u \in C_0^0(D); u \geq 0\}$. We start with the following decomposition lemma.

Lemma 2.2 For any $u \in P(Q)$, there exists a unique Borel measure μ on D such that

$$u(x,t) = \int_D p(x,t;y,0)\,d\mu(y) + v(x,t), \quad (x,t) \in Q, \tag{2.3}$$

where v is a nonnegative solution of the equation $(\partial_t + L)v = 0$ in $D \times (-\infty, T)$ which vanishes on $D \times (-\infty, 0]$. (Thus, if [UP] holds for (1.1), then (2.3) holds with $v = 0$.) Furthermore, for any $\varphi \in C_0^0(D)$,

$$\lim_{t\downarrow 0} \int_D u(x,t)\varphi(x)m(x)\,d\nu(x) = \int_D \varphi(y)\,d\mu(y). \tag{2.4}$$

In addition, if $u \in P_\beta(Q)$, then $\int_D \beta\left(p(\,\cdot\,;y,0)\right)\,d\mu(y) < \infty$ and $v \in P_\beta(Q)$.

Proof The first assertion of this lemma is a direct consequence of Theorem 4.2 of [6], and the third assertion follows from it by the Fubini theorem. We only show (2.4), which also implies the uniqueness of the representing measure μ. It suffices to show (2.4) for any $\varphi \in C_0^{0,+}(D)$ with Supp $\varphi \subset D_k$ for some k. Put

$$\Phi(y,t) = \int_D p(x,t;y,0)\varphi(x)m(x)\,d\nu(x), \quad (y,t) \in Q.$$

We claim that Φ is bounded on $D_k \times (0, T/2]$, and

$$\lim_{t\downarrow 0} \Phi(y,t) = \varphi(y), \quad y \in D_k.$$

Indeed, the extension principle and the local regularity of solutions show (see Lemma 8 of [8] and [28]) that

$$\psi(y,t) = \int_{D_k} p(x,t;y,0)m(x)\,d\nu(x)$$

is bounded on $D_k \times (0, T/2]$, and for any $y \in D_k$ and $\delta > 0$,

$$\lim_{t \downarrow 0} \psi(y,t) = 1,$$
$$\lim_{t \downarrow 0} \int_{D_k \setminus B(y,\delta)} p(x,t;y,0) m(x)\, d\nu(x) = 0.$$

This implies the claim. Put

$$w(x,t) = \int_{D \setminus D_k} p(x,t;y,0)\, d\mu(y) + v(x,t), \quad (x,t) \in D \times (0,T),$$
$$w(x,t) = 0, \qquad (x,t) \in D_k \times (-\infty, 0].$$

Then w is a continuous function on $D \times (0,T) \cup D_k \times (-\infty, 0]$. We have

$$\int_D u(x,t)\varphi(x)m(x)\, d\nu(x) = \int_{D_k} \Phi(y,t)\, d\mu(y) + \int_{D_k} w(x,t)\varphi(x)m(x)\, d\nu(x).$$

Since $\mu(D_k) < \infty$, the Lebesgue dominated convergence theorem together with the above claim shows (2.4). □

We put

$$P^0_\beta(Q) = \left\{ v \in P_\beta(Q); \lim_{t \downarrow 0} v(x,t) = 0 \text{ on } D \right\}.$$

By Lemma 2.2, for any $u \in P_\beta(Q)$ there exists a unique Borel measure κ on D such that $d\kappa(y) = \beta\left(p(\,\cdot\,;y,0)\right) d\mu(y)$, $\kappa(D) < \infty$, and a function v defined by

$$v(x,t) = u(x,t) - \int_D \frac{p(x,t;y,0)}{\beta\left(p(\,\cdot\,;y,0)\right)}\, d\kappa(y)$$

belongs to $P^0_\beta(Q)$. Thus it suffices to show (2.1) for $u \in P^0_\beta(Q)$. Obviously, if [UP] holds for (1.1), then $P^0_\beta(Q) = \{0\}$; and so Lemma 2.2 already shows Theorem 2.1.

Lemma 2.3 Let $u \in P^0_\beta(Q)$. For any $j < l$, there exists a finite Borel measure $\lambda_{j,l}$ on $\partial D_j \times [0,T)$ such that

$$u(x,t) = \int_{\partial D_j \times [0,T)} \frac{p_l(x,t;y,s)}{\beta\left(p_l(\,\cdot\,;y,s)\right)}\, d\lambda_{j,l}(y,s), \quad (x,t) \in D_j \times (0,T), \tag{2.5}$$

and $\lambda_{j,l}\left(\partial D_j \times [0,T)\right) = \beta(u)$. Here p_l is the Green function for $\partial_t + L$ on $D_l \times (0,T)$.

Proof For the time being, we make a temporary assumption that the coefficients of L and the Riemannian metric are smooth. Then u is smooth on $D \times [0, T)$. (For basic results on parabolic equations, see [8], [22], [28], [32], [33].) Let v be the solution of the initial and boundary value problem:

$$\begin{aligned}
&(\partial_t + L)v = 0 && \text{in} \quad (D_l \setminus \overline{D_j}) \times (0, T), \\
&v(x, 0) = 0 && \text{on} \quad D_l \setminus \overline{D_j}, \\
&v = u && \text{on} \quad \partial D_j \times [0, T), \\
&v = 0 && \text{on} \quad \partial D_l \times [0, T).
\end{aligned}$$

By the maximum principle, $0 \le v \le u$ on $(\overline{D_l} \setminus D_j) \times [0, T)$. Define w by

$$\begin{aligned}
&w = u - v && \text{in} \quad (D_l \setminus \overline{D_j}) \times [0, T), \\
&w = 0 && \text{on} \quad \left(\overline{D_j} \times (-\infty, T)\right) \bigcup \left((D_l \setminus \overline{D_j}) \times (-\infty, 0)\right).
\end{aligned}$$

Let us show that

$$(\partial_t + L)w \le 0 \quad \text{in} \quad E_l \equiv D_l \times (-\infty, T).$$

For $\varepsilon > 0$, put $w_\varepsilon = (w^2 + \varepsilon^2)^{1/2}$. Let ψ be any nonnegative function in $C_0^\infty(E_l)$. Since

$$\frac{w}{w_\varepsilon}\psi \in C_0^1\left((-\infty, T); L^2\left(D_l \setminus \overline{D_j}\right)\right) \bigcap C_0^0\left((-\infty, T); H_0^1\left(D_l \setminus \overline{D_j}\right)\right)$$

and w satisfies $(\partial_t + L)w = 0$ in $\left(D_l \setminus \overline{D_j}\right) \times (-\infty, T)$, we have

$$\begin{aligned}
0 &= \iint_{E_l} [(\partial_t + L)w] \left(\frac{w}{w_\varepsilon}\psi\right) m \, d\nu dt \\
&= \iint_{E_l} \left\{-w\partial_t\left(\frac{w}{w_\varepsilon}\psi\right) + \left\langle A\nabla w, \nabla\left(\frac{w}{w_\varepsilon}\psi\right)\right\rangle + Vw\left(\frac{w}{w_\varepsilon}\psi\right)\right\} m \, d\nu dt \\
&\ge \iint_{E_l} \left\{-\left[\frac{\varepsilon^2}{w^2 + \varepsilon^2}\frac{w}{w_\varepsilon}\partial_t w\right]\psi - \frac{w}{w_\varepsilon}(w\partial_t\psi) + \langle A\nabla w_\varepsilon, \nabla\psi\rangle \right. \\
&\qquad\qquad \left. + Vw\left(\frac{w}{w_\varepsilon}\psi\right)\right\} m \, d\nu dt.
\end{aligned}$$

Here we have used the pointwise inequality

$$\left\langle A\nabla w, \nabla\left(\frac{w}{w_\varepsilon}\psi\right)\right\rangle \geq \langle A\nabla w_\varepsilon, \nabla\psi\rangle .$$

We see that $0 \leq w/w_\varepsilon \leq 1$, $0 \leq \varepsilon^2/(w^2+\varepsilon^2) \leq 1$, $\lim_{\varepsilon\to 0} w/w_\varepsilon = 1$ and $\lim_{\varepsilon\to 0}\varepsilon^2/(w^2+\varepsilon^2) = 0$ on $\{w > 0\}$. Thus, by the Lebesgue dominated convergence theorem

$$0 \geq \iint_{E_l} \{-w\partial_t\psi + \langle A\nabla w, \nabla\psi\rangle + Vw\psi\}\, m\, d\nu dt.$$

Hence $(\partial_t + L)w \leq 0$ in the weak sense; and $\mu = (\partial_t + L)(u-w)$ is a Borel measure on $D_l \times (-\infty, T)$ supported by $\partial D_j \times [0,T)$. Therefore,

$$(u-w)(x,t) = \int_{\partial D_j\times[0,T)} p_l(x,t;y,s)\, d\mu(y,s), \quad (x,t) \in D_l \times (0,T).$$

Since $u - w = u$ in $D_j \times [0,T)$, this yields (2.5) under the smoothness assumption. By the Fubini theorem, (2.5) implies $\lambda_{j,l}(\partial D_j \times [0,T)) = \beta(u) < \infty$.

In order to treat the general case, we use the regularization argument as in [8] and [32]. Consider a series of elliptic operators $\{L^k\}_{k=1}^\infty$ whose coefficients are smooth in D_{l+2} and converge to those of L *a.e.* in D_{l+2} as $k \to \infty$. Let u^k be a solution of the initial and boundary value problem:

$$\begin{aligned} &(\partial_t + L^k)u^k = 0 && \text{in} \quad D_{l+1} \times (0,T), \\ &u^k(x,0) = 0 && \text{on} \quad D_{l+1}, \\ &u^k = u && \text{on} \quad \partial D_{l+1} \times [0,T). \end{aligned}$$

Let p_l^k be the Green function for $\partial_t + L^k$ on $D_l \times (0,T)$. Then the above argument shows that there exists a Borel measure $\lambda_{j,l}^k$ on $\partial D_j \times [0,T)$ such that

$$u^k(x,t) = \int_{\partial D_j\times[0,T)} \frac{p_l^k(x,t;y,s)}{\beta\left(p_l^k(\,\cdot\,;y,s)\right)}\, d\lambda_{j,l}^k(y,s), \quad (x,t) \in D_j \times (0,T), \quad (2.6)$$

and $\lambda_{j,l}^k(\partial D_j \times [0,T)) = \beta(u)$. Choose a subsequence of $\left\{\lambda_{j,l}^k\right\}_{k=1}^\infty$ which converges weakly on $\partial D_j \times [0, T-\delta]$ for any $\delta > 0$. For simplicity, we also denote the subsequence by $\left\{\lambda_{j,l}^k\right\}_{k=1}^\infty$. Note that with (x,t) fixed, $\lim_{k\to\infty} u^k(x,t) =$

$u(x,t)$, and $\left\{p_l^k(x,t;\,\cdot\,,\,\cdot\,)\right\}_{k=1}^{\infty}$ converges uniformly to $p_l(x,t;\,\cdot\,,\,\cdot\,)$ as $k \to \infty$ on $\partial D_j \times [0,t]$. Furthermore, $\left\{\beta\left(p_l^k(\,\cdot\,;y,s)\right)\right\}_{k=1}^{\infty}$ converges uniformly to $\beta\left(p_l(\,\cdot\,;y,s)\right)$ as $k \to \infty$ on $\partial D_j \times [0, T-\delta]$ for any $\delta > 0$. Thus, letting $k \to \infty$ in (2.6), we get (2.5) in the general case. This completes the proof. □

Proposition 2.4 For any $u \in P_\beta^0(Q)$, there exists a finite Borel measure λ on $\partial_M^\beta Q$ such that λ is supported by $\partial_M^\beta Q \setminus (D \times \{0\} \cup \{\varpi\})$,

$$u(x,t) = \int_{\partial_M^\beta Q} K_\beta(x,t;\Xi)\, d\lambda(\Xi), \quad (x,t) \in Q, \tag{2.7}$$

and $\lambda(\partial_M^\beta Q) = \beta(u)$.

Proof Letting $l \to \infty$ in (2.5), we obtain that there exists a finite Borel measure λ_j on $\partial D_j \times [0,T)$ such that

$$u(x,t) = \int_{\partial D_j \times [0,T)} K_\beta(x,t;y,s)\, d\lambda_j(y,s), \quad (x,t) \in D_j \times (0,T), \tag{2.8}$$

and $\lambda_j\left(\partial D_j \times [0,T)\right) = \beta(u)$. Then, by choosing a subsequence of $\{\lambda_j\}_{j=1}^{\infty}$ if necessary, we get (2.7) from (2.8). □

Put $P_{\beta,1}^0(Q) = \left\{u \in P_\beta^0(Q); \beta(u) \le 1\right\}$. Then we see that $P_{\beta,1}^0(Q)$ is compact and convex. Let $ex\, P_{\beta,1}^0(Q)$ be the set of extreme points of $P_{\beta,1}^0(Q)$. It is known that $ex\, P_{\beta,1}^0(Q)$ is a Borel subset of $P_{\beta,1}^0(Q)$. The following lemma says that $ex\, P_{\beta,1}^0(Q)$ is described in terms of β-Martin kernels.

Lemma 2.5 $ex\, P_{\beta,1}^0(Q) \setminus \{0\} = \left\{K_\beta(\,\cdot\,;\Xi); \Xi \in \mathcal{L}_m^\beta(Q)\right\}$.

Proof For self-containedness we give a proof. We first claim that

$$ex\, P_{\beta,1}^0(Q) \setminus \{0\} = \left\{u \in P_\beta^0(Q); u \text{ is minimal, and } \beta(u) = 1\right\}. \tag{2.9}$$

Suppose that $u \in ex\, P_{\beta,1}^0(Q) \setminus \{0\}$. Let $\alpha = \beta(u)$. Then $\alpha = 1$; if not, $0 < \alpha < 1$ and $u = \alpha(u/\alpha) + (1-\alpha)0$; which is a contradiction. Let us show that if $v \in P_\beta^0(Q)$ satisfies $v \le u$ in Q, then $v = \gamma u$ with $\gamma = \beta(u)$. When $\gamma = 0$, $v = 0$ and $v = \gamma u$. When $0 < \gamma < 1$,

$$u = (1-\gamma)\left(\frac{u-v}{1-\gamma}\right) + \gamma\left(\frac{v}{\gamma}\right).$$

Since u is extreme, $v/\gamma = u$. Thus $v = \gamma u$. When $\gamma = 1$, $\beta(u - v) = 0$; and $v = u$. Conversely, suppose that $u \in P^0_\beta(Q)$ is minimal and $\beta(u) = 1$. Assume that there exist $0 < \alpha < 1$ and $v, w \in P^0_{\beta,1}(Q)$ such that $u = \alpha v + (1 - \alpha)w$. Since $\alpha v \le u$, there exists a constant C such that $\alpha v = Cu$. But $\beta(v) = 1$, for $1 = \beta(u) = \alpha\beta(v) + (1 - \alpha)\beta(w)$. Thus $\alpha = C$, and $v = u$. Similarly, $w = u$. Hence $u \in ex\, P^0_{\beta,1}(Q)$. This completes the proof of the claim.

Let $u \in P^0_\beta(Q)$ be minimal and $\beta(u) = 1$. By Proposition 2.4, there exists a probability measure λ satisfying (2.7). We claim that $\mathrm{Supp}\,\lambda$, the support of λ, consists of a single point. Let us show the claim along the line given in the proof of Lemma 12.12 of [24] (see also Proposition 1.4 of [49]). Since $\lambda(\partial^\beta_M Q) = 1$ and $\beta\left(K_\beta(\,\cdot\,;\Xi)\right) \le 1$, λ is supported by

$$E = \left\{\Xi \in \partial^\beta_M Q; \beta\left(K_\beta(\,\cdot\,;\Xi)\right) = 1\right\} \setminus \left(D \times \{0\} \cup \{\varpi\}\right).$$

Let $\xi \in \mathrm{Supp}\,\lambda$. For any natural number j, let $U_j = \left\{\eta \in Q^{\beta *}; \delta_\beta(\xi, \eta) < j^{-1}\right\}$ and $\lambda_j = \lambda|_{U_j}$. Then $\lambda_j(U_j) > 0$ and λ_j is supported by $E \cap U_j$. By the minimality of u,

$$u(x,t) = \lambda_j(U_j)^{-1} \int_{U_j} K_\beta(x,t;\Xi)\, d\lambda_j(\Xi).$$

Since $\{\lambda_j(U_j)^{-1}\lambda_j\}_{j=1}^\infty$ converges vaguely to the measure concentrated on one point ξ, $u = K_\beta(\,\cdot\,;\xi)$. Now, suppose that there exists $\eta \in \mathrm{Supp}\,\lambda \setminus \{\xi\}$. Then $K_\beta(\,\cdot\,;\eta) = u = K_\beta(\,\cdot\,;\xi)$, which is a contradiction. This proves the claim. Hence $u = K_\beta(\,\cdot\,;\Xi)$ for some $\Xi \in \mathcal{L}^\beta_m Q$. This implies the right hand side of (2.9) is equal to the set $\left\{K_\beta(\,\cdot\,;\Xi); \Xi \in \mathcal{L}^\beta_m Q\right\}$. The proof of Lemma 2.5 is now complete. □

Lemma 2.6 The cone $P^0_\beta(Q)$ is a lattice, i.e., for any $u, v \in P^0_\beta(Q)$ there exist the greatest lower bound $u \wedge v \in P^0_\beta(Q)$ and the least upper bound $u \vee v \in P^0_\beta(Q)$ of u and v.

Proof We only show the existence of $u \wedge v$. For $j = 1, 2, \cdots$, let w_j be a solution of the initial and boundary value problem:

$$\begin{aligned} &(\partial_t + L)w_j = 0 && \text{in} \quad D_j \times (0,T), \\ &w_j = \min(u,v) && \text{on} \quad \partial D_j \times (0,T), \\ &w_j = 0 && \text{on} \quad D_j \times \{0\}. \end{aligned}$$

By the maximum principle, $w_j \geq w_{j+1}$ in $D_j \times (0,T)$. By the a priori estimates, the sequence $\{w_j\}_{j=1}^{\infty}$ converges to a solution $w \in P_\beta^0(Q)$ uniformly on any compact subset of $D \times [0,T)$. By the maximum principle, w is the greatest lower bound in $P_\beta^0(Q)$ of u and v. □

We are now ready to complete a proof of Theorem 2.1.

Proof of Theorem 2.1 It suffices to show (2.1) and (2.2) for $u \in P_\beta^0(Q)$ with $\beta(u) = 1$, since any $v \in P_\beta^0(Q) \setminus \{0\}$ satisfies $\beta(v/\beta(v)) = 1$. Put $X = \{u \in P_\beta^0(Q); \beta(u) \leq 1\}$ and $F = C^0\left(D \times [0,T)\right)$. Let $\widetilde{X}$ be the cone generated by $X \times \{1\}$ in $F \times \mathbf{R}$. Then, by the proof of Proposition 11.3 of [49], Lemma 2.6 implies that $\widetilde{X}$ is a lattice. By virtue of Choquet's Theorem on p. 70 of [49], for each u in X with $\beta(u) = 1$ there exists a unique probability measure λ on X which is supported by $ex\, X \setminus \{0\}$, and represents $(u,1) \in \widetilde{X}$, i.e.,

$$f(u,1) = \int_X f(v,1)\, d\lambda(v)$$

for any continuous linear functional f on $F \times \mathbf{R}$. Note that for any $(x,t) \in Q$ the functional

$$f(v,r) = v(x,t), \qquad (v,r) \in F \times \mathbf{R},$$

is a continuous linear functional on $F \times \mathbf{R}$. Thus we have

$$u(x,t) = \int_X v(x,t)\, d\lambda(v).$$

By Lemma 2.5, λ is supported by $\left\{K_\beta(\,\cdot\,;\Xi); \Xi \in \mathcal{L}_m^\beta Q\right\}$. By identifying $ex\, X \setminus \{0\}$ with $\mathcal{L}_m^\beta Q$, we obtain that

$$u(x,t) = \int_{\mathcal{L}_m^\beta Q} K_\beta(x,t;\Xi)\, d\lambda(\Xi).$$

This proves (2.1) and (2.2) for $u \in P_\beta^0(Q)$ with $\beta(u) = 1$. □

We conclude this section with a remark on another proof of Theorem 2.1 via the axiomatic potential theory.

Remark 2.7 J. Bliedner informed the author through K. Janssen that the space associated with solutions of the parabolic equation $(\partial_t + L)u = 0$ can be shown to be a Bauer harmonic space by combining results on the

Brelot harmonic space for elliptic equations (cf. [17] and [25]) and results on the product of the semigroup e^{-tL} and the uniform motion on $\mathbf{R}$ (cf. [13]), and that such a proof can not be found in the literature. If the space for $\partial_t + L$ is the Bauer harmonic space, then Theorem 2.1 can be derived from Theorem 5.1 of [34] by the potential theoretic method.

3 [IU] implies [SP]

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 Let $a < \lambda_0$. Let G be the Green function of $L - a$ on D. For any $\delta > 0$, put

$$G_\delta(x,y) = \int_\delta^\infty e^{at} p(x,y,t)\,dt,$$
$$G^\delta(x,y) = \int_0^\delta e^{at} p(x,y,t)\,dt.$$

Then $G = G_\delta + G^\delta$. Let us show that

$$\int_D G(x,y)G^\delta(y,z)\,d\mu(y) \le \delta G(x,z), \quad x,z \in D, \tag{3.1}$$

where $d\mu(y) = m(y)\,d\nu(y)$. We have

$$\begin{aligned}
&\int_D G(x,y)G^\delta(y,z)\,d\mu(y)\\
&= \int_D d\mu(y) \int_0^\infty e^{at} p(x,y,t)\,dt \int_0^\delta e^{ar} p(y,z,r)\,dr\\
&= \int_0^\delta dr \int_0^\infty dt\, e^{a(t+r)} \int_D p(x,y,t)p(y,z,r)\,d\mu(y)\\
&= \int_0^\delta dr \int_0^\infty dt\, e^{a(t+r)} p(x,z,t+r)\\
&\le \delta G(x,z).
\end{aligned}$$

Here, note that we have not used the condition [IU] in proving (3.1), and (3.1) holds for any subcritical operator. Similarly,

$$\int_D G^\delta(x,y)G(y,z)\,d\mu(y) \le \delta G(x,z), \quad x,z \in D. \tag{3.2}$$

Thus, for any compact subset K of D

$$\begin{aligned}
&\int_{D\setminus K} G(x,y)G(y,z)\,d\mu(y) \\
&= \int_{D\setminus K} \left(G^\delta(x,y) + G_\delta(x,y)\right)\left(G^\delta(y,z) + G_\delta(y,z)\right)\,d\mu(y) \\
&\le 2\delta G(x,z) + \int_{D\setminus K} G_\delta(x,y)G_\delta(y,z)\,d\mu(y).
\end{aligned}$$

By Theorem 3.2 of [19], for any $\delta > 0$ there exists a positive constant C_δ such that

$$C_\delta^{-1}\phi_0(z)\phi_0(y) \le p(z,y,\delta) \le C_\delta\phi_0(z)\phi_0(y), \quad z,y \in D.$$

Let $t > \delta$. Multiply these by $p(x,z,t-\delta)$, and integrate them. Then we get

$$C_\delta^{-1}e^{-\lambda_0(t-\delta)}\phi_0(x)\phi_0(y) \le p(x,y,t) \le C_\delta e^{-\lambda_0(t-\delta)}\phi_0(x)\phi_0(y)$$

for any $x,y \in D$ and $t \ge \delta$. Thus,

$$\begin{aligned}
&C_\delta^{-1}I_\delta \le \frac{G_\delta(x,y)}{\phi_0(x)\phi_0(y)} \le C_\delta I_\delta, \quad x,y \in D, \\
&I_\delta = \int_\delta^\infty e^{at}e^{-\lambda_0(t-\delta)}\,dt.
\end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned}
&\int_{D\setminus K} G_\delta(x,y)G_\delta(y,z)\,d\mu(y) \\
&\le (C_\delta I_\delta)^2\,\phi_0(x)\phi_0(z)\int_{D\setminus K}\phi_0(y)^2\,d\mu(y) \\
&\le \left[(C_\delta I_\delta)^2\,C_1 I_1^{-1}\right]G(x,z)\int_{D\setminus K}\phi_0(y)^2\,d\mu(y).
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
&\int_{D\setminus K} G(x,y)G(y,z)\,d\mu(y) \\
&\le 2\delta G(x,z) + A_\delta G(x,z)\int_{D\setminus K}\phi_0(y)^2\,d\mu(y),
\end{aligned}$$

where $A_\delta = (C_\delta I_\delta)^2 C_1 I_1^{-1}$. For any $\varepsilon > 0$, let $\delta = \varepsilon/3$ and choose K such that

$$A_\delta \int_{D\setminus K} \phi_0(y)^2 \, d\mu(y) < \varepsilon/3.$$

Then

$$\int_{D\setminus K} G(x,y)G(y,z)\, d\mu(y) < \varepsilon G(x,z), \quad x, z \in D.$$

That is, 1 is a small perturbation of $L - a$ on D. □

For $b < \lambda_0$, denote by $G(x, y; b)$ the Green function of $L - b$ on D.

Proposition 3.1 Suppose that [IU] holds. Then, for any $b < \lambda_0$, there exists a positive constant C such that

$$C^{-1} G(x, y; a) \le G(x, y; b) \le C G(x, y; a), \quad x, y \in D. \tag{3.4}$$

Proof It is known that if 1 is a small perturbation of $L - a$ on D, then $G(x, y; b)$ is comparable with $G(x, y; a)$. Thus this proposition follows from Theorem 1.1. But we give a short and direct proof. By (3.3),

$$C^{-1} \le \frac{\int_1^\infty e^{at} p(x, y, t)\, dt}{\int_1^\infty e^{bt} p(x, y, t)\, dt} \le C \tag{3.5}$$

for some positive constant C. On the other hand,

$$e^{-|a-b|} \le \frac{\int_0^1 e^{at} p(x, y, t)\, dt}{\int_0^1 e^{bt} p(x, y, t)\, dt} \le e^{|a-b|}. \tag{3.6}$$

The inequalities (3.4) follow from (3.5) and (3.6). □

Remark 3.2 The converse of Theorem 1.1 does not hold. By Theorem 6.1 of [45], for any domain D in $\mathbf{R}^2$ with finite area 1 is a small perturbation of $-\Delta$ on D. But there exists bounded planar domains for which the heat semigroups are not intrinsically ultracontractive (see Example 1 on p.371 of [19] and Section 4 of [10]). Thus, for such planar domains, [IU] does not hold but [SP] holds.

4 Semi-concrete integral representations

Throughout this section we assume that the condition [IU] is satisfied, and prove Theorem 1.2. Theorem 1.2 gives explicit integral representations of nonnegative solutions of (1.1) provided that the Martin boundary $\partial_M D$ of D for $L - a$, $a < \lambda_0$, is determined explicitly. For simplicity of notations, we assume without loss of generality that $\lambda_0 > 0$ and $a = 0$.

Let $p(x, y, t)$ be the minimal fundamental solution for $\partial_t + L$ on $D \times (0, \infty)$. Extend $p(x, y, t)$ to $\{t \leq 0\}$ by $p(x, y, t) = 0$ there. We see that

$$(\partial_t + L_x)\, p(x, y, t) = 0 \quad \text{in} \quad D \times \mathbf{R} \setminus \{(y, 0)\}.$$

Lemma 4.1 For any $\xi \in \partial_M D$ there exists the limit

$$\lim_{D \ni y \to \xi} \frac{p(x, y, t)}{\phi_0(y)} \equiv q(x, \xi, t), \quad x \in D,\ t \in \mathbf{R}. \tag{4.1}$$

Furthermore, as functions of (x, t), $\{p(x, y, t)/\phi_0(y)\}_y$ converges to $q(x, \xi, t)$ as $y \to \xi$ uniformly on $K \times \mathbf{R}$ for any compact subset K of D.

Proof Obviously, for $t \leq 0$, (4.1) holds with $q(x, \xi, t) = 0$. We have only to show (4.1) for $t > 0$. We have

$$\frac{p(x, y, t)}{\phi_0(y)} = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \left[\phi_j(y)/\phi_0(y)\right].$$

By [IU], for any $\delta > 0$ there exists a constant C such that $p(x, y, \delta/3) \leq C\phi_0(x)\phi_0(y)$ for any $x, y \in D$. With $d\mu(x) = m(x)\, d\nu(x)$, we have for any $j = 1, 2, \cdots$

$$\begin{aligned} e^{-\lambda_j \delta/3} |\phi_j(y)| &= \left| \int_D p(x, y, \delta/3) \phi_j(x) d\mu(x) \right| \\ &\leq C \int_D \phi_0(x)\phi_0(y) |\phi_j(x)| d\mu(x) \leq C\phi_0(y). \end{aligned}$$

Thus, $|\phi_j(y)|/\phi_0(y) \leq C e^{\lambda_j \delta/3}$, $y \in D$, $j = 1, 2, \cdots$. Furthermore,

$$\sum_{j=0}^{\infty} e^{-\lambda_j \delta/3} < \infty.$$

Therefore, for any $t \geq \delta$ and $x, y \in D$

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \left|\phi_j(x)\phi_j(y)\right| / \phi_0(x)\phi_0(y)$$

$$\leq C^2 \sum_{j=0}^{\infty} e^{-\lambda_j t} e^{2\lambda_j \delta/3} = C^2 \sum_{j=0}^{\infty} e^{-\lambda_j \delta/3} < \infty. \qquad (4.2)$$

This implies that for any compact subset K of D

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} \left|\phi_j(x)\phi_j(y)\right| / \phi_0(y)$$

$$\leq C^2 \left(\sup_K \phi_0\right) \sum_{j=0}^{\infty} e^{-\lambda_j \delta/3}, \qquad x \in K,\ y \in D,\ t \geq \delta. \qquad (4.3)$$

By Theorem 1.1 and Theorem 6.3 of [53], ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ to the Martin compactification D^* of D with respect to L. Hence, by (4.3)

$$q(x, \xi, t) \equiv \lim_{D \ni y \to \xi} \frac{p(x, y, t)}{\phi_0(y)}$$

$$= \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \left[\phi_j/\phi_0\right](\xi), \qquad x \in D,\ t > 0, \qquad (4.4)$$

where the series in (4.4) converges uniformly on $K \times \partial_M D \times [\delta, \infty)$ for any compact subset K of D and $\delta > 0$. Furthermore, as functions of (x, t), $\{p(x, y, t)/\phi_0(y)\}_y$ converges to $q(x, \xi, t)$ uniformly on $K \times [\delta, \infty)$. Thus, it remains to show that $\{p(x, y, t)/\phi_0(y)\}_y$ converges uniformly on $K \times [0, 1/2]$. Let U and W be domains such that $K \Subset U \Subset W \Subset D$. Let us show that there exist positive constants C and α such that

$$p(x, y, t) \leq C e^{-\alpha/t} \phi_0(x)\phi_0(y), \quad x \in U,\ y \in D \setminus W,\ 0 < t < 1. \qquad (4.5)$$

By Theorem 4.1 of [43], there exist positive constants C_1 and α such that

$$p(x, y, t) \leq C_1 e^{-\alpha/t}, \quad x \in U,\ y \in \partial W,\ 0 < t < 1.$$

This implies that for some constant C

$$p(x, y, t) \leq C e^{-\alpha/t} \phi_0(x)\phi_0(y), \quad x \in U,\ y \in \partial W,\ 0 < t < 1. \qquad (4.6)$$

We see that with $x \in U$ fixed

$$\begin{aligned}
&(\partial_t + L_y)\, p(x,y,t) = 0 && \text{in} \quad \big(D \setminus \overline{W}\big) \times (0,1), \\
&p(x,y,0) = 0 && \text{on} \quad D \setminus \overline{W}, \\
&(\partial_t + L_y)\, \big(e^{-\alpha/t}\phi_0(y)\big) > 0 && \text{in} \quad \big(D \setminus \overline{W}\big) \times (0,1), \\
&\lim_{t \downarrow 0} e^{-\alpha/t}\phi_0(y) = 0 && \text{on} \quad D \setminus \overline{W}.
\end{aligned}$$

Since p is the minimal fundamental solution. the maximum principle together with (4.6) yields (4.5). Thus, the family $\{p(x,y,t)/\phi_0(y)\}_{y \in D \setminus W}$ of solutions in the variable $(x,t) \in U \times (-1,1)$ is uniformly bounded. Therefore, for any sequence $\{y^j\}_{j=1}^{\infty}$ in $D \setminus W$ converging to ξ there exists a subsequence $\{y^{j_k}\}_k$ such that $\{p(x,y^{j_k},t)/\phi_0(y^{j_k})\}_k$ converges uniformly on $K \times [0,1/2]$ to a nonnegative solution. But this limit must be $q(x,\xi,t)$, which is determined uniquely by ξ. Hence $\{p(x,y,t)/\phi_0(y)\}_y$ converges to $q(x,\xi,t)$ as $y \to \xi$ uniformly on $K \times [0,1/2]$. This completes the proof of Lemma 4.1. □

Lemma 4.2 (i) The function q on $D \times \partial_M D \times \mathbf{R}$ is continuous, $q(x,\xi,t) = 0$ for $t \le 0$, and $q(x,\xi,t) > 0$ for $t > 0$. Furthermore, $q(x,\xi,t)$ for $t > 0$ has the series expansion (4.4) which converges uniformly on $K \times \partial_M D \times [\delta, \infty)$ for any compact subset K of D and $\delta > 0$.
(ii) For any $\xi \in \partial_M D$ fixed, the function $q(x,\xi,t)$ satisfies the equation

$$(\partial_t + L_x)\, q(x,\xi,t) = 0 \quad \text{in} \quad D \times \mathbf{R}.$$

(iii) For any $\delta > 0$ there exists a constant C such that

$$C^{-1} \le \frac{q(x,\xi,t)}{e^{-\lambda_0 t}\phi_0(x)} \le C, \quad t \ge \delta,\ x \in D,\ \xi \in \partial_M D. \tag{4.7}$$

(iv) For any compact subset K of D there exist positive constants C and α such that

$$q(x,\xi,t) \le C e^{-\alpha/t}\phi_0(x), \quad x \in K,\ \xi \in \partial_M D,\ 0 < t < 1. \tag{4.8}$$

Proof By [IU], for any $\delta > 0$ there exists a positive constant C_δ such that

$$C_\delta^{-1}\phi_0(z)\phi_0(y) \le p(z,y,\delta) \le C_\delta \phi_0(z)\phi_0(y), \quad z, y \in D$$

(see [19]). From these inequalities we get, with another constant C_δ,

$$C_\delta^{-1} \le \frac{e^{\lambda_0 t} p(x,y,t)}{\phi_0(x)\phi_0(y)} \le C_\delta, \quad x, y \in D,\ t \ge \delta. \tag{4.9}$$

This shows (iii), which implies that $q(x,\xi,t) > 0$ for $t > 0$. The other assertions in (i) and (ii) follow from Lemma 4.1 and its proof. Finally, (4.5) implies (iv). □

Remark 4.3 In many cases, ϕ_0 is bounded on D. If so, (4.2) implies that the series expansion (4.4) converges uniformly on $D \times \partial_M D \times [\delta, \infty)$ for any $\delta > 0$.

A direct consequence of (4.5) and (4.9) is worth mentioning.

Lemma 4.4 For any domains U and W with $U \Subset W \Subset D$, there exist positive constants C and α such that

$$p(x,y,t) \le C f(t)\phi_0(x)\phi_0(y), \quad x \in U,\ y \in D \setminus W,\ t > 0, \tag{4.10}$$

where $f(t) = e^{-\alpha/t}$ for $0 < t < 1$, and $f(t) = e^{-\lambda_0 t}$ for $t \ge 1$. Furthermore,

$$G(x,y) \le C\phi_0(x)\phi_0(y), \quad x \in U,\ y \in D \setminus W, \tag{4.11}$$

where G is the Green function of L on D.

Proof Obviously, (4.10) follows from (4.5) and (4.9). Since

$$G(x,y) = \int_0^\infty p(x,y,t)\,dt,$$

(4.11) follows from (4.10). □

Let $K(x,\xi)$ be the Martin kernel for L on D with reference point $x^0 \in D$, i.e., $K(x^0,\xi) = 1,\ \xi \in \partial_M D$. The following lemma gives a relation between K and q.

Lemma 4.5 For any $\xi \in \partial_M D$,

$$\lim_{D \ni y \to \xi} \frac{G(x,y)}{\phi_0(y)} = \int_0^\infty q(x,\xi,t)\,dt, \quad x \in D, \tag{4.12}$$

$$K(x,\xi) = \frac{\int_0^\infty q(x,\xi,t)\,dt}{\int_0^\infty q(x^0,\xi,t)\,dt}, \quad x \in D. \tag{4.13}$$

Proof By Lemma 4.1 and (4.10), we obtain (4.12); which implies (4.13), since $K(x,\xi) = \lim_{y\to\xi} G(x,y)/G(x^0,y)$. $\square$

Lemma 4.6 Let $\xi, \eta \in \partial_M D$, $0 \le s, r < T$ and $C > 0$. If

$$q(x,\xi,t-s) = Cq(x,\eta,t-r), \quad (x,t) \in Q,$$

then $\xi = \eta,\ s = r$ and $C = 1$.

Proof By Lemma 4.2 (i), $s = r$. Thus, by (4.4) and (4.7), $[\phi_j/\phi_0](\xi) = C[\phi_j/\phi_0](\eta)$ for any $j = 0, 1, \cdots$; which implies that $C = 1$ and

$$[\phi_j/\phi_0](\xi) = [\phi_j/\phi_0](\eta), \quad j = 0, 1, \cdots . \tag{4.14}$$

By (4.14), $q(x,\xi,t) = q(x,\eta,t)$ for $x \in D$ and $t > 0$. Thus, by (4.13), $K(\,\cdot\,,\xi) = K(\,\cdot\,,\eta)$ on D. Hence $\xi = \eta$. $\square$

The following proposition is not used in proving Theorem 1.2. But it is of independent interest, and worth mentioning.

Proposition 4.7 Let κ and μ be finite Borel measures on $\partial_M D$ supported by the minimal Martin boundary $\partial_m D$. Suppose that

$$\int_{\partial_M D} [\phi_j/\phi_0]\, d\kappa = \int_{\partial_M D} [\phi_j/\phi_0]\, d\mu, \quad j = 0, 1, 2, \cdots . \tag{4.15}$$

Then $\kappa = \mu$.

Proof We have

$$
\begin{aligned}
&\int_{\partial_M D} K(x,\xi)\left(\int_0^\infty q(x^0,\xi,t)\,dt\right)\,d\kappa(\xi)\\
&=\int_0^\infty dt\int_{\partial_M D} q(x,\xi,t)\,d\kappa(\xi)\\
&=\int_0^\infty dt\sum_{j=0}^\infty e^{-\lambda_j t}\phi_j(x)\int_{\partial_M D}\left[\phi_j/\phi_0\right](\xi)\,d\kappa(\xi)\\
&=\int_0^\infty dt\sum_{j=0}^\infty e^{-\lambda_j t}\phi_j(x)\int_{\partial_M D}\left[\phi_j/\phi_0\right](\xi)\,d\mu(\xi)\\
&=\int_0^\infty dt\int_{\partial_M D} q(x,\xi,t)\,d\mu(\xi)\\
&=\int_{\partial_M D} K(x,\xi)\left(\int_0^\infty q(x^0,\xi,t)\,dt\right)\,d\mu(\xi).
\end{aligned}
$$

Recall that the Martin representation theorem for positive solutions of the elliptic equation $Lu = 0$ in D says that any positive solution is represented uniquely by the integral of the Martin kernel $K(x,\xi)$ with respect to a finite Borel measure on $\partial_M D$ supported by $\partial_m D$. Hence $\kappa = \mu$. □

We note that results related to Proposition 4.7 were shown by more involved method in [43] (see Proposition 9.7, Theorem 9.9 and Lemma 9.10 therein).

Now, let β be the measure on $Q = D \times (0,T)$ as in Section 2, i.e., $d\beta(x,t) = a(x)b(t)m(x)\,d\nu(x)\,dt$. The following proposition determines the β-Martin boundary $\partial_M^\beta Q$, β-Martin compactification $Q^{\beta *}$, and β-Martin kernel K_β for $(\partial_t + L, Q)$. Recall that $p(x,t;y,s) = p(x,y,t-s)$ and $K_\beta(\,\cdot\,;y,s) = p(\,\cdot\,;y,s)/\beta\left(p(\,\cdot\,;y,s)\right)$. We write

$$
q(x,t;\xi,s) = q(x,\xi,t-s)
$$

for $\xi \in \partial_M D$ and $0 \le s < T$.

Proposition 4.8 (i) The β-Martin boundary $\partial_M^\beta Q$ of Q for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial_M D \times [0,T)$ and the one point set $\{\varpi\}$:

$$
\partial_M^\beta Q = D \times \{0\} \cup \partial_M D \times [0,T) \cup \{\varpi\}. \tag{4.16}
$$

In particular, $\partial_M^\beta Q$ does not depend on β.
(ii) The β-Martin compactification $Q^{\beta *}$ of Q for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0,T)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of ϖ is given by the family $\{\varpi\} \cup D^* \times (T-\varepsilon, T),\ 0 < \varepsilon < T/2$. In particular, $Q^{\beta *}$ does not depend on β.

(iii) The β-Martin kernel K_β is given as follows: For $(x,t) \in Q$,

$$K_\beta(x,t;y,0) = \frac{p(x,t;y,0)}{\beta\left(p(\,\cdot\,;y,0)\right)}, \quad (y,0) \in D \times \{0\}, \tag{4.17}$$

$$K_\beta(x,t;\xi,s) = \frac{q(x,t;\xi,s)}{\beta\left(q(\,\cdot\,;\xi,s)\right)}, \quad (\xi,s) \in \partial_M D \times [0,T), \tag{4.18}$$

and $K_\beta(x,t;\varpi) = 0$.

Proof We see that any sequence $\{(y^j,s^j)\}_{j=1}^\infty$ in Q with no accumulation points in Q has a subsequence $\{(z^k,r^k)\}_{k=1}^\infty$ satisfying at least one of the following three conditions:

(1) $\lim_{k\to\infty} z^k = y \in D$ and $\lim_{k\to\infty} r^k = 0$;

(2) $\lim_{k\to\infty} z^k = \xi \in \partial_M D$ and $\lim_{k\to\infty} r^k = s \in [0,T)$;

(3) $\lim_{k\to\infty} r^k = T$.

We see that for a sequence $\{(z^k,r^k)\}_k$ satisfying (1)

$$\lim_{k\to\infty} K_\beta(x,t;z^k,r^k) = \frac{p(x,t;y,0)}{\beta\left(p(\,\cdot\,;y,0)\right)},$$

and for a sequence satisfying (3)

$$\lim_{k\to\infty} K_\beta(x,t;z^k,r^k) = 0.$$

For a sequence $\{(z^k,r^k)\}_k$ satisfying (2), we have by Lemma 4.1

$$\lim_{k\to\infty} K_\beta(x,t;z^k,r^k) = \frac{q(x,t;\xi,s)}{\beta\left(q(\,\cdot\,;\xi,s)\right)}.$$

Let X be the set on the right hand side of (4.16). For $\Xi \in X$, denote by $J_\beta(\,\cdot\,;\Xi)$ the function on the right hand side of (4.17) or (4.18) or 0 depending on Ξ. Then, by Lemmas 4.2 and 4.6, $J_\beta(\,\cdot\,;\Xi) \neq J_\beta(\,\cdot\,;\Xi')$ if $\Xi \neq \Xi'$. Hence any fundamental sequence $\{(z^k, r^k)\}_k$ satisfies only one of the above three conditions, and the assertions (i), (ii) and (iii) of Proposition 4.8 hold. □

From Proposition 4.8 and Theorem 2.1 we obtain the following theorem which is a weak version of Theorem 1.2 and does not claim the uniqueness of representing measures.

Theorem 4.9 For any nonnegative solution u of (1.1) there exists a pair of Borel measures μ on D and λ on $\partial_M D \times [0, T)$ such that (1.4) holds.

Proof Let u be a nonnegative solution of (1.1) which is not identically zero. Choose a measure β such that $\beta(u) = 1$. By Theorem 2.1 and Proposition 4.8, there exists a pair of Borel measures μ on D and λ on $\partial_M D \times [0, T)$ satisfying (1.4) and

$$1 = \int_D \beta\left(p(\,\cdot\,; y, 0)\right) \, d\mu(y) + \int_{\partial_M D \times [0,T)} \beta\left(q(\,\cdot\,; \xi, s)\right) \, d\lambda(\xi, s).$$

□

In order to complete the proof of Theorem 1.2, we need to identify the set

$$\mathcal{L}^\beta_m Q = \partial^\beta_m Q \setminus \left(D \times \{0\} \cup \{\varpi\}\right),$$

where

$$\partial^\beta_m Q = \left\{\Xi \in \partial^\beta_M Q; K_\beta(\,\cdot\,;\Xi) \text{ is minimal, and } \beta\left(K_\beta(\,\cdot\,;\Xi)\right) = 1\right\}.$$

By Proposition 4.8,

$$\partial^\beta_M Q \setminus \left(D \times \{0\} \cup \{\varpi\}\right) = \partial_M D \times [0, T)$$

and for $\Xi \in \partial_M D \times [0, T)$

$$K_\beta(\,\cdot\,;\Xi) = \frac{q(\,\cdot\,;\Xi)}{\beta\left(q(\,\cdot\,;\Xi)\right)}, \quad \beta\left(K_\beta(\,\cdot\,;\Xi)\right) = 1.$$

Thus

$$\mathcal{L}^\beta_m Q = \left\{(\xi, s) \in \partial_M D \times [0, T); q(\,\cdot\,; \xi, s) \text{ is minimal}\right\}. \tag{4.19}$$

In the rest of this section we shall show that $\mathcal{L}^\beta_m Q = \partial_m D \times [0, T)$.

Lemma 4.10 Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, T)$. Then $q(\,\cdot\,; \xi, s)$ is not minimal.

Proof We claim that there exists a finite Borel measure γ on $\partial_M D$ supported by $\partial_m D$ such that

$$q(\,\cdot\,; \xi, s) = \int_{\partial_m D} q(\,\cdot\,; \eta, s)\, d\gamma(\eta). \tag{4.20}$$

Before showing this claim, we show that if it holds, then $q(\,\cdot\,; \xi, s)$ is not minimal. Indeed, suppose that $q(\,\cdot\,; \xi, s)$ is minimal. Then, as in the latter half of the proof of Lemma 2.5, the support of γ consists of a single point. Thus, for some $\eta \in \partial_m D$ and constant C

$$q(\,\cdot\,; \xi, s) = Cq(\,\cdot\,; \eta, s).$$

Hence, by Lemma 4.6, $\xi = \eta$; which is a contradiction. Therefore, we have only to show the claim. By the elliptic Martin representation theorem, there exists a unique finite Borel measure μ on $\partial_M D$ supported by $\partial_m D$ such that

$$K(x, \xi) = \int_{\partial_m D} K(x, \eta)\, d\mu(\eta).$$

Put

$$H(x, \eta) = \int_0^\infty q(x, \eta, t)\, dt.$$

By Lemma 4.5, $K(x, \eta) = H(x, \eta)/H(x^0, \eta)$. Thus

$$H(x, \xi) = \int_{\partial_m D} H(x, \eta)\, d\gamma(\eta), \tag{4.21}$$

where $d\gamma(\eta) = [H(x^0, \xi)/H(x^0, \eta)]\, d\mu(\eta)$. For $\alpha > 0$, denote by G_α the Green function of $L + \alpha$ on D. Since

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(x, y, t)\, dt,$$

the same argument as in the proof of Lemma 4.5 shows that for any $\eta \in \partial_M D$

$$\lim_{D \ni y \to \eta} \frac{G_\alpha(x, y)}{\phi_0(y)} = \int_0^\infty e^{-\alpha t} q(x, \eta, t)\, dt. \tag{4.22}$$

We denote by $H_\alpha(x,\eta)$ the right hand side of (4.22). By the resolvent equation,

$$G_\alpha(x,y) = G(x,y) - \alpha \int_D G_\alpha(x,z)G(z,y)\,d\lambda(z),$$

where $d\lambda(z) = m(z)\,d\nu(z)$. Fix $x \in D$. By Theorem 1.1, the constant function 1 is a semismall perturbation of L on D: for any $\varepsilon > 0$ there exists a compact subset K of D such that

$$\int_{D\setminus K} G_\alpha(x,z)G(z,y)\,d\lambda(z) \le \varepsilon G(x,y), \quad y \in D \setminus K.$$

By Fatou's lemma,

$$\begin{aligned}
&\int_{D\setminus K} G_\alpha(x,z)H(z,\eta)\,d\lambda(z)\\
&\le \limsup_{y\to\eta} \int_{D\setminus K} G_\alpha(x,z)\frac{G(z,y)}{\phi_0(y)}\,d\lambda(z)\\
&\le \varepsilon \lim_{y\to\eta} \frac{G(x,y)}{\phi_0(y)} = \varepsilon H(x,\eta).
\end{aligned}$$

Thus, with $F(x,z,\eta) = G_\alpha(x,z)H(z,\eta)$, we have

$$\begin{aligned}
&\left| H_\alpha(x,\eta) - H(x,\eta) + \alpha \int_D F(x,z,\eta)\,d\lambda(z) \right|\\
&\le \left| H_\alpha(x,\eta) - H(x,\eta) + \alpha \int_K F(x,z,\eta)\,d\lambda(z) \right| + \alpha \int_{D\setminus K} F(x,z,\eta)\,d\lambda(z)\\
&\le 2\alpha\varepsilon H(x,\eta).
\end{aligned}$$

Hence

$$H_\alpha(x,\eta) = H(x,\eta) - \alpha \int_D G_\alpha(x,z)H(z,\eta)\,d\lambda(z).$$

By (4.21),

$$\int_{\partial_m D} H_\alpha(x,\eta)\,d\gamma(\eta) = H(x,\xi) - \alpha \int_D G_\alpha(x,z)H(z,\xi)\,d\lambda(z) = H_\alpha(x,\xi).$$

This together with (4.22) implies

$$\int_0^\infty e^{-\alpha t} \left(\int_{\partial_m D} q(x,\eta,t)\,d\gamma(\eta) \right) dt = \int_0^\infty e^{-\alpha t} q(x,\xi,t)\,dt.$$

Thus the Laplace transforms of $e^{\alpha s} q(x,t;\xi,s)$ and $e^{\alpha s} \int_{\partial_m D} q(x,t;\eta,s)\, d\gamma(\eta)$ coincide; and so (4.20) holds. $\square$

Lemma 4.11 Let $(\xi, s) \in \partial_m D \times [0,T)$. Then $q(\,\cdot\,;\xi,s)$ is minimal.

For proving this lemma, we need an integral representation theorem for nonnegative solutions of the equation

$$(\partial_t + L)\, w = 0 \qquad \text{in} \quad D \times (0,\infty). \tag{4.23}$$

Theorem 4.12 For any nonnegative solution w of (4.23) there exists a pair of Borel measures μ on D and λ on $\partial_M D \times [0,\infty)$ such that

$$w(x,t) = \int_D p(x,y,t)\, d\mu(y) + \int_{\partial_M D \times [0,t)} q(x,\xi,t-s)\, d\lambda(\xi,s) \tag{4.24}$$

for any $(x,t) \in D \times (0,\infty)$.

This theorem is an analogue to Theorem 4.9, and can be shown in the same way as Theorem 4.9.

Proof of Lemma 4.11 Let $(\xi,s) \in \partial_m D \times [0,T)$. Let u be a nonnegative solution of (1.1) which is not identically zero. Suppose that $u(\,\cdot\,) \le q(\,\cdot\,;\xi,s)$ on Q. Put $v(x,t) = u(x,t+s)$. Then $v(x,t) \le q(x,t;\xi,0)$ for $x \in D$ and $0 < t < T-s$. Fix any $0 < S < T-s$. Define $w(x,t)$ by

$$\begin{aligned} w(x,t) &= v(x,t), && (x,t) \in D \times (0,S], \\ w(x,t) &= \int_D p(x,y,t-S) v(y,S) m(y)\, d\nu(y), && (x,t) \in D \times (S,\infty). \end{aligned} \tag{4.25}$$

The integral in (4.25) converges and w is well-defined, since Lemmas 4.1 and 4.2 imply

$$\int_D p(x,y,t-S) q(y,\xi,S) m(y)\, d\nu(y) = q(x,\xi,t), \quad x \in D,\ t > S. \tag{4.26}$$

Furthermore, w is a nonnegative solution of (4.23) such that $w(\,\cdot\,) \le q(\,\cdot\,;\xi,0)$ on $D \times (0,\infty)$. By Theorem 4.12, there exists a Borel measure λ on $\partial_M D \times [0,\infty)$ such that

$$w(\,\cdot\,) = \int_{\partial_M D \times [0,\infty)} q(\,\cdot\,;\eta,s)\, d\lambda(\eta,s).$$

For any $0 < \delta < 1$, put

$$w_\delta(\,\cdot\,) = \int_{\partial_M D \times [\delta, \delta^{-1}]} q(\,\cdot\,; \eta, s)\, d\lambda(\eta, s).$$

Then w_δ satisfies

$$\begin{aligned} &(\partial_t + L)\, w_\delta = 0 && \text{in} \quad D \times (0, \infty), \\ &w_\delta = 0 && \text{on} \quad D \times [0, \delta], \\ &0 \le w_\delta(x,t) \le q(x,\xi,t) \le Ce^{-\lambda_0 t}\phi_0(x), && x \in D,\ t \ge \delta, \end{aligned}$$

where C is a positive constant depending only on δ. Put

$$h(x) = \int_0^\infty w_\delta(x,t)\, dt.$$

Then, $Lh = 0$ in D and $h \le C\phi_0$ in D. By [IU], there exists a constant C such that $\phi_0(x) \le CG(x, x^0)$. Thus $h(x) \le CG(x, x^0)$, $x \in D$. Hence $h = 0$. This implies that $w_\delta = 0$, since w_δ is a nonnegative continuous function. Hence $\lambda\,(\partial_M D \times [\delta, \delta^{-1}]) = 0$ for any $\delta > 0$; and so $\lambda\,(\partial_M D \times (0, \infty)) = 0$. Define a measure μ on $\partial_M D$ by $\mu(B) = \lambda\,(B \times \{0\})$ for any Borel subset B of $\partial_M D$. Then

$$w(x,t) = \int_{\partial_M D} q(x, \eta, t)\, d\mu(\eta). \tag{4.27}$$

We have

$$\begin{aligned} &\left(\int_0^\infty q(x^0, \xi, t)\, dt\right) K(x, \xi) = \int_0^\infty q(x, \xi, t)\, dt \\ &\ge \int_0^\infty w(x,t)\, dt = \int_0^\infty dt \int_{\partial_M D} q(x, \eta, t)\, d\mu(\eta) \\ &= \int_{\partial_M D} K(x, \eta) \left(\int_0^\infty q(x^0, \eta, t)\, dt\right) d\mu(\eta) \equiv \int_{\partial_M D} K(x, \eta) d\gamma(\eta). \end{aligned}$$

Since $\xi \in \partial_m D$, this implies that

$$\int_{\partial_M D} K(x, \eta)\, d\gamma(\eta) = CK(x, \xi)$$

for a positive constant C. Then the same argument as in the proof of Lemma 12.12 of [24] shows that the support of γ consists of a single point ζ. Thus $K(x,\zeta) = K(x,\xi)$, since $C = \gamma(\partial_M D)$. Hence $\zeta = \xi$ and $\gamma = C\delta_\xi$, where δ_ξ is the probability measure concentrated on the point ξ. Therefore

$$\mu = C\left(\int_0^\infty q(x^0,\xi,t)\,dt\right)\delta_\xi.$$

This together with (4.27) implies that $w(x,t) = Cq(x,\xi,t)$ for some constant C. Thus, for $x \in D$ and $t \in (s, S+s]$

$$u(x,t) = v(x,t-s) = w(x,t-s) = Cq(x,\xi,t-s).$$

Since S is any positive number less than $T-s$, it follows from this that

$$u(\,\cdot\,) = Cq(\,\cdot\,;\xi,s) \quad \text{on } Q.$$

Hence $q(\,\cdot\,;\xi,s)$ is minimal. □

Completion of the proof of Theorem 1.2 By (4.19), Lemmas 4.10 and 4.11,

$$\mathcal{L}_m^\beta Q = \partial_m D \times [0,T). \tag{4.28}$$

This together with Theorem 2.1 and Proposition 4.8 shows Theorem 1.2 in the same way as in the proof of Theorem 4.9. □

We conclude this section with a remark on the assumption of Theorem 1.2.

Remark 4.13 It is an open problem whether the conclusions of Theorem 1.2 still hold true even if the assumption [IU] is replaced by [SP].

5 Examples

In the Introduction we have given concrete examples as applications of Theorem 1.2. In this section we give further examples.

Example 5.1 Let M be a Riemannian manifold of dimension $n \geq 2$ such that it is complete, simply connected, and its sectional curvatures are bounded between two negative constants. Let L_0 be a uniformly elliptic operator on M of the form

$$L_0 u = -\mathrm{div}\,(A\nabla u)\,,$$

where A is a symmetric measurable section of End $(T(M))$ satisfying

$$\lambda|\xi|^2 \le \langle A_x \xi, \xi\rangle \le \lambda^{-1}|\xi|^2, \quad (x,\xi) \in TM,$$

for some positive constant λ. Denote by $d(x)$ the Riemannian distance between $x \in M$ and a point x^0 fixed in M. Let $\alpha > 1$,

$$m(x) = \left(d(x)^2 + 1\right)^{-\alpha/2},$$

and $L = m(x)^{-1}L_0$ on $D = M$. Then the condition [IU] is satisfied (which will be shown later). Furthermore the Martin boundary of M for L_0 is homeomorphic to $S_\infty(M)$, the sphere at infinity of M, and every Martin boundary point is minimal (see [4], [7]). This implies that the Martin boundary and minimal Martin boundary of M for L coincide, and they are homeomorphic to $S_\infty(M)$. Hence the conclusions of Theorem 1.2 hold true with $\partial_M D = \partial_m D = S_\infty(M)$.

Now, let us show that [IU] is satisfied. It is well known that for some positive constant δ the inequality $\delta \le L_0$ holds as quadratic forms on $C_0^\infty(M)$ with respect to the Riemannian measure $d\nu$. Let G_0 and G be the Green functions of L_0 and L on M with respect to the reference measure $d\nu$, respectively. Then we see that $G(x,y) = G_0(x,y)m(y)$. By Corollary 6.1 of [5], $m(x)$ is a small perturbation of L_0 on M; and so 1 is a small perturbation of L on M, i.e., the condition [SP] is satisfied with $a = 0$. Then the spectrum of the self-adjoint operator L_M on $L^2(M,\ m\,d\nu)$ associated with L consists of discrete eigenvalues with finite multiplicity (see Theorem 6.3 of [53] and Theorem 5.12 of [43]). Let ϕ_0 be the normalized positive eigenfunction for the first eigenvalue λ_0. By [SP], $\lambda_0 > 0$ and there exists a positive constant C such that

$$C^{-1}G_0(x,x^0) \le \phi_0(x) \le CG_0(x,x^0), \quad x \in M,\ d(x) \ge 1$$

(see Theorem 1.5 of [42]). By Remark 2.1 of [4],

$$C^{-1}d(x) \le -\log G_0(x,x^0) \le Cd(x), \quad x \in M,\ d(x) \ge R,$$

for some positive constants C and R. Thus, with $\langle d(x)\rangle = (d(x)^2+1)^{1/2}$,

$$-\log\phi_0(x) \le C_0\langle d(x)\rangle, \quad x \in M, \tag{5.1}$$

for some positive constant C_0. Since $L_0 \geq \delta$ as quadratic forms with respect to the measure $d\nu$ and $L = m(x)^{-1} L_0$,

$$L \geq \delta m(x)^{-1} = \delta \langle d(x) \rangle^{\alpha} \tag{5.2}$$

as quadratic forms with respect to the measure $m\, d\nu$. Since

$$C_0 \langle d(x) \rangle \leq \varepsilon\delta \langle d(x) \rangle^{\alpha} + C_0 \left(\frac{C_0}{\varepsilon\delta} \right)^{1/(\alpha-1)}, \quad \varepsilon > 0,$$

we have by (5.1) and (5.2)

$$-\log \phi_0 \leq \varepsilon L + C_1 \varepsilon^{-1/(\alpha-1)}, \quad \varepsilon > 0, \tag{5.3}$$

for some positive constant C_1. Now, by using a covering by balls as in the proof of Proposition 2.1 of [5], we can show the following Sobolev inequality

$$\left(\int_M v^{2p}\, d\nu \right)^{1/p} \leq C_2 \int_M |\nabla v|^2\, d\nu, \quad v \in C_0^{\infty}(M),$$

where $p = n/(n-2)$ for $n \geq 3$ and $1 < p < \infty$ for $n = 2$ (see also the proof of Proposition 2.3 on pp.191–192 of [29]). Let q be the conjugate exponent of p. For any $g \in L^q(M; m\, d\nu)$, we have

$$\begin{aligned}
\int_M g v^2 m\, d\nu &\leq \left(\int_M |g|^q m\, d\nu \right)^{1/q} \left(\int_M v^{2p} m\, d\nu \right)^{1/p} \\
&\leq \|g\|_q \left(\int_M v^{2p}\, d\nu \right)^{1/p} \leq C_2 \|g\|_q \int_M |\nabla v|^2\, d\nu.
\end{aligned}$$

This implies the quadratic form inequality

$$g \leq C_2 \|g\|_q L. \tag{5.4}$$

Then, by the same argument as in the proof of Rosen's lemma (see Lemma 4.4.1 and Corollary 4.4.2 of [18]), we get from (5.3) and (5.4) the logarithmic Sobolev inequality

$$\int_M \left(u^2 \log u \right) \phi_0^2 m\, d\nu \leq \varepsilon Q_{\phi_0}(u) + \beta(\varepsilon) \|u\|_2^2 + \|u\|_2^2 \log \|u\|_2$$

for all $\varepsilon > 0$ and nonnegative $u \in L^1 \cap L^\infty \cap \mathrm{Dom}\,(Q_{\phi_0})$, where

$$Q_{\phi_0}(u) = \int_M \langle A\nabla u, \nabla u\rangle \phi_0^2\, d\nu + \int_M \lambda_0 u^2 \phi_0^2 m\, d\nu,$$
$$\beta(\varepsilon) = C_3 \varepsilon^{-1/(\alpha-1)} - (q/2)\log\varepsilon + C_4$$

for some positive constants C_3 and C_4. This implies [IU] (see Corollary 2.2.8 and Example 2.3.4 of [18]). More precisely,

$$p(x,y,t) \le C\exp\left[Ct^{-1/(\alpha-1)}\right]\phi_0(x)\phi_0(y), \quad 0 < t < 1,\ x,\, y \in M, \tag{5.5}$$

for some positive constant C; which together with the semigroup property of the minimal fundamental solution p implies

$$p(x,y,t) \le C\exp\left[C - \lambda_0(t-1)\right]\phi_0(x)\phi_0(y), \quad t \ge 1,\ x,\, y \in M. \tag{5.6}$$

Example 5.2 Suppose that the manifold M, noncompact domain D, and operator L in Section 1 are of the form

$$M = M_1 \times M_2, \quad D = D_1 \times D_2, \quad L = L^1 + L^2,$$

where D_i with $i = 1$ or 2 is a domain of a Riemannian manifold M_i, and L^i is an elliptic operator on D_i of the form (1.2) with obvious modification of notations. Suppose that [IU] is satisfied for (L^i, D_i), $i = 1, 2$, i.e.,

$$p^i(x_i, y_i, t) \le C_t^i \phi_0^i(x_i)\phi_0^i(y_i), \quad x_i,\, y_i \in D_i, \quad t > 0.$$

Here ϕ_0^i is the normalized positive eigenfunction for the lowest eigenvalue λ_0^i of $L^i_{D_i}$, p^i is the minimal fundamental solution for $\partial_t + L^i$ on $D_i \times (0,\infty)$, and C_t^i is a positive constant depending only on t. Then [IU] is satisfied for (L, D) with $\lambda_0 = \lambda_0^1 + \lambda_0^2$, $\phi_0(x_1, x_2) = \phi_0^1(x_1)\phi_0^2(x_2)$, and $p(x_1, x_2, y_1, y_2, t) = p^1(x_1, y_1, t)p^2(x_2, y_2, t)$. Thus the conclusions of Theorem 1.2 hold.

In order to identify the Martin boundary $\partial_M D$ of D with respect to $L - a$ $(a < \lambda_0)$, we further assume that either $D_2 = M_2$ is a compact manifold or

$$\lim_{t\to 0} t \log C_t^2 = 0$$

and $\partial_M D_2 = \partial_m D_2$, i.e., every point in the Martin boundary $\partial_M D_2$ of D_2 for $L^2 - a_2$ $(a - \lambda_0^1 < a_2 < \lambda_0^2)$ is minimal. (Recall that if D_2 is compact, [IU] is satisfied with $\lim_{t\to 0} t\log C_t^2 = 0$.) Then we obtain by Theorem 1.1 and

Theorem 4.2 of [45] that the Martin compactification D^* of D for $L - a$ is homeomorphic to $D_1^* \times D_2^*$, where $D_2^* = D_2$ when D_2 is compact, otherwise D_2^* is the Martin compactification of D_2 for $L^2 - a_2$, and D_1^* is the Martin compactification of D_1 for $L^1 - (a - a_2)$. Furthermore,

$$\partial_M D = (\partial_M D_1 \times D_2^*) \cup (D_1 \times \partial_M D_2),$$
$$\partial_m D = (\partial_m D_1 \times D_2^*) \cup (D_1 \times \partial_M D_2),$$

where $\partial_M D_2 = \emptyset$ when D_2 is compact.

The additional conditions imposed on (L^2, D_2) are satisfied, for example, in the case where (L^2, D_2) is as in Example 5.1 with $\alpha > 2$ or Example 1.5 with $\partial_M D_2 = \partial_m D_2$ or Example 5.3 to be stated below. For further examples, see Sections 9 and 10 of [43].

Let $L_0 = -\sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j)$ be a uniformly elliptic operator on $D = \mathbf{R}^n$ as (1.13).

Example 5.3 Let $L = \langle x\rangle^\alpha L_0$ on $D = \mathbf{R}^n$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\alpha > 2$ and $n \geq 3$. Then [IU] is satisfied, and the conclusions of Theorem 1.2 hold true with

$$\partial_M D = \partial_m D = \{\infty\},$$

where ∞ is the point at infinity of the one point compactification of $\mathbf{R}^n$.

The Martin compactification of $\mathbf{R}^n$ for L_0 is known to be the one point compactification of $\mathbf{R}^n$; and so that for L is also equal to $\mathbf{R}^n \cup \{\infty\}$. Let us show that [IU] is satisfied. By the Hardy inequality, there exists a positive constant C_0 such that

$$\langle x\rangle^{-2} \leq C_0 L_0$$

as quadratic forms with respect to the Lebesgue measure dx. Thus

$$\langle x\rangle^{\alpha-2} \leq C_0 \langle x\rangle^\alpha L_0 = C_0 L$$

as quadratic forms with respect to the measure $\langle x\rangle^{-\alpha}\, dx$. Since the Green function $G_0(x, y)$ of L_0 on $\mathbf{R}^n$ is comparable with $|x-y|^{2-n}$, $\langle x\rangle^{-\alpha}$ is a small perturbation of L_0 on $\mathbf{R}^n$. Thus the positive eigenfunction ϕ_0 of L satisfies

$$-\log \phi_0(x) \leq C_1 \log 2\langle x\rangle, \quad x \in \mathbf{R}^n,$$

for some positive constant C_1. Hence the same argument as in Example 5.1 shows that [IU] is satisfied with

$$p(x,y,t) \le Ct^{-\gamma}\phi_0(x)\phi_0(y), \qquad\qquad 0<t<1,\ x,y\in\mathbf{R}^n,$$
$$p(x,y,t) \le C\exp\left[-\lambda_0(t-1)\right]\phi_0(x)\phi_0(y), \quad t\ge 1,\ x,\,y\in\mathbf{R}^n,$$

where C and γ are positive constants.

Example 5.4 Let $\beta > 1$ and $L = \langle x\rangle^{\beta}\,(L_0 + V(x))$ on $D = \mathbf{R}^n$, where $n \ge 2$, the coefficients $a_{ij}(x)$ and $V(x)$ are $\mathbb{Z}^n$-periodic, i.e., $a_{ij}(x+z) = a_{ij}(x)$ and $V(x+z) = V(x)$ for any $x \in \mathbf{R}^n$ and $z \in \mathbb{Z}^n$, and V is a function in $L^p_{loc}(\mathbf{R}^n)$, $p > n/2$, satisfying $C^{-1} \le V(x) \le C$ for a positive constant C. Then [IU] is satisfied; and the conclusions of Theorem 1.2 hold true with

$$\partial_M D = \partial_m D = \infty S^{n-1},$$

where ∞S^{n-1} is the sphere at infinity of $\mathbf{R}^n$ (see [47]). As for [IU], we can show it by the same argument as in Example 5.1, because Theorem 5.1 of [43] and Theorem 1.1 of [47] imply that the positive eigenfunction $\phi_0(x)$ of L decays exponentially as $x \to \infty$.

6 The case $D \times (-\infty, 0)$

In this section we give an integral representation theorem for nonnegative solution of the equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times (-\infty, 0). \tag{6.1}$$

Theorem 6.1 Assume [IU]. Then, for any nonnegative solution u of (6.1) there exists a unique pair of a nonnegative constant α and a Borel measure λ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_m D \times (-\infty, 0)$ such that

$$u(x,t) = \alpha e^{-\lambda_0 t}\phi_0(x) + \int_{\partial_M D\times(-\infty,t)} q(x,\xi,t-s)\,d\lambda(\xi,s) \tag{6.2}$$

for any $(x,t) \in D \times (-\infty, 0)$.

Conversely, for any nonnegative constant α and a Borel measure λ on $\partial_M D \times (-\infty, 0)$ such that it is supported by $\partial_m D \times (-\infty, 0)$ and

$$\int_{\partial_M D\times(-\infty,t)} q(x^0,\xi,t-s)\,d\lambda(\xi,s) < \infty, \quad -\infty < t < 0, \tag{6.3}$$

the right hand side of (6.2) is a nonnegative solutions of (6.1).

Proof By Theorem 4.2.5 of [18],

$$\lim_{s\to-\infty} e^{\lambda_0(t-s)}p(x,y,t-s)/\phi_0(y)=\phi_0(x)$$

with the convergence uniform in $(x,y,t)\in D\times D\times(-T,0)$ for any $T>0$. Thus the same argument as in the proof of Theorem 1.2 shows Theorem 6.1. □

For results related to Theorem 6.1, see [31] and [50]. We conclude this section with several remarks.

Remark 6.2 It follows from Theorem 6.1 that $e^{-\lambda_0 t}\phi_0(x)$ is minimal in the set of all nonnegative solutions of (6.1); which is related to Conjecture 3.6 of [52] and Problem 1.2 of [15].

Remark 6.3 By the same argument as in the proof of Proposition 4.8, we can show that the β-Martin compactification of $D\times(-\infty,0)$ for ∂_t+L is homeomorphic to the disjoint union of the topological product $D^*\times(-\infty,0)$ and two points, the "bottom" and the "top", which correspond to $e^{-\lambda_0 t}\phi_0(x)$ and 0, respectively.

Remark 6.4 We can also establish an integral representation theorem for nonnegative solution of the equation

$$(\partial_t+L)u=0 \quad \text{in} \quad D\times(-\infty,\infty), \tag{6.4}$$

which is completely analogous to Theorem 6.1.

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