Quantum Deformations of Catalan's Constant, Mahler's Measure, and Hölder-Shintani's Double Sine Function

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1 Introduction

The Catalan constant ([C])

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190\cdots \tag{1.1}$$

is a famous mysterious constant appearing in many places in mathematics and physics. Using Euler's dilogarithm

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

G can be written as

$$G = \frac{1}{2i}\left(\mathrm{Li}_2(i) - \mathrm{Li}_2(-i)\right). \tag{1.2}$$

The integral representation

$$G = \int_0^1 \frac{\tan^{-1}(x)}{x} dx \tag{1.3}$$

follows from the expression

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

The basic interpretation of G is considering it as the special value of the zeta function

$$G = L(2, \chi_{-4}),$$

where

$$L(s,\chi_{-4})=\sum_{n=1}^{\infty}\frac{\chi_{-4}(n)}{n^s} \tag{1.4}$$

is the Dirichlet L-function for the non-principal Dirichlet character χ_{-4} modulo 4:

$$\chi_{-4}(n)=\begin{cases}1 & \text{if } n\equiv 1 \pmod 4\\ -1 & \text{if } n\equiv 3 \pmod 4\\ 0 & \text{otherwise.}\end{cases}$$

Then, the algebraic nature of G would be clarified via the $K_3(\mathbf{Z}[\sqrt{-1}])$-regulator according to the Lichtenbaum-Beilinson conjecture since

$$G=\frac{\zeta_{\mathbf{Q}(\sqrt{-1})}(2)}{\zeta(2)}=\frac{6}{\pi^2}\zeta_{\mathbf{Q}(\sqrt{-1})}(2).$$

Recently, Rivoal and Zudilin [RZ] made a progress towards the irrationality problem for G: they proved that one of $L(2,\chi_{-4})(=G)$, $L(4,\chi_{-4})$, $\cdots$, $L(14,\chi_{-4})$ is irrational.

In previous papers [K1] [KK] [KOW] we showed that

$$G=2\pi\log\left(F\left(\frac{1}{4}\right)2^{-\frac{1}{8}}\right), \tag{1.5}$$

where

$$F(x)=e^x\prod_{n=1}^{\infty}\left(\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^n e^{2x}\right) \tag{1.6}$$

is the double sine function defined by Hölder [H] in 1886. This is a quasi-periodic function satisfying

$$F(x+1)=F(x)(-2\sin\pi x).$$

We refer to Manin [Man] for an excellent survey of multiple sine functions. The formula (1.5) comes from

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^2} &= \frac{1}{2i}\left(\mathrm{Li}_2(e^{2\pi ix})-\mathrm{Li}_2(e^{-2\pi ix})\right)\\ &= 2\pi\log\left(F(x)(2\sin\pi x)^{-x}\right) \qquad (1.7)\end{aligned}$$

valid for $0<x<1$, where the left hand side is the Clausen function (see Lewin [L] and Milnor [Mil]).

On the other hand, there is a formula due to Smyth [S] for G using the Mahler measure originating from the theory of transcendental numbers (Mahler [Mah]):

$$G = \frac{\pi}{2} m(x + y - xy + 1). \tag{1.8}$$

See [Mah] [S] [LSW] [D] [S] [EW] for background concerning the Mahler measure. We recall that the Mahler measure of a rational function $f(x_1, \cdots, x_n) \in \mathbf{C}(x_1, \cdots, x_n)$ is given by

$$\begin{aligned} m(f) &= \int_0^1 \cdots \int_0^1 \log \left| f(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}) \right| d\theta_1 \cdots d\theta_n \\ &= \mathrm{Re} \int_0^1 \cdots \int_0^1 \log \left(f(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}) \right) d\theta_1 \cdots d\theta_n. \end{aligned} \tag{1.9}$$

Thus we have a triangle:

$$\begin{array}{ccc} G & = & \dfrac{\pi}{2} m(x + y - xy + 1) \\ \bigg\| & & \bigg/\!\!\bigg/ \\ & 2\pi \log \left(F\left(\dfrac{1}{4}\right) 2^{-\frac{1}{8}} \right). & \end{array} \tag{1.10}$$

The purpose of this paper is to make a quantum deformation (or a q-deformation) of this triangle to

$$\begin{array}{ccc} G_q & = & \dfrac{\pi}{2} m_q \left(\dfrac{x-1}{x+1} y + 1 \right) \\ \bigg\| & & \bigg/\!\!\bigg/ \\ & \dfrac{i}{2} \log F_q \left(\dfrac{1}{4}\right) & \\ & (\mathrm{mod}\ \pi\mathbf{Z}) & \end{array} \tag{1.11}$$

for $0 < q < 1$. (Actually we treat all $q \geq 0$.) When letting $q \uparrow 1$ (the "classical limit") we recover the original triangle: note that

$$\begin{aligned} m\left(\frac{x-1}{x+1} y + 1 \right) &= m((x-1)y + x + 1) && (\text{since} \quad m(x+1) = 0) \\ &= m(xy - y + x + 1) && (\text{taking} \quad y \longmapsto -y) \\ &= m(-xy + y + x + 1). \end{aligned}$$

On the other hand taking the "crystal limit" $q \downarrow 0$ we have

$$\begin{array}{ccc} G_0 & = & \frac{\pi}{2} m_0\left(\frac{x-1}{x+1}y+1\right) \\ \backslash\backslash & & // \\ & \frac{i}{2}\log F_0\left(\frac{1}{4}\right) & \end{array} \tag{1.12}$$

with $G_0 = \frac{\pi}{4}$, $m_0(\frac{x-1}{x+1}y+1) = \frac{1}{2}$, and $F_0\left(\frac{1}{4}\right) = -i$.

The q-deformations are defined as follows. We define

$$G_q = \int_0^1 \frac{\tan^{-1}(x)}{x} d_q x \tag{1.13}$$

by the Jackson integral. For simplicity, here we restrict to the case $0 < q < 1$ (see the text for the general case), then the Jackson integral is given as

$$\int_0^1 f(x) d_q x = \sum_{n=0}^{\infty} f(q^n)(q^n - q^{n+1}). \tag{1.14}$$

It is shown that

$$G_q = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n[n]_q} \tag{1.15}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[2n+1]_q}, \tag{1.16}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Using the quantum dilogarithm

$$\mathrm{Li}_{2,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n[n]_q} \tag{1.17}$$

(see Kirillov [Kir] and Faddeev-Kashaev [FK]) we see that

$$G_q = \frac{1}{2i}\left(\mathrm{Li}_{2,q}(i) - \mathrm{Li}_{2,q}(-i)\right). \tag{1.18}$$

The q-Mahler measure was introduced in a previous paper [K2] as

$$m_q(f) = \mathrm{Re} \int_0^1 \cdots \int_0^1 l_q\left(f(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n})\right) d\theta_1 \cdots d\theta_n \tag{1.19}$$

for a rational function $f(x_1, ..., x_n) \in \mathbf{C}(x_1, ..., x_n)$ with the q-logarithm

$$l_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_q} \tag{1.20}$$

originally defined for $|x-1| < 1$ and analytically continued to $x \in \mathbf{C}$ via

$$l_q(x) = (1-q) \sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}}. \tag{1.21}$$

Lastly, we put

$$F_q(x) = \prod_{n=0}^{\infty} \left(\frac{1-q^n e^{2\pi i x}}{1-q^n e^{-2\pi i x}} \right)^{1-q}, \tag{1.22}$$

which is expressed also as

$$F_q(x) = \exp\left(-2i \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n[n]_q} \right) \tag{1.23}$$

and

$$F_q(x) = \exp\left(\mathrm{Li}_{2,q}(e^{-2\pi i x}) - \mathrm{Li}_{2,q}(e^{2\pi i x}) \right). \tag{1.24}$$

In the text, we make also another deformation of $F(x)$ using the regularized double sine function $F(x, (1, \tau))$ introduced by Shintani [Shi] to investigate Kronecker's Jugendtraum for real quadratic fields.

Concerning the parameter q, we investigate certain dualities such as $G_q \longleftrightarrow G_{q^{-1}}$, $m_q(f) \longleftrightarrow m_{q^{-1}}(f)$, and $F_q \leftrightarrow F_{q^{-1}}$.

Lastly we notice that our result can be generalized to some extent to other polynomials and rational functions. We refer to the paper [KO] for related matters.

2 Quantum Catalan constant

We already defined G_q for $0 < q < 1$. In the case $q > 1$ we define also

$$G_q = \int_0^1 \frac{\tan^{-1}(x)}{x} d_q x$$

via the Jackson integral

$$\int_0^1 f(x) d_q x = \sum_{n=1}^{\infty} f(q^{-n})(q^{1-n} - q^{-n}).$$

We see easily the following properties of G_q.

Theorem 1. *(1)* $G_q = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)[2n+1]_q}$ *for* $0 < q < 1$ *and* $q > 1$.

(2) $\lim_{q \uparrow 1} G_q = G$.

(3) $\lim_{q \downarrow 0} G_q = \frac{\pi}{4}$.

(4) $\lim_{q \uparrow +\infty} G_q = 1$.

Proof. (1) Let $0 < q < 1$. At this point we must be careful, because of the non-absolute convergence of $\tan^{-1}(1)$. We have

$$
\begin{aligned}
G_q &= (1-q) \sum_{n=0}^{\infty} \tan^{-1}(q^n) \\
&= (1-q)\tan^{-1}(1) + (1-q) \sum_{n=1}^{\infty} \tan^{-1}(q^n) \\
&= \frac{\pi}{4}(1-q) + (1-q) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{(2m+1)n}}{2m+1} \\
&= (1-q) \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} + (1-q) \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m+1}}{(2m+1)(1-q^{2m+1})} \\
&= (1-q) \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \left(1 + \frac{q^{2m+1}}{1-q^{2m+1}}\right) \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)[2m+1]_q}.
\end{aligned}
$$

The case $q > 1$ is exactly similar.

(2)(3)(4): These follow from (1) by noting the uniform convergence of the series for G_q on taking the following limits:

$$
\begin{aligned}
\lim_{q \uparrow 1} [n]_q &= n, \\
\lim_{q \downarrow 0} [n]_q &= 1,
\end{aligned}
$$

and

$$
\lim_{q \uparrow \infty} [n]_q = \begin{cases} 1 & \text{if } n = 1 \\ +\infty & \text{if } n > 1. \end{cases}
$$

□

3 Quantum Mahler measure

We prove the following result:

Theorem 2. *For all $q > 0$,*

$$G_q = \frac{\pi}{2} m_q \left(\frac{x-1}{x+1} y + 1 \right).$$

We fix the notation by putting

$$l_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{[n]_q}$$

which is absolutely convergent in

$$|x-1| < \max\{1, q\},$$

where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1. \end{cases}$$

The function $l_1(x) = \log(x)$ is the well-known logarithm. For $q \neq 1$, the analytic continuation of $l_q(x)$ to all $x \in \mathbf{C}$ is given by

$$l_q(x) = \begin{cases} (1-q) \displaystyle\sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}} & \text{if } 0 < q < 1 \\ (q-1) \displaystyle\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{m}} & \text{if } q > 1. \end{cases}$$

Both calculations are similar and easy. For example, when $0 < q < 1$

$$\begin{aligned} l_q(x) &= (1-q) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{1-q^n} \\ &= (1-q) \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^{n-1}(x-1)^n q^{nm} \\ &= (1-q) \sum_{m=0}^{\infty} \frac{(x-1)q^m}{1+(x-1)q^m} \\ &= (1-q) \sum_{m=0}^{\infty} \frac{x-1}{x-1+q^{-m}}. \end{aligned}$$

Thus $l_q(x)$ is meromorphic on $\mathbf{C}$ for $0 < q < 1$ or $q > 1$. We notice that

$$l_0(x) = 1 - \frac{1}{x}$$

and

$$l_\infty(x) = x - 1$$

are obtained as $\lim_{q\downarrow 0} l_q(x)$ and $\lim_{q\uparrow +\infty} l_q(x)$.

Proof of Theorem 2: We calculate

$$m_q = m_q\left(a\frac{x-1}{x+1}y + 1\right)$$

for $0 < a \leq 1$, and show that

$$\begin{aligned} m_q &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_q} \\ &= \frac{1}{\pi i}\left(\mathrm{Li}_{2,q}(ia) - \mathrm{Li}_{2,q}(-ia)\right). \end{aligned} \tag{3.1}$$

Then we see easily that

$$m_q\left(\frac{x-1}{x+1}y + 1\right) = \frac{2}{\pi}G_q$$

from Theorem 1(1). We prove (3.1) for $0 < q < 1$. The calculation is similar for $q > 1$. By definition

$$\begin{aligned} m_q &= \mathrm{Re}\int_0^1\int_0^1 l_q\left(a\frac{e^{2\pi i\theta_1} - 1}{e^{2\pi i\theta_1} + 1}e^{2\pi i\theta_2} + 1\right)d\theta_1 d\theta_2 \\ &= (1-q)\mathrm{Re}\sum_{m=0}^{\infty}\int_0^1\int_0^1\frac{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2} + q^{-m}}d\theta_1 d\theta_2. \end{aligned}$$

We show that

$$\int_0^1\int_0^1\frac{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2} + q^{-m}}d\theta_1 d\theta_2 = \frac{2}{\pi}\tan^{-1}(aq^{-m}). \tag{3.2}$$

Then, using the absolutely convergent series

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

for $|x| < 1$ we have

$$\begin{aligned} m_q &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\tan^{-1}(aq^{-m}) \\ &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(aq^{-m})^{2n+1}}{2n+1} \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)[2n+1]_q}, \end{aligned}$$

which is valid for $0 < a < 1$. In the case $a = 1$ we again take care, because of the non-absolutely converging series $\tan^{-1}(1)$:

$$\begin{aligned} m_q &= \frac{2}{\pi}(1-q)\sum_{m=0}^{\infty}\tan^{-1}(q^m) \\ &= \frac{2}{\pi}(1-q)\tan^{-1}(1) + \frac{2}{\pi}(1-q)\sum_{m=1}^{\infty}\tan^{-1}(q^m) \\ &= \frac{1}{2}(1-q) + \frac{2}{\pi}(1-q)\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(2n+1)m}}{2n+1} \\ &= \frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} + \frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}}{(2n+1)(1-q^{2n+1})} \\ &= \frac{2}{\pi}(1-q)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(1+\frac{q^{2n+1}}{1-q^{2n+1}}\right) \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)[2n+1]_q} \\ &= \frac{2}{\pi}G_q. \end{aligned}$$

Now, we return to the calculation of the integral (3.2):

$$\begin{aligned} I_m &= \int_0^1\int_0^1 \frac{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}+q^{-m}}d\theta_1 d\theta_2 \\ &= 1 - q^{-m}\int_0^1\int_0^1\frac{d\theta_1 d\theta_2}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}+q^{-m}}. \end{aligned}$$

Notice that

$$\int_0^1 \frac{d\theta_2}{ai\tan(\pi\theta_1)e^{2\pi i\theta_2}+q^{-m}} = \begin{cases} q^m & \text{if } |ai\tan(\pi\theta_1)| < q^{-m} \\ 0 & \text{if } |ai\tan(\pi\theta_1)| > q^{-m}. \end{cases}$$

Take the number α_m in $0 < \alpha_m < \frac{1}{2}$ satisfying

$$\cot(\pi\alpha_m) = aq^m.$$

Then,

$$\begin{aligned} |ai\tan(\pi\theta_1)| < q^{-m} &\iff |\cot(\pi\theta_1)| > aq^m \\ &\iff 0 < \theta_1 < \alpha_m \text{ or } 1-\alpha_m < \theta_1 < 1. \end{aligned}$$

Hence

$$\begin{aligned} I_m &= 1 - q^{-m}\left(\int_0^{\alpha_m} q^m d\theta_1 + \int_{1-\alpha_m}^1 q^m d\theta_1\right) \\ &= 1 - 2\alpha_m \\ &= 1 - \frac{2}{\pi}\cot^{-1}(aq^m) \\ &= \frac{2}{\pi}\tan^{-1}(aq^m). \end{aligned}$$

□

Here we introduce a reciprocity $G_q \longleftrightarrow G_{q^{-1}}$.

Theorem 3. $\dfrac{G_q - \frac{\pi}{4}}{\sqrt{q}}$ *is invariant under* $q \longmapsto q^{-1}$.

Proof. This can be proved directly from the definition of G_q, but we prefer to obtain it from a more general reciprocity $m_q(f) \longleftrightarrow m_{q^{-1}}(f)$ shown in Theorem 4 below, using Theorem 2. □

Theorem 4. $\dfrac{m_q(f) - m_0(f)}{\sqrt{q}}$ *is invariant under* $q \longmapsto q^{-1}$.

Proof. From the definition of $m_q(f)$ it is sufficient to show the invariance of

$$\frac{l_q(x) - l_0(x)}{\sqrt{q}}$$

under $q \longmapsto q^{-1}$. This is obvious for $q = 1$. Let $0 < q < 1$. Then

$$l_q(x) = (1-q)\left(1 - \frac{1}{x}\right) + (1-q)\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{-m}}$$

and

$$l_{q^{-1}}(x) = (q^{-1}-1)\sum_{m=1}^{\infty} \frac{x-1}{x-1+q^{-m}}.$$

Hence

$$l_q(x) = (1-q)l_0(x) + ql_{q^{-1}}(x).$$

Thus

$$\frac{l_q(x) - l_0(x)}{\sqrt{q}} = \frac{l_{q^{-1}}(x) - l_0(x)}{\sqrt{q^{-1}}}.$$

□

4 Quantum Hölder's double sine function

Before introducing into the quantum double sine function, we recall briefly the expression

$$F(x) = (2\sin \pi x)^x \exp\left(\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{n^2}\right) \tag{4.1}$$

in $0 < x < 1$ for Hölder's double sine function $F(x)$. There are several ways to reach this, and we refer to [KK] [KOW] for a general treatment containing multiple sine functions; we notice that $F(x) = \mathscr{S}_2(x)$ in [KK] [KOW] and we treated $\mathscr{S}_r(x)$ for integres $r \geq 1$.

A simple way to see (4.1) is as follows. From the defintion (1.6) of $F(x)$ we have

$$\log F(x) = x + \sum_{n=1}^{\infty}\left(n\left(\log\left(1-\frac{x}{n}\right) - \log\left(1+\frac{x}{n}\right)\right) + 2x\right)$$

and

$$\begin{aligned} \frac{F'(x)}{F(x)} &= 1 + \sum_{n=1}^{\infty}\frac{2x}{x^2-n^2} \\ &= \pi x \cot(\pi x). \end{aligned}$$

Hence, noting $F(0) = 1$, we see

$$\begin{aligned} F(x) &= \exp\left(\int_0^x \pi t \cot(\pi t) dt\right) \\ &= \exp\left(\Big[t\log(\sin \pi t)\Big]_0^x - \int_0^x \log(\sin \pi t) dt\right) \\ &= \exp\left(x \log(\sin \pi x) - \int_0^x \log(\sin \pi t) dt\right). \end{aligned}$$

Here, we use

$$\log(\sin \pi x) = -\log 2 - \sum_{n=1}^{\infty}\frac{\cos(2\pi n x)}{n}$$

for $0 < x < 1$, then by uniform convergence

$$\begin{aligned}\int_0^x \log(\sin \pi t)dt &= -x \log 2 - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^x \cos(2\pi n t)dt \\ &= -x \log 2 - \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^2}.\end{aligned}$$

Thus, we obtain (4.1). Hence, letting $x = \frac{1}{4}$ we get

$$\begin{aligned}G &= 2\pi \log \left(F\left(\frac{1}{4}\right) \left(2 \sin \frac{\pi}{4}\right)^{-\frac{1}{4}} \right) \\ &= 2\pi \log \left(F\left(\frac{1}{4}\right) 2^{-\frac{1}{8}} \right)\end{aligned}$$

as in (1.5).

Now, define the quantum Hölder's double sine function:

$$F_q(x) = \begin{cases} \prod_{n=1}^{\infty} \left(\frac{1 - q^{-n} e^{2\pi i x}}{1 - q^{-n} e^{-2\pi i x}} \right)^{q-1} & \text{for } q > 1, \\ (F(x)(2 \sin \pi x)^{-x})^{-4\pi i} & \text{for } q = 1, \\ \prod_{n=0}^{\infty} \left(\frac{1 - q^{n} e^{2\pi i x}}{1 - q^{n} e^{-2\pi i x}} \right)^{1-q} & \text{for } 0 < q < 1, \\ -e^{2\pi i x} & \text{for } q = 0. \end{cases}$$

Theorem 5. *(1)*

$$\begin{aligned}F_q(x) &= \exp \left(-2i \sum_{m=1}^{\infty} \frac{\sin(2\pi m x)}{m[m]_q} \right) \\ &= \exp \left(\mathrm{Li}_{2,q}(e^{-2\pi i x}) - \mathrm{Li}_{2,q}(e^{2\pi i x}) \right).\end{aligned}$$

(2)

$$G_q \equiv \frac{i}{2} \log F_q \left(\frac{1}{4} \right) \qquad \mod \pi\mathbf{Z}.$$

Proof. We prove the case $0 < q < 1$. The case $q > 1$ is similar.

(1) From the definition of $F_q(x)$,

$$\begin{aligned}
\log F_q(x) &= (1-q)\sum_{n=0}^{\infty}\left(\log(1-q^n e^{2\pi ix}) - \log(1-q^n e^{-2\pi ix})\right) \\
&= (1-q)\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-2\pi imx}-e^{2\pi imx}}{m}q^{nm} \\
&= (1-q)\sum_{m=1}^{\infty}\frac{e^{-2\pi imx}-e^{2\pi imx}}{m(1-q^m)} \\
&= \sum_{m=1}^{\infty}\frac{e^{-2\pi imx}-e^{2\pi imx}}{m[m]_q} \\
&= \mathrm{Li}_{2,q}(e^{-2\pi ix}) - \mathrm{Li}_{2,q}(e^{2\pi ix}) \\
&= -2i\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m[m]_q}.
\end{aligned}$$

(2) This follows from (1) by putting $x = \frac{1}{4}$.

□

Theorem 6. *The function*

$$\frac{\log F_q(x) - \log F_0(x)}{\sqrt{q}}$$

is invariant under $q \longmapsto q^{-1}$.

Proof. This follows from Theorem 5(1) with some calculation. Alternatively, we may proceed as follows. For a suitable series $\{a(n) \mid n = 1, 2, ...\}$ and $0 \le q \le \infty$ define

$$A_q = \sum_{n=1}^{\infty}\frac{a(n)}{[n]_q}$$

with

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{for } q \neq 1, \infty \\ n & \text{for } q = 1 \\ \delta_{n,1} & \text{for } q = \infty. \end{cases}$$

Then $\frac{A_q - A_0}{\sqrt{q}}$ is invariant under $q \longmapsto q^{-1}$. Actually, the proof is simple: let $0 < q < 1$. Then

$$\begin{aligned}
A_{q^{-1}} &= \sum_{n=1}^{\infty}\frac{a(n)}{[n]_{q^{-1}}} \\
&= q^{-1}\sum_{n=1}^{\infty}\frac{q^n a(n)}{[n]_q},
\end{aligned}$$

so

$$\begin{aligned} A_q - qA_{q^{-1}} &= \sum_{n=1}^{\infty} \frac{(1-q^n)a(n)}{[n]_q} \\ &= (1-q)\sum_{n=1}^{\infty} a(n) \\ &= (1-q)A_0. \end{aligned}$$

□

There is another quantization of $F(x)$ using the regularized double sine function

$$F(x,(\omega_1,\omega_2)) = \frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))}$$

due to Shintani [Shi], where $\Gamma_2(x,(\omega_1,\omega_2))$ is the (regularized) double gamma function of Barnes [Bar]; we refer to [KK] for a general treatment of regularized multiple sine functions $S_r(x,(\omega_1,\cdots,\omega_r))$ generalizing $S_2(x,(\omega_1,\omega_2)) = F(x,(\omega_1,\omega_2))$ and $S_1(x,\omega) = 2\sin(\frac{\pi x}{\omega})$.

Theorem 7. *Let $0 < q < 1$ and put $\tau = \frac{\log q}{2\pi i}$. Then*

$$G_q = \frac{i}{2}(1-q)\log\left(\frac{F\left(\frac{1}{4},(1,\tau)\right)F\left(\frac{1}{4},(1,-\tau)\right)}{(1+i)\sin\left(\frac{\pi}{4\tau}\right)}\right).$$

Proof. From Shintani [Shi] (Proposition 5) we have

$$\log F(x,(1,\tau)) = -\sum_{m=1}^{\infty}\frac{e^{2\pi imx}}{m(1-q^m)} + \sum_{m=1}^{\infty}\frac{q'^m e^{2\pi imx/\tau}}{m(1-q^m)} + \frac{\pi i}{2}\left(\frac{x^2}{\tau} - \left(\frac{1}{\tau}+1\right)x\right) + \frac{\pi i}{4} + \frac{\pi i}{2}\left(\frac{1}{\tau}+\tau\right),$$

where

$$q = e^{2\pi i\tau} \qquad \text{and} \qquad q' = e^{-2\pi i/\tau}.$$

Similarly

$$\log F\left(y,\left(1,-\frac{1}{\tau}\right)\right) = -\sum_{m=1}^{\infty}\frac{e^{2\pi imy}}{m(1-q'^m)} + \sum_{m=1}^{\infty}\frac{q^m e^{-2\pi imy}}{m(1-q^m)} + \frac{\pi i}{2}\left(-\tau y^2 - (-\tau+1)y\right) + \frac{\pi i}{4} + \frac{\pi i}{2}\left(-\tau - \frac{1}{\tau}\right).$$

Hence,

$$\begin{aligned}
&\log F(x,(1,\tau)) + \log F\left(\frac{x}{\tau},\left(1,-\frac{1}{\tau}\right)\right) \\
= &-\sum_{m=1}^{\infty}\frac{e^{2\pi imx}-q^m e^{-2\pi imx}}{m(1-q^m)} - \sum_{m=1}^{\infty}\frac{e^{2\pi imx/\tau}}{m} - \frac{\pi ix}{\tau} + \frac{\pi i}{2} \\
= &-\sum_{m=1}^{\infty}\frac{e^{2\pi imx}-e^{-2\pi imx}}{m(1-q^m)} - \sum_{m=1}^{\infty}\frac{e^{-2\pi imx}}{m} - \sum_{m=1}^{\infty}\frac{e^{2\pi imx/\tau}}{m} - \frac{\pi ix}{\tau} + \frac{\pi i}{2} \\
= &-2i\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m(1-q^m)} + \log(1-e^{-2\pi ix}) + \log(1-e^{2\pi ix/\tau}) - \frac{\pi ix}{\tau} + \frac{\pi i}{2}.
\end{aligned}$$

Then letting $x = \frac{1}{4}$ and multiplying $\frac{i}{2}(1-q)$ we get

$$\begin{aligned}
&\frac{i(1-q)}{2}\log\left(F\left(\frac{1}{4},(1,\tau)\right)F\left(\frac{1}{4\tau},\left(1,-\frac{1}{\tau}\right)\right)\right) \\
&\qquad\qquad = G_q + \frac{i}{2}(1-q)\left(\frac{\log 2}{2} + \frac{3\pi i}{4} - \frac{\pi i}{4\tau} + \log(1-e^{\frac{\pi i}{2\tau}})\right).
\end{aligned}$$

Then, as the homogeneity

$$F(cx,(c\omega_1,c\omega_2)) = F(x,(\omega_1,\omega_2))$$

proved in [KK] implies

$$\begin{aligned}
F\left(\frac{1}{4\tau},\left(1,-\frac{1}{\tau}\right)\right) &= F\left(\frac{1}{4},(\tau,-1)\right) \\
&= F\left(\frac{1}{4},(-1,\tau)\right),
\end{aligned}$$

we obtain Theorem 7. □

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