ON THE HYPOELLIPTICITY WITH A BIG LOSS OF DERIVATIVES

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Abstract. We study, for a model class of classical pseudodifferential operators with symplectic characteristics of multiplicity k, necessary and sufficient conditions for the hypoellipticity with loss of $r+k/2$ derivatives ($r>0$).

1. Introduction and setting of the problem

Consider, for $n \geq 2$ and $1 \leq \nu < n$,

$$T^*\mathbb{R}^n = \mathbb{R}^n_z \times \mathbb{R}^n_\zeta = \left(\mathbb{R}^\nu_x \times \mathbb{R}^{n-\nu}_y\right) \times \left(\mathbb{R}^\nu_\xi \times \mathbb{R}^{n-\nu}_\eta\right),$$

and define

$$\Sigma = \{(z,\zeta) \in T^*\mathbb{R}^n \setminus 0;\ x = \xi = 0\}.$$

Suppose we are given in $\mathbb{R}^n$ a (properly supported) pseudodifferential operator (ψdo) $A = A(z, D_z)$ of order $m \in \mathbb{R}$, $A \in \mathrm{OPS}^m_{\mathrm{cl}}(\mathbb{R}^n)$. By this we mean that the symbol $a(z,\zeta)$ admits an asymptotic semi-regular expansion of the kind

$$a(z,\zeta) \sim \sum_{j\geq 0} a_{m-j/2}(z,\zeta)$$

where $a_{m-j/2}$ is smooth (i.e. C^∞) on $T^*\mathbb{R}^n \setminus 0$ and (positively) homogeneous of degree $m - j/2$ in the fiber variable ζ. Suppose furthermore that for a given integer $k \geq 1$ we have

$$|a_{m-j/2}(z,\zeta)| \lesssim |\zeta|^{m-j/2}\mathrm{dist}_\Sigma(z,\zeta)^{k-j}, \quad j = 0,\dots,k, \tag{1}$$

$$|\zeta|^m \mathrm{dist}_\Sigma(z,\zeta)^k \lesssim |a_m(z,\zeta)|, \tag{2}$$

where $\mathrm{dist}_\Sigma(z = (x,y), \zeta = (\xi,\eta)) = |x| + |\xi|/|\zeta|$ denotes the distance of $(z, \zeta/|\zeta|)$ to Σ, and the notation $f \lesssim g$ means that for any given conic set $\Gamma \subset T^*\mathbb{R}^n \setminus 0$ with compact base and any given $\varepsilon > 0$ there exists $C = C_{\Gamma,\varepsilon} > 0$ such that

$$f(z,\zeta) \leq C g(z,\zeta), \quad \forall (z,\zeta) \in \Gamma, \quad |\zeta| \geq \varepsilon.$$

As usual, we rephrase condition (1) by saying that A *vanishes to order k at Σ*, and condition (2) by saying that A *is transversally elliptic (with respect to Σ)*.

This paper was completed while the second author was Ryuudou Visiting Professor at the Tokyo Institute of Technology. He wishes to thank the TIT for the excellent hospitality during his stay.

We are interested in studying the *hypoellipticity* of A. More precisely, we shall stick to the following notions of hypoellipticity.

Definition 1.1. *We say that A is hypoelliptic at $\rho_0 \in T^*\mathbb{R}^n \setminus 0$ with loss of $r \geq 0$ derivatives, resp. hypoelliptic at ρ_0, if for every distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ and any given $s \in \mathbb{R}$ we have*

$$Au \in H^s \quad at \ \ \rho_0 \implies u \in H^{s+m-r} \quad at \ \ \rho_0, \tag{3}$$

resp. for all $u \in \mathcal{D}'(\mathbb{R}^n)$

$$\rho_0 \notin \mathrm{WF}(Au) \implies \rho_0 \notin \mathrm{WF}(u).$$

Furthermore, we say that A is hypoelliptic at $z_0 \in \mathbb{R}^n$ with loss of $r \geq 0$ derivatives, resp. hypoelliptic at z_0, if A is hypoelliptic with loss of r derivatives, resp. hypoelliptic, at $\rho_0 = (z_0, \zeta)$ for all $\zeta \neq 0$.

Remark 1.2. *Recall that a distribution $v \in \mathcal{D}'(\mathbb{R}^n)$ belongs to the Sobolev space H^t at ρ_0 iff there exists $v' \in H^t_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\rho_0 \notin \mathrm{WF}(v - v')$. Equivalently, with $\rho_0 = (z_0, \zeta_0)$, there exist a neighborhood U of z_0 and a conic neighborhood Γ of ζ_0, such that*

$$\int_\Gamma (1 + |\zeta|^2)^t |\widehat{\varphi v}(\zeta)|^2 d\zeta < +\infty, \quad \forall \varphi \in C_0^\infty(U).$$

When $\rho_0 \notin \Sigma$ we have $a_m(\rho_0) \neq 0$ and by classical elliptic theory A is hypoelliptic at ρ_0 with loss of 0 derivatives. Thus the problem is the hypoellipticity of A at the points of Σ.

When $\rho_0 \in \Sigma$, it is known after Sjöstrand [19] that A cannot be hypoelliptic at ρ_0 with loss of $r < k/2$ derivatives. The hypoellipticity with loss of $k/2$ derivatives has been completely described by Boutet, Grigis and Helffer in [4] (where they actually treat the general case of Σ closed and conic submanifold of $T^*\mathbb{R}^n \setminus 0$).

The main ingredient to detect hypoellipticity is the so called *localized operator A_ρ of A at $\rho \in \Sigma$*. In our case, for $\rho = (0, y, 0, \eta \neq 0)$, A_ρ can be defined as follows. Upon setting

$$a_\rho(x, \xi) := \sum_{|\alpha|+|\beta|+j=k} \frac{1}{\alpha!\beta!} \Big(\partial_x^\alpha \partial_\xi^\beta a_{m-j/2}\Big)(\rho) x^\alpha \xi^\beta, \tag{4}$$

we define

$$A_\rho = \mathrm{Op}(a_\rho)(x, D_x), \tag{5}$$

where Op denotes the usual quantization in the (x, ξ)-variables.

Remark 1.3. *It is important to note that A_ρ is a polynomial-coefficient differential operator of order k in $\mathbb{R}^\nu$ which, by virtue of the transversal ellipticity, is* **globally elliptic** *in the sense of Shubin* [18] *(see also* [8]*). As a consequence, A_ρ maps continuously $\mathcal{S}(\mathbb{R}^\nu)$ (resp. $\mathcal{S}'(\mathbb{R}^\nu)$) into itself and*

$$f \in \mathcal{S}'(\mathbb{R}^\nu), \ A_\rho f \in \mathcal{S}(\mathbb{R}^\nu) \implies f \in \mathcal{S}(\mathbb{R}^\nu).$$

Throughout the paper, we shall tacitly consider $A_\rho\colon L^2(\mathbb{R}^\nu) \longrightarrow L^2(\mathbb{R}^\nu)$ as an unbounded operator with domain

$$B^k(\mathbb{R}^\nu) := \left\{ f \in \mathcal{S}'(\mathbb{R}^\nu);\ \|f\|_{B^k(\mathbb{R}^\nu)} := \left(\sum_{|\alpha|+|\beta|\le k} \left\| x^\alpha D_x^\beta f \right\|^2_{L^2(\mathbb{R}^\nu)} \right)^{1/2} < +\infty \right\}.$$

Recall that $B^k(\mathbb{R}^\nu)$ is dense in $L^2(\mathbb{R}^\nu)$ with compact embedding.

The global ellipticity of A_ρ yields that A_ρ has a finite dimensional kernel and a closed range with finite codimension (so that A_ρ is a Fredholm operator). Moreover, $\mathrm{Spec}(A_\rho)$*, the spectrum of A_ρ, is either the whole complex plane or a discrete set, made of a sequence of eigenvalues of finite multiplicities.*

This latter case, via Shubin theory, occurs if the numerical range (i.e. the set of values) of the polynomial

$$\tilde{a}_\rho(x,\xi) := \sum_{|\alpha|+|\beta|=k} \frac{1}{\alpha!\beta!} \left(\partial_x^\alpha \partial_\xi^\beta a_m \right)(\rho) x^\alpha \xi^\beta, \tag{6}$$

which is a closed connected cone of $\mathbb{C}$, is **not** *the whole complex plane $\mathbb{C}$.*

Boutet-Grigis-Helffer's theorem reads as follows.

Theorem 1.4. *A is hypoelliptic at $\rho_0 \in \Sigma$ with loss of $k/2$ derivatives iff*

$$\mathrm{Ker}(A_{\rho_0}) = \{0\}. \tag{7}$$

In this paper we consider cases in which condition (7) above is violated at some point ρ_0 of Σ, and our purpose is to give (*inter alia*) necessary and sufficient conditions for the hypoellipticity of A with a loss of derivatives greater than $k/2$.

More precisely, on supposing that $\mathrm{Ker}(A_{\rho_0}) \neq \{0\}$ for some $\rho_0 \in \Sigma$ and that A_ρ has discrete spectrum for ρ near ρ_0, we shall show that it is possible to microlocally associate with A, pseudodifferential square-systems $\Lambda = \Lambda(y, D_y)$ in $\mathbb{R}^{n-\nu}$ of order $m - k/2$ and size $\dim \mathrm{Ker}\,(A_{\rho_0})$ (how Λ is associated with A is made precise in Definition 4.5 and Theorem 4.6 below) so that, after canonically identifying Σ with $T^*\mathbb{R}^{n-\nu} \setminus 0$, the hypoellipticity of A at ρ_0 with loss of $r + k/2$ derivatives ($r > 0$) is **equivalent** to the hypoellipticity of each Λ at ρ_0 with loss of r derivatives. The **complete symbol** of **every** system Λ will be **explicitly constructed**.

The very idea of associating the systems Λ with A goes back to the seminal work of Sjöstrand [19], and was further exploited by Helffer [7] in his study of hypoellipticity with loss of 3/2 derivatives for operators with symplectic double-characteristics (i.e. $k = 2$, $r = 1/2$), and by Grigis and Rothschild [5] in their study of analytic (and C^∞) hypoellipticity of a class of differential operators with polynomial coefficients. In this respect, the present work is a natural continuation of these papers.

We point out that a serious limitation of our approach is the lack of invariance (see Section 7). Because of this, in treating examples (see Section

6), we are forced to consider only Grushin-type operators. Nevertheless, the examples we analyze seem to be new and also kind-of-unexpected (at least to us). In particular, in Example 6.5 we show an instance of a transversally elliptic operator with characteristics of order 4 that, depending upon the choice of the lower order terms, is hypoelliptic with loss of 2, 3, 4 or 5 derivatives.

2. Hypotheses on the localized operator A_ρ and related constructions

As mentioned in the Introduction, we suppose that $\mathrm{Ker}(A_{\rho_0}) \neq \{0\}$ for some $\rho_0 \in \Sigma$ and make our main hypothesis **(H1)**:

(H1): *The localized operator*

$$A_\rho \colon L^2(\mathbb{R}^\nu) \longrightarrow L^2(\mathbb{R}^\nu), \quad \mathrm{Dom}(A_\rho) = B^k(\mathbb{R}^\nu),$$

has **discrete** *spectrum for all* $\rho \in \Sigma$.

In the sequel we shall always identify Σ with $T^*\mathbb{R}^{n-\nu} \setminus 0$.

It is crucial to observe the following homogeneity property of A_ρ. Consider, for $t > 0$, the dilations

$$(\mathsf{M}_t f)(x) := t^{\nu/4} f(t^{1/2}x), \tag{8}$$

that are **isometries** of $L^2(\mathbb{R}^\nu)$ (and automorphisms of $\mathcal{S}(\mathbb{R}^\nu)$, resp. $\mathcal{S}'(\mathbb{R}^\nu)$). Since by (4)

$$a_{(y,t\eta)}(t^{-1/2}x, t^{1/2}\xi) = t^{m-k/2} a_{(y,\eta)}(x,\xi), \tag{9}$$

we have

$$A_{(y,t\eta)}\Big(\mathsf{M}_t f\Big) = t^{m-k/2}\mathsf{M}_t\Big(A_{(y,\eta)} f\Big). \tag{10}$$

As a consequence

$$\mathsf{M}_t\Big(\mathrm{Ker}(A_{(y,\eta)})\Big) = \mathrm{Ker}(A_{(y,t\eta)}), \quad \forall t > 0. \tag{11}$$

We now give a few consequences of hypothesis **(H1)** that we gather in the next proposition.

Proposition 2.1. *1) Also A_ρ^* has discrete spectrum and, obviously,* $\mathrm{Spec}(A_\rho^*) = \overline{\mathrm{Spec}(A_\rho)}$, *for all* $\rho \in \Sigma$.

2) One has $\mathrm{ind}(A_\rho) = \mathrm{ind}(A_\rho^*) = 0$, *for all* $\rho \in \Sigma$.

3) The order of vanishing k is **even**.

Proof. Since the ψdo A^* (the formal adjoint of A) vanishes to order k at Σ and is transversally elliptic, and since, as one can easily see, $(A^*)_\rho = A_\rho^*$, point 1) is well-known.

As regards point 2), we have that if $\lambda \in \mathbb{C} \setminus \mathrm{Spec}(A_\rho)$, then $A_\rho - \lambda \colon B^k(\mathbb{R}^\nu) \longrightarrow L^2(\mathbb{R}^\nu)$ is an isomorphism, whence we obviously have $\mathrm{ind}(A_\rho -$

$\lambda) = 0$. Now, $A_\rho = A_\rho - \lambda\,\mathrm{id} + \lambda\,\mathrm{id}$, and since $\mathrm{id}\colon B^k(\mathbb{R}^\nu) \longrightarrow L^2(\mathbb{R}^\nu)$ is compact, we get that $\mathrm{ind}(A_\rho) = \mathrm{ind}(A_\rho - \lambda) = 0$. In particular, we have that

$$\dim \mathrm{Ker}\,(A_\rho - \lambda) = \dim \mathrm{Ker}\,\left(A_\rho^* - \bar{\lambda}\right), \quad \forall \lambda \in \mathbb{C},\ \forall \rho \in \Sigma.$$

As regards point 3), it is well-known that when $\nu > 1$ then k must be even. In case $\nu = 1$ we immediately observe that $\mathrm{ind}(A_\rho) = \mathrm{ind}(\mathrm{Op}(\tilde{a}_\rho)(x, D_x))$, where $\tilde{a}_\rho$ was defined in (6). In fact, $A_\rho - \mathrm{Op}(\tilde{a}_\rho)(x, D_x)$ is a global differential operator in $\mathbb{R}^\nu$ of order $k-1$ and hence it is a compact operator from $B^k(\mathbb{R}^\nu)$ into $L^2(\mathbb{R}^\nu)$. Now, if one puts

$$k_\pm := \mathrm{card}(\{\zeta \in \mathbb{C};\ \pm\mathrm{Im}(\zeta) > 0,\ \tilde{a}_\rho(1, \zeta) = 0\}),$$

we have

$$\tilde{a}_\rho(x, \xi) = x^k \tilde{a}_\rho(1, \xi/x) = c \prod_{j=1}^{k_+} (\xi - \zeta_j^+ x) \prod_{j=1}^{k_-} (\xi - \zeta_j^- x),$$

where $c = \dfrac{1}{k!}(\partial_\xi^k a_m)(\rho) \neq 0$ and $\tilde{a}_\rho(1, \zeta_j^\pm) = 0$, $\pm\mathrm{Im}\,(\zeta_j^\pm) > 0$. As a consequence,

$$0 = \mathrm{ind}(\mathrm{Op}(\tilde{a}_\rho)) = \sum_{j=1}^{k_+} \mathrm{ind}(D_x - \zeta_j^+ x) + \sum_{j=1}^{k_-} \mathrm{ind}(D_x - \zeta_j^- x) = k_+ - k_-,$$

and since $k_+ + k_- = k$ we have the statement. □

Remark 2.2. *As a consequence of point 2) of the above proposition, we have that if* $\rho_0 \in \Sigma$ *with* $\mathrm{Ker}(A_{\rho_0}) \neq \{0\}$, *then also* $\mathrm{Ker}(A_{\rho_0}^*) \neq \{0\}$, *and the two spaces have the* **same** *dimension. (Because of this, the systems* Λ *mentioned in the introduction are square-systems of size* $\dim \mathrm{Ker}\,(A_{\rho_0})$*.)*
As already remarked (see Remark 1.3), as a consequence of Shubin's theory [18]

$$\tilde{a}_\rho(T^*\mathbb{R}^\nu) \neq \mathbb{C} \implies A_\rho \ \text{ has discrete spectrum.}$$

After Sjöstrand [19], *it is known that this condition is fulfilled when* $k = 2$ *and* $\nu > 1$, *and it is also fulfilled for* $k = 2$ *and* $\nu = 1$ *when* $k_+ = k_-$ *(*$\tilde{a}_\rho(T^*\mathbb{R}^\nu)$ *is then a closed strictly convex cone of* $\mathbb{C}$*).*

Let now $\rho_0 \in \Sigma$ be a point where $\mathrm{Ker}(A_{\rho_0}) \neq \{0\}$. We may suppose, by (11), that $\rho_0 \in \mathbb{S}^*\Sigma = \{(y, \eta) \in \Sigma;\ |\eta| = 1\}$. We set

$$1 \leq d = \dim \mathrm{Ker}\,(A_{\rho_0}) = \dim \mathrm{Ker}\,(A_{\rho_0}^*).$$

Since $A_\rho^* A_\rho$ and $A_\rho A_\rho^*$ are globally elliptic, self-adjoint and nonnegative differential operators of order $2k$, with discrete spectrum contained in $[0, +\infty)$, and since

$$\mathrm{Ker}\,(A_\rho) = \mathrm{Ker}\,(A_\rho^* A_\rho), \quad \mathrm{Ker}\,(A_\rho^*) = \mathrm{Ker}\,(A_\rho A_\rho^*),\ \forall \rho \in \Sigma,$$

it follows that

$$0 \in \mathrm{Spec}(A_{\rho_0}^* A_{\rho_0}) \cap \mathrm{Spec}(A_{\rho_0} A_{\rho_0}^*)$$

and

$$\dim \operatorname{Ker}(A^*_{\rho_0} A_{\rho_0}) = \dim \operatorname{Ker}(A_{\rho_0} A^*_{\rho_0}) = d.$$

Definition 2.3. *Let $\tilde{U} \subset \mathbb{S}^*\Sigma$ be a neighborhood of $\rho_0 \in \mathbb{S}^*\Sigma$. We say that $\tilde{U}$ is* **admissible** *if for some $\varepsilon > 0$ we have:*

(i) $\varepsilon \notin \operatorname{Spec}(A^*_\rho A_\rho) \cup \operatorname{Spec}(A_\rho A^*_\rho)$, *for all $\rho \in \tilde{U}$;*

(ii) *Upon setting*

$$V_1(\rho) := \bigoplus_{0 \le \lambda < \varepsilon,\ \lambda \in \operatorname{Spec}(A^*_\rho A_\rho)} \operatorname{Ker}(A^*_\rho A_\rho - \lambda) \subset \mathcal{S}(\mathbb{R}^\nu),$$

$$V_2(\rho) := \bigoplus_{0 \le \lambda < \varepsilon,\ \lambda \in \operatorname{Spec}(A_\rho A^*_\rho)} \operatorname{Ker}(A_\rho A^*_\rho - \lambda) \subset \mathcal{S}(\mathbb{R}^\nu)$$

($\oplus$ is **finite and orthogonal***), we have*

$$\dim V_1(\rho) = \dim V_2(\rho) = d, \quad \forall \rho \in \tilde{U};$$

(iii) *The Hermitian (with respect to the $L^2(\mathbb{R}^\nu)$-inner product) vector-bundles of rank d,*

$$V_j := \bigsqcup_{\rho \in \tilde{U}} V_j(\rho) \longrightarrow \tilde{U}, \quad j = 1, 2,$$

are **trivial**.

Remark 2.4. *1) The fact that V_1 and V_2 are indeed Hermitian vector-bundles of rank d follows from the fact that $V_j(\rho) = \operatorname{Im}(\pi_j(\rho))$, $j = 1, 2$, where $\pi_j(\rho)$ are the orthogonal projectors defined by*

$$\pi_1(\rho) = \frac{1}{2\pi i} \oint_{|\zeta| = \varepsilon} (\zeta - A^*_\rho A_\rho)^{-1} d\zeta, \pi_2(\rho) = \frac{1}{2\pi i} \oint_{|\zeta| = \varepsilon} (\zeta - A_\rho A^*_\rho)^{-1} d\zeta, \quad \rho \in \tilde{U}.$$

2) It is obvious that if conditions (i) and (ii) are satisfied, then upon possibly shrinking $\tilde{U}$ we may always suppose that also (iii) holds. Hence, the existence of an admissible neighborhood of ρ_0 is always granted.

We now list two fundamental consequences.

Lemma 2.5. *Let $\tilde{U}$ be admissible. One has*

$$A_\rho \colon V_1(\rho) \longrightarrow V_2(\rho), \quad A^*_\rho \colon V_2(\rho) \longrightarrow V_1(\rho), \ \forall \rho \in \tilde{U}.$$

Proof. We start off by observing that

$$A_\rho(\zeta - A^*_\rho A_\rho) = (\zeta - A_\rho A^*_\rho) A_\rho, \quad \forall \zeta \in \mathbb{C}, \ \forall \rho \in \Sigma,$$

so that

$$(\zeta - A_\rho A^*_\rho)^{-1} A_\rho = A_\rho (\zeta - A^*_\rho A_\rho)^{-1}, \ \forall \rho \in \tilde{U}, \ |\zeta| = \varepsilon,$$

whence

$$A_\rho \pi_1(\rho) = \pi_2(\rho) A_\rho, \quad A^*_\rho \pi_2(\rho) = \pi_1(\rho) A^*_\rho, \quad \forall \rho \in \tilde{U},$$

which concludes the proof. ∎

Remark 2.6. *When* $\rho = \rho_0$, $V_1(\rho_0) = \operatorname{Ker}(A_{\rho_0})$, *and* $A_{\rho_0}(V_1(\rho_0)) = \{0\}$. *Now let* $\phi'_j(\rho;\cdot) \in \mathcal{S}(\mathbb{R}^\nu)$, $1 \le j \le d$, *and* $\phi_j(\rho;\cdot) \in \mathcal{S}(\mathbb{R}^\nu)$, $1 \le j \le d$, *respectively, be* $L^2(\mathbb{R}^\nu)$*-orthonormal and smooth in* $\rho \in \tilde{U}$ *bases of* $V_1(\rho)$ *and* $V_2(\rho)$, *respectively. Then the matrix representing* $A_\rho \colon V_1(\rho) \longrightarrow V_2(\rho)$ *in the bases* ϕ', ϕ, *is*

$$\Lambda(\rho) = \Big(\Lambda^{(jr)}_{\phi,\phi'}(\rho)\Big)_{1\le j,r\le d} = \bigg(\Big(A_\rho\phi'_r(\rho;\cdot), \phi_j(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\bigg)_{1\le j,r\le d}.$$

This matrix is **smooth** *in* ρ *and* **vanishes** *at* ρ_0.
Notice that it is not required that the ϕ'_j *and the* ϕ_j, *respectively, be eigenfunctions of* $A^*_\rho A_\rho$ *and* $A_\rho A^*_\rho$, *respectively.*

Lemma 2.7. *Let* $\tilde{U}$ *be admissible and let* $V_j(\rho)^\perp$, $\rho \in \tilde{U}$, *be the* $L^2(\mathbb{R}^\nu)$*-orthogonal of* $V_j(\rho)$, $j = 1, 2$. *Then*

$$A_\rho \colon B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp \longrightarrow V_2(\rho)^\perp, \text{ and } \quad A^*_\rho \colon B^k(\mathbb{R}^\nu) \cap V_2(\rho)^\perp \longrightarrow V_1(\rho)^\perp,$$

are **isomorphisms** *for all* $\rho \in \tilde{U}$.

Proof. We just prove the property for A_ρ, since the proof for A^*_ρ is analogous. Let $f \in B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp$, and let $g \in V_2(\rho)$. Hence, from Lemma 2.5 we get $A^*_\rho g \in V_1(\rho)$ and hence

$$(A_\rho f, g)_{L^2(\mathbb{R}^\nu)} = (f, A^*_\rho g)_{L^2(\mathbb{R}^\nu)} = 0.$$

Thus $A_\rho\Big(B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp\Big) \subset V_2(\rho)^\perp$, for all $\rho \in \tilde{U}$.

We now observe that

- $B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp$ is **closed** in $B^k(\mathbb{R}^\nu)$ (which is obvious);
- $B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp$ is $L^2(\mathbb{R}^\nu)$-orthogonal to $\operatorname{Ker}(A_\rho)$ (which follows from $\operatorname{Ker}(A_\rho) = \operatorname{Ker}(A^*_\rho A_\rho) \subset V_1(\rho)$).

Now, because of the global ellipticity of A_ρ, $\rho \in \Sigma$, we have for A_ρ the a-priori estimate

$$\|f\|_{B^k(\mathbb{R}^\nu)} \le C\Big(\|A_\rho f\|_{L^2(\mathbb{R}^\nu)} + \|f\|_{L^2(\mathbb{R}^\nu)}\Big), \quad \forall f \in B^k(\mathbb{R}^\nu). \tag{12}$$

As a consequence, we claim that if $T \subset B^k(\mathbb{R}^\nu)$ is closed in $B^k(\mathbb{R}^\nu)$ and $T \perp_{L^2} \operatorname{Ker}(A_\rho)$, then $A_\rho(T)$ is closed in $L^2(\mathbb{R}^\nu)$. To see this, we need only prove that if $g \in \overline{A_\rho(T)}$, $g \ne 0$, then $g \in A_\rho(T)$. Let hence $\{f_j\}_{j\ge1} \subset T$ be a sequence such that $A_\rho f_j \xrightarrow{L^2} g$. Then it must be

$$\sup_{j\ge1} \|f_j\|_{B^k(\mathbb{R}^\nu)} < +\infty.$$

Otherwise, by possibly passing to a subsequence, we would have $\|f_j\|_{B^k(\mathbb{R}^\nu)} \to +\infty$, and hence $A_\rho\Big(f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)}\Big) \xrightarrow{L^2} 0$. Since $\Big\|f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)}\Big\|_{B^k(\mathbb{R}^\nu)} = 1$, being $B^k(\mathbb{R}^\nu)$ compactly embedded in $L^2(\mathbb{R}^\nu)$, we have (again possibly for some subsequence) $f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)} \xrightarrow{L^2} \varphi \in L^2(\mathbb{R}^\nu)$, thereby yielding $A_\rho\Big(f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)}\Big) \xrightarrow{\mathcal{S}'} A_\rho\varphi$, and finally that $A_\rho\varphi = 0$, i.e. $\varphi \in \mathrm{Ker}\,(A_\rho)$. On the other hand, since by hypothesis $f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)} \in T$ and $T \perp_{L^2} \mathrm{Ker}\,(A_\rho)$, we get $\varphi \in \mathrm{Ker}\,(A_\rho)^\perp$, so that $\varphi = 0$. But then from (12) we obtain that $f_j/\|f_j\|_{B^k(\mathbb{R}^\nu)} \xrightarrow{B^k} 0$, which is a contradiction. Hence

$$\sup_{j\geq 1} \|f_j\|_{B^k(\mathbb{R}^\nu)} < +\infty.$$

Now, by compactness and by possibly passing to a subsequence, we may suppose that $\{f_j\}_{j\geq 1}$ be convergent in $L^2(\mathbb{R}^\nu)$. Then, again by virtue of inequality (12), we get that $\{f_j\}_{j\geq 1}$ is a Cauchy sequence in $B^k(\mathbb{R}^\nu)$, and finally conclude that $A_\rho\Big(B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp\Big)$ *is a closed subspace of* $V_2(\rho)^\perp$. We now prove that in fact

$$A_\rho\Big(B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp\Big) = V_2(\rho)^\perp.$$

Otherwise, in case $A_\rho\Big(B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp\Big) \neq V_2(\rho)^\perp$, we would find in the Hilbert space $V_2(\rho)^\perp$ an element $g \neq 0$ such that

$$(g, A_\rho f)_{L^2(\mathbb{R}^\nu)} = 0, \quad \forall f \in B^k(\mathbb{R}^\nu) \cap V_1(\rho)^\perp.$$

On the other hand, by Lemma 2.5 we already know that

$$(g, A_\rho f')_{L^2(\mathbb{R}^\nu)} = 0, \quad \forall f' \in V_1(\rho).$$

Hence we must have

$$(g, A_\rho f)_{L^2(\mathbb{R}^\nu)} = 0, \quad \forall f \in B^k(\mathbb{R}^\nu).$$

In particular we must have $(g, A_\rho f)_{L^2(\mathbb{R}^\nu)} = 0$ for all $f \in \mathcal{S}(\mathbb{R}^\nu)$, and hence that $A_\rho^* g = 0$ in $\mathcal{S}'(\mathbb{R}^\nu)$. But then $g \in \mathcal{S}(\mathbb{R}^\nu)$ and $A_\rho^* g = 0$, so that also $A_\rho A_\rho^* g = 0$, that is $g \in \mathrm{Ker}\,(A_\rho A_\rho^*) \subset V_2(\rho)$. Therefore $0 \neq g \in V_2(\rho)^\perp \cap V_2(\rho)$, a contradiction which proves the lemma. ☐

As a consequence, we have that the map

$$\Big(A_\rho\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\Big)^{-1} : V_2(\rho)^\perp \longrightarrow V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu), \quad \rho \in \tilde{U},$$

is **continuous**. Upon defining $E_\rho\colon L^2(\mathbb{R}^\nu) \longrightarrow B^k(\mathbb{R}^\nu)$, $\rho \in \tilde{U}$, by

$$E_\rho = \begin{cases} 0, \text{ in } V_2(\rho) \\ \\ \Big(A_\rho\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\Big)^{-1}, \text{ in } V_2(\rho)^\perp, \end{cases} \tag{13}$$

then E_ρ is **continuous**, for all $\rho \in \tilde{U}$, and

$$A_\rho E_\rho = 1 - \pi_2(\rho).$$

As a global ψdo in $\mathbb{R}^\nu$, $\pi_2(\rho)$ is smoothing, for its (x,ξ)-symbol is, with $\{\phi_j(\rho;\cdot)\}_{1\leq j\leq d}$ an $L^2(\mathbb{R}^\nu)$-orthonormal basis of $V_2(\rho)$,

$$\sigma_{(x,\xi)}(\pi_2(\rho)) = \sum_{j=1}^{d} e^{-ix\cdot\xi}\phi_j(\rho;x)\overline{\hat{\phi}_j(\rho;\xi)}.$$

It follows that E_ρ is a globally elliptic ψdo of order $-k$. We also have that

$$E_\rho A_\rho = 1 - \pi_1(\rho),$$

and that E_ρ may be thought of as a continuous map from $\mathcal{S}(\mathbb{R}^\nu)$ (resp. $\mathcal{S}'(\mathbb{R}^\nu)$) into itself. Finally, exactly as in Helffer [7], one proves that E_ρ depends in a C^∞ fashion on $\rho \in \tilde{U}$.

A particularly meaningful case in which the bundles V_1 and V_2 can be explicitly described is the following.

Suppose

$$[A_\rho, A_\rho^*] = 0 \;\text{ for all } \rho\in\Sigma \text{ (or for } \rho \text{ conically near } \rho_0). \tag{14}$$

We then have the following further properties:

- $\mathrm{Ker}\,(A_\rho - \zeta) = \mathrm{Ker}\,(A_\rho^* - \bar{\zeta})$, $\forall \zeta\in\mathbb{C}$;
- $\zeta\neq\zeta' \implies \mathrm{Ker}\,(A_\rho-\zeta) \perp_{L^2} \mathrm{Ker}\,(A_\rho-\zeta')$;
- $L^2(\mathbb{R}^\nu) = \bigoplus_{\zeta\in\mathrm{Spec}(A_\rho)} \mathrm{Ker}\,(A_\rho-\zeta)$ (orthogonal sum).

We now take $\delta>0$ and a neighborhood $\tilde{U}\subset\mathbb{S}^*\Sigma$ of ρ_0 such that:

(i) $\mathrm{Spec}(A_\rho)\subset\{\zeta\in\mathbb{C};\ |\zeta|<\delta\}\cup\{\zeta\in\mathbb{C};\ |\zeta|>\delta\}$, for all $\rho\in\tilde{U}$;

(ii) Upon setting

$$W(\rho) := \bigoplus_{|\zeta|<\delta,\ \zeta\in\mathrm{Spec}(A_\rho)} \mathrm{Ker}\,(A_\rho-\zeta)\subset\mathcal{S}(\mathbb{R}^\nu) \;\; \text{(orthogonal sum)},$$

we require that $\dim W(\rho)=d$ for all $\rho\in\tilde{U}$, so that, in particular, $W(\rho_0)=\mathrm{Ker}\,(A_{\rho_0})$. We let $W=\bigsqcup_{\rho\in\tilde{U}} W(\rho)\longrightarrow\tilde{U}$ be the resulting Hermitian vector-bundle of rank d.

Suppose furthermore that *the morphism* $A_\rho\colon W(\rho)\longrightarrow W(\rho)$ *is* **regular** *at* ρ_0 (see [17]), that is to say, upon possibily shrinking $\tilde{U}$, we may find smooth sections ϕ_j of $W\to\tilde{U}$, $1\leq j\leq d$, such that

$$\Big(\phi_j(\rho;\cdot),\phi_{j'}(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} = \delta_{jj'},\ 1\leq j,j'\leq d,\ \forall\rho\in\tilde{U}, \tag{a}$$

$$A_\rho\phi_j(\rho;\cdot) = \lambda_j(\rho)\phi_j(\rho;\cdot),\ 1\leq j\leq d,\ \forall\rho\in\tilde{U}, \tag{b}$$

for some functions $\lambda_j\in C^\infty(\tilde{U};\{\zeta\in\mathbb{C};\ |\zeta|<\delta\})$, with $\lambda_j(\rho_0)=0$, $1\leq j\leq d$.

Since

$$A_\rho^* A_\rho \phi_j(\rho;\cdot) = |\lambda_j(\rho)|^2 \phi_j(\rho;\cdot),\ 1 \le j \le d,\ \rho \in \tilde{U},$$

we have that $A_\rho^* A_\rho = A_\rho A_\rho^* \colon W(\rho) \longrightarrow W(\rho)$, and that

$$\mathrm{Spec}(A_\rho^* A_\rho\big|_{W(\rho)}) = \{|\lambda_j(\rho)|^2;\ 1 \le j \le d\} \subset [0,\delta^2),\ \rho \in \tilde{U}.$$

On the other hand, $A_\rho^* A_\rho \colon W(\rho)^\perp \cap B^{2k}(\mathbb{R}^\nu) \longrightarrow W(\rho)^\perp$ and we claim that

$$\mathrm{Spec}(A_\rho^* A_\rho\big|_{W(\rho)^\perp}) \subset (\delta^2, +\infty), \quad \rho \in \tilde{U}.$$

In fact, take $0 \ne \psi \in W(\rho)^\perp$ and write

$$\psi = \sum_{|\zeta|>\delta,\ \zeta \in \mathrm{Spec}(A_\rho)} \psi_\zeta \quad \text{(orthogonal sum)}.$$

Then $A_\rho \psi = \sum_{|\zeta|>\delta} \zeta \psi_\zeta$, so that

$$\|A_\rho \psi\|^2_{L^2(\mathbb{R}^\nu)} = \Big(\sum_{|\zeta|>\delta} \zeta \psi_\zeta, \sum_{|\zeta'|>\delta} \zeta' \psi_{\zeta'} \Big)_{L^2(\mathbb{R}^\nu)} =$$

$$= \sum_{|\zeta|>\delta} |\zeta|^2 \, \|\psi_\zeta\|^2_{L^2(\mathbb{R}^\nu)} > \delta^2 \sum_{|\zeta|>\delta} \|\psi_\zeta\|^2_{L^2(\mathbb{R}^\nu)} = \delta^2 \, \|\psi\|^2_{L^2(\mathbb{R}^\nu)} .$$

Hence, if $A_\rho^* A_\rho \psi = \mu \psi$ for some $\mu > 0$ and $0 \ne \psi \in W(\rho)^\perp$, we obtain

$$\mu \, \|\psi\|^2_{L^2(\mathbb{R}^\nu)} = \Big(A_\rho^* A_\rho \psi, \psi \Big)_{L^2(\mathbb{R}^\nu)} = \|A_\rho \psi\|^2_{L^2(\mathbb{R}^\nu)} > \delta^2 \, \|\psi\|^2_{L^2(\mathbb{R}^\nu)} ,$$

which proves the claim.
Hence $\tilde{U}$ is *admissible* and, furthermore,

$$V_1(\rho) = V_2(\rho) = W(\rho), \quad \forall \rho \in \tilde{U}.$$

It is important to observe that in this case the matrix representing A_ρ with respect to the basis $\{\phi_j(\rho;\cdot)\}_{1 \le j \le d}$ is the diagonal matrix

$$\mathrm{diag}(\lambda_1(\rho), \ldots, \lambda_d(\rho)), \quad \rho \in \tilde{U}.$$

Remark 2.8. *A sufficient condition which ensures (14) is that the* ψdo $[A, A^*]$ *has order* $2m - k - 1/2$.

All constructions above were carried out on $\tilde{U} \subset \mathbb{S}^*\Sigma$. Let $U \subset \Sigma$ be a conic neighborhood of ρ_0 which projects onto an admissible $\tilde{U}$ (we will say that such U is itself **admissible**). Define for $(y, \eta) \in U$, using (10),

$$V_1(y,\eta) := \mathsf{M}_{|\eta|}\Big(V_1(y, \eta/|\eta|) \Big), \quad V_2(y,\eta) := \mathsf{M}_{|\eta|}\Big(V_2(y, \eta/|\eta|) \Big).$$

We hence obtain Hermitian vector bundles of rank d (that we keep calling V_1, V_2) with fibers $V_j(y,\eta) \subset \mathcal{S}(\mathbb{R}^\nu)$, $j = 1, 2$,

$$V_j := \bigsqcup_{(y,\eta) \in U} V_j(y,\eta) \longrightarrow U, \quad j = 1, 2,$$

and get consequences (15) to (17) below:

(15) $$A_\rho\colon V_1(\rho) \longrightarrow V_2(\rho), \quad A_\rho^*\colon V_2(\rho) \longrightarrow V_1(\rho), \quad \forall \rho \in U;$$

(16) $$A_\rho\colon V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu) \longrightarrow V_2(\rho)^\perp, A_\rho^*\colon V_2(\rho)^\perp \cap B^k(\mathbb{R}^\nu) \longrightarrow V_1(\rho)^\perp$$

are **isomorphisms** for all $\rho \in U$;

(17) E_ρ (see (13)) is now defined for all $\rho \in U$.

From now on, we shall work on U, without further mention.

In the sequel V will stand, when no risk of confusion is present, for either bundle V_1 and V_2.

Definition 2.9. *A* **gauge** *of V is a vector $\phi = (\phi_1, \dots, \phi_d)$ of smooth* **orthonormal** *sections of V having the following homogeneity property*

(18)
$$\phi_j((y, t\eta); t^{-1/2}x) = t^{\nu/4}\phi_j((y,\eta); x), \ 1 \le j \le d, \ t > 0, \ (y,\eta) \in U, \ x \in \mathbb{R}^\nu.$$

Note that since V is a trivial bundle, the existence of a gauge is always ensured.

Once we are given a gauge ϕ of V, we may define the following operators.

Definition 2.10. *Put, for every $\rho \in U$:*

(*i*) $$h_\phi^-(\rho)\colon \mathbb{C}^d \longrightarrow V(\rho), \quad h_\phi^-(\rho)\theta = \sum_{j=1}^d \theta_j \phi_j(\rho;\cdot), \ \theta \in \mathbb{C}^d;$$

(*ii*) $$h_\phi^+(\rho)\colon \mathcal{S}'(\mathbb{R}^\nu) \longrightarrow \mathbb{C}^d, \quad h_\phi^+(\rho) f = \begin{bmatrix} \langle f, \overline{\phi_1(\rho;\cdot)}\rangle_{\mathcal{S}',\mathcal{S}} \\ \vdots \\ \langle f, \overline{\phi_d(\rho;\cdot)}\rangle_{\mathcal{S}',\mathcal{S}} \end{bmatrix}.$$

(Here $\langle\cdot,\cdot\rangle_{\mathcal{S}',\mathcal{S}}$ denotes the $\mathcal{S}'$-$\mathcal{S}$ duality.)

We collect in the next lemma (whose proof is straightforward) a few useful properties of the $h_\phi^\pm$.

Lemma 2.11. (*a*) *Let ϕ, ψ be gauges of V (not necessarily the same V). Then $h_\psi^+(\rho) \circ h_\phi^-(\rho)\colon \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is given by*

$$h_\psi^+(\rho) \circ h_\phi^-(\rho)\colon \theta \longmapsto \begin{bmatrix} \sum_{j=1}^d (\phi_j(\rho;\cdot), \psi_1(\rho;\cdot))_{L^2(\mathbb{R}^\nu)}\theta_j \\ \vdots \\ \sum_{j=1}^d (\phi_j(\rho;\cdot), \psi_d(\rho;\cdot))_{L^2(\mathbb{R}^\nu)}\theta_j \end{bmatrix}.$$

(*b*) *Upon denoting by* $\pi(\rho)\colon L^2(\mathbb{R}^\nu) \longrightarrow V(\rho)$ *the orthogonal projection onto* $V(\rho)$, *we have*

$$h_\phi^-(\rho) \circ h_\phi^+(\rho) = \pi(\rho), \quad \forall \rho \in U.$$

(*c*) *Let* ϕ, ψ *be gauges of the* **same** V. *Then* $h_\phi^-(\rho) = h_\psi^-(\rho) \circ \gamma(\rho)$, $h_\phi^+(\rho) = \gamma(\rho)^* \circ h_\psi^+(\rho)$, *where* $\gamma(\rho)$ *is the* $d \times d$ *smooth (in* ρ*) unitary matrix defined by the relations*

$$\phi_j(\rho;\cdot) = \sum_{j'=1}^{d} \gamma_{j'j}(\rho)\psi_{j'}(\rho;\cdot), \quad j = 1, \dots, d.$$

Following Sjöstrand and Helffer, we next define on U an object which takes care of both the localized operator A_ρ **and** the subspaces $V_1(\rho)$, $V_2(\rho)$, for $\rho \in U$.

Definition 2.12. *For a given gauge* ϕ *of* V_2 *and* ϕ' *of* V_1, *define for every* $\rho \in U$

$$\mathcal{A}_{\rho,\phi,\phi'} = \begin{bmatrix} A_\rho & h_\phi^-(\rho) \\ h_{\phi'}^+(\rho) & 0 \end{bmatrix} : \begin{matrix} \mathcal{S}(\mathbb{R}^\nu) \\ \times \\ \mathbb{C}^d \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\mathbb{R}^\nu) \\ \times \\ \mathbb{C}^d \end{matrix}, \tag{19}$$

$$\mathcal{A}_{\rho,\phi,\phi'} \begin{bmatrix} f \\ \theta \end{bmatrix} := \begin{bmatrix} A_\rho f + h_\phi^-(\rho)\theta \\ h_{\phi'}^+(\rho) f \end{bmatrix}. \tag{20}$$

We have the following crucial result.

Theorem 2.13. $\mathcal{A}_{\rho,\phi,\phi'}$ *is* **invertible** *for all* $\rho \in U$, *with inverse given by*

$$\mathcal{E}_{\rho,\phi,\phi'} = \begin{bmatrix} E_\rho & h_{\phi'}^-(\rho) \\ h_\phi^+(\rho) & -\Lambda_{\phi,\phi'}(\rho) \end{bmatrix} : \begin{matrix} \mathcal{S}(\mathbb{R}^\nu) \\ \times \\ \mathbb{C}^d \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\mathbb{R}^\nu) \\ \times \\ \mathbb{C}^d \end{matrix}, \tag{21}$$

where:

$E_\rho\colon L^2(\mathbb{R}^\nu) \longrightarrow B^k(\mathbb{R}^\nu)$, $\rho \in U$, *is the operator defined in (13) and (17), that is, upon writing* $f = f'_\rho + f''_\rho$ *with* $f'_\rho \in V_2(\rho)$ *and* $f''_\rho \in V_2(\rho)^\perp$,

$$E_\rho f = \Big(A_\rho\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\Big)^{-1} f''_\rho; \tag{22}$$

$\Lambda_{\phi,\phi'}(\rho)$ *is the* $d \times d$ *smooth matrix defined by*

$$\Lambda_{\phi,\phi'}(\rho) = \Big(h_\phi^+(\rho) \circ A_\rho \circ h_{\phi'}^-(\rho)\Big), \quad \rho \in U. \tag{23}$$

Proof. To prove **injectivity**, we suppose $A_\rho f + h_\phi^-(\rho)\theta = 0$ and $h_{\phi'}^+(\rho) f = 0$. Then $f \in V_1(\rho)^\perp$ so that $A_\rho f \in V_2(\rho)^\perp$. Since $h_\phi^-(\rho)\theta \in V_2(\rho)$, we conclude $A_\rho f = 0 = h_\phi^-(\rho)\theta$, whence $f = 0$, $\theta = 0$.

To prove **surjectivity**, suppose we are given $g \in \mathcal{S}(\mathbb{R}^\nu)$ and $\omega \in \mathbb{C}^d$. We have to solve the system

$$\begin{cases} A_\rho f + h_\phi^-(\rho)\theta = g \\ \\ h_{\phi'}^+(\rho) f = \omega. \end{cases} \tag{24}$$

We look for $f = h_{\phi'}^-(\rho)\omega + f'$, with $f' \in V_1(\rho)^\perp$ to be determined. By definition, it is then true that $h_{\phi'}^+(\rho) f = \omega$. The first equation in (24) is now

$$A_\rho f + h_\phi^-(\rho)\theta = \Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega + A_\rho f' + h_\phi^-(\rho)\theta = g.$$

To solve this equation, we impose the condition

$$g - \Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega - h_\phi^-(\rho)\theta \in V_2(\rho)^\perp,$$

that is

$$h_\phi^+(\rho)\Big(g - \Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega - h_\phi^-(\rho)\theta\Big) = 0,$$

which gives

$$\Big(h_\phi^+(\rho) \circ h_\phi^-(\rho)\Big)\theta = \theta = h_\phi^+(\rho) g - \Big(h_\phi^+(\rho) \circ A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega,$$

and hence we can find the unique $\theta \in \mathbb{C}^d$ satisfying the above equation. Notice that $g - \Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega - h_\phi^-(\rho)\theta \in \mathcal{S}(\mathbb{R}^\nu)$. As a consequence, we have

$$f' = \Big(A_\rho\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\Big)^{-1}\Big(g - \Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega - h_\phi^-(\rho)\theta\Big).$$

Remark also that $f' \in \mathcal{S}(\mathbb{R}^\nu)$ by virtue of the ellipticity of A_ρ.

Since

$$E_\rho = \begin{cases} 0, & \text{on } V_2(\rho) \\ \\ \Big(A_\rho\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\Big)^{-1}, & \text{on } V_2(\rho)^\perp, \end{cases}$$

and being $\Big(A_\rho \circ h_{\phi'}^-(\rho)\Big)\omega + h_\phi^-(\rho)\theta \in V_2(\rho)$, we finally have

$$\begin{cases} f = E_\rho g + h_{\phi'}^-(\rho)\omega \\ \\ \theta = h_\phi^+(\rho) g - \Lambda_{\phi,\phi'}(\rho)\omega. \end{cases}$$

This concludes the proof. □

Remark 2.14. *1) In exactly the same way the theorem was proven, one shows that $\mathcal{E}_{\rho,\phi,\phi'}$ is the inverse of $\mathcal{A}_{\rho,\phi,\phi'}$ also when the latter is considered as acting on $\mathcal{S}'(\mathbb{R}^\nu) \times \mathbb{C}^d$ into iteself.*

2) Note that if $\tilde{\phi}$, resp. $\tilde{\phi}'$, is another gauge of V_2, resp. V_1, then

$$\mathcal{E}_{\rho,\tilde{\phi},\tilde{\phi}'} = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \gamma(\rho)^{-1} \end{bmatrix} \mathcal{E}_{\rho,\phi,\phi'} \begin{bmatrix} \mathrm{id} & 0 \\ 0 & (\gamma'(\rho)^{-1})^* \end{bmatrix},$$

where γ, resp. γ', is the unitary matrix for which $h^-_{\tilde{\phi}}(\rho) = h^-_{\phi}(\rho) \circ \gamma(\rho)$, resp. $h^-_{\tilde{\phi}'}(\rho) = h^-_{\phi'}(\rho) \circ \gamma'(\rho)$.

3) The $d \times d$ matrix $\Lambda_{\phi,\phi'}(\rho) = h^+_{\phi}(\rho) \circ A_\rho \circ h^-_{\phi'}(\rho)$ is smooth in ρ and it has entries

$$\Lambda^{(jj')}_{\phi,\phi'}(\rho) = \Big(A_\rho \phi'_{j'}(\rho;\cdot), \phi_j(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}, \quad 1 \le j, j' \le d.$$

Notice that $\Lambda_{\phi,\phi'}(\rho_0) = 0$. In general the above matrix may be **singular** *also for $\rho \neq \rho_0$, $\rho \in U$. As a matter of fact,*

$$\mathrm{Ker}\big(\Lambda_{\phi,\phi'}(\rho)\big) = \{\theta \in \mathbb{C}^d;\ h^-_{\phi'}(\rho)\theta \in \mathrm{Ker}(A_\rho)\}.$$

Hence, the above matrix in invertible at $\rho \neq \rho_0$ exactly when $\mathrm{Ker}(A_\rho) = \{0\}$.

4) Since

$$\begin{bmatrix} A_\rho & h^-_{\phi}(\rho) \\ h^+_{\phi'}(\rho) & 0 \end{bmatrix}^* = \begin{bmatrix} A^*_\rho & h^-_{\phi'}(\rho) \\ h^+_{\phi}(\rho) & 0 \end{bmatrix},$$

then $\begin{bmatrix} E^*_\rho & h^-_{\phi}(\rho) \\ h^+_{\phi'}(\rho) & -\Lambda_{\phi,\phi'}(\rho)^* \end{bmatrix}$ *is the inverse of* $\begin{bmatrix} A^*_\rho & h^-_{\phi'}(\rho) \\ h^+_{\phi}(\rho) & 0 \end{bmatrix}$.

We end this section by giving an idea of the strategy that will be followed below. Exactly in the same way the invertibility of the localized operators A_ρ allows a construction of an actual parametrix of A, the invertibility of the matrices $\mathcal{A}_{\rho,\phi,\phi'}$ will allow the construction of a parametrix of a suitable system of ψdo's in **all** the variables, of which $\mathcal{A}_{\rho,\phi,\phi'}$ is the associated **localized** operator. This construction requires a proper pseudodifferential calculus. Fortunately, the core of the calculus has been already developed by Boutet de Monvel in [3] (see also Helffer [7]), and in the next section we just recall Boutet's calculus emphasizing the homogeneity properties and the related asymptoptic expansions.

Remark 2.15. *At this point, at last, we are in a position to comment on Helffer's setting in* [7]*, that of Grigis and Rothschild in* [5]*, and to motivate our own setting.*

In [7] *Helffer considers an a-priori more general case, namely a system of the form (19) where he only requires that ϕ, resp. ϕ',* **at** *ρ_0 be bases of* $\mathrm{Ker}\,(A^*_{\rho_0})$*, resp.* $\mathrm{Ker}\,(A_{\rho_0})$ *(spaces that he does* **not** *require to have the same dimension). From the invertibility of $\mathcal{A}_{\rho_0,\phi,\phi'}$ a* **perturbation argument** *yields the invertibility of $\mathcal{A}_{\rho,\phi,\phi'}$ in a neighborhood of ρ_0. The inverse has the form $\mathcal{E}_{\rho,\psi,\psi'}$ for suitable ψ, ψ', which, however, in general coincide with ϕ, ϕ', resp.,* **only** *at ρ_0. At any rate, this more general setup ensures control only on the* **principal symbol** *$\Lambda_{m-k/2}(y,\eta)$ of the systems $\Lambda(y, D_y)$, allowing him to obtain hypoellipticity results with loss of $\frac{k}{2} + \frac{r}{r+1}$ derivatives, r being a positive integer. In particular, it seems to us that in his setting cases in which $\Lambda_{m-k/2}(y,\eta)$ vanishes identically or vanishes to order ≥ 2 on some submanifold of Σ are out of reach. Our motivation is precisely to be able to* *treat these latter cases, which in principle require knowledge of the complete symbol of* $\Lambda(y, D_y)$. *To get the full expansion of the symbol at each "order of homogeneity", we have to solve systems of the form* $\mathcal{A}_{\rho,\phi,\phi'}$ **exactly**, *and for that we have decided to introduce hypothesis* **(H1)** *and work with the bundles* V_1 *and* V_2. *A consequence (and maybe a drawback) of our approach is that we will obtain hypoellipticity of* A *and* A^* *at the same time.*

As for Grigis and Rothschild, our approach is similar (at least in spirit) to theirs; however, they still use a perturbation-argument which is strictly bound to treating "translation invariant" operators (that is, operators whose symbol does not depend on y*).*

3. The calculus

3.1. $S^{m,k}$ **classes.** We start off by recalling what Boutet's $S^{m,k}$ classes (see [3]) are in our situation.

Definition 3.1. *Let* $m, k \in \mathbb{R}$. *By* $S^{m,k}$ *we denote the class of all smooth functions* $a(z = (x, y), \zeta = (\xi, \eta)) = a\colon T^*\mathbb{R}^n \longrightarrow \mathbb{C}$ *such that the following inequalities hold for all multi-indices* $\alpha, \beta, \gamma, \delta$

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \partial_\eta^\delta a(z,\zeta)\right| \lesssim |\zeta|^{m-|\gamma|-|\delta|} \left(|x| + \frac{|\xi|}{|\zeta|} + \frac{1}{|\zeta|^{1/2}}\right)^{k-|\alpha|-|\gamma|}. \tag{25}$$

By $a(z, D_z)$ *we denote the corresponding (properly supported)* ψdo, *and by* $\mathrm{OPS}^{m,k}$ *the corresponding class of* ψdo*'s. We put*

$$S^{m,\infty} := \bigcap_{k\in\mathbb{R}} S^{m,k}.$$

Remark 3.2. *One has the following useful properties (see* [3]*).*

$$(i) \qquad S^{m,k} \subset S^{m+k_-/2}_{1/2,1/2}, \quad k_- = \max\{0, -k\};$$

$$(ii) \qquad S^{m,k} \subset S^{m',k'} \iff m \le m' \ \text{ and } \ m - \frac{k}{2} \le m' - \frac{k'}{2}.$$

We will use the "homogeneous" analogue of the classes $S^{m,k}$.

Definition 3.3. *a) Let* $k \in \mathbb{R}$. *By* $\mathbf{S}^k$ *we denote the class of all smooth functions* $b(x, \xi) = b\colon T^*\mathbb{R}^\nu \longrightarrow \mathbb{C}$ *satisfying the following* **global** *inequalities for all multi-indices* α, β

$$\left|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)\right| \le C_{\alpha\beta}\,(1 + |x| + |\xi|)^{k-|\alpha|-|\beta|} \tag{26}$$

(see Schubin [18], *or Helffer* [8]*).*

By OPS^k *we denote the class of the corresponding global* ψdo*'s (as acting from* $\mathcal{S}(\mathbb{R}^\nu)$, *resp.* $\mathcal{S}'(\mathbb{R}^\nu)$, *into itself). Given* $a(x, D_x) \in \mathrm{OP}\mathbf{S}^k$ *and* $b(x, D_x) \in \mathrm{OP}\mathbf{S}^{k'}$, *it is well-known that their composition, hereafter denoted* *by* $a(x,D_x)\bullet b(x,D_x)$*, belongs to* $\mathrm{OP}\mathbf{S}^{k+k'}$*. Moreover, the symbol of the composition, hereafter denoted by* $\sigma_{(x,\xi)}(a\bullet b)$*, is given (as an oscillatory integral) by*

$$\sigma_{(x,\xi)}(a\bullet b)=\iint e^{-is\cdot\tau}a(x,\xi+\tau)b(x+s,\xi)ds\,đ\tau,\quad (x,\xi)\in T^*\mathbb{R}^\nu. \tag{27}$$

Here $đ\tau=(2\pi)^{-\nu}d\tau$.

b) Let $m,k\in\mathbb{R}$*. By* $S^{m,k}_{\mathrm{hom}}$ *we denote the class of all smooth functions* $a(y,\eta;x,\xi)=a\colon (T^*\mathbb{R}^{n-\nu}\setminus 0)\times T^*\mathbb{R}^\nu\longrightarrow\mathbb{C}$ *such that*

$$(i)\qquad a(y,\eta;\cdot)\in\mathbf{S}^k,\quad \forall(y,\eta)\in T^*\mathbb{R}^{n-\nu}\setminus 0;$$

$$(ii)\qquad a(y,t\eta;t^{-1/2}x,t^{1/2}\xi)=t^{m-k/2}a(y,\eta;x,\xi),\ t>0.$$

It is important to notice that any given symbol $a\in S^{m,k}_{\mathrm{hom}}$*, when multiplied by any* **excision function** $\chi=\chi(\eta)$ *(i.e.* χ *is smooth, vanishes in a bounded neighborhood of* 0 *and is identically* 1 *outside a bounded neighborhood of* 0*), gives a symbol* $\chi a\in S^{m,k}$*. This fact will hereafter be denoted simply by* $S^{m,k}_{\mathrm{hom}}\subset S^{m,k}$*, so that when thinking of* $a\in S^{m,k}_{\mathrm{hom}}$ *as an element of* $S^{m,k}$*, we shall tacitly assume it was previously multiplied by our favorite excision function.*

c) By $\mathrm{OPS}^{m,k}_{\mathrm{cl}}$ *we denote che class of operators* $a(z,D_z)\in\mathrm{OPS}^{m,k}$ *such that there exists a formal series* $\sum_{j\geq 0}a_{m-j/2}(y,\eta;x,\xi)$ *of symbols* $a_{m-j/2}\in S^{m,k+j}_{\mathrm{hom}}$ *such that*

$$a\sim\sum_{j\geq 0}a_{m-j/2},$$

i.e.

$$a-\sum_{j=0}^{N-1}a_{m-j/2}\in S^{m,k+N},\quad \forall N\geq 1. \tag{28}$$

The following lemma is crucial (see [3]).

Lemma 3.4. *Given a formal series* $\sum_{j\geq 0}a_{m-j/2}$*,* $a_{m-j/2}\in S^{m,k+j}_{\mathrm{hom}}$*, there exists* $a(z,D_z)\in\mathrm{OPS}^{m,k}_{\mathrm{cl}}$*,* **uniquely determined modulo** $\mathrm{OPS}^{m,\infty}$*, such that* $a\sim\sum_{j\geq 0}a_{m-j/2}$*.*

The main motivation for introducing $\mathrm{OPS}^{m,k}_{\mathrm{cl}}$ is given by the following remark.

Remark 3.5. *Suppose we are given a* ψdo $A\in\mathrm{OPS}^m_{\mathrm{cl}}(\mathbb{R}^n)$ *with symbol* $\sigma(A)$ *having semi-regular asymptotic expansion* $\sigma(A)\sim\sum_{r\geq 0}a_{m-r/2}(z,\zeta)$*, and such that, for some* $k\in\mathbb{Z}_+$*,* $a_{m-r/2}$ *vanishes at* Σ *to order* $k-r$*,* $0\leq r\leq k$*. Then it is readily checked that* $A\in\mathrm{OPS}^{m,k}_{\mathrm{cl}}$ *with*

$$\sigma(A)\sim\sum_{j\geq 0}\tilde{a}_{m-j/2}(y,\eta;x,\xi),$$

where

(29)
$$\tilde{a}_{m-j/2}(y,\eta;x,\xi)=\sum_{|\alpha|+|\beta|+r=k+j}\frac{1}{\alpha!\beta!}(\partial_x^\alpha\partial_\xi^\beta a_{m-r/2})(x=0,y,\xi=0,\eta)x^\alpha\xi^\beta.$$

It is worth noting that (at least when $k\geq 1$) $\tilde{a}_m(y,\eta;x,\xi)$ is $a_{(y,\eta)}(x,\xi)$, as defined in (4).

The following composition result will be systematically used throughout.

Lemma 3.6. *Let $a(z,D_z)\in \mathrm{OPS}_{\mathrm{cl}}^{m,k}$, $b(z,D_z)\in \mathrm{OPS}_{\mathrm{cl}}^{m',k'}$, where $a\sim\sum_{j\geq 0}a_{m-j/2}$, $b\sim\sum_{j\geq 0}b_{m'-j/2}$. Then, modulo $\mathrm{OPS}^{m+m',\infty}$,*

$$a(z,D_z)b(z,D_z)=c(z,D_z)\in \mathrm{OPS}_{\mathrm{cl}}^{m+m',k+k'},$$

with $c\sim\sum_{r\geq 0}c_{m+m'-r/2}$, and the $c_{m+m'-r/2}$ given by

(30)
$$c_{m+m'-r/2}(y,\eta;x,\xi)=$$
$$=\sum_{2|\alpha|+j+j'=r}\frac{1}{\alpha!i^{|\alpha|}}\sigma_{(x,\xi)}\Big((\partial_\eta^\alpha a_{m-j/2})(y,\eta;x,D_x)\bullet(\partial_y^\alpha b_{m'-j'/2})(y,\eta;x,D_x)\Big),$$

where, recalling Definition 3.3 (a)-(b), $\sigma_{(x,\xi)}\Big(\partial_\eta^\alpha a_{m-j/2}\bullet\partial_y^\alpha b_{m'-j'/2}\Big)$ denotes the symbol in $\mathbf{S}^{k+k'+j+j'}$ of the composition.

We omit the proof of the lemma.

3.2. **Hermite operators.** We start out this sub-section by recalling, from Boutet's paper [3], the definition of another important class of operators.

Definition 3.7. *Let $\mu\in\mathbb{R}$. By $\mathbf{H}^\mu$ we denote the class of all smooth functions $\varphi(y,\eta;x)=\varphi\colon T^*\mathbb{R}^{n-\nu}\times\mathbb{R}^\nu\longrightarrow\mathbb{C}$ such that the following inequalities hold for all multi-indices α,β,γ, and all integers $N\in\mathbb{Z}_+$*

(31)
$$\left|\partial_x^\alpha\partial_y^\beta\partial_\eta^\gamma\varphi(y,\eta;x)\right|\lesssim|\eta|^{\mu+\nu/4-|\gamma|+|\alpha|/2-N/2}\left(|x|+\frac{1}{|\eta|^{1/2}}\right)^{-N}.$$

By

$$\varphi(y,D_y;x)\colon C_0^\infty(\mathbb{R}^{n-\nu})\longrightarrow C^\infty(\mathbb{R}^n)$$

we denote the ψdo

(32)
$$\varphi(y,D_y;x)f=\int e^{iy\cdot\eta}\varphi(y,\eta;x)\hat{f}(\eta)đ\eta,$$

and by $\mathrm{OP}\mathbf{H}^\mu$ the class of ψdo's of the form $\varphi(y,D_y;x)+R$, where $\varphi\in\mathbf{H}^\mu$ and $R\colon\mathcal{E}'(\mathbb{R}^{n-\nu})\longrightarrow C^\infty(\mathbb{R}^n)$ is a smoothing operator.

As in the former sub-section, we have a corresponding "homogeneous" version.

Definition 3.8. *Let $\mu \in \mathbb{R}$. By $\mathbf{H}^{\mu}_{\mathrm{hom}}$ we denote the class of all smooth functions $\varphi(y,\eta;x) = \varphi\colon T^*\mathbb{R}^{n-\nu} \setminus 0 \times \mathbb{R}^{\nu} \longrightarrow \mathbb{C}$ such that*

$$(i) \qquad \varphi(y,\eta;\cdot) \in \mathcal{S}(\mathbb{R}^{\nu}), \quad \forall (y,\eta) \in T^*\mathbb{R}^{n-\nu} \setminus 0;$$

$$(ii) \qquad \varphi(y,t\eta;t^{-1/2}x) = t^{\mu+\nu/4}\varphi(y,\eta;x), \quad t > 0.$$

Observe that $\mathbf{H}^{\mu}_{\mathrm{hom}} \subset \mathbf{H}^{\mu}$, in the sense that $\varphi \in \mathbf{H}^{\mu}_{\mathrm{hom}}$ yields $\chi(\eta)\varphi(y,\eta;x) \in \mathbf{H}^{\mu}$ for every excision function χ. By $\mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$ we denote those operators $\varphi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu}$ for which there exists a formal series $\sum_{j\geq 0}\varphi_{\mu-j/2}$, with $\varphi_{\mu-j/2} \in \mathbf{H}^{\mu-j/2}_{\mathrm{hom}}$, such that $\varphi \sim \sum_{j\geq 0}\varphi_{\mu-j/2}$, i.e.

$$\varphi - \sum_{j=0}^{N-1}\varphi_{\mu-j/2} \in \mathbf{H}^{\mu-N/2}, \quad \forall N \geq 1.$$

We call φ_{μ} the **principal symbol** *of $\varphi(y,D_y;x)$.*

There is an analogue of Lemma 3.4, which we state without proof.

Lemma 3.9. *Given any formal series $\sum_{j\geq 0}\varphi_{\mu-j/2}$, $\varphi_{\mu-j/2} \in \mathbf{H}^{\mu-j/2}_{\mathrm{hom}}$, there exists $\varphi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$ with $\varphi \sim \sum_{j\geq 0}\varphi_{\mu-j/2}$.*

As the attentive reader may have already noticed, the motivation for introducing the classes $\mathbf{H}^{\mu}_{\mathrm{hom}}$ is that the eigenfunctions of the localized operator $A_{(y,\eta)}$ of the Introduction are in fact smooth in (y,η), $\eta \neq 0$, and rapidly decreasing in x (see also Definition 2.9). Moreover, the Hermite operators $\varphi(y,D_y;x)$ are the natural *quantized version* **in all the variables** of the operators h^-_{ϕ} of Section 2.

If we want to quantize *in all the variables* an eigenvalue equation

$$A_{(y,\eta)}(x,D_x)\Big(\phi(y,\eta;\cdot)\Big) = \lambda(y,\eta)\phi(y,\eta;x),$$

we need to understand the composition between $\mathrm{OPS}^{m,k}_{\mathrm{cl}}$ and $\mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$, and the composition between $\mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$ and $\mathrm{OPS}^{m-k/2}_{\mathrm{cl}}(\mathbb{R}^{n-\nu})$. This is taken care of by the following lemma (whose proof is left to the reader).

Lemma 3.10. *1) Let $a(z,D_z) \in \mathrm{OPS}^{m,k}_{\mathrm{cl}}$ with $a \sim \sum_{j\geq 0}a_{m-j/2}$, and let $\varphi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$, with $\varphi \sim \sum_{j\geq 0}\varphi_{\mu-j/2}$. Then*

$$a(z,D_z)\varphi(y,D_y;x) = \psi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu+m-k/2}_{\mathrm{cl}},$$

with $\psi \sim \sum_{\ell\geq 0}\psi_{\mu+m-k/2-\ell/2}$, where

$$(33) \qquad \psi_{\mu+m-k/2-\ell/2}(y,\eta;x) =$$

$$= \sum_{2|\alpha|+j+r=\ell}\frac{1}{\alpha! i^{|\alpha|}}(\partial^{\alpha}_{\eta}a_{m-j/2})(y,\eta;x,D_x)\Big(\partial^{\alpha}_{y}\varphi_{\mu-r/2}(y,\eta;\cdot)\Big).$$

2) Let $\varphi(y, D_y; x)$ be as in 1) above. Let $b(y, D_y) \in \mathrm{OPS}^{m'}_{\mathrm{cl}}(\mathbb{R}^{n-\nu})$ with symbol $b(y,\eta) \sim \sum_{j\geq 0} b_{m'-j/2}(y,\eta)$. Then

$$\varphi(y, D_y; x) b(y, D_y) = \psi(y, D_y; x) \in \mathrm{OP}\mathbf{H}^{\mu+m'}_{\mathrm{cl}},$$

with $\psi \sim \sum_{\ell\geq 0} \psi_{\mu+m'-\ell/2}$, where

$$\psi_{\mu+m'-\ell/2}(y,\eta;x) = \tag{34}$$

$$= \sum_{2|\alpha|+j+r=\ell} \frac{1}{\alpha! i^{|\alpha|}} (\partial_\eta^\alpha \varphi_{\mu-j/2})(y,\eta;x)(\partial_y^\alpha b_{m'-r/2})(y,\eta).$$

As mentioned earlier, the class $\mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$ represents the natural quantization of the operators h^-_ϕ. As regards quantization of the operators h^+_ϕ we proceed as follows.

Definition 3.11. *Let $\mu \in \mathbb{R}$ and let $\varphi \in \mathbf{H}^\mu$ have the asymptotic expansion $\varphi \sim \sum_{\ell\geq 0} \varphi_{\mu-\ell/2}$, $\varphi_{\mu-\ell/2} \in \mathbf{H}^{\mu-\ell/2}_{\mathrm{hom}}$. By*

$$\varphi^*(y, D_y; x) \colon C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-\nu})$$

we denote the operator

$$C_0^\infty(\mathbb{R}^\nu_x \times \mathbb{R}^{n-\nu}_y) \ni g \longmapsto \varphi^*(y, D_y; x) g = \iint e^{iy\cdot\eta} \overline{\varphi(y,\eta;x)} \hat{g}(x,\eta) đ\eta dx \tag{35}$$

($\hat{g}(x,\eta)$ denotes partial Fourier transform with respect to y).
By $\mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}}$ we denote the class of all operators of the form $\varphi^*(y, D_y; x) + R$, where $R \colon \mathcal{E}'(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^{n-\nu})$ is a smoothing operator, and we call φ_μ the **principal symbol** *of $\varphi^*(y, D_y; x)$.*

In the following lemma we gather some composition results that will be crucial in the sequel. (The proof is of course left to the reader.)

Lemma 3.12. *1) Let $a(z, D_z) \in \mathrm{OPS}^{m,k}_{\mathrm{cl}}$ with $a \sim \sum_{j\geq 0} a_{m-j/2}$, and let $\varphi^*(y, D_y; x) \in \mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}}$ with $\varphi \sim \sum_{\ell\geq 0} \varphi_{\mu-\ell/2}$. Then*

$$\varphi^*(y, D_y; x) a(z, D_z) = \psi^*(y, D_y; x) \in \mathrm{OP}\mathbf{H}^{*\mu+m-k/2}_{\mathrm{cl}},$$

with $\psi \sim \sum_{r\geq 0} \psi_{\mu+m-k/2-r/2}$, where

$$\psi_{\mu+m-k/2-r/2}(y,\eta;x) = \tag{36}$$

$$= \sum_{2|\alpha|+j+\ell=r} \overline{\frac{i^{|\alpha|}}{\alpha!} (\partial_y^\alpha a_{m-j/2})(y,\eta;x,D_x)^* \Big((\partial_\eta^\alpha \varphi_{\mu-\ell/2})(y,\eta;\cdot) \Big)},$$

where $(\partial_y^\alpha a_{m-j/2})(y,\eta;x,D_x)^* \in \mathrm{OP}\mathbf{S}^{k+j}$ is the formal adjoint of the operator $(\partial_y^\alpha a_{m-j/2})(y,\eta;x,D_x)$.

2) Let $\varphi^*(y, D_y; x)$ be as above, and let $b(y, D_y) \in \mathrm{OPS}^{m'}_{\mathrm{cl}}(\mathbb{R}^{n-\nu})$ with $b \sim \sum_{j\geq 0} b_{m'-j/2}$. Then

$$b(y, D_y)\varphi^*(y, D_y; x) = \psi^*(y, D_y; x) \in \mathrm{OP}\mathbf{H}^{*\mu+m'}_{\mathrm{cl}},$$

with $\psi \sim \sum_{r\geq 0} \psi_{\mu+m'-r/2}$, *where*

(37)
$$\psi_{\mu+m'-r/2}(y,\eta;x) = \sum_{2|\alpha|+j+\ell=r} \frac{i^{|\alpha|}}{\alpha!} \overline{(\partial_\eta^\alpha b_{m'-j/2})(y,\eta)} (\partial_y^\alpha \varphi_{\mu-\ell/2})(y,\eta;x).$$

3) Let $\varphi^*(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}}$ *be as above. Let* $\psi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu'}_{\mathrm{cl}}$ *with* $\psi \sim \sum_{r\geq 0} \psi_{\mu'-r/2}$. *Then*

$$\varphi^*(y,D_y;x)\psi(y,D_y;x) = b(y,D_y) \in \mathrm{OPS}^{\mu+\mu'}_{\mathrm{cl}}(\mathbb{R}^{n-\nu}),$$

with $b \sim \sum_{j\geq 0} b_{\mu+\mu'-j/2}$, *where*

(38)
$$b_{\mu+\mu'-j/2}(y,\eta) = \sum_{2|\alpha|+\ell+r=j} \frac{1}{\alpha! i^{|\alpha|}} \int \overline{\partial_\eta^\alpha \varphi_{\mu-\ell/2}(y,\eta;x)} \partial_y^\alpha \psi_{\mu'-r/2}(y,\eta;x) dx.$$

4) Given $\varphi(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$ *with* $\varphi \sim \sum_{j\geq 0} \varphi_{\mu-j/2}$, *then for all* $g \in C_0^\infty(\mathbb{R}^{n-\nu})$ *and* $f \in C_0^\infty(\mathbb{R}^n)$ *we have*

$$\Big(\varphi(y,D_y;x)g, f\Big)_{L^2(\mathbb{R}^n)} = \Big(g, \psi^*(y,D_y;x)f\Big)_{L^2(\mathbb{R}^\nu)}, \tag{39}$$

where $\psi^*(y,D_y;x) \in \mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}}$ *with* $\psi \sim \sum_{j\geq 0} \psi_{\mu-j/2}$ *and*

$$\psi_{\mu-j/2}(y,\eta;x) = \sum_{2|\alpha|+\ell=j} \frac{i^{|\alpha|}}{\alpha!} \overline{\partial_y^\alpha \partial_\eta^\alpha \varphi_{\mu-\ell/2}(y,\eta;x)}. \tag{40}$$

Point 3) in the above lemma takes care of a composition $\varphi^*\psi$. We need also to control $\psi\varphi^*$, and are thus forced to introduce the appropriate classes of symbols and operators (following Boutet de Monvel [3]).

3.3. $\mathcal{H}^{m,k}$ classes.

Definition 3.13. *For* $m,k \in \mathbb{R}$ *we put*

$$\mathcal{H}^{m,k} := \bigcap_{j\geq 0} S^{m-j,k-2j}, \tag{41}$$

and $\mathrm{OP}\mathcal{H}^{m,k}$ *denotes the corresponding class of* ψdo*'s. Note that* $\mathcal{H}^{m,k} = \mathcal{H}^{m-k/2,0}$.

We need the corresponding homogeneous version.

Definition 3.14. *Let* $m,k \in \mathbb{R}$. *By* $\mathcal{H}^{m,k}_{\mathrm{hom}}$ *we denote the class of all smooth functions* $a(y,\eta;x,\xi) = a\colon T^*\mathbb{R}^{n-\nu}\setminus 0 \times T^*\mathbb{R}^\nu \longrightarrow \mathbb{C}$ *such that*

(i) $\quad a(y,\eta;\cdot) \in \mathcal{S}(\mathbb{R}^{2\nu}_{(x,\xi)}), \quad \forall (y,\eta) \in T^*\mathbb{R}^{n-\nu}\setminus 0;$

(ii) $\quad a(y,t\eta;t^{-1/2}x, t^{1/2}\xi) = t^{m-k/2} a(y,\eta;x,\xi), \ t>0.$

Note that $\mathcal{H}^{m,k}_{\mathrm{hom}} = \bigcap_{j\geq 0} S^{m-j,k-2j}_{\mathrm{hom}}$, *and that, furthermore,* $\mathcal{H}^{m,k}_{\mathrm{hom}} \subset \mathcal{H}^{m,k}$ *(by means of excision functions as before). By* $\mathrm{OP}\mathcal{H}^{m,k}_{\mathrm{cl}}$ *we denote the class of*

ψdo*'s* $a(z,D_z)\in \mathrm{OP}\mathcal{H}^{m,k}$ *such that* $a\sim\sum_{j\geq 0}a^{(m,k)}_{j/2}(y,\eta;x,\xi)$ *with* $a^{(m,k)}_{j/2}\in \mathcal{H}^{m,k+j}_{\mathrm{hom}}$, *and where the expansion means*

$$a-\sum_{j=0}^{N-1}a^{(m,k)}_{j/2}\in\mathcal{H}^{m,k+N},\quad \forall N\geq 1. \tag{42}$$

We have the following lemma.

Lemma 3.15. *1) Let* $\varphi^*(y,D_y;x)\in \mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}}$, $\psi(y,D_y;x)\in\mathrm{OP}\mathbf{H}^{\mu'}_{\mathrm{cl}}$, *with* $\varphi\sim\sum_{\ell\geq 0}\varphi_{\mu-\ell/2}$, $\psi\sim\sum_{r\geq 0}\psi_{\mu'-r/2}$. *Then*

$$\psi(y,D_y;x)\varphi^*(y,D_y;x)=a(z,D_z)\in\mathrm{OP}\mathcal{H}^{\mu+\mu',0}_{\mathrm{cl}},$$

with $a\sim\sum_{j\geq 0}a^{(\mu+\mu',0)}_{j/2}$, *where*

$$a^{(\mu+\mu',0)}_{j/2}(y,\eta;x,\xi)=\sum_{2|\alpha|+\ell+r=j}\frac{e^{-ix\cdot\xi}}{\alpha! i^{|\alpha|}}\partial_\eta^\alpha\psi_{\mu'-r/2}(y,\eta;x)\overline{\partial_y^\alpha\hat{\varphi}_{\mu-\ell/2}(y,\eta;\xi)}. \tag{43}$$

2) Let $a(z,D_z)\in\mathrm{OP}\mathcal{H}^{m,k}_{\mathrm{cl}}$, $\varphi(y,D_y;x)\in\mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$, *with* $a\sim\sum_{j\geq 0}a^{(m,k)}_{j/2}$, $\varphi\sim\sum_{j\geq 0}\varphi_{\mu-j/2}$. *Then*

$$a(z,D_z)\varphi(y,D_y;x)=\psi(y,D_y;x)\in\mathrm{OP}\mathbf{H}^{m-k/2+\mu}_{\mathrm{cl}},$$

with $\psi\sim\sum_{r\geq 0}\psi_{m-k/2+\mu-r/2}(y,\eta;x)$ *where*

$$\psi_{m-k/2+\mu-r/2}(y,\eta;x)= \tag{44}$$
$$=\sum_{2|\alpha|+j+\ell=r}\frac{1}{\alpha! i^{|\alpha|}}(\partial_\eta^\alpha a^{(m,k)}_{j/2})(y,\eta;x,D_x)\Big(\partial_y^\alpha\varphi_{\mu-\ell/2}(y,\eta;\cdot)\Big),\ r\geq 0.$$

As usual, we omit the proof.

3.4. **Continuity properties and** WF**-relations.** In this subsection we focus on the continuity properties in Sobolev spaces and the wave-front-set relations of the operators considered above.

First of all, we recall some standard notation (see [10], Vol.I) concerning the wave-front-set relations.

Suppose we are given a linear continuous map $P\colon\mathcal{E}'(X)\longrightarrow\mathcal{D}'(Y)$, with $X\subset\mathbb{R}^n_x$, $Y\subset\mathbb{R}^m_y$ given open sets (just to fix ideas), and let $\mathsf{k}_P(y,x)=\mathsf{k}_P\in\mathcal{D}'(Y\times X)$ be the Schwartz kernel of P. We define

$$\mathrm{WF}'(P):=\big\{\big((y,\eta),(x,\xi)\big)\in\big(T^*Y\times T^*X\big)\setminus 0;\ (y,x,\eta,-\xi)\in\mathrm{WF}(\mathsf{k}_P)\big\}.$$

With this notation we have the following lemma.

Lemma 3.16. *One has the following facts.*

1) $$A\in\mathrm{OPS}^{m,k}\implies \mathrm{WF}'(A)\subset\mathrm{diag}\big(T^*\mathbb{R}^n\setminus 0\times T^*\mathbb{R}^n\setminus 0\big);$$

2) $$A\in\mathrm{OP}\mathcal{H}^{m,k}\implies\mathrm{WF}'(A)\subset\mathrm{diag}\big(\Sigma\times\Sigma\big);$$

3) $\qquad A \in \mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}} \implies$

$$\mathrm{WF}'(A) \subset \big\{\big((x=0,y,\xi=0,\eta),(y,\eta)\big);(y,\eta)\in T^*\mathbb{R}^{n-\nu}\setminus 0\big\}\,;$$

4) $\qquad A \in \mathrm{OP}\mathbf{H}^{*\mu}_{\mathrm{cl}} \implies$

$$\mathrm{WF}'(A) \subset \big\{\big((y,\eta),(x=0,y,\xi=0,\eta)\big);(y,\eta)\in T^*\mathbb{R}^{n-\nu}\setminus 0\big\}\,.$$

The proofs of 1) and 2) are already in [3]. Points 3) and 4) follow by direct inspection of the kernel k_A.

As for the continuity in Sobolev spaces, we shall limit ourselves to what is strictly needed for the following lemma.

Lemma 3.17. *1) Let $A \in \mathrm{OPS}^{m,-k}$, $k \geq 0$, be properly supported. Then*

$$A\colon H^t_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow H^{t-m-k/2}_{\mathrm{loc}}(\mathbb{R}^n)$$

is continuous for all $t \in \mathbb{R}$.

2) Let $A \in \mathrm{OP}\mathcal{H}^{m,k}$ be properly supported. Then

$$A\colon H^t_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow H^{t-m+k/2}_{\mathrm{loc}}(\mathbb{R}^n)$$

is continuous for all $t \in \mathbb{R}$.

3) Let $A \in \mathrm{OP}\mathbf{H}^{\mu}_{\mathrm{cl}}$, $B \in \mathrm{OP}\mathbf{H}^{*\mu'}_{\mathrm{cl}}$ be properly supported. Then

$$A\colon H^t_{\mathrm{loc}}(\mathbb{R}^{n-\nu}) \longrightarrow H^{t-\mu}_{\mathrm{loc}}(\mathbb{R}^n),\ B\colon H^t_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow H^{t-\mu'}_{\mathrm{loc}}(\mathbb{R}^{n-\nu}),$$

are continuous for all $t \in \mathbb{R}$.

Proof. 1) By Remark 3.2, if $k \geq 0$ then $\mathrm{OPS}^{m,-k} \subset \mathrm{OPS}^{m+k/2}_{1/2,1/2}(\mathbb{R}^n)$. Hence the result follows by the Calderón-Vaillancourt theorem.

2) The result is a consequence of point 1) since $\mathrm{OP}\mathcal{H}^{m,k} \subset \mathrm{OPS}^{m-k/2,0}$.

3) This part is actually a consequence of the general framework of Boutet de Monvel's paper [3]. However, in our particular setting, a more direct proof can be given by the standard procedure of considering in the first place symbols which **do not** depend on $y \in \mathbb{R}^{n-\nu}$, and then in the second place by perturbation. When considering symbols of the form $\varphi(\eta;x) = \varphi \in \mathbf{H}^{\mu}_{\mathrm{hom}}$, the continuity of $\varphi(D_y;x)\colon H^t(\mathbb{R}^{n-\nu}) \longrightarrow H^{t-\mu}(\mathbb{R}^n)$ follows by observing that

$$\int_{\mathbb{R}^\nu} |\chi(\eta)\hat{\varphi}(\eta;\xi)|^2 d\xi = |\eta|^{2\mu}\int_{\mathbb{R}^\nu} |\chi(\eta)\hat{\varphi}(\frac{\eta}{|\eta|};\xi')|^2 d\xi' \leq C(1+|\eta|^2)^{\mu},$$

χ being an excision function. □

A final observation which will be crucial later on is the following.

Lemma 3.18. *1) If $A \in \mathrm{OPS}^{m,\infty}$ and $B \in \mathrm{OP}\mathcal{H}^{m',k'}$ (properly supported), then both AB and BA are smoothing operators.*

2) If $A \in \mathrm{OPS}^{m,\infty}$, $B \in \mathrm{OP}\mathbf{H}^{*\mu}$, $C \in \mathrm{OP}\mathbf{H}^{\mu'}$ (properly supported), then AC and BA are smoothing operators.

Proof. 1) For all $N \geq 0$ one has $A \in \mathrm{OPS}^{m,N}$, so that by a general result of Boutet de Monvel's [3] $AB, BA \in \mathrm{OP}\mathcal{H}^{m+m'-k'/2-N/2,0}$. It follows from Lemma 3.17, 2), that

$$AB, BA \colon H^t_{\mathrm{loc}}(\mathbb{R}^n) \longrightarrow H^{t-m-m'+k'/2+N/2}_{\mathrm{loc}}(\mathbb{R}^n)$$

are continuous, for all $t \in \mathbb{R}$. The conclusion follows by the arbitrariness of N.

2) Since $A \in \mathrm{OPS}^{m,N}$ for all $N \geq 0$, by ([3],(5.11)) we have $BA \in \bigcap_{N\geq 0} \mathrm{OP}\mathbf{H}^{*\mu+m-N/2}$, which proves that BA is smoothing. The other case is similar. □

4. Parametrices

As promised in the Introduction, given our initial operator A for which $\mathrm{dimKer}(A_{\rho_0}) = d \geq 1$ and satisfying **(H1)**, we want to construct a $d \times d$ system $\Lambda(y, D_y)$ of order $m - k/2$, whose hypoellipticity at ρ_0 is equivalent to the hypoellipticity of A at ρ_0. This construction will require the whole machinery developed in Section 3.

In order to simplify the exposition, we shall first make a "global" construction, that is we suppose that $\tilde{U} = \mathbb{S}^*\Sigma = \mathbb{S}^*\mathbb{R}^{n-\nu}$ is *admissible*, so that, in particular, the vector bundles V_1 and V_2 on Σ are **trivial**.

We shall show afterwards how to microlocalize the construction.

We consider systems of the form

$$\mathcal{A} = \begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}, \tag{45}$$

where A is our initial operator, $H^- = [H_1^- \; H_2^- \; \ldots \; H_d^-]$ is a $1 \times d$ matrix of operators $H_j^- \in \mathrm{OP}\mathbf{H}^0_{\mathrm{cl}}$ with

$$\sigma(H_j^-) \sim \sum_{\ell \geq 0} \phi^{(j)}_{-\ell/2}(y, \eta; x), \quad 1 \leq j \leq d.$$

We will always **suppose** that $\phi := (\phi_0^{(1)}, \phi_0^{(2)}, \ldots, \phi_0^{(d)})$ be a gauge of V_2. Furthermore, $H^+ = \begin{bmatrix} H_1^+ \\ \vdots \\ H_d^+ \end{bmatrix}$ is a $d \times 1$ matrix of operators $H_j^+ \in \mathrm{OP}\mathbf{H}^{*0}_{\mathrm{cl}}$ with

$$\sigma(H_j^+) \sim \sum_{\ell \geq 0} {\phi'}^{(j)}_{-\ell/2}(y, \eta; x), \quad 1 \leq j \leq d.$$

We will always **suppose** that $\phi' := ({\phi'}_0^{(1)}, {\phi'}_0^{(2)}, \ldots, {\phi'}_0^{(d)})$ be a gauge of V_1. Note that

$$\mathcal{A}\colon \begin{array}{c} C_0^\infty(\mathbb{R}^n) \\ \times \\ C_0^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d) \end{array} \longrightarrow \begin{array}{c} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d) \end{array}.$$

By definition, the **localized** operator associated with $\mathcal{A}$ at $\rho = (y,\eta) \in \Sigma$ is the system $\mathcal{A}_{\rho,\phi,\phi'} = \begin{bmatrix} A_\rho & h_\phi^-(\rho) \\ h_{\phi'}^+(\rho) & 0 \end{bmatrix}$. The core of this paper is the following theorem.

Theorem 4.1. *Suppose* **(H1)** *holds and that* $\mathbb{S}^*\mathbb{R}^{n-\nu}$ *is* **admissible**. *Then, given any system* $\mathcal{A}$ *as in (45), there exist:*

- (i) $E \in \mathrm{OPS}_{\mathrm{cl}}^{-m,-k}$;
- (ii) $K^- = [K_1^- \ldots K_d^-]$ *a* $1 \times d$ *matrix of operators* $K_j^- \in \mathrm{OP}\mathbf{H}_{\mathrm{cl}}^0$ *with* $\sigma(K_j^-) \sim \sum_{\ell\geq 0} {\psi'}^{(j)}_{-\ell/2}(y,\eta;x)$, $1 \leq j \leq d$, **where** ${\psi'}_0^{(j)} = {\phi'}_0^{(j)}$,$1 \leq j \leq d$;
- (iii) $K^+ = \begin{bmatrix} K_1^+ \\ \vdots \\ K_d^+ \end{bmatrix}$ *a* $d \times 1$ *matrix of operators* $K_j^+ \in \mathrm{OP}\mathbf{H}_{\mathrm{cl}}^{*0}$, *with* $\sigma(K_j^+) \sim \sum_{\ell\geq 0} \psi^{(j)}_{-\ell/2}(y,\eta;x)$, $1 \leq j \leq d$, **where** $\psi_0^{(j)} = \phi_0^{(j)}$, $1 \leq j \leq d$;
- (iv) *a pseudodifferential* $d\times d$ *system* $\Lambda = \Lambda(y,D_y) = \Big(\Lambda^{(rj)}(y,D_y)\Big)_{1\leq r,j\leq d}$, *whose entries* $\Lambda^{(rj)} \in \mathrm{OPS}_{\mathrm{cl}}^{m-k/2}(\mathbb{R}^{n-\nu})$, *with*
$$\sigma(\Lambda^{(rj)}) \sim \sum_{\ell\geq 0} \Lambda^{(rj)}_{m-k/2-\ell/2}(y,\eta),$$
where

$$\Big(\Lambda^{(rj)}_{m-k/2}(\rho)\Big)_{1\leq r,j\leq d} = \Lambda_{\phi,\phi'}(\rho) = h_\phi^+(\rho) \circ A_\rho \circ h_{\phi'}^-(\rho), \quad \forall \rho \in \Sigma, \tag{46}$$

such that the system

$$\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix} \tag{47}$$

is a two-sided parametrix of $\mathcal{A}$, *i.e.* $\mathcal{A}\mathcal{E} - \mathrm{id}$ *and* $\mathcal{E}\mathcal{A} - \mathrm{id}$ *map* $\mathcal{E}'(\mathbb{R}^n) \times \mathcal{E}'(\mathbb{R}^{n-\nu};\mathbb{C}^d)$ *into* $C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d)$.

Remark 4.2. *The full expansion of the symbol of* Λ *will be computed in the proof, and put in evidence at the end.*

Proof. **First step.** Suppose we have already found a system $\mathcal{E}$ as in (47) such that

$$\mathcal{A}\mathcal{E} = \begin{bmatrix} \mathrm{id} + R & \text{smoothing} \\ \text{smoothing} & \mathrm{id} + \text{smoothing} \end{bmatrix}, \tag{48}$$

where $R \in \mathrm{OPS}^{0,\infty}$. Using the argument of [7] (see also [4]) gives the existence of $Q \in \mathrm{OPS}^{-m,-k}$ for which $AQ = \mathrm{id} + L$, $L \in \mathrm{OP}\mathcal{H}^{0,0}$, so that $AQR = R + LR$. It follows from Lemma 3.18 that LR is smoothing. Hence

$$\begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix} \begin{bmatrix} E - QR & K^- \\ K^+ & -\Lambda \end{bmatrix} = \begin{bmatrix} \mathrm{id} + \text{smoothing} & \text{smoothing} \\ -H^+QR + \text{smoothing} & \mathrm{id} + \text{smoothing} \end{bmatrix}.$$

Now, $E - QR \in \mathrm{OPS}^{-m,-k}_{\mathrm{cl}}$ (since $QR \in \mathrm{OPS}^{-m,\infty}$), and H^+QR is smoothing as a consequence of Lemma 3.18.

By denoting $E - QR$ by E again, we have a system of the form (47) which is a right parametrix of $\mathcal{A}$.

The same argument applies if we have already found a system $\mathcal{E}'$ of the form (47) such that

$$\mathcal{E}'\mathcal{A} = \begin{bmatrix} \mathrm{id} + R' & \text{smoothing} \\ \text{smoothing} & \mathrm{id} + \text{smoothing} \end{bmatrix}, \tag{49}$$

with $R' \in \mathrm{OPS}^{0,\infty}$.

Once we have a right parametrix $\mathcal{E}$ and a left parametrix $\mathcal{E}'$ of $\mathcal{A}$, we immediately conclude that $\mathcal{E} - \mathcal{E}' = \text{smoothing}$.

Second step. We now construct a system $\mathcal{E}$ satisfying (48). The construction of a system $\mathcal{E}'$ satisfying (49) proceeds by the same arguments (e.g. by constructing a right parametrix for $\mathcal{A}^*$), and is left to the reader.

In order to carry out the construction, by Remark 3.5 we think of A as an operator in $\mathrm{OPS}^{m,k}_{\mathrm{cl}}$. If $\sigma(A) \sim \sum_{r\geq 0} a_{m-r/2}(z,\zeta)$ is the semiregular asymptotic expansion of the symbol of A, with $a_{m-r/2}$ vanishing at Σ to order $k-r$, $0 \leq r \leq k$, then modulo $S^{m,\infty}$ we have

$$\sigma(A) \sim \sum_{j\geq 0} \tilde{a}_{m-j/2}(y,\eta;x,\xi),$$

where

$$\tilde{a}_{m-j/2}(y,\eta;x,\xi) = \sum_{|\alpha|+|\beta|+r=k+j} \frac{1}{\alpha!\beta!}\partial_x^\alpha \partial_\xi^\beta a_{m-r/2}(0,y,0,\eta)x^\alpha\xi^\beta$$

belongs to $S^{m,k+j}_{\mathrm{hom}}$, for all $j \geq 0$. We put

$$A^{(k+j)}_{\rho=(y,\eta)} := \mathrm{Op}(\tilde{a}_{m-j/2})(y,\eta;x,D_x) \in \mathrm{OP}\mathbf{S}^{k+j}, \quad j \geq 0. \tag{50}$$

Note that $A^{(k)}_\rho$ is **our** localized operator A_ρ.

Since we look for $E \in \mathrm{OPS}_{\mathrm{cl}}^{-m,-k}$, we look for the asymptotic expansion (modulo $S^{-m,\infty}$) of its symbol

$$\sigma(E) \sim \sum_{j\geq 0} \tilde{e}_{-m-j/2}(y,\eta;x,\xi), \quad \tilde{e}_{-m-j/2} \in S_{\mathrm{hom}}^{-m,-k+j},\ j \geq 0.$$

In analogy with (50), we put

$$E_{\rho=(y,\eta)}^{(-k+j)} := \tilde{e}_{-m-j/2}(y,\eta;x,D_x) \in \mathrm{OP}\mathbf{S}^{-k+j}, \quad j \geq 0. \tag{51}$$

Having already fixed the principal symbols of $K^{\pm}$ and Λ, by Theorem 2.13 we are forced to take

$$E_\rho^{(-k)} := E_\rho = \begin{cases} 0, & \text{on } V_2(\rho) \\ \left(A_\rho^{(k)}\big|_{V_1(\rho)^\perp \cap B^k(\mathbb{R}^\nu)}\right)^{-1}, & \text{on } V_2(\rho)^\perp. \end{cases}$$

That $E_\rho^{(-k)} \in \mathrm{OP}\mathbf{S}^{-k}$ with symbol $\tilde{e}_{-m}(y,\eta;x,\xi) \in S_{\mathrm{hom}}^{-m,-k}$, is a consequence of its very definition. We are therefore left with determining all the symbols

$$\tilde{e}_{-m-\ell/2}, \quad \ell \geq 0,$$

$$\psi_{-\ell/2}^{(j)},\ {\psi'}_{-\ell/2}^{(j)}, \quad 1 \leq j \leq d,\ \ell \geq 0,$$

$$\Lambda_{m-k/2-\ell/2}^{(rj)}, \quad 1 \leq r,j \leq d,\ \ell \geq 0.$$

Imposing (48) amounts to solving the following system of equations

1 $$AE + \sum_{j=1}^{d} H_j^- K_j^+ = \mathrm{id} \mod \mathrm{OPS}^{0,\infty};$$

2 $$H_j^+ E = \text{smoothing}, \quad 1 \leq j \leq d;$$

3 $$AK_j^- - \sum_{r=1}^{d} H_r^- \Lambda^{(rj)} = \text{smoothing}, \quad 1 \leq j \leq d;$$

4 $$H_j^+ K_r^- - \delta_{jr} = \text{smoothing}, \quad 1 \leq j,r \leq d.$$

In view of the composition rules recalled in Section 3, **1** through **4** are satisfied iff the following relations are fulfilled for each degree of homogeneity $\ell \geq 0$:

1′ $$\sum_{2|\alpha|+p+q=\ell} \frac{1}{\alpha! i^{|\alpha|}} \Bigg[\sigma_{(x,\xi)}\left(\partial_\eta^\alpha A_{(y,\eta)}^{(k+p)} \bullet \partial_y^\alpha E_{(y,\eta)}^{(-k+q)}\right) + \\ + \sum_{j=1}^{d} e^{-ix\cdot\xi} \partial_\eta^\alpha \phi_{-p/2}^{(j)}(y,\eta;x) \overline{\partial_y^\alpha \hat{\psi}_{-q/2}^{(j)}(y,\eta;\xi)}\Bigg] = \begin{cases} 1, & \text{if } \ell = 0 \\ 0, & \text{if } \ell \geq 1; \end{cases}$$

2′
$$\sum_{2|\alpha|+p+q=\ell} \frac{i^{|\alpha|}}{\alpha!} \overline{\left(\partial_y^\alpha E^{(-k+p)}_{(y,\eta)}\right)^* \partial_\eta^\alpha {\phi'}^{(j)}_{-q/2}(y,\eta;\cdot)} = 0, \quad 1 \le j \le d;$$

3′
$$\sum_{2|\alpha|+p+q=\ell} \frac{1}{\alpha! i^{|\alpha|}} \Bigg[\partial_\eta^\alpha A^{(k+p)}_{(y,\eta)}\ \partial_y^\alpha {\psi'}^{(j)}_{-q/2}(y,\eta;\cdot) + \\ - \sum_{r=1}^{d} \partial_\eta^\alpha \phi^{(r)}_{-p/2}(y,\eta;x) \partial_y^\alpha \Lambda^{(rj)}_{m-k/2-q/2}(y,\eta) \Bigg] = 0,\ 1 \le j \le d;$$

4′
$$\sum_{2|\alpha|+p+q=\ell} \frac{1}{\alpha! i^{|\alpha|}} \left(\partial_y^\alpha {\psi'}^{(r)}_{-q/2}(y,\eta;\cdot), \partial_\eta^\alpha {\phi'}^{(j)}_{-p/2}(y,\eta;\cdot) \right)_{L^2(\mathbb{R}^\nu)} = \begin{cases} \delta_{jr},\ \text{if } \ell = 0 \\ 0,\ \text{if } \ell \ge 1. \end{cases}$$

By the very definition of $E^{(-k)}_{(y,\eta)}$, $\psi^{(j)}_0$, ${\psi'}^{(j)}_0$, $1 \le j \le d$, and $\Lambda_{m-k/2}(y,\eta)$, the above relations are *automatically satisfied when* $\ell = 0$, as we are going to show next.

- **4′** is just a reformulation of the gauge-orthonormality condition $h^+_{\phi'}(\rho) \circ h^-_{\phi'}(\rho) = I_{\mathbb{C}^d}$, for all $\rho \in \Sigma$;
- **3′** reads as $A^{(k)}_\rho {\psi'}^{(j)}_0(\rho;\cdot) = \sum_{r=1}^{d} \Lambda^{(rj)}_{m-k/2}(\rho) \phi^{(r)}_0(\rho;\cdot)$. Since, by definition,
$$\Lambda_{m-k/2}(\rho) = h^+_\phi(\rho) \circ A^{(k)}_\rho \circ h^-_{\phi'}(\rho),$$
and
$$A^{(k)}_\rho {\psi'}^{(j)}_0(\rho;\cdot) = A^{(k)}_\rho {\phi'}^{(j)}_0(\rho;\cdot) = \sum_{r=1}^{d} \left(A^{(k)}_\rho {\phi'}^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot) \right)_{L^2(\mathbb{R}^\nu)} \phi^{(r)}_0(\rho;\cdot),$$
3′ is obviously true;
- As regards **1′**, by the definition of $E^{(-k)}_\rho$ and the fact that $h^+_\phi(\rho) \circ h^-_\phi(\rho) = I_{\mathbb{C}^d}$, we have that for all $f \in \mathcal{S}(\mathbb{R}^\nu)$
$$A^{(k)}_\rho \bullet E^{(-k)}_\rho f = f - \left(h^-_\phi(\rho) \circ h^+_\phi(\rho) \right) f = f - \pi_2(\rho) f, \quad \forall \rho \in \Sigma.$$
Now recall that the operator
$$f \longmapsto \left(h^-_\phi(\rho) \circ h^+_\phi(\rho) \right) f = \sum_{j=1}^{d} \left(f, \phi^{(j)}_0(\rho;\cdot) \right)_{L^2(\mathbb{R}^\nu)} \phi^{(j)}_0(\rho;\cdot),$$
has (x,ξ)-symbol (as a smoothing operator in the x-variables) precisely given by
$$\sum_{j=1}^{d} e^{-ix\cdot\xi} \phi^{(j)}_0(\rho;x) \overline{\hat{\phi}^{(j)}_0(\rho;\xi)},$$
which yields **1′** for $\ell = 0$;

- As regards $\mathbf{2'}$, we have to show that for $1 \leq j \leq d$, ${\phi'}_0^{(j)}(\rho;\cdot) \in \mathrm{Ker}((E_\rho^{(-k)})^*)$, that is ${\phi'}_0^{(j)}(\rho;\cdot) \in \mathrm{Im}(E_\rho^{(-k)})^\perp = V_1(\rho)$, which is true by hypothesis.

We next proceed in solving **1'-4'** at step $\ell = 1$:

$$\mathbf{1'} \quad \sigma_{(x,\xi)}\Big(A^{(k)}_{(y,\eta)} \bullet E^{(-k+1)}_{(y,\eta)}\Big) + \sigma_{(x,\xi)}\Big(A^{(k+1)}_{(y,\eta)} \bullet E^{(-k)}_{(y,\eta)}\Big) =$$

$$= -\sum_{j=1}^{d} e^{-ix\cdot\xi}\Big[\phi^{(j)}_{-1/2}(y,\eta;x)\overline{\hat{\phi}^{(j)}_0(y,\eta;\xi)}\Big] +$$

$$-\sum_{j=1}^{d} e^{-ix\cdot\xi}\Big[\phi^{(j)}_0(y,\eta;x)\overline{\hat{\psi}^{(j)}_{-1/2}(y,\eta;\xi)}\Big];$$

$$\mathbf{2'} \qquad (E^{(-k+1)}_{(y,\eta)})^*{\phi'}^{(j)}_0(y,\eta;\cdot) + (E^{(-k)}_{(y,\eta)})^*{\phi'}^{(j)}_{-1/2}(y,\eta;\cdot) = 0, \quad 1 \leq j \leq d;$$

$$\mathbf{3'} \quad A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1/2}(y,\eta;\cdot) + A^{(k+1)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot) =$$

$$= \sum_{r=1}^{d}\Big[\phi^{(r)}_0(y,\eta;x)\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta) + \phi^{(r)}_{-1/2}(y,\eta;x)\Lambda^{(rj)}_{m-k/2}(y,\eta)\Big],$$

for $1 \leq j \leq d$;

$$\mathbf{4'} \; \Big({\psi'}^{(r)}_{-1/2}(y,\eta;\cdot), {\phi'}^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \Big({\phi'}^{(r)}_0(y,\eta;\cdot), {\phi'}^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} = 0,$$

for $1 \leq j, r \leq d$.

We first solve **3'** and **4'** for ${\psi'}^{(j)}_{-1/2}$ and $\Lambda^{(rj)}_{m-k/2-1/2}$.

We look for ${\psi'}^{(j)}_{-1/2}$ in the form

$$(52) \qquad {\psi'}^{(j)}_{-1/2}(y,\eta;x) = {\psi'}^{(j)}_{-1/2,1}(y,\eta;x) + {\psi'}^{(j)}_{-1/2,2}(y,\eta;x),$$

where ${\psi'}^{(j)}_{-1/2,1}(y,\eta;\cdot) \in V_1(y,\eta)$ and ${\psi'}^{(j)}_{-1/2,2}(y,\eta;\cdot) \in V_1(y,\eta)^\perp$. From **4'** we get

$$(53) \quad \Big({\psi'}^{(r)}_{-1/2,1}(y,\eta;\cdot), {\phi'}^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} = -\Big({\phi'}^{(r)}_0(y,\eta;\cdot), {\phi'}^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)},$$

for all $j, r = 1, \ldots, d$, which determines uniquely the components ${\psi'}^{(j)}_{-1/2,1}$, $j = 1, \ldots, d$.

Plugging (52) into **3'** yields

$$(54) \quad A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1/2,2}(y,\eta;\cdot) = \sum_{r=1}^{d}\phi^{(r)}_0(y,\eta;x)\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta) +$$

$$-A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1/2,1}(y,\eta;x) + \sum_{r=1}^{d}\phi^{(r)}_{-1/2}(y,\eta;x)\Lambda^{(rj)}_{m-k/2}(y,\eta) +$$

$$-A^{(k+1)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot) =: F_j(y,\eta;x), \quad 1\le j\le d.$$

The obvious idea is to look for $\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta)$ such that $F_j(y,\eta;\cdot)\in V_2(y,\eta)^\perp$. Taking the $L^2(\mathbb{R}^\nu)$-inner product of $F_j(y,\eta;\cdot)$ with the functions $\phi^{(s)}_0(y,\eta;\cdot)$, $1\le s\le d$, and using the identity $h^+_\phi(y,\eta)\circ h^-_\phi(y,\eta)=I_{\mathbb{C}^d}$, we obtain

$$\begin{aligned}\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta) &= \Big(A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1/2,1}(y,\eta;\cdot),\phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &+\Big(A^{(k+1)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot),\phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &-\sum_{s=1}^d\Lambda^{(sj)}_{m-k/2}(y,\eta)\Big(\phi^{(s)}_{-1/2}(y,\eta;\cdot),\phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}.\end{aligned} \tag{55}$$

Since $A^{(k)}_{(y,\eta)}\colon V_1(y,\eta)^\perp\cap B^k(\mathbb{R}^\nu)\xrightarrow{\sim}V_2(y,\eta)^\perp$, we take

$${\psi'}^{(j)}_{-1/2,2}(y,\eta;\cdot)=E^{(-k)}_{(y,\eta)}F_j(y,\eta;\cdot),\quad 1\le j\le d. \tag{56}$$

We now turn to **1'** and **2'**.
Define

$$R^{(1)}_{(y,\eta)}\colon \mathcal{S}(\mathbb{R}^\nu)\ni f\longmapsto\sum_{j=1}^d\Big(f,\psi^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_0(y,\eta;\cdot)\in\mathcal{S}(\mathbb{R}^\nu), \tag{57}$$

$$S^{(1)}_{(y,\eta)}\colon \mathcal{S}(\mathbb{R}^\nu)\ni f\longmapsto\sum_{j=1}^d\Big(f,\phi^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_{-1/2}(y,\eta;\cdot)\in\mathcal{S}(\mathbb{R}^\nu), \tag{58}$$

$$\pi_2(y,\eta)\colon \mathcal{S}(\mathbb{R}^\nu)\ni f\longmapsto\sum_{j=1}^d\Big(f,\phi^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_0(y,\eta;\cdot)\in\mathcal{S}(\mathbb{R}^\nu). \tag{59}$$

Hence **1'** reduces to the operator-equation

$$A^{(k)}_{(y,\eta)}\bullet E^{(-k+1)}_{(y,\eta)}=-\Big(A^{(k+1)}_{(y,\eta)}\bullet E^{(-k)}_{(y,\eta)}+S^{(1)}_{(y,\eta)}\Big)-R^{(1)}_{(y,\eta)}=:-T^{(1)}_{(y,\eta)}-R^{(1)}_{(y,\eta)}, \tag{60}$$

with unknowns $E^{(-k+1)}_{(y,\eta)}$ and $R^{(1)}_{(y,\eta)}$. Recall that, by the very definition,

$$A^{(k)}_{(y,\eta)}\bullet E^{(-k)}_{(y,\eta)}=\mathrm{id}-\pi_2(y,\eta) \tag{61}$$

(and that $\pi_2(y,\eta)^2=\pi_2(y,\eta)$ since $h^+_\phi(y,\eta)\circ h^-_\phi(y,\eta)=I_{\mathbb{C}^d}$). We take as $E^{(-k+1)}_{(y,\eta)}$ the operator defined by

(62)

$$E^{(-k+1)}_{(y,\eta)}\colon\mathcal{S}(\mathbb{R}^\nu)\ni f\longmapsto -E^{(-k)}_{(y,\eta)}\bullet\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(1)}_{(y,\eta)}f+$$

$$+\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}{\phi'}^{(j)}_{0}(y,\eta;\cdot)\in\mathcal{S}(\mathbb{R}^\nu),$$

for some $\gamma^{(j)}_{-1/2}\in\mathbf{H}^{-1/2}_{\rm hom}$ to be determined. Plugging (62) into (60) gives

$$A^{(k)}_{(y,\eta)}\bullet E^{(-k+1)}_{(y,\eta)}f=-\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(1)}_{(y,\eta)}f+ \tag{63}$$

$$+\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_{0}(y,\eta;\cdot)=-T^{(1)}_{(y,\eta)}f-R^{(1)}_{(y,\eta)}f,$$

whence we must have

(64)

$$\pi_2(y,\eta)\bullet T^{(1)}_{(y,\eta)}f+\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_{0}(y,\eta;\cdot)+R^{(1)}_{(y,\eta)}f=0,$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$, that is to say

$$\sum_{j=1}^{d}\Big(f,(T^{(1)}_{(y,\eta)})^*\phi^{(j)}_0(y,\eta;\cdot)+\psi^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_0(y,\eta;\cdot)+ \tag{65}$$

$$+\sum_{j,r=1}^{d}\Big(f,\overline{\Big(A^{(k)}_{(y,\eta)}{\phi'}^{(r)}_{0}(y,\eta;\cdot),\phi^{(j)}_0(y,\eta;\cdot)\Big)}_{L^2(\mathbb{R}^\nu)}\gamma^{(r)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_0(y,\eta;\cdot)=0,$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$. As a consequence, we obtain

$$\psi^{(j)}_{-1/2}(y,\eta;x)=-\Bigg[(T^{(1)}_{(y,\eta)})^*\phi^{(j)}_0(y,\eta;\cdot)+ \tag{66}$$

$$+\sum_{r=1}^{d}\overline{\Big(A^{(k)}_{(y,\eta)}{\phi'}^{(r)}_{0}(y,\eta;\cdot),\phi^{(j)}_0(y,\eta;\cdot)\Big)}_{L^2(\mathbb{R}^\nu)}\gamma^{(r)}_{-1/2}(y,\eta;x)\Bigg],\ 1\le j\le d.$$

It remains to pick the $\gamma^{(r)}_{-1/2}$, $1\le r\le d$. To this purpose we finally use **2'**, that is

$$\Big({\phi'}^{(j)}_{0}(y,\eta;\cdot),E^{(-k+1)}_{(y,\eta)}f\Big)_{L^2(\mathbb{R}^\nu)}+\Big({\phi'}^{(j)}_{-1/2}(y,\eta;\cdot),E^{(-k)}_{(y,\eta)}f\Big)_{L^2(\mathbb{R}^\nu)}=0, \tag{67}$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$ and all $j=1,\dots,d$, i.e., using (62),

$$\Bigg({\phi'}^{(j)}_{0}(y,\eta;\cdot),-E^{(-k)}_{(y,\eta)}\bullet\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(1)}_{(y,\eta)}f+$$

$$+\sum_{r=1}^{d}\Big(f,\gamma^{(r)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}{\phi'}^{(r)}_{0}(y,\eta;\cdot)\Bigg)_{L^2(\mathbb{R}^\nu)}+\Big({\phi'}^{(j)}_{-1/2}(y,\eta;\cdot),E^{(-k)}_{(y,\eta)}f\Big)_{L^2(\mathbb{R}^\nu)}=0,$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$ and all $j=1,\dots,d$, which can be rewritten as

$$\Bigg(f,(E^{(-k)}_{(y,\eta)})^*{\phi'}^{(j)}_{-1/2}(y,\eta;\cdot)-\Big[E^{(-k)}_{(y,\eta)}\bullet\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(1)}_{(y,\eta)}\Big]^*{\phi'}^{(j)}_{0}(y,\eta;\cdot)+$$

$$+\gamma^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} = 0,\ \forall f\in\mathcal{S}(\mathbb{R}^\nu),$$

which determines the $\gamma^{(j)}_{-1/2}$ uniquely. This completes step $\ell = 1$.

An important observation at this point is the following.

Remark 4.3. *Suppose that the following* **parity** *assumption is satisfied:*

(H2) The gauge $\phi = (\phi_1, \dots, \phi_d)$ of V_2 and the gauge $\phi' = (\phi'_1, \dots, \phi'_d)$ of V_1 **have the same parity in** x, that is, for every (y, η) the functions $\phi_j(y,\eta;\cdot)$, $\phi'_r(y,\eta;\cdot)$, $1 \le j, r \le d$, are either **all** even or **all** odd in x.

Since k is **even** *(see point 3) of Proposition 2.1), $A^{(k+1)}_{(y,\eta)}$ flips the parity, so that all the terms $\left(A^{(k+1)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot), \phi^{(r)}_0(y,\eta;\cdot)\right)_{L^2(\mathbb{R}^\nu)}$ in (55) vanish. Furthermore, if the terms $\phi_{-1/2}$ and $\phi'_{-1/2}$, respectively, in the asymptotic expansion of the symbols of the operators H^- and H^+, respectively,* **are chosen to be zero**, *it turns out that $\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta) = 0$ for all (y,η) and all $1 \le j, r \le d$.*

We now consider equations **1'-4'** at step $\ell = 2$. It will then be clear how to proceed by induction, and this will be left to the reader. To simplify notation, expressions of the form $\sum_{|\alpha|=1} \partial^\alpha_\eta F \partial^\alpha_y G$ will be shortened to $\partial_\eta F \partial_y G$. Also, equation **1'** will be written in operator-form. We hence have:

$$\mathbf{1'}\qquad A^{(k)}_{(y,\eta)} \bullet E^{(-k+2)}_{(y,\eta)} f =$$

$$= -\left[A^{(k+1)}_{(y,\eta)} \bullet E^{(-k+1)}_{(y,\eta)} + A^{(k+2)}_{(y,\eta)} \bullet E^{(-k)}_{(y,\eta)} + \frac{1}{i}\partial_\eta A^{(k)}_{(y,\eta)} \bullet \partial_y E^{(-k)}_{(y,\eta)} + S^{(2)}_{(y,\eta)}\right] f +$$

$$-R^{(2)}_{(y,\eta)} f =: -T^{(2)}_{(y,\eta)} f - R^{(2)}_{(y,\eta)} f,\ \forall f \in \mathcal{S}(\mathbb{R}^\nu),$$

where

$$R^{(2)}_{(y,\eta)} = f \longmapsto \sum_{j=1}^{d} \Big(f, \psi^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \phi^{(j)}_0(y,\eta;\cdot), \tag{68}$$

$$S^{(2)}_{(y,\eta)} f = \sum_{j=1}^{d}\Bigg[\Big(f, \psi^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \phi^{(j)}_{-1/2}(y,\eta;\cdot) + \tag{69}$$

$$+\Big(f, \phi^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \phi^{(j)}_{-1}(y,\eta;\cdot) + \frac{1}{i}\Big(f, \partial_y\phi^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \partial_\eta\phi^{(j)}_0(y,\eta;\cdot)\Bigg];$$

$$\mathbf{2'}\qquad (E^{(-k+2)}_{(y,\eta)})^* {\phi'}^{(j)}_0(y,\eta;\cdot) + (E^{(-k+1)}_{(y,\eta)})^* {\phi'}^{(j)}_{-1/2}(y,\eta;\cdot) +$$

$$+(E^{(-k)}_{(y,\eta)})^* {\phi'}^{(j)}_{-1}(y,\eta;\cdot) + \frac{1}{i}(\partial_y E^{(-k)}_{(y,\eta)})^* \partial_\eta {\phi'}^{(j)}_0(y,\eta;\cdot) = 0,\ 1 \le j \le d;$$

3′ $A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1}(y,\eta;\cdot)+A^{(k+1)}_{(y,\eta)}{\psi'}^{(j)}_{-1/2}(y,\eta;\cdot)+$

$$+A^{(k+2)}_{(y,\eta)}{\phi'}^{(j)}_{0}(y,\eta;\cdot)+\frac{1}{i}\partial_\eta A^{(k)}_{(y,\eta)}\partial_y{\phi'}^{(j)}_{0}(y,\eta;\cdot)=$$

$$=\sum_{r=1}^{d}\Big[\phi^{(r)}_0(y,\eta;x)\Lambda^{(rj)}_{m-k/2-1}(y,\eta)+\phi^{(r)}_{-1/2}(y,\eta;x)\Lambda^{(rj)}_{m-k/2-1/2}(y,\eta)+$$

$$+\phi^{(r)}_{-1}(y,\eta;x)\Lambda^{(rj)}_{m-k/2}(y,\eta)+\frac{1}{i}\partial_\eta\phi^{(r)}_0(y,\eta;x)\partial_y\Lambda^{(rj)}_{m-k/2}(y,\eta)\Big],$$

for $1\le j\le d$;

4′ $\Big({\psi'}^{(r)}_{-1}(y,\eta;\cdot),{\phi'}^{(j)}_{0}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\Big({\psi'}^{(r)}_{-1/2}(y,\eta;\cdot),{\phi'}^{(j)}_{-1/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+$

$$+\Big({\phi'}^{(r)}_{0}(y,\eta;\cdot),{\phi'}^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\frac{1}{i}\Big(\partial_y{\phi'}^{(r)}_{0}(y,\eta;\cdot),\partial_\eta{\phi'}^{(j)}_{0}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}=0,$$

for all $1\le j,r\le d$.
As before, we begin by solving **3'** and **4'** for ${\psi'}^{(j)}_{-1}$ and $\Lambda^{(rj)}_{m-k/2-1}$.
We look for ${\psi'}^{(j)}_{-1}$ in the form

$$\psi'^{(j)}_{-1}(y,\eta;x)={\psi'}^{(j)}_{-1,1}(y,\eta;x)+{\psi'}^{(j)}_{-1,2}(y,\eta;x), \tag{70}$$

where ${\psi'}^{(j)}_{-1,1}(y,\eta;\cdot)\in V_1(y,\eta)$ and ${\psi'}^{(j)}_{-1,2}(y,\eta;\cdot)\in V_1(y,\eta)^\perp$. From **4'** we get

$$\Big({\psi'}^{(r)}_{-1,1}(y,\eta;\cdot),{\phi'}^{(j)}_{0}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}=\text{ given function}, \tag{71}$$

for all $j,r=1,\dots,d$, which determines uniquely the components ${\psi'}^{(j)}_{-1,1}$, $j=1,\dots,d$.
Plugging (70) into **3'** yields

$$A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1,2}(y,\eta;\cdot)=\sum_{r=1}^{d}\phi^{(r)}_0(y,\eta;x)\Lambda^{(rj)}_{m-k/2-1}(y,\eta)+ \tag{72}$$

$$-A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1,1}(y,\eta;x)+G_j(y,\eta;x)=:F_j(y,\eta;x),\quad 1\le j\le d,$$

where the G_j are given. Once again we look for $\Lambda^{(rj)}_{m-k/2-1}(y,\eta)$ such that $F_j(y,\eta;\cdot)\in V_2(y,\eta)^\perp$. Taking the $L^2(\mathbb{R}^\nu)$-inner product of $F_j(y,\eta;\cdot)$ with the functions $\phi^{(s)}_0(y,\eta;\cdot)$, $1\le s\le d$, yields

$$\Lambda^{(rj)}_{m-k/2-1}(y,\eta)= \tag{73}$$

$$=\Big(A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-1,1}(y,\eta;\cdot),\phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}-\Big(G_j(y,\eta;\cdot),\phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)},$$

for all $1\le j,r\le d$, and, as a consequence, we also get

$$\psi'^{(j)}_{-1,2}(y,\eta;x)=E^{(-k)}_{(y,\eta)}F_j(y,\eta;\cdot),\quad 1\le j\le d. \tag{74}$$

To solve **1'** and **2'**, we take as $E^{(-k+2)}_{(y,\eta)}$ the operator defined by

$$(75)\qquad E^{(-k+2)}_{(y,\eta)}\colon \mathcal{S}(\mathbb{R}^\nu)\ni f\longmapsto -E^{(-k)}_{(y,\eta)}\bullet\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(2)}_{(y,\eta)}f+ \\ +\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}{\phi'}^{(j)}_0(y,\eta;\cdot)\in\mathcal{S}(\mathbb{R}^\nu),$$

for some $\gamma^{(j)}_{-1}\in\mathbf{H}^{-1}_{\mathrm{hom}}$ to be determined. Plugging (75) into **1'** yields

$$(76)\qquad A^{(k)}_{(y,\eta)}\bullet E^{(-k+2)}_{(y,\eta)}f=-\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(2)}_{(y,\eta)}f+ \\ +\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot)=-T^{(2)}_{(y,\eta)}f-R^{(2)}_{(y,\eta)}f,$$

whence we must have

$$(77)\qquad \pi_2(y,\eta)\bullet T^{(2)}_{(y,\eta)}f+\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot)+R^{(2)}_{(y,\eta)}f=0,$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$, that is to say

$$(78)\qquad \sum_{j=1}^{d}\Big(f,(T^{(2)}_{(y,\eta)})^*\phi^{(j)}_0(y,\eta;\cdot)+\psi^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\phi^{(j)}_0(y,\eta;\cdot)+ \\ +\sum_{j=1}^{d}\Big(f,\gamma^{(j)}_{-1}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot)=0,\ \forall f\in\mathcal{S}(\mathbb{R}^\nu).$$

As before, we obtain

$$(79)\qquad \psi^{(j)}_{-1}(y,\eta;x)=-\Bigg[(T^{(2)}_{(y,\eta)})^*\phi^{(j)}_0(y,\eta;\cdot)+ \\ +\sum_{r=1}^{d}\overline{\Big(A^{(k)}_{(y,\eta)}{\phi'}^{(r)}_0(y,\eta;\cdot),\phi^{(j)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}}\gamma^{(r)}_{-1}(y,\eta;x)\Bigg],\ 1\le j\le d.$$

It remains to pick the $\gamma^{(r)}_{-1}$, $1\le r\le d$. To this purpose we finally use **2'**, that is

$$(80)\qquad \Big({\phi'}^{(j)}_0(y,\eta;\cdot),E^{(-k+2)}_{(y,\eta)}f\Big)_{L^2(\mathbb{R}^\nu)}+\Big(L_j(y,\eta;\cdot),f\Big)_{L^2(\mathbb{R}^\nu)}=0,$$

for all $f\in\mathcal{S}(\mathbb{R}^\nu)$ and all $j=1,\ldots,d$, where the functions $L_j(y,\eta;\cdot)$ are given. As before, we get

$$\Bigg(f,L_j(y,\eta;\cdot)-\Big[E^{(-k)}_{(y,\eta)}\bullet\big(\mathrm{id}-\pi_2(y,\eta)\big)\bullet T^{(2)}_{(y,\eta)}\Big]^*{\phi'}^{(j)}_0(y,\eta;\cdot)+ \\ +\gamma^{(j)}_{-1}(y,\eta;\cdot)\Bigg)_{L^2(\mathbb{R}^\nu)}=0,\ \forall f\in\mathcal{S}(\mathbb{R}^\nu),$$

which determines the $\gamma_{-1}^{(j)}$ uniquely. This completes the proof of the theorem. $\square$

From the above proof, we may now, as promised, spell out the matrices $\Lambda^{(rj)}_{m-k/2-\ell/2}(y,\eta)$, $1 \le r, j \le d$, $\ell \ge 0$, in the asymptotic expansion of the symbol of $\Lambda(y, D_y)$.

As regards the principal symbol we have

$$\Lambda^{(rj)}_{m-k/2}(y,\eta) = \Big(A^{(k)}_{(y,\eta)}{\phi'}^{(j)}_0(y,\eta;\cdot), \phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}. \tag{81}$$

For the lower order terms $\Lambda^{(rj)}_{m-k/2-\ell/2}$, $\ell \ge 1$, on supposing that the ${\psi'}^{(j)}_{-q/2}$ and the $\Lambda^{(rj)}_{m-k/2-q/2}$, $0 \le q < \ell$, have already been determined, we write

$${\psi'}^{(j)}_{-\ell/2}(y,\eta;\cdot) = {\psi'}^{(j)}_{-\ell/2,1}(y,\eta;\cdot) + {\psi'}^{(j)}_{-\ell/2,2}(y,\eta;\cdot) \in V_1(y,\eta) \oplus V_1(y,\eta)^\perp,$$

for $1 \le j \le d$. The components ${\psi'}^{(j)}_{-\ell/2,1}$ are uniquely determined by the relations

$$\Big({\psi'}^{(j)}_{-\ell/2,1}(y,\eta;\cdot), {\phi'}^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} = \tag{82}$$

$$= - \sum_{\substack{2|\alpha|+p+q=\ell\\ 0\le q<\ell}} \frac{1}{\alpha! i^{|\alpha|}} \Big(\partial_y^\alpha {\psi'}^{(j)}_{-q/2}(y,\eta;\cdot), \partial_\eta^\alpha {\phi'}^{(r)}_{-p/2}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}.$$

As for $\Lambda_{m-k/2-\ell/2}(y,\eta)$, $1 \le r, j \le d$, we have

$$\Lambda^{(rj)}_{m-k/2-\ell/2}(y,\eta) = \Big(A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-\ell/2,1}(y,\eta;\cdot), \phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \tag{83}$$

$$- \sum_{s=1}^{d} \sum_{\substack{2|\alpha|+p+q=\ell\\ 0\le q<\ell}} \frac{1}{\alpha! i^{|\alpha|}} \partial_y^\alpha \Lambda^{(sj)}_{m-k/2-q/2}(y,\eta) \Big(\partial_\eta^\alpha \phi^{(s)}_{-p/2}(y,\eta;x), \phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} +$$

$$+ \sum_{\substack{2|\alpha|+p+q=\ell\\ 0\le q<\ell}} \frac{1}{\alpha! i^{|\alpha|}} \Big(\partial_\eta^\alpha A^{(k+p)}_{(y,\eta)} \partial_y^\alpha {\psi'}^{(j)}_{-q/2}(y,\eta;\cdot), \phi^{(r)}_0(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}.$$

As for ${\psi'}^{(j)}_{-\ell/2,2}$, $1 \le j \le d$, we have

$${\psi'}^{(j)}_{-\ell/2,2}(y,\eta;\cdot) = E^{(-k)}_{(y,\eta)}\Big[-A^{(k)}_{(y,\eta)}{\psi'}^{(j)}_{-\ell/2,1}(y,\eta;x) + \tag{84}$$

$$+ \sum_{r=1}^{d} \phi^{(r)}_0(y,\eta;\cdot)\Lambda^{(rj)}_{m-k/2-\ell/2}(y,\eta) +$$

$$+\sum_{r=1}^{d}\sum_{\substack{2|\alpha|+p+q=\ell\\ 0\le q<\ell}}\frac{1}{\alpha! i^{|\alpha|}}\partial_\eta^\alpha\phi^{(r)}_{-p/2}(y,\eta;x)\partial_y^\alpha\Lambda^{(rj)}_{m-k/2-q/2}(y,\eta)+$$

$$-\sum_{\substack{2|\alpha|+p+q=\ell\\ 0\le q<\ell}}\frac{1}{\alpha! i^{|\alpha|}}\partial_\eta^\alpha A^{(k+p)}_{(y,\eta)}\partial_y^\alpha\psi'^{(j)}_{-q/2}(y,\eta;\cdot)\Bigg].$$

Although (82), (83) and (84) are extremely complicated, there are instances in which they may be somewhat simplified, as it will be seen later on in Section 6.

Remark 4.4. *If*

$$\tilde H^- = H^-R,\qquad \tilde H^+ = SH^+,$$

where $R, S\in \mathrm{OPS}^0_{\mathrm{cl}}(\mathbb{R}^{n-\nu}; d\times d)$ *with unitary principal symbols, then the parametrix* $\begin{bmatrix}\tilde E & \tilde K^-\\ \tilde K^+ & -\tilde\Lambda\end{bmatrix}$ *of* $\begin{bmatrix}A & \tilde H^-\\ \tilde H^+ & 0\end{bmatrix}$ *is given by*

$$\begin{bmatrix}\mathrm{id} & 0\\ 0 & R^{-1}\end{bmatrix}\begin{bmatrix}E & K^-\\ K^+ & -\Lambda\end{bmatrix}\begin{bmatrix}\mathrm{id} & 0\\ 0 & S^{-1}\end{bmatrix},\tag{85}$$

where R^{-1}, *resp.* S^{-1}, *is a two-sided parametrix of* R, *resp.* S. *Hence, in particular, we have* $\tilde\Lambda = R^{-1}\Lambda S^{-1}$.

Note, however, that **in general** *for two systems of the form (45),* $\begin{bmatrix}A & H^-\\ H^+ & 0\end{bmatrix}$, $\begin{bmatrix}A & \tilde H^-\\ \tilde H^+ & 0\end{bmatrix}$, *we* **do not** *have a* (ψdo) *conjugation-relation as in (85).*

We end this section by explaining how the above arguments **can be microlocalized** to a conic admissible neighborhood $U\subset T^*\mathbb{R}^{n-\nu}\setminus 0$ of a point $(y_0,\eta_0)\in\Sigma$ where $\mathrm{Ker}(A^{(k)}_{(y_0,\eta_0)})\neq\{0\}$. It is convenient to give the following definition.

Definition 4.5. *Given our operator* A *satisfying* **(H1)**, *let* $\rho_0=(y_0,\eta_0)\in\Sigma$ *be a point where* $\dim\mathrm{Ker}(A^{(k)}_{\rho_0}) = d\ge 1$, *and let* $U\subset\Sigma$ *be an admissible conic neighborhood of* ρ_0. *We say that a system* $\mathcal{A}=\begin{bmatrix}A & H^-\\ H^+ & 0\end{bmatrix}$ *is a* **system associated with** A **in** U *if:*

- $H^-=[H_1^-\ \dots\ H_d^-]$, $H_j^-\in\mathrm{OP}\mathbf{H}^0_{\mathrm{cl}}$ *with* $\sigma(H_j^-)\sim\sum_{\ell\ge 0}\phi^{(j)}_{-\ell/2}(y,\eta;x)$, $1\le j\le d$;

• $H^+ = \begin{bmatrix} H_1^+ \\ \vdots \\ H_d^+ \end{bmatrix}$, $H_j^+ \in \mathrm{OP}\mathbf{H}^{*0}_{\mathrm{cl}}$ *with* $\sigma(H_j^-) \sim \sum_{\ell\geq 0} {\phi'}^{(j)}_{-\ell/2}(y,\eta;x)$, $1 \leq j \leq d$;

• *The globally defined* $\phi := (\phi_0^{(1)}, \dots, \phi_0^{(d)})$, *resp.* $\phi' := ({\phi'}_0^{(1)}, \dots, {\phi'}_0^{(d)})$, **when** $(y,\eta) \in U$, *is a* **gauge** *of* $V_2 \longrightarrow U$, *resp.* $V_1 \longrightarrow U$.

We may now prove the following crucial result.

Theorem 4.6. *Given any system* $\mathcal{A} = \begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}$ *associated with* A *in* U, *there exists a system* $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$, *where* $E \in \mathrm{OPS}^{-m,-k}_{\mathrm{cl}}$, $\Lambda = \Lambda(y, D_y) = \Big(\Lambda^{(rj)}(y,D_y)\Big)_{r,j=1,\dots,d}$ *is a* $d \times d$ *pseudodifferential system whose entries* $\Lambda^{(rj)} \in \mathrm{OPS}^{m-k/2}_{\mathrm{cl}}(\mathbb{R}^{n-\nu})$, $K^- = [K_1^- \dots K_d^-]$, $K_j^- \in \mathrm{OP}\mathbf{H}^0_{\mathrm{cl}}$, $K^+ = \begin{bmatrix} K_1^+ \\ \vdots \\ K_d^+ \end{bmatrix}$, $K_j^+ \in \mathrm{OP}\mathbf{H}^{*0}_{\mathrm{cl}}$, $1 \leq j \leq d$, *which is a two-sided microlocal parametrix for* $\mathcal{A}$ *at* ρ_0, *i.e. there is a conic neighborhood* $\Gamma \subset\subset U$ *of* (y_0, η_0) *such that for any given* $f \in \mathcal{E}'(\mathbb{R}^n)$ *with* $\mathrm{WF}(f) \subset \tilde{\Gamma} := \{(x,y,\xi,\eta);\ (y,\eta) \in \Gamma\}$ *and any given* $g = \begin{bmatrix} g_1 \\ \vdots \\ g_d \end{bmatrix} \in \mathcal{E}'(\mathbb{R}^{n-\nu}; \mathbb{C}^d)$ *with* $\mathrm{WF}(g) := \bigcup_{j=1}^{d} \mathrm{WF}(g_j) \subset \Gamma$, *we have*

$$\mathcal{E}\mathcal{A}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix},\ \mathcal{A}\mathcal{E}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \in \begin{matrix} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu}; \mathbb{C}^d) \end{matrix}.$$

Proof. We start by taking $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$ as in the statement with symbols $\sigma(\Lambda^{(rj)}) \sim \sum_{\ell\geq 0} \Lambda^{(rj)}_{m-k/2-\ell/2}(y,\eta)$, $\sigma(K_j^-) \sim \sum_{\ell \geq 0} {\psi'}^{(j)}_{-\ell/2}(y,\eta;x)$, and $\sigma(K_j^+) \sim \sum_{\ell\geq 0} \psi^{(j)}_{-\ell/2}(y,\eta;x)$, $1 \leq j, r \leq d$, **where** ${\psi'}^{(j)}_0 = {\phi'}^{(j)}_0$ and $\psi^{(j)}_0 = \phi^{(j)}_0$, $1 \leq j \leq d$. Write

$$\mathcal{A}\mathcal{E} = \begin{bmatrix} AE + H^-K^+ & AK^- - H^-\Lambda \\ H^+E & H^+K^- \end{bmatrix}$$

and note that by the calculus of Section 3 the full symbols of the entries of $\mathcal{A}\mathcal{E}$ are given by the left-hand-sides of **1'** through **4'**.

Fix now a conic neighborhood $\Gamma \subset\subset U$ of (y_0, η_0). The proof of Theorem 4.1 shows that we may globally construct the operators $E^{(-k+q)}_{(y,\eta)}$, and the symbols $\psi^{(j)}_{-q/2}(y,\eta;x)$, ${\psi'}^{(j)}_{-q/2}(y,\eta;x)$, $\Lambda^{(rj)}_{m-k/2-q/2}(y,\eta)$, $q \geq 0$ and $1 \leq j, r \leq d$, in such a way that **equations 1'-4' are satisfied** in $\tilde{\Gamma} = \{(x, y, \xi, \eta);\ (y, \eta) \in \Gamma\}$. As a consequence, keeping into account Lemma 3.16, given any f and g as in the statement, we have

$$H^+Ef \in C^\infty(\mathbb{R}^n; \mathbb{C}^d),$$
$$\Big(AK^- - H^-\Lambda\Big)g \in C^\infty(\mathbb{R}^n),$$
$$(H^+K^- - \mathrm{id})g \in C^\infty(\mathbb{R}^{n-\nu}; \mathbb{C}^d),$$
$$\Big(AE + \sum_{j=1}^{d} H^-_j K^+_j - \mathrm{id}\Big)f - Rf \in C^\infty(\mathbb{R}^n),$$

where the symbol of the ψdo R belongs to $S^{0,\infty}(\tilde{\Gamma})$. As in the first step of the proof of Theorem 4.1, we can reabsorb R into E in such a way that

$$\Big(AE + \sum_{j=1}^{d} H^-_j K^+_j - \mathrm{id}\Big)f \in C^\infty(\mathbb{R}^n).$$

By a similar argument we can construct a system $\mathcal{E}'$ which is a microlocal left-parametrix for $\mathcal{A}$ at ρ_0. As usual, $\mathcal{E}$ and $\mathcal{E}'$ coincide microlocally. □

Remark 4.7. *Observe that the full symbol of* Λ **in** Γ *is given by relations (81) and (83).*

5. Hypoellipticity results and other applications

As an immediate consequence of Theorem 4.6 we have the following hypoellipticity result.

Theorem 5.1. *Given our operator* A *satisfying* **(H1)***, let* $\rho_0 = (y_0, \eta_0) \in \Sigma$ *be a point where* $\mathrm{Ker}(A^{(k)}_{\rho_0}) \neq \{0\}$*, let* $\mathcal{A} = \begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}$ *be any system associated with* A *in* U*, and let the system* $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$ *be a two-sided (microlocal) parametrix of* $\mathcal{A}$ *at* ρ_0*.*
Then A *is hypoelliptic at* ρ_0 *with loss of* $\frac{k}{2} + r$ *derivatives* ($r > 0$)*, resp. hypoelliptic at* ρ_0*, iff* Λ *is hypoelliptic at* ρ_0 *with loss of* r *derivatives, resp. hypoelliptic at* ρ_0*.*

Proof. **Sufficiency.** By hypothesis there is a conic neighborhood $\Gamma \subset\subset U$ of ρ_0 such that

$$\mathcal{E}\mathcal{A}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \in \begin{matrix} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu}; \mathbb{C}^d) \end{matrix},$$

whenever $f \in \mathcal{E}'(\mathbb{R}^n)$, $g \in \mathcal{E}'(\mathbb{R}^{n-\nu};\mathbb{C}^d)$ are such that $\mathrm{WF}(f) \subset \tilde{\Gamma}$, $\mathrm{WF}(g) \subset \Gamma$. Suppose now that $Af \in H^t$ at ρ_0 (see Remark 1.2). From Lemmas 3.16, 3.17 we have $EAf \in H^{t+m-k/2}$ at ρ_0, and since $EAf + K^-H^+f - f \in C^\infty(\mathbb{R}^n)$, we get $f - K^-H^+f \in H^{t+m-k/2}$ at ρ_0. On the other hand $K^+Af \in H^t$ at ρ_0, and since $K^+Af - \Lambda H^+ f \in C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d)$, we obtain $\Lambda H^+ f \in H^t$ at ρ_0. Thus $H^+f \in H^{t+m-k/2-r}$ at ρ_0. By continuity $K^-H^+f \in H^{t+m-k/2-r}$ at ρ_0. Hence $f \in H^{t+m-k/2-r}$ at ρ_0.

Necessity. By Hypothesis we have also

$$\mathcal{A}\mathcal{E}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \in \begin{matrix} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d) \end{matrix},$$

whenever f and g are as above. Suppose now that $\Lambda g \in H^t$ at ρ_0. Then (again by Lemmas 3.16, 3.17) we have $H^-\Lambda g \in H^t$ at ρ_0, and since $AK^-g - H^-\Lambda g \in C^\infty(\mathbb{R}^n)$, we get $AK^-g \in H^t$ at ρ_0. Thus $K^-g \in H^{t+m-k/2-r}$ at ρ_0. As $H^+K^-g - g \in C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d)$, we conclude that $g \in H^{t+m-k/2-r}$ at ρ_0.

The same arguments show that A is hypoelliptic at ρ_0 iff Λ is hypoelliptic at ρ_0. □

In the next theorem we collect some interesting results that are trivial consequences of the proof of Theorem 5.1.

Theorem 5.2. *Let $\mathcal{A}$ be any fixed system associated with A in U, and let $\mathcal{E}$ be a two-sided microlocal parametrix for $\mathcal{A}$ as in Theorem 4.6*

(a) *(Existence of a micro-parametrix for A.) Suppose that for some $r > 0$ there exists a $d \times d$ system of ψdo's $G = (G_{jl})_{1\le j,l\le d}$ with $G_{jl} \in \mathrm{OPS}^{-(m-k/2)+r}_{1/2,1/2}(\mathbb{R}^{n-\nu})$, $1 \le j,l \le d$, such that for all $g \in \mathcal{E}'(\mathbb{R}^{n-\nu};\mathbb{C}^d)$ with $\mathrm{WF}(g) \subset \Gamma$ we have*

$$\Lambda G g - g \in C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d)$$

(resp. $G\Lambda g - g \in C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d)$). Then for all $f \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{WF}(f) \subset \tilde{\Gamma}$ we have

$$A(E + K^-GK^+)f - f \in C^\infty(\mathbb{R}^n)$$

(resp. $(E + K^-GK^+)Af - f \in C^\infty(\mathbb{R}^n)$).

(b) *(Propagation of singularities.) Suppose there exists a set $T \subset \Gamma \cap \mathrm{Char}(\Lambda) = \{(y,\eta) \in \Gamma;\ \det \Lambda_{m-k/2}(y,\eta) = 0\}$, such that $(y_0,\eta_0) \in T \cap \left(\overline{T \setminus \{(y_0,\eta_0)\}}\right)$, and suppose that Λ satisfies the following property: For all $g \in \mathcal{E}'(\mathbb{R}^{n-\nu};\mathbb{C}^d)$*

$$\mathrm{WF}(\Lambda g) \cap \Gamma = \emptyset,\ (y_0,\eta_0) \in \mathrm{WF}(g) \implies T \subset \mathrm{WF}(g).$$

Then, upon denoting by

$$\Gamma_0 := \{(0,y,0,\eta);\ (y,\eta) \in \Gamma\},\ T_0 := \{(0,y,0,\eta);\ (y,\eta) \in T\},$$

we have: For all $f \in \mathcal{E}'(\mathbb{R}^n)$

$$\mathrm{WF}(Af) \cap \Gamma_0 = \emptyset,\ (0, y_0, 0, \eta_0) \in \mathrm{WF}(f) \implies T_0 \subset \mathrm{WF}(f).$$

Proof. **(a)** Suppose $\Lambda Gg - g \in C^\infty(\mathbb{R}^{n-\nu}; \mathbb{C}^d)$ whenever $\mathrm{WF}(g) \subset \Gamma$. Since $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$ is a right-parametrix for $\mathcal{A}$, we have $AEf = f - H^-K^+f + C^\infty(\mathbb{R}^n)$ whenever $\mathrm{WF}(f) \subset \tilde{\Gamma}$. On the other hand,

$$AK^-GK^+f = H^-\Lambda GK^+f + C^\infty(\mathbb{R}^n) = H^-K^+f + C^\infty(\mathbb{R}^n).$$

Hence $E + K^-GK^+$ is a right-parametrix for A. In the same way one proves the other case.
(b) We use the fact that $\mathcal{E}$ is a left-parametrix for $\mathcal{A}$. Take any $f \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{WF}(Af) \cap \Gamma_0 = \emptyset$ and $(0, y_0, 0, \eta_0) \in \mathrm{WF}(f)$. Since $\mathrm{WF}(EAf) \subset \mathrm{WF}(Af)$, we have $\mathrm{WF}(EAf) \cap \Gamma_0 = \emptyset$. As $EAf = f - K^-H^+f + C^\infty(\mathbb{R}^n)$, we obtain

$$\mathrm{WF}(f) \cap \Gamma_0 = \mathrm{WF}(K^-H^+f) \cap \Gamma_0.$$

The WF′-relation for K^- yields

$$\mathrm{WF}(K^-H^+f) \cap \Gamma_0 \subset \{(0, y, 0, \eta);\ (y, \eta) \in \Gamma \cap \mathrm{WF}(H^+f)\}.$$

On the other hand, the WF′-relation for H^+ gives

$$\{(0, y, 0, \eta);\ (y, \eta) \in \Gamma \cap \mathrm{WF}(H^+f)\} \subset \mathrm{WF}(f) \cap \Gamma_0.$$

Hence

$$(0, y, 0, \eta) \in \mathrm{WF}(f) \cap \Gamma_0 \iff (y, \eta) \in \mathrm{WF}(H^+f) \cap \Gamma. \tag{86}$$

By hypothesis $\mathrm{WF}(K^+Af) \cap \Gamma = \emptyset$, and since $K^+Af = \Lambda H^+f + C^\infty(\mathbb{R}^{n-\nu}; \mathbb{C}^d)$, we get $\mathrm{WF}(\Lambda H^+f) \cap \Gamma = \emptyset$. On the other hand, by hypothesis $(0, y_0, 0, \eta_0) \in \mathrm{WF}(f)$, whence (86) yields $(y_0, \eta_0) \in \mathrm{WF}(H^+f)$. By the propagation property of Λ, we conclude that $T \subset \mathrm{WF}(H^+f)$ whence, again by (86), $T_0 \subset \mathrm{WF}(f)$. □

It is worth noting that not only is the hypoellipticity/propagation of singularities of A related to that of Λ, but also lower bounds of A are related to lower bounds of Λ. We make this precise in a simplified setup (which, however, gives the idea of more general possible results).
Suppose that $A_\rho^{(k)} = (A_\rho^{(k)})^* \geq 0$ for all $\rho \in \Sigma$ (so that **(H1)** holds), and suppose we may find an **admissible** (conic) set U of the form $U = \Omega \times (\mathbb{R}^{n-\nu} \setminus \{0\})$, $y_0 \in \Omega \subset \mathbb{R}^{n-\nu}$ being open. Note that in this case $V_1(\rho) = V_2(\rho)$ for all $\rho \in U$, and that the $d \times d$ matrix $\Lambda_{m-k/2}(y, \eta)$ is Hermitian and ≥ 0. Consider now a system $\mathcal{A} = \begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}$ associated with A on U, **where** $H^+ = (H^-)^*$ **and the terms of order** $-j/2$, $j \geq 1$, **in the symbol of** H^- **are chosen to be zero**. Theorem 4.6 ensures the

existence of a system $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$ such that for some neighborhood ω of y_0, $\omega \subset\subset \Omega$, we have

$$\mathcal{A}\mathcal{E}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix},\ \mathcal{E}\mathcal{A}\begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \in \begin{matrix} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu};\mathbb{C}^d) \end{matrix},$$

for all $f \in \mathcal{E}'(\mathbb{R}^n)$ and $g \in \mathcal{E}'(\mathbb{R}^{n-\nu};\mathbb{C}^d)$ with

$\mathrm{WF}(g) \subset \omega \times (\mathbb{R}^{n-\nu} \setminus \{0\}) =: \Gamma,\ \mathrm{WF}(f) \subset \{(x,y,\xi,\eta);\ y \in \omega, \eta \neq 0\} =: \tilde{\Gamma}.$

One then has the following result.

Theorem 5.3. *Suppose that for any given $\omega' \subset\subset \omega$ and any $\mu < (m-k/2-1)/2$ there are constants $c > 0$, $C \geq 0$ such that*

$$\operatorname{Re}\big(\Lambda g, g\big) \geq c\,\|g\|^2_{(m-k/2-1)/2} - C\,\|g\|^2_\mu,\ \forall g \in C_0^\infty(\omega';\mathbb{C}^d). \tag{87}$$

Then for any ω' and μ as above there are constants $c' > 0$, $C' \geq 0$ such that

$$\operatorname{Re}\big(Af, f\big) \geq c'\,\|f\|^2_{(m-k/2-1)/2} - C'\,\|f\|^2_\mu,\ \forall f \in C_0^\infty(\mathbb{R}^\nu \times \omega'). \tag{88}$$

Proof. Write $f = K^-H^+f + (\mathrm{id} - K^-H^+)f$. Then

$$\operatorname{Re}\big(Af,f\big) = \operatorname{Re}\big(AK^-H^+f, K^-H^+f\big) + 2\operatorname{Re}\big(AK^-H^+f, f - K^-H^+f\big) + \\ + \operatorname{Re}\big(A(f - K^-H^+f), f - K^-H^+f\big) = (1) + (2) + (3).$$

Estimating (1)**.** We have $AK^-H^+f = H^-\Lambda H^+f + \mathrm{smoothing}(f)$, so that

$$(1) = \operatorname{Re}\big(H^-\Lambda H^+f + \mathrm{smoothing}(f), K^-H^+f\big) = \\ = \operatorname{Re}\big(\Lambda H^+f, H^+K^-H^+f\big) + O(\|f\|^2_\mu).$$

Since $H^+K^- - \mathrm{id}$ is smoothing on ω we get

$$(1) = \operatorname{Re}\big(\Lambda H^+f, H^+f\big) + O(\|f\|^2_\mu) \geq c\,\big\|H^+f\big\|^2_{(m-k/2-1)/2} - C'\,\|f\|^2_\mu$$

for some $C' \geq 0$ (we will use the same C' to denote a suitable nonnegative constant, possibly different at each occurance). By the continuity of K^- we finally have

$$(1) = \operatorname{Re}\big(AK^-H^+f, K^-H^+f\big) \geq c_1\,\big\|K^-H^+f\big\|^2_{(m-k/2-1)/2} - C'\,\|f\|^2_\mu,$$

where $c_1 > 0$.

Estimating (2)**.** We have

$$\big(AK^-H^+f, f - K^-H^+f\big) = \\ = \big(H^-\Lambda H^+f + \mathrm{smoothing}(f), EAf + \mathrm{smoothing}(f)\big) = \\ = \big(\Lambda H^+f, H^+EAf\big) + O(\|f\|^2_\mu).$$

Since H^+E is smoothing on $\mathbb{R}^\nu \times \omega$, we conclude that

$$(2) = 2\operatorname{Re}\big(AK^-H^+f, f - K^-H^+f\big) = O(\|f\|^2_\mu).$$

Estimating (3)**.** Since $K^-H^+(\mathrm{id}-K^-H^+)$ is smoothing on $\mathbb{R}^\nu\times\omega$, we may write

$$(3)=\mathrm{Re}\Big((A+K^-H^+)(f-K^-H^+f),f-K^-H^+f\Big)+O(\|f\|_\mu^2).$$

By Proposition 7.1 of [16], we have

$$\mathrm{Re}\Big((A+K^-H^+)(f-K^-H^+f),f-K^-H^+f\Big)\geq$$
$$\geq c_2\left\|f-K^-H^+f\right\|^2_{(m-k/2-1)/2}-C'\left\|f\right\|^2_\mu,$$

where $c_2>0$. Since

$$\|f\|^2_{(m-k/2-1)/2}=$$
$$=\left\|K^-H^+f\right\|^2_{(m-k/2-1)/2}+\left\|f-K^-H^+f\right\|^2_{(m-k/2-1)/2}+O(\|f\|^2_\mu),$$

we obtain (88). □

Remark 5.4. *Inequality (87) is usually called* strong Melin's inequality*. We may call inequality (88)* a strong Hörmander's inequality*, which when $k=2$ is a sharpened form of the classical Hörmander inequality (see* [10]*, Vol. III, Thm. 22.3.2), and for $k\geq 4$ is the generalization dealt with in* [16]*. Actually, in* [16] *(see also* [17]*) we have given, in a more general setup, sufficient conditions in order to have the weaker inequality*

$$\mathrm{Re}\big(Af,f\big)\geq -c'\left\|f\right\|^2_{(m-k/2-1)/2},\quad c'>0. \tag{89}$$

In the present setting, inequality (88) is indeed much stronger, so that it is natural to ask when (87) holds. We do not know the answer in general, but there is at least one classical and meaningful case which we can handle (see [2] *and* [4]*). Suppose that the principal symbol of $H^\pm$ on U (admissible) is a gauge $\phi=(\phi_0^{(1)},\dots,\phi_0^{(d)})$ of $V_1=V_2\longrightarrow U$, such that the $\phi_0^{(j)}(y,\eta;\cdot)$, $1\leq j\leq d$, are either* **all** *even or* **all** *odd in the variable x. Suppose finally that $A^{(k)}_{(y,\eta)}\big|_{V_1(y,\eta)}=\lambda(y,\eta)\,\mathrm{id}_{V_1(y,\eta)}$. In this case, by virtue of the choice of H^- and Remark 4.3), we have $\Lambda_{m-k/2-1/2}(y,\eta)=0$ on U, so that the symbol of Λ has an expansion on Γ of the form*

$$\sigma(\Lambda)(y,\eta)=\Lambda_{m-k/2}(y,\eta)+\Lambda_{m-k/2-1}(y,\eta)+\dots,$$

where

$$\Lambda_{m-k/2}(y,\eta)=\lambda(y,\eta)I_{\mathbb{C}^d},\ \forall(y,\eta)\in U.$$

Hence

$$\mathrm{Char}(\Lambda)\cap\Gamma=\{(y,\eta)\in\Gamma;\ \lambda(y,\eta)=0\}=:\Sigma_\lambda.$$

Suppose furthermore that the set Σ_λ is a smooth closed conic submanifold of $T^*\mathbb{R}^{n-\nu}\setminus 0$ (on which the canonical 1-form $\sum_{j=1}^{n-\nu}\eta_j dy_j$ does not vanish indentically), and that λ vanishes **exactly** *to order 2 on Σ_λ. Under all these assumptions, in* [14] *and* [16] *(see also* [2] *and* [4]*), it is shown how to associate with Λ, at every $\rho\in\Sigma_\lambda$, a localized operator $\Lambda_\rho^{(2)}$ which depends*

only *on* $\Lambda_{m-k/2}$ *and* $\Lambda_{m-k/2-1}$*. In* [14] *and* [16] *it is proved that inequality (87) holds iff*

$$\mathrm{Re}(\Lambda_\rho^{(2)}) = \frac{\Lambda_\rho^{(2)} + (\Lambda_\rho^{(2)})^*}{2} > 0,\ \forall \rho \in \Sigma_\lambda.$$

Hence, the content of Theorem 5.3 is that if we have a stronger information on the way λ *vanishes, and the highly nontrivial information that* $\mathrm{Re}(\Lambda_\rho^{(2)}) > 0$, *we obtain the stronger inequality (88).*

Remark 5.5. *All the preceding constructions were carried out on supposing that* A *is transversally elliptic, i.e. condition (2) holds. On the other hand, it is immediate to see that the validity of (2) is equivalent to the validity for all* $\rho \in \Sigma$ *of*

$$(2')_\rho \qquad (x,\xi) \in T^*\mathbb{R}^\nu \setminus \{(0,0)\} \implies \tilde{a}_\rho(x,\xi) \neq 0$$

($\tilde{a}_\rho$ *having been defined in (6)). Note that if* $(2')_{\rho_0}$ *holds, then by homogeneity* $(2')_\rho$ *holds for all* ρ *in a conic neighborhood of* ρ_0 *in* Σ.
Moreover, condition $(2')_\rho$ *is exactly what ensures that the localized operator* A_ρ *be* **globally elliptic** *in the sense of Shubin. Now Theorem 1.4 can be restated in the following form.*

Theorem 1.4'. Let A satisfy (1) and suppose that $(2')_{\rho_0}$ holds for some $\rho_0 \in \Sigma$. Then A is hypoelliptic at ρ_0 with loss of $k/2$ derivatives iff $\mathrm{Ker}\,(A_{\rho_0}) = \{0\}$.

We can hence take A *to satisfy (1) and (2')*$_{\rho_0}$*, and suppose* **(H1)** *be fulfilled in a neighborhood* $U \subset \Sigma$ *of* ρ_0 *(that we may suppose admissible). Then Theorem 4.6, and Theorems 5.1 to 5.3 still hold true.*

At this point, we can easily remove the restriction of having A *defined in the whole* $\mathbb{R}^n_z = \mathbb{R}^\nu_x \times \mathbb{R}^{n-\nu}_y$*, and suppose that* A *is defined only on some open set* $\Omega \subset \mathbb{R}^n$ *such that* $\Omega \cap \{x = 0\} \neq \emptyset$*, condition (1) holds on* $T^*\Omega \setminus 0$ *and with* $\Sigma = \{(z,\zeta) \in T^*\Omega \setminus 0;\ x = \xi = 0\}$*. Suppose that for some* $\rho_0 \in \Sigma$ *condition* $(2')_{\rho_0}$ *holds and that* $\mathrm{Ker}\,(A_{\rho_0}) \neq \{0\}$*. We extend* A *to an operator* $\tilde{A}$ *defined in the whole* $\mathbb{R}^n$ *in the following way. Define* $B(z, D_z)$ *as follows (recall that* k *must be even):*

$$B(z,D_z) := |D_z|^{(m-k)/2} \left(\sum_{j=1}^{\nu} \Big(D_{x_j}^2 + x_j^2 |D_y|^2 \Big) \right)^{k/2} |D_z|^{(m-k)/2}.$$

Notice that $B(z,D_z) \in \mathrm{OPS}^m_{\mathrm{cl}}(\mathbb{R}^n)$ *vanishes to order* k *for* $x = \xi = 0$*, and is transversally elliptic. Fix now a cut-off* $\chi = \chi(z,\zeta)$*, homogeneous of degree* 0 *in* ζ*,* $0 \le \chi \le 1$*, such that* χ *is supported in a full conic neighborhood of* ρ_0 *contained in* $T^*\Omega \setminus 0$*, and* $\chi \equiv 1$ *in a smaller full conic neighborhood of* ρ_0*. Put for* $f \in C_0^\infty(\mathbb{R}^n)$*,*

$$\tilde{A}(z,D_z)f := A(z,D_z)\Big(\chi(z,D_z)f\Big) + B(z,D_z)\Big[\Big(1 - \chi(z,D_z)\Big)f\Big].$$

Obviously $\tilde{A} \in \mathrm{OPS}^m_{\mathrm{cl}}(\mathbb{R}^n)$, *vanishes to order* k *for* $x = \xi = 0$, *and the hypoellipticity of* A *at* ρ_0 *is* **equivalent** *to the hypoellipticity of* $\tilde{A}$ *at* ρ_0. *Hence, on supposing that* **(H1)** *holds for* A *in a neighborhood* $\tilde{U} \subset \{(y,\eta) \in T^*\mathbb{R}^{n-\nu};\ (0,y) \in \Omega, |\eta| = 1\}$, *all the mentioned results still hold true for* A.

Remark 5.6. *All the contructions above deal with the* C^∞*-hypoellipticity. Suppose now that* A *is an analytic* ψdo *(see, for instance, Treves' book* [21]*). It is known after Métivier* [13] *that when* A_ρ *is injective then* A *is microlocally analytic hypoelliptic at* ρ. *The natural question is therefore the following:*

- Given a system $\begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}$ associated with A, and its parametrix $\begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$, is it true that Λ is an analytic system of ψdo's? And furthermore, if this is the case, is the analytic hypoellipticity of A at ρ equivalent to the analytic hypoellipticity of Λ at ρ?

We do not know the answer. However, the results of Stein [20] *and Grigis-Rothschild* [5] *(see also Kwon* [12]*) allow one to conjecture that the answer is indeed positive.*

6. Examples

In this section we collect a number of simple, yet meaningful, examples in which the machinery developed earlier is put at work.

In all the examples below, we shall deal with situations in which $[A_\rho, A^*_\rho] = 0$, so that the vector bundles V_1 and V_2 (see Definition 2.3) coincide. Furthermore, we shall choose systems $\mathcal{A} = \begin{bmatrix} A & H^- \\ H^+ & 0 \end{bmatrix}$ associated with A (see Definition 4.5) where for all $j = 1, \ldots, d = \dim \operatorname{Ker}(A_{\rho_0})$, we take $\sigma(H^\pm_j) = \phi^{(j)}_0(y,\eta;x) = {\phi'}^{(j)}_0(y,\eta;x)$, with $\phi = (\phi^{(1)}_0, \ldots, \phi^{(d)}_0)$ an orthonormal gauge of $V_1 = V_2$. Hence, if $\mathcal{E} = \begin{bmatrix} E & K^- \\ K^+ & -\Lambda \end{bmatrix}$ is the microlocal parametrix of $\mathcal{A}$ constructed in Theorem 4.6, the principal symbol of $K^\pm_j$ is $\phi^{(j)}_0$, $1 \le j \le d$.

In the sequel, with the exception of Example 6.5, we will limit ourselves to considering hypoellipticity with loss of $1 + k/2$ derivatives, so that only the first three terms in the symbol of $\Lambda(y, D_y)$ will matter. For convenience, we thus recall how formula (83) reads in the present setting for $\ell = 0, 1, 2$.

We have

(90)

$$\begin{cases} \Lambda^{(rj)}_{m-k/2}(\rho) = \Big(A^{(k)}_\rho \phi^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}, \\ \\ \Lambda^{(rj)}_{m-k/2-1/2}(\rho) = \Big(A^{(k+1)}_\rho \phi^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}, \\ \\ \Lambda^{(rj)}_{m-k/2-1}(\rho) = \Big(A^{(k)}_\rho {\psi'}^{(j)}_{-1,1}(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \\ \quad + i\displaystyle\sum_{s=1}^{d}\sum_{|\alpha|=1} \partial^\alpha_y \Lambda^{(sj)}_{m-k/2}(\rho)\Big(\partial^\alpha_\eta \phi^{(s)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \\ + \Big(A^{(k+1)}_\rho {\psi'}^{(j)}_{-1/2}(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \Big(A^{(k+2)}_\rho \phi^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} + \\ \qquad + \dfrac{1}{i}\displaystyle\sum_{|\alpha|=1}\Big(\partial^\alpha_\eta A^{(k)}_\rho \partial^\alpha_y \phi^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}, \end{cases}$$

where $1 \le j, r \le d$, and

(91)

$$\begin{cases} {\psi'}^{(j)}_{-1,1}(\rho;\cdot) = -\dfrac{1}{i}\displaystyle\sum_{r=1}^{d}\sum_{|\alpha|=1}\Big(\partial^\alpha_y \phi^{(j)}_0(\rho;\cdot), \partial^\alpha_\eta \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \phi^{(r)}_0(\rho;\cdot), \\ \\ {\psi'}^{(j)}_{-1/2}(\rho;\cdot) = {\psi'}^{(j)}_{-1/2,2}(\rho;\cdot) = \\ = E^{(-k)}_\rho\Big[\displaystyle\sum_{r=1}^{d}\Big(A^{(k+1)}_\rho \phi^{(j)}_0(\rho;\cdot), \phi^{(r)}_0(\rho;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \phi^{(r)}_0(\rho;\cdot) - A^{(k+1)}_\rho \phi^{(j)}_0(\rho;\cdot)\Big]. \end{cases}$$

6.1. **Example.** Consider in $\mathbb{R}^2$ the differential operator

(92) $$A = D_x^2 + \mu^2 x^2 D_y^2 + a(x,y)D_x + b(x,y)D_y + c(x,y),$$

where $\mu > 0$ is constant and $a, b, c \in C^\infty(\mathbb{R}^2;\mathbb{C})$. In this case the localized operator $A^{(2)}_{\rho=(y,\eta)}$ is given by

$$A^{(2)}_{(y,\eta)} = D_x^2 + \mu^2 x^2 \eta^2 + b(0,y)\eta.$$

Note that $[A^{(2)}_{(y,\eta)}, (A^{(2)}_{(y,\eta)})^*] = 0$, because

$$(A^{(2)}_{(y,\eta)})^* = D_x^2 + \mu^2\eta^2 x^2 + \overline{b(0,y)}\eta.$$

As it is well-known,

(93) $$\mathrm{Spec}\big(A^{(2)}_{(y,\eta)}\big) = \{\lambda_k(y,\eta) := (2k+1)\mu|\eta| + b(0,y)\eta;\ k \in \mathbb{Z}_+\}.$$

Each λ_k has multiplicity 1 and relative normalized eigenfunction

(94) $$f_k(y,\eta;x) = f_k(\eta;x) = (\mu|\eta|)^{1/4} h_k(\sqrt{\mu|\eta|}\,x),$$

where

$$h_k(t) = \frac{1}{\pi^{1/4}\sqrt{2^k k!}}\Big(\frac{d}{dt} - t\Big)^k (e^{-t^2/2}),\ k \in \mathbb{Z}_+, \tag{95}$$

are the classical Hermite functions.

For definiteness, we study the hypoellipticity of A at the origin $(0,0) \in \mathbb{R}^2$. From the Boutet-Grigis-Helffer Theorem 1.4 we know that A is hypoelliptic at $(0,0)$ with loss of 1 derivative iff

$$b(0,0) \notin \{\pm(2k+1)\mu;\ k \in \mathbb{Z}_+\}. \tag{96}$$

We hence study the case in which for some $j \in \mathbb{Z}_+$ one has

$$b(0,0) = \pm(2j+1)\mu.$$

We treat two possible situations:

$(I)_\pm$ $b(0,0) = \pm(2j+1)\mu$ and the function $y \longmapsto (2j+1)\mu \mp b(0,y)$ vanishes exactly to order $r \geq 1$ at $y = 0$.

$(II)_\pm$ The function $y \longmapsto (2j+1)\mu \mp b(0,y)$ vanishes identically near $y = 0$ (for $|y| < \delta$, say).

Remark 6.1. *1) Notice that if $y \longmapsto b(0,y)$ is analytic near $y = 0$, and $b(0,0) = \pm(2j+1)\mu$, then $(I)_\pm$ and $(II)_\pm$ are the only possibilities to be considered.*

2) Notice that if $b(0,0) = \pm(2j+1)\mu$, then A is already hypoelliptic at $(x = 0, y = 0, \xi = 0, \pm\eta > 0)$ with loss of 1 derivative. We may hence microlocally construct the scalar operator $\Lambda(y, D_y) \in \mathrm{OPS}^1_{\mathrm{cl}}(\mathbb{R})$ in the conic region

$$\Gamma^{\mp}_\delta := \{(y,\eta);\ |y| < \delta,\ \pm\eta < 0\}.$$

We then take as the ϕ_0 in the beginning of this section, the eigenfunction f_j. Since f_j is independent of y, we immediately have from (91) that $\psi'_{-1,1} \equiv 0$. Moreover, since $A^{(3)}_{(y,\eta)}$ flips the parity, we have from (91) that

$$\psi'_{-1/2}(y,\eta;\cdot) = -E^{(-2)}_{(y,\eta)} A^{(3)}_{(y,\eta)} f_j(\eta;\cdot).$$

As a consequence, keeping again into account that f_j is independent of y, formula (90), for $m = k = 2$, reads as

$$\begin{cases} \Lambda_1(y,\eta) = \lambda_j(y,\eta), \\ \Lambda_{1/2}(y,\eta) = 0, \\ \Lambda_0(y,\eta) = \Big(A^{(4)}_{(y,\eta)} f_j(\eta;\cdot), f_j(\eta;\cdot)\Big)_{L^2(\mathbb{R})} + \\ \qquad -\Big(E^{(-2)}_{(y,\eta)} A^{(3)}_{(y,\eta)} f_j(\eta;\cdot), (A^{(3)}_{(y,\eta)})^* f_j(\eta;\cdot)\Big)_{L^2(\mathbb{R})} + \\ \qquad + i\dfrac{\partial\lambda_j}{\partial y}(y,\eta)\Big(\dfrac{\partial f_j}{\partial\eta}(\eta;\cdot), f_j(\eta;\cdot)\Big)_{L^2(\mathbb{R})}, \end{cases} \tag{97}$$

with $(y,\eta)\in\Gamma^{\mp}_{\delta}$, where

$$
\begin{cases}
A^{(3)}_{(y,\eta)} = a(0,y)D_x + \dfrac{\partial b}{\partial x}(0,y)x\eta \\[2ex]
A^{(4)}_{(y,\eta)} = \dfrac{\partial a}{\partial x}(0,y)xD_x + \dfrac{1}{2}\dfrac{\partial^2 b}{\partial x^2}(0,y)x^2\eta + c(0,y).
\end{cases}
\tag{98}
$$

It is convenient to rewrite $A^{(3)}$ and $A^{(4)}$ in terms of the creation/annihilation operators

$$
\frac{d}{dx} \mp \mu|\eta|x,
\tag{99}
$$

keeping into account that for all $k\in\mathbb{Z}_+$

$$
\Big(\frac{d}{dx} - \mu|\eta|x\Big)f_k(\eta;x) = \sqrt{2(k+1)\mu|\eta|}\, f_{k+1}(\eta;x),
$$

$$
\Big(\frac{d}{dx} + \mu|\eta|x\Big)f_k(\eta;x) = -\sqrt{2k\mu|\eta|}\, f_{k-1}(\eta;x)
$$

(as usual, $f_{-1} \overset{\text{def}}{=} 0$). Hence a computation gives

$$
A^{(3)}_{(y,\pm\eta<0)} = p\Big(\frac{d}{dx} - \mu|\eta|x\Big) - q\sqrt{\mu|\eta|}\,x,
\tag{100}
$$

where

$$
p = \frac{1}{i}a(0,y), \qquad q = i\sqrt{\mu|\eta|}\,a(0,y) \pm \sqrt{\frac{|\eta|}{\mu}}\,\frac{\partial b}{\partial x}(0,y),
\tag{101}
$$

and

$$
A^{(4)}_{(y,\pm\eta<0)} = \alpha\sqrt{\mu|\eta|}\,x\Big(\frac{d}{dx} - \mu|\eta|x\Big) - \beta(\sqrt{\mu|\eta|}\,x)^2 + \gamma,
\tag{102}
$$

where

$$
\alpha = \frac{1}{i\sqrt{\mu|\eta|}}\frac{\partial a}{\partial x}(0,y), \qquad \beta = i\frac{\partial a}{\partial x}(0,y) \pm \frac{1}{2\mu}\frac{\partial^2 b}{\partial x^2}(0,y), \quad \gamma = c(0,y).
\tag{103}
$$

Note that

$$
(A^{(3)}_{(y,\pm\eta<0)})^* = -\bar{p}\Big(\frac{d}{dx} + \mu|\eta|x\Big) - \bar{q}\sqrt{\mu|\eta|}\,x.
$$

We next compute $\Lambda_0(y,\pm\eta<0)$. We have:

$$
\Big(\alpha\sqrt{\mu|\eta|}\,x\Big(\frac{d}{dx} - \mu|\eta|x\Big)f_j, f_j\Big)_{L^2(\mathbb{R})} =
$$

$$
= \alpha\sqrt{2(j+1)\mu|\eta|}\big(h_{j+1}, th_j\big)_{L^2(\mathbb{R})} = -\alpha(j+1)\sqrt{\mu|\eta|},
$$

because

$$
th_j = \frac{1}{2}\Big(\frac{d}{dt} + t\Big)h_j - \frac{1}{2}\Big(\frac{d}{dt} - t\Big)h_j = -\frac{1}{2}\Big(\sqrt{2j}h_{j-1} + \sqrt{2(j+1)}h_{j+1}\Big).
$$

Moreover,

$$
-\beta\Big((\sqrt{\mu|\eta|}\,x)^2 f_j, f_j\Big)_{L^2(\mathbb{R})} = -\beta\big(th_j, th_j\big)_{L^2(\mathbb{R})} = -\frac{2j+1}{2}\beta.
$$

In conclusion

$$\left(A^{(4)}_{(y,\pm\eta<0)}f_j, f_j\right)_{L^2(\mathbb{R})} = \gamma - \frac{2j+1}{2}\beta - (j+1)\alpha\sqrt{\mu|\eta|}. \tag{104}$$

On the other hand,

$$-\left(E^{(-2)}_{(y,\eta)}A^{(3)}_{(y,\pm\eta<0)}f_j(\eta;\cdot), (A^{(3)}_{(y,\pm\eta<0)})^* f_j(\eta;\cdot)\right)_{L^2(\mathbb{R})} = \tag{105}$$

$$= -\sum_{k\neq j}\frac{1}{\lambda_k(y,\pm\eta<0)}\left(A^{(3)}_{(y,\pm\eta<0)}f_j(\eta;\cdot), f_k(\eta;\cdot)\right)_{L^2}\overline{\left((A^{(3)}_{(y,\pm\eta<0)})^* f_j(\eta;\cdot), f_k(\eta;\cdot)\right)_{L^2}}.$$

Now

$$(A^{(3)}f_j, f_k)_{L^2} = p\left((\frac{d}{dx} - \mu|\eta|x)f_j, f_k\right)_{L^2} - q(\sqrt{\mu|\eta|}\,xf_j, f_k)_{L^2} = \tag{106}$$

$$= p\sqrt{2(j+1)\mu|\eta|}\,\delta_{j+1,k} - q(th_j, h_k)_{L^2} =$$

$$= \left[p\sqrt{2(j+1)\mu|\eta|} + q\frac{\sqrt{2(j+1)}}{2}\right]\delta_{j+1,k} + q\frac{\sqrt{2j}}{2}\delta_{j-1,k},$$

and

$$((A^{(3)})^* f_j, f_k)_{L^2} = -\bar{p}\left((\frac{d}{dx} + \mu|\eta|x)f_j, f_k\right)_{L^2} - \bar{q}(\sqrt{\mu|\eta|}\,xf_j, f_k)_{L^2} = \tag{107}$$

$$= \bar{p}\sqrt{2j\mu|\eta|}\,\delta_{j-1,k} - \bar{q}(th_j, h_k)_{L^2} =$$

$$= \bar{q}\frac{\sqrt{2(j+1)}}{2}\delta_{j+1,k} + \left[\bar{p}\sqrt{2j\mu|\eta|} + \bar{q}\frac{\sqrt{2j}}{2}\right]\delta_{j-1,k}.$$

Thus

$$-\left(E^{(-2)}A^{(3)}f_j, (A^{(3)})^* f_j\right)_{L^2} = \tag{108}$$

$$= -\sum_{k\neq j}\frac{1}{\lambda_k}\left\{\left[p\sqrt{2(j+1)\mu|\eta|} + q\frac{\sqrt{2(j+1)}}{2}\right]\delta_{j+1,k} +\right.$$

$$\left. + q\frac{\sqrt{2j}}{2}\delta_{j-1,k}\right\}\left\{q\frac{\sqrt{2(j+1)}}{2}\delta_{j+1,k} + \left[p\sqrt{2j\mu|\eta|} + q\frac{\sqrt{2j}}{2}\right]\delta_{j-1,k}\right\} =$$

$$= -\left(\frac{j+1}{\lambda_{j+1}} + \frac{j}{\lambda_{j-1}}\right)\left(\frac{q}{2} + p\sqrt{\mu|\eta|}\right)q.$$

Finally, for $(y,\eta) \in \Gamma^{\mp}_{\delta}$ we have that

$$\Lambda_1(y,\eta) = \lambda_j(y,\eta) = (2j+1)\mu|\eta| + b(0,y)\eta, \tag{109}$$

$$\Lambda_0(y,\eta) = \gamma - \frac{2j+1}{2}\beta - (j+1)\sqrt{\mu|\eta|}\,\alpha + \tag{110}$$

$$-\left(\frac{j+1}{\lambda_{j+1}} + \frac{j}{\lambda_{j-1}}\right)\left(\frac{q}{2} + \sqrt{\mu|\eta|}\right)q + i\frac{\partial\lambda_j}{\partial y}(y,\eta)\left(\frac{\partial f_j}{\partial\eta}(\eta;\cdot), f_j(\eta;\cdot)\right)_{L^2(\mathbb{R})}.$$

Since f_j is **real-valued** and $\|f_j(\eta;\cdot)\|^2_{L^2} = 1$, we have

$$\Big(\frac{\partial f_j}{\partial\eta}(\eta;\cdot), f_j(\eta;\cdot)\Big)_{L^2(\mathbb{R})} = 0, \tag{111}$$

so that the last term in Λ_0 vanishes.

Suppose now we are in situation $(II)_\pm$. Then λ_j vanishes identically on $\Gamma^\mp_\delta$. As a consequence, if $\Lambda_0(y=0,\pm\eta<0)\neq 0$ then $\Lambda(y,D_y)$ is hypoelliptic at $(y=0,\pm\eta<0)$ with loss of 1 derivative. Now, since

$$\lambda_{j+1}(0,\pm\eta<0) = 2\mu|\eta|, \qquad \lambda_{j-1}(0,\pm\eta<0) = -2\mu|\eta|,$$

we obtain

$$\Lambda_0(y=0,\pm\eta<0) = \gamma(0) - \frac{2j+1}{2}\beta(0) - (j+1)\sqrt{\mu|\eta|}\,\alpha(0,\pm\eta<0) +$$
$$-\frac{1}{2\mu|\eta|}\left(\frac{q(0,\pm\eta<0)}{2} + \sqrt{\mu|\eta|}\,p(0)\right) q(0,\pm\eta<0).$$

From (101) and (103) we have

$$\gamma(0) - \frac{2j+1}{2}\beta(0) - (j+1)\sqrt{\mu|\eta|}\,\alpha(0,\pm\eta<0) =$$
$$= c(0,0) + \frac{i}{2}\frac{\partial a}{\partial x}(0,0) \mp \frac{2j+1}{4\mu}\frac{\partial^2 b}{\partial x^2}(0,0),$$

and

$$-\frac{1}{2\mu|\eta|}\left(\frac{q(0,\pm\eta<0)}{2} + \sqrt{\mu|\eta|}\,p(0)\right) q(0,\pm\eta<0) = -\frac{1}{2\mu|\eta|}\Big(\frac{i}{2}\sqrt{\mu|\eta|}\,a(0,0) +$$
$$\pm\frac{1}{2}\sqrt{\frac{|\eta|}{\mu}}\frac{\partial b}{\partial x}(0,0) - i\sqrt{\mu|\eta|}\,a(0,0)\Big)\Big(i\sqrt{\mu|\eta|}\,a(0,0) \pm \sqrt{\frac{|\eta|}{\mu}}\frac{\partial b}{\partial x}(0,0)\Big) =$$
$$= \frac{1}{4}\left(ia(0,0) \mp \frac{1}{\mu}\frac{\partial b}{\partial x}(0,0)\right)\left(ia(0,0) \pm \frac{1}{\mu}\frac{\partial b}{\partial x}(0,0)\right) =$$
$$= -\frac{1}{4}\Big(a(0,0)^2 + \frac{1}{\mu^2}\frac{\partial b}{\partial x}(0,0)^2\Big).$$

At last, when $(II)_\pm$ holds, we have

$$\Lambda_0(0,\pm\eta<0) = \tag{112}$$
$$= c(0,0) + \frac{i}{2}\frac{\partial a}{\partial x}(0,0) \mp \frac{2j+1}{4\mu}\frac{\partial^2 b}{\partial x^2}(0,0) - \frac{1}{4}\Big(a(0,0)^2 + \frac{1}{\mu^2}\frac{\partial b}{\partial x}(0,0)^2\Big) =: T_\pm.$$

We have finally proved the following proposition.

Proposition 6.2. *If $(II)_\pm$ holds and $T_\pm \neq 0$, then A is hypoelliptic at $(0,0)$ with loss of 2 derivatives.*

We next pass to considering the case in which $(I)_\pm$ holds. There are two quite different possibilities:

(a) $b(0,0) = \pm(2j+1)\mu$ *and* $\frac{d\lambda_j}{dy}(0, \pm\eta < 0) \neq 0$*;*

(b) $\frac{d^\ell \lambda_j}{dy^\ell}(0, \pm\eta < 0) = 0$, $\ell = 0, 1, \dots, r-1$, *and* $\frac{d^r \lambda_j}{dy^r}(0, \pm\eta < 0) \neq 0$ *for some* $r \geq 2$.

In any event, note that

$$\Lambda_1(y, \eta) = \theta_\pm(y) y^r |\eta|,$$

where $\theta_\pm$ is smooth near $y = 0$ and

$$\theta_\pm(0) = \frac{1}{r!} \frac{d^r \lambda_j}{dy^r}(0, \mp 1) \neq 0.$$

Thanks to classical results on principal type operators (see [10], Vol.IV, Thm. 26.1.1) we have the following proposition.

Proposition 6.3. *Suppose* $(I)_\pm$ *and* **(a)** *hold. Then* A *is* **not** *hypoelliptic at* $(0,0)$*. In fact, one may find a distribution* $u_\pm \in \mathcal{D}'(\mathbb{R}^2)$ *such that* $Au_\pm \in C^\infty$ *at* $(0,0)$ *and* $(x = 0, y = 0, \xi = 0, \pm\eta < 0) \in \mathrm{WF}(u_\pm)$.

When **(b)** holds, the situation can be much more complicated. In particular, $\Lambda(y, D_y)$ can still be hypoelliptic at $(y = 0, \pm\eta < 0)$ with loss of 1 derivative, as the following proposition shows.

Proposition 6.4. *Suppose* $(I)_\pm$ *and* **(b)** *hold, and that the following conditions be satisfied*

$$\begin{cases} \tau + \dfrac{T_\pm}{\theta_\pm(0)} \neq 0, \ \forall \tau \geq 0, & \textit{when } r \textit{ is even,} \\ \\ \tau + \dfrac{T_\pm}{\theta_\pm(0)} \neq 0, \ \forall \tau \in \mathbb{R}, & \textit{when } r \textit{ is odd.} \end{cases} \tag{113}$$

Then A *is hypoelliptic at* $(0,0)$ *with loss of 2 derivatives.*

Proof. The symbol of $\frac{1}{\theta_\pm(y)}\Lambda(y, D_y)$ is, modulo $S^{-1/2}$, of the form

$$y^r|\eta| + \frac{1}{\theta_\pm(y)}\Lambda_0(y, \pm\eta < 0). \tag{114}$$

The hypothesis ensures that

$$\left| y^r|\eta| + \frac{T_\pm}{\theta_\pm(0)} \right| \geq \text{const.} > 0, \ \forall y \in \mathbb{R}, \ \forall \eta \neq 0. \tag{115}$$

Since $\Lambda_0(0, \pm\eta < 0) = T_\pm$, there exist $0 < \delta' < \delta$ sufficiently small and $c > 0$ such that, with $\sigma(\frac{1}{\theta_\pm(y)}\Lambda(y, D_y)) =: g(y, \eta)$, we have

$$|g(y, \eta)| \geq c, \ \forall y, |y| \leq \delta', \ \forall \eta, \pm\eta < 0, |\eta| \geq \frac{1}{\delta'}. \tag{116}$$

On the other hand, it is easy to show that for all $p, q \in \mathbb{Z}_+$

$$\left|\frac{\partial_y^p \partial_\eta^q g(y,\eta)}{g(y,\eta)}\right| \le C_{pq}|\eta|^{-q+p/r}, \ \forall y, |y| \le \delta', \ \forall \eta, \pm\eta < 0, \ |\eta| \ge \frac{1}{\delta'}.$$

As a consequence, $\Lambda(y, D_y)$ has a microlocal two-sided parametrix in the (microlocal) Hörmander class $\mathrm{OPS}^0_{1,1/r}(\mathbb{R})$, which ensures the hypoellipticity of $\Lambda(y, D_y)$ at $(y = 0, \pm\eta < 0)$ with loss of 1 derivative, and hence that of A at $(0,0)$ with loss of 2 derivatives. □

Remark 6.5. *1) Note that if* $a(0,0) = \frac{\partial a}{\partial x}(0,0) = \frac{\partial b}{\partial x}(0,0) = \frac{\partial^2 b}{\partial x^2}(0,0) = 0$, *then* $T_\pm = c(0,0)$, *so that in Proposition 6.2 there is hypoellipticity whenever* $c(0,0) \ne 0$.

2) In Proposition 6.4, the hypothesis is obviously satisfied if $\mathrm{Im}(\frac{T_\pm}{\theta_\pm(0)}) \ne 0$.

When $T_\pm = 0$, *the hypoellipticity or non-hypoellipticity of* $\Lambda(y, D_y)$ *is decided by a precise knowledge of the terms* $\Lambda_{-1/2}, \Lambda_{-1}, \ldots$.

Remark 6.6. *We can consider the seemingly more general case of an operator in* $\mathbb{R}^2$

$$A' = D_{x'}^2 + 2\alpha x' D_{x'} D_{y'} + \beta x'^2 D_{y'}^2 + p(x',y')D_{x'} + q(x',y')D_{y'} + r(x',y'),$$

$p, q, r \in C^\infty(\mathbb{R}^2; \mathbb{C})$, $\alpha, \beta \in \mathbb{R}$ *with* $\beta > \alpha^2$.
In fact, by the change of variables $x = x'$, $y = y' - \alpha x'^2/2$, A' *goes over to the operator* A *defined in (92), with* $\mu^2 = \beta - \alpha^2$ *and*

$$\begin{cases} a(x,y) = p(x, y + \alpha x^2/2), \\ b(x,y) = q(x, y + \alpha x^2/2) - x\alpha p(x, y + \alpha x^2/2) + i\alpha, \\ c(x,y) = r(x, y + \alpha x^2/2). \end{cases}$$

6.2. **Example.** Consider in $\mathbb{R}^n = \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ $(n \ge 3)$ the following 4th-order differential operator

$$A = M^2 + aD_x^2 + \langle b, D_y\rangle D_x + \langle cD_y, D_y\rangle + \alpha D_x + \langle \beta, D_y\rangle + \gamma, \tag{117}$$

where

$$M = D_x^2 + \mu^2 x^2 |D_y|^2, \quad \mu > 0,$$

$a, \alpha, \gamma \in C^\infty(\mathbb{R}^n; \mathbb{C})$, $b, \beta \in C^\infty(\mathbb{R}^n; \mathbb{C}^{n-1})$, $c = {}^t c$ is a smooth $(n-1)\times(n-1)$ symmetric complex matrix.

Again, we consider the hypoellipticity of A at the origin $(0,0)$. The localized operator $A^{(4)}_{(y,\eta)}$ is

$$A^{(4)}_{(y,\eta)} = \Big(D_x^2 + \mu^2 x^2 |\eta|^2\Big)^2 + \langle c(0,y)\eta, \eta\rangle. \tag{118}$$

For simplicity suppose that $c(0, y)$ be **self-adjoint**, so that $A^{(4)}_{(y,\eta)} = (A^{(4)}_{(y,\eta)})^*$. Of course,

$$\mathrm{Spec}(A^{(4)}_{(y,\eta)}) = \{\lambda_k(y, \eta);\ k \in \mathbb{Z}_+\},$$

where

$$\lambda_k(y, \eta) = \langle q_k(y)\eta, \eta\rangle, \quad q_k(y) := c(0, y) + (2k + 1)^2\mu^2 I_{n-1}. \tag{119}$$

The λ_k have multiplicity 1 and the corresponding normalized eigenfunctions are the $f_k(\eta; x)$ defined in (94) (now $\eta \in \mathbb{R}^{n-1} \setminus \{0\}$).

Put

$$I := \{k \in \mathbb{Z}_+;\ q_k(0) \ \text{ is either positive-definite or negative-definite}\},$$

and let

$$J := \mathbb{Z}_+ \setminus I.$$

Notice that J is a finite, possibly empty, set. The Boutet-Grigis-Helffer Theorem 1.4 then says that A *is hypoelliptic at* $(0, 0)$ *with loss of 2 derivatives iff* $J = \emptyset$ (note that $\mathbb{R}^{n-1}_\eta \setminus \{0\}$ is connected).

Suppose now $J \neq \emptyset$. There is a case which is trivial to handle, namely when for some $k \in J$ and some $\eta_0 \neq 0$ we have

$$\lambda_k(0, \eta_0) = 0, \quad \nabla_\eta\lambda_k(0, \eta_0) = 2q_k(0)\eta_0 \neq 0. \tag{120}$$

If this happens, we have the following result.

Proposition 6.7. *Suppose (120) holds. Then A is* **not** *hypoelliptic at* $(0, 0)$*. Furthermore, upon denoting by $\psi(t) = (y(t), \eta(t))$, $t \in \mathbb{R}$, the integral curves of $H_{\lambda_k} = (\nabla_\eta\lambda_k, -\nabla_y\lambda_k)$ with $\psi(0) = (0, \eta_0)$, then for any given $u \in \mathcal{D}'(\mathbb{R}^n)$ for which*

$$(x = 0, y = 0, \xi = 0, \eta = \eta_0) \in \mathrm{WF}(u) \setminus \mathrm{WF}(Au),$$

one has

$$(x = 0, y = y(t), \xi = 0, \eta = \eta(t)) \in \mathrm{WF}(u) \setminus \mathrm{WF}(Au),$$

for all t sufficiently small.

Proof. For all $k' \neq k$ we have $\lambda_{k'}(0, \eta_0) \neq 0$, for

$$\lambda_k(y, \eta) - \lambda_{k'}(y, \eta) = \Big((2k + 1)^2 - (2k' + 1)^2\Big)\mu^2|\eta|^2. \tag{121}$$

We may hence find a conic neighborhood Γ of $(0, \eta_0)$ on which only λ_k vanishes and $H_{\lambda_k} \neq 0$. The corresponding operator $\Lambda(y, D_y) \in \mathrm{OPS}^2_{\mathrm{cl}}(\mathbb{R}^{n-1})$ is therefore of real principal type in Γ by virtue of the fact that $\Lambda_{3/2}(y, \eta) = 0$ on Γ. Once more, the conclusion follows from Thm. 26.1.1 in Hörmander's book [10], Vol. IV, and Theorem 5.2 above. □

Notice that a possible example is $\lambda_k(0, \eta) = \eta_1^2 - \eta_2^2$ in $\mathbb{R}^3_\eta$.

We are thus left with considering the case in which the functions $0 \neq \eta \longmapsto \lambda_k(0,\eta)$, $k \in J$, have only zeros of multiplicity 2, that is, we may suppose

(122)
$$k \in J \implies \begin{cases} q_k(0) \ \text{ is either positive-semidefinite or negative-semidefinite,} \\ \\ \text{Ker } q_k(0) \neq \{0\}. \end{cases}$$

Hence,

(123) $$\text{either } \ J = \{k\}, \quad \text{or } \ J = \{k, k+1\}, \quad \text{for some } k \in \mathbb{Z}_+,$$

and in the latter case we necessarily have, by (121),

$$q_k(0) \leq 0, \quad q_{k+1}(0) \geq 0.$$

It is important to observe that the sole semi-definiteness of $q_k(0)$ does not give sufficient information on the zeros of $\lambda_k(y,\eta)$ for y near 0.

To focus on one possible case, let us assume that there exists a neighborhood $U \subset \mathbb{R}^{n-1}$ of the origin and for any given $k \in J$ a smooth submanifold $\Sigma_k \subset U \times (\mathbb{R}^{n-1} \setminus \{0\})$ such that $\lambda_k(y,\eta)$, $(y,\eta) \in U \times (\mathbb{R}^{n-1} \setminus \{0\})$, vanishes **exactly** to second order on Σ_k.

A crucial observation is that, by possibly shrinking U, we may suppose $\Sigma_{k'} \cap \Sigma_k = \emptyset$ when $k \neq k'$ (again by virtue of (121)). Hence, we are reduced to studying the hypoellipticity of A at a point $(x = 0, y = 0, \xi = 0, \eta = \eta_0)$ with $(0,\eta_0) \in \Sigma_k$ for a unique $k \in J$. One can therefore find a conic neighborhood $\Gamma \subset T^*\mathbb{R}^{n-1} \backslash 0$ of $(0,\eta_0)$ where only λ_k vanishes. Once again, we microlocally construct $\Lambda(y, D_y) \in \mathrm{OPS}^2_{\mathrm{cl}}(\mathbb{R}^{n-1})$ with (now $m = k = 4$)

$$\begin{cases} \Lambda_2(y,\eta) = \lambda_k(y,\eta), \\ \\ \Lambda_{3/2}(y,\eta) = 0, \\ \\ \Lambda_1(y,\eta) = \Big(A^{(6)}_{(y,\eta)} f_k(\eta;\cdot), f_k(\eta;\cdot)\Big)_{L^2(\mathbb{R})} + \\ \qquad\qquad - \Big(E^{(-4)}_{(y,\eta)} A^{(5)}_{(y,\eta)} f_k(\eta;\cdot), (A^{(5)}_{(y,\eta)})^* f_k(\eta;\cdot)\Big)_{L^2(\mathbb{R})} + \\ \qquad\qquad + i \displaystyle\sum_{j=1}^{n-1} \frac{\partial \lambda_k}{\partial y_j}(y,\eta) \Big(\frac{\partial f_k}{\partial \eta_j}(\eta;\cdot), f_k(\eta;\cdot)\Big)_{L^2(\mathbb{R})} + \\ \qquad\qquad + \displaystyle\frac{1}{i} \sum_{j=1}^{n-1} \Big(\frac{\partial A^{(4)}_{(y,\eta)}}{\partial \eta_j} \frac{\partial f_k}{\partial y_j}(\eta;\cdot), f_k(\eta;\cdot)\Big)_{L^2(\mathbb{R})}, \end{cases}$$

where

(124) $$A^{(5)}_{(y,\eta)} = \langle b(0,y), \eta\rangle D_x + \langle \frac{\partial c}{\partial x}(0,y)\eta, \eta\rangle x = \\ = \ell(y,\eta)\left(\frac{d}{dx} - \mu|\eta| x\right) + m(y,\eta)\sqrt{\mu|\eta|}\, x,$$

(125) $$\begin{cases} \ell(y,\eta) = \dfrac{1}{i}\langle b(0,y),\eta\rangle, \\ m(x,y) = \dfrac{1}{i}\langle b(0,y),\eta\rangle\sqrt{\mu|\eta|} + \dfrac{1}{\sqrt{\mu|\eta|}}\langle \dfrac{\partial c}{\partial x}(0,y)\eta,\eta\rangle, \end{cases}$$

(126)
$$A^{(6)}_{(y,\eta)} = a(0,y)D_x^2 + \langle \frac{\partial b}{\partial x}(0,y),\eta\rangle x D_x + \frac{1}{2}\langle \frac{\partial^2 c}{\partial x^2}(0,y)\eta,\eta\rangle x^2 + \langle \beta(0,y),\eta\rangle =$$
$$= P(y)\left(\frac{d}{dx} - \mu|\eta|x\right)^2 + Q(y,\eta)\sqrt{\mu|\eta|}\,x\left(\frac{d}{dx} - \mu|\eta|x\right) +$$
$$+R(y,\eta)(\sqrt{\mu|\eta|}\,x)^2 + S(y,\eta),$$

(127) $$\begin{cases} P(y) = -a(0,y), \\ Q(y,\eta) = \dfrac{1}{i\sqrt{\mu|\eta|}}\langle \dfrac{\partial b}{\partial x}(0,y),\eta\rangle - 2a(0,y)\sqrt{\mu|\eta|}, \\ R(y,\eta) = -a(0,y)\mu|\eta| + \dfrac{1}{i}\langle \dfrac{\partial b}{\partial x}(0,y),\eta\rangle + \dfrac{1}{2\mu|\eta|}\langle \dfrac{\partial^2 c}{\partial x^2}(0,y)\eta,\eta\rangle, \\ S(y,\eta) = \langle \beta(0,y),\eta\rangle - a(0,y)\mu|\eta|. \end{cases}$$

Notice that exactly as in Example 6.1, we have
$$(\frac{\partial f_k}{\partial \eta_j}, f_k)_{L^2(\mathbb{R})} = 0, \quad 1 \le j \le n-1,$$
and that the last term in Λ_1 vanishes for f_k does not depend on y.
By analogous computations as in Example 6.1, we obtain
(128)
$$\Lambda_1(y,\eta) = \left(\frac{k}{\lambda_{k-1}(y,\eta)} + \frac{k+1}{\lambda_{k+1}(y,\eta)}\right)\Big(\sqrt{\mu|\eta|}\,m(y,\eta)\ell(y,\eta) - \frac{m(y,\eta)^2}{2}\Big) +$$
$$-(k+1)\sqrt{\mu|\eta|}\,Q(y,\eta) + \frac{2k+1}{2}R(y,\eta) + S(y,\eta).$$
To conclude about hypoellipticity of $\Lambda(y,D_y)$ with loss of 1 derivative at $(y=0,\eta=\eta_0)$, we just need to apply Thm. 22.4.15 of Hörmander [10], Vol. III, bearing in mind that
$$\text{either}\quad \lambda_k(y,\eta) \ge 0, \quad \text{or}\quad \lambda_k(y,\eta) \le 0,$$
and in all cases it vanishes exactly to second order on Σ_k.

Put
$$\mathrm{Spec}(F_{\lambda_k}(0,\eta_0)) \setminus \{0\} = \{\pm i\theta_\ell;\ \theta_\ell > 0,\ \ell = 1,\dots,r\}$$
$(r \ge 0)$. Then the hypoellipticity condition reads as

(129) $$\Lambda_1^s(0,\eta_0) \pm \sum_{\ell=1}^{r}(2h_\ell+1)\theta_\ell + \sigma(v, F_{\lambda_k}(0,\eta_0)v) \ne 0$$

for all $h \in \mathbb{Z}_+^r$, for all $v \in \mathrm{Ker}\,(F_{\lambda_k}(0,\eta_0)^2)$, Λ_1^s being the subprincipal symbol of Λ, and $\sigma(\cdot,\cdot)$ the canonical symplectic form. In (129) we take $+\sum(2h_\ell+1)\theta_\ell$, respectively $-\sum(2h_\ell+1)\theta_\ell$, according to whether $\lambda_k \geq 0$ or $\lambda_k \leq 0$, respectively.

Remark 6.8. *We have supposed that $\lambda_k(y,\eta)$, $k \in J$, vanishes exactly to second order on some submanifold Σ_k. Of course, there are many more different possibilities. We put in evidence a few of them.*

1) *$\lambda_k(y,\eta) = 0$ identically for y near 0. As in Example 6.1 one has that $\Lambda(y,D_y)$ is hypoelliptic at $y=0$ with loss of 1 derivative provided that $\Lambda_1(0,\eta) \neq 0$ for all $\eta \neq 0$.*
2) *For y near 0, $\lambda_k(y,\eta) = |y|^{2p}\langle\Phi(y)\eta,\eta\rangle$, for some integer $p \geq 1$ and some smooth matrix $\Phi(y) = \Phi(y)^* > 0$.*
   In this case the condition
   $$\Lambda_1(0,\frac{\eta}{|\eta|}) + \tau \neq 0, \quad \forall \eta \neq 0,\ \forall \tau \geq 0,$$
   implies that $\Lambda(y,D_y)$ has a microlocal parametrix in $\mathrm{OPS}^{-1}_{1,\frac{1}{2p}}(\mathbb{R}^{n-1})$, which yields again the hypoellipticity of Λ with loss of 1 derivative. (The proof goes as in the corresponding case of Example 6.1.)
3) *For y near 0,*
   $$\text{either}\ \ \lambda_k(y,\eta) = \eta_1^2 - y_1^{2p}|\eta'|^2, \quad \text{or}\ \ \lambda_k(y,\eta) = \eta_1^2 - y_1^{2p-1}|\eta'|^2,$$
   $\eta' = (\eta_2,\dots,\eta_{n-1})$, for some integer $p \geq 1$. In either case, we have that $\Lambda(y,D_y)$, and therefore A, is **not** *hypoelliptic at 0, and a propagation of singularities occurs in the characteristic set of A (see Ivrii* [11]*, and Hörmander* [10]*, Vol. III, Section 23.4).*

6.3. **Example.** In the examples above, the characteristic manifold Σ had codimension 2. We now wish to treat Grushin-type operators with Σ of codimension greater than 2. In this case $\Lambda(y,D_y)$ can be a genuine system.

Let $n \geq 4$, $2 \leq \nu < n$, and let $Y \subset \mathbb{R}^{n-\nu}$ be open. On $\mathbb{R}^\nu \times Y$ consider the following operator

$$A = \sum_{j,k=1}^{\nu}\Big[P_{jk}(y,D_y)x_jx_k + Q_{jk}(y,D_y)x_kD_{x_j} + R_{jk}(y,D_y)D_{x_k}D_{x_j}\Big] + T(y,D_y),$$

where $P_{jk} = P_{kj}$, $R_{jk} = R_{kj}$, Q_{jk} and T are linear differential operators in Y, with smooth coefficients, of order m, $m-2$, $m-1$, $m-1$, respectively.

For $(y,\eta) \in T^*Y \setminus 0$, denote by $p(y,\eta)$, $r(y,\eta)$, $q(y,\eta)$, respectively, the $\nu\times\nu$ matrices of the principal symbols of P_{jk}, R_{jk} and Q_{jk}, respectively, and likewise denote by $t(y,\eta)$ the principal symbol of T. Remark that the above matrices are **invariantly defined** on T^*Y.

We suppose that p, r, q are **all real matrices**, and that the quadratic form

$$\left\langle \begin{bmatrix} p(y,\eta) & \frac{1}{2}\,{}^t q(y,\eta) \\ \frac{1}{2} q(y,\eta) & r(y,\eta) \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}, \begin{bmatrix} x \\ \xi \end{bmatrix} \right\rangle_{\mathbb{R}^{2\nu}}$$

is positive definite for all $(y,\eta) \in T^*Y \setminus 0$.

As a consequence, the principal symbol $a(x,y,\xi,\eta)$ of A satisfies

$$a(x,y,\xi,\eta) \lesssim |\eta|^m \left(|x|^2 + \frac{|\xi|^2}{|\eta|^2} \right), \quad \text{and} \quad |\eta|^m \left(|x|^2 + \frac{|\xi|^2}{|\eta|^2} \right) \lesssim a(x,y,\xi,\eta),$$

so that the characteristic manifold $\Sigma = \{(x,y,\xi,\eta) \in T^*(\mathbb{R}^\nu \times Y) \setminus 0;\ x = \xi = 0\}$, can be identified with $T^*Y \setminus 0$.

For $(y,\eta) \in T^*Y \setminus 0$,

(130)

$$A^{(2)}_{(y,\eta)}(x, D_x) = \sum_{j,k=1}^{\nu} \Big[p_{jk}(y,\eta) x_j x_k + q_{jk}(y,\eta) x_k D_{x_j} + r_{jk}(y,\eta) D_{x_k} D_{x_j} \Big] + t(y,\eta).$$

The problem is to determine the spectrum of $A^{(2)}_{(y,\eta)}$ as an unbounded operator in $L^2(\mathbb{R}^\nu)$ (with domain $B^2(\mathbb{R}^\nu)$). To this end, consider the fundamental matrix $F(y,\eta)$ restricted to $\mathbb{C} \otimes T_{(y,\eta)}\Sigma^\sigma \simeq \mathbb{C}^{2\nu}$

$$F(y,\eta) := \begin{bmatrix} \frac{1}{2} q(y,\eta) & r(y,\eta) \\ -p(y,\eta) & -\frac{1}{2}\,{}^t q(y,\eta) \end{bmatrix} : \mathbb{C}^{2\nu} \longrightarrow \mathbb{C}^{2\nu}.$$

It is well-known that

$$\mathrm{Spec}(F(y,\eta)) = \{\pm i\mu_1(y,\eta), \ldots, \pm i\mu_\nu(y,\eta)\},$$

for some $\mu_j(y,\eta) > 0$, $\mu_j(y,t\eta) = t^{m-1}\mu_j(y,\eta)$, $j = 1, \ldots, \nu$, for all $(y,\eta) \in T^*Y \setminus 0$, for all $t > 0$.

We make the following assumption

(131)

$$\begin{cases} A^{(2)}_{(y,\eta)} = (A^{(2)}_{(y,\eta)})^*, \quad \text{i.e.} \\ \qquad a'(y,\eta) := t(y,\eta) + \frac{i}{2} \mathrm{Tr}\, q(y,\eta) \in \mathbb{R},\ \forall (y,\eta) \in T^*Y \setminus 0, \\ \mu_j \in C^\infty(T^*Y \setminus 0; \mathbb{R}_+),\ j = 1, \ldots, \nu. \end{cases}$$

We therefore have (see [9]) that

$$\mathrm{Spec}(A^{(2)}_{(y,\eta)}) = \{\gamma_{(y,\eta)}(\alpha);\ \alpha \in \mathbb{Z}^\nu_+\},$$

with

$$\gamma_{(y,\eta)}(\alpha) = 2\langle \alpha, \mu(y,\eta) \rangle_{\mathbb{R}^\nu} + \mathrm{Tr}^+ F(y,\eta) + a'(y,\eta). \tag{132}$$

Let $\rho_0 = (y_0, \eta_0) \in T^*Y \setminus 0$, and suppose that

$$J := \{\alpha \in \mathbb{Z}_+^\nu;\ \gamma_{\rho_0}(\alpha) = 0\} \ \text{ has } \ \mathrm{card}(J) = d \geq 1$$

(if $d > 1$, the $\mu_j(\rho_0)$, $1 \leq j \leq d$, are necessarily rationally dependent).

We further assume that there exist:

- A conic neighborhood $\Gamma \subset T^*Y \setminus 0$ of ρ_0;
- Functions $\psi_1, \dots, \psi_d \in C^\infty(\Gamma; \mathcal{S}(\mathbb{R}^\nu))$ such that

$$\psi_j(y, t\eta; t^{-1/2}x) = t^{\nu/4}\psi_j(y, \eta; x), \quad \Big(\psi_j(y, \eta; \cdot), \psi_{j'}(y, \eta; \cdot)\Big)_{L^2(\mathbb{R}^\nu)} = \delta_{jj'},$$

  for all $1 \leq j, j' \leq d$, $(y, \eta) \in \Gamma$, $t > 0$, $x \in \mathbb{R}^\nu$.
- A bijection $\theta\colon \{1, 2, \dots, d\} \longrightarrow J$ such that

$$A^{(2)}_{(y,\eta)}\psi_j(y, \eta; \cdot) = \lambda_j(y, \eta)\psi_j(y, \eta; \cdot),\ j = 1, \dots, d, (y, \eta) \in \Gamma,$$

  with

$$\lambda_j(y, \eta) := \gamma_{(y,\eta)}(\theta(j)).$$
- A smooth symplectic submanifold $\Sigma' \subset \Gamma$ of codimension $2\nu'$, $1 \leq \nu' < n - \nu$, $\rho_0 \in \Sigma'$, such that every λ_j is either ≥ 0 of ≤ 0 in Γ, and vanishes exactly to second order on Σ'.

For $(y, \eta) \in \Sigma'$, let $F_j(y, \eta)\colon \mathbb{C} \otimes (T_{(y,\eta)}\Sigma')^\sigma \longrightarrow \mathbb{C} \otimes (T_{(y,\eta)}\Sigma')^\sigma$, $1 \leq j \leq d$, be the fundamental matrix of λ_j at (y, η), $j = 1, \dots, d$. Put

$$\mathrm{Spec}(F_j(y, \eta)) = \{\pm i h_1^{(j)}(y, \eta), \dots, \pm i h_{\nu'}^{(j)}(y, \eta)\},$$

where $h_\ell^{(j)}(y, \eta) > 0$, $\ell = 1, \dots, \nu'$, and define for $(y, \eta) \in \Sigma'$ and $\beta = (\beta^{(1)}, \dots, \beta^{(d)}) \in (\mathbb{Z}_+^{\nu'})^d$, the matrix

$$\Delta_\beta(y, \eta) = \mathrm{diag}\Big(\epsilon_j \langle \beta^{(j)}, h^{(j)}(y, \eta)\rangle_{\mathbb{R}^{\nu'}} + \epsilon_j \mathrm{Tr}^+ F_j(y, \eta)\Big)_{j=1,\dots,d}, \tag{133}$$

where $\epsilon_j = 1$, resp. $\epsilon_j = -1$, if $\lambda_j \geq 0$, resp. $\lambda_j \leq 0$.

Our purpose is now to prove the following result.

Proposition 6.9. *In the above hypotheses, there exists a smooth $d \times d$ matrix $\gamma(y, \eta)$, $(y, \eta) \in \Sigma'$ (see (138) below), such that A is hypoelliptic at ρ_0 with loss of 2 derivatives iff for all $\beta = (\beta^{(1)}, \dots, \beta^{(d)}) \in (\mathbb{Z}_+^{\nu'})^d$ the matrix*

$$\Delta_\beta(\rho_0) + \gamma(\rho_0)$$

is **invertible**.

Proof. We start out by observing that

$$\sigma_{m-1}(A)(x, y, \xi, \eta) = \langle \sigma_{m-1}(P)(y, \eta)x, x\rangle + \langle \sigma_{m-2}(Q)(y, \eta)x, \xi\rangle + \\ + \langle \sigma_{m-3}(R)(y, \eta)\xi, \xi\rangle + t(y, \eta), \tag{134}$$

$$\sigma_{m-2}(A)(x, y, \xi, \eta) = \langle \sigma_{m-2}(P)(y, \eta)x, x\rangle + \langle \sigma_{m-3}(Q)(y, \eta)x, \xi\rangle + \\ + \langle \sigma_{m-4}(R)(y, \eta)\xi, \xi\rangle + \sigma_{m-2}(T)(y, \eta), \tag{135}$$

$(x, \xi) \in T^*\mathbb{R}^\nu$, $(y, \eta) \in T^*Y \setminus 0$.

As for the localized operators $A^{(3)}$, and $A^{(4)}$, we have

$$\begin{cases} A^{(3)}_{(y,\eta)}(x,\xi)=0, \\ A^{(4)}_{(y,\eta)}(x,\xi)=\langle\sigma_{m-1}(P)(y,\eta)x,x\rangle+\langle\sigma_{m-2}(Q)(y,\eta)x,\xi\rangle+ \\ \qquad\qquad +\langle\sigma_{m-3}(R)(y,\eta)\xi,\xi\rangle+\sigma_{m-2}(T)(y,\eta). \end{cases}$$

For the corresponding $d\times d$ (microlocal) system $\Lambda(y,D_y)$ of order $m-1$ in Y we have from (90)

$$\Lambda_{m-1}(y,\eta)=\mathrm{diag}\Big(\lambda_j(y,\eta)\Big)_{j=1,\dots,d},\quad \Lambda_{m-3/2}(y,\eta)=0,$$

and

$$\begin{aligned}\Lambda^{(j'j)}_{m-2}(y,\eta)&=\Big(A^{(4)}_{(y,\eta)}(x,D_x)\psi_j(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &+i\sum_{\ell=1}^{n-\nu}\frac{\partial\lambda_j}{\partial y_\ell}(y,\eta)\Big(\frac{\partial\psi_j}{\partial\eta_\ell}(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &+\Big(A^{(2)}_{(y,\eta)}(x,D_x)({\psi'}_j)_{-1,1}(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &+\frac{1}{i}\sum_{\ell=1}^{n-\nu}\Big(\frac{\partial A^{(2)}_{(y,\eta)}}{\partial\eta_\ell}(x,D_x)\frac{\partial\psi_j}{\partial y_\ell}(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}.\end{aligned}$$

A crucial remark is that whenever $(y,\eta)\in\Sigma'$, we have

$$\begin{aligned}\Lambda^{(j'j)}_{m-2}(y,\eta)&=\Big(A^{(4)}_{(y,\eta)}(x,D_x)\psi_j(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}+\\ &+\frac{1}{i}\sum_{\ell=1}^{n-\nu}\Big(\frac{\partial A^{(2)}_{(y,\eta)}}{\partial\eta_\ell}(x,D_x)\frac{\partial\psi_j}{\partial y_\ell}(y,\eta;\cdot),\psi_{j'}(y,\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)},\ 1\le j',j\le d.\end{aligned}\tag{136}$$

Fortunately enough, we know what the necessary and sufficient conditions for the hypoellipticity of $\Lambda(y,D_y)$ at $(y_0,\eta_0)\in\Sigma'$ with loss of 1 derivative are, thanks to the Boutet-Grigis-Helffer Theorem in the system-case (see [4], and also [2]). Namely, the conditions read:

The matrix

$$\Lambda_{m-2}(y_0,\eta_0)+\frac{i}{2}(\langle\partial_y,\partial_\eta\rangle\Lambda_{m-1})(y_0,\eta_0)+\Delta_\beta(y_0,\eta_0)\tag{137}$$

is invertible for all $\beta\in(\mathbb{Z}_+^{\nu'})^d$.

This concludes the proof, once we define for $(y,\eta)\in\Sigma'$

$$\gamma(y,\eta):=\Lambda_{m-2}(y,\eta)+\frac{i}{2}(\langle\partial_y,\partial_\eta\rangle\Lambda_{m-1})(y,\eta).\tag{138}$$

□

Remark 6.10. *We have supposed that the eigenvalues* $\lambda_1, \dots, \lambda_d$ *vanish on the* **same** *manifold* Σ'.

What happens (in case $d > 1$*) if the* λ_j *vanish exactly to second order (for instance) on different* Σ'_j *(with* $(y_0, \eta_0) \in \bigcap_{j=1}^{d} \Sigma'_j$*)?*

To our knowledge, the conditions on Λ_{m-2} *and the geometry of the* Σ'_j *which ensure the hypoellipticity of* $\Lambda(y, D_y)$ *have not yet been found.*

6.4. **Example.** We consider here a source of possible examples, all connected with the Heisenberg group (see also, for instance, Beals and Greiner [1]).

Consider in $\mathbb{R}^\nu_t \times \mathbb{R}^\nu_s \times \mathbb{R}_{y'}$ the vector fields

$$X_j = D_{t_j} + 2s_j D_{y'}, \quad Y_j = D_{s_j} - 2t_j D_{y'}, \ j = 1, \dots, \nu,$$

and recall that $X_j = X_j^*$, $Y_j = Y_j^*$ and

$$[X_j, Y_k] = 4i\delta_{jk} D_{y'}, \quad 1 \le j, k \le \nu.$$

It is well-known (see, e.g., [5]), that one can find a unitary Fourier integral operator F of order 0 such that

$$X_j F = F D_{x_j}, \quad Y_j F = F x_j D_y, \ j = 1, \dots, \nu,$$

so that

$$D_{y'} F = F\Big(-\frac{1}{4} D_y\Big).$$

Now, with any given polynomial (with complex coefficients) $p = p(t, s, y')$, we can associate the left-invariant differential operator $p(X, Y, D_{y'})$. Hence

$$p(X, Y, D_{y'})F = Fp(D_x, xD_y, -\frac{1}{4} D_y),$$

so that if p satisfies suitable algebraic assumptions, the differential operator $p(D_x, xD_y, -D_y/4)$ fits the framework we developed here, allowing us to give hypoellipticity results for $p(X, Y, D_{y'})$.

For instance, just to mention one possibility, consider

$$P = \sum_{j=1}^{\nu} (X_j^2 + \mu_j^2 Y_j^2) + \sum_{j=1}^{\nu} (\alpha_j X_j + \beta_j Y_j) + \gamma D_{y'} + \delta,$$

with $\mu_j > 0, \alpha_j, \beta_j, \gamma, \delta \in \mathbb{C}$, $j = 1, \dots, \nu$. Then $PF = FA$, where

$$A = \sum_{j=1}^{\nu} (D_{x_j}^2 + \mu_j^2 x_j^2 D_y^2) + \sum_{j=1}^{\nu} (\alpha_j D_{x_j} + \beta_j x_j D_y) - \frac{\gamma}{4} D_y + \delta.$$

Then for the localized operator $A^{(2)}_{(y,\eta)} = A^{(2)}_\eta = \sum_{j=1}^{\nu}(D^2_{x_j} + \mu_j^2 x_j^2\eta^2) - \frac{\gamma}{4}\eta$, we have

$$\mathrm{Spec}(A^{(2)}_\eta) = \{\sum_{j=1}^{\nu}(2k_j+1)\mu_j|\eta| - \frac{\gamma}{4}\eta;\ k \in \mathbb{Z}^\nu_+\}.$$

Hence, P is hypoelliptic with loss of 1 derivative iff

$$\frac{\gamma}{4} \notin \{\pm\sum_{j=1}^{\nu}(2k_j+1)\mu_j;\ k \in \mathbb{Z}^\nu_+\}.$$

Consider now the case in which there are $d \geq 1$ multiindices $k^{(1)}, \dots, k^{(d)} \in \mathbb{Z}^\nu_+$ such that

$$\text{either}\ \ \frac{\gamma}{4} = \sum_{j=1}^{\nu}(2k_j^{(\ell)}+1)\mu_j, \quad \ell = 1, \dots, d,$$

$$\text{or}\ \ \frac{\gamma}{4} = -\sum_{j=1}^{\nu}(2k_j^{(\ell)}+1)\mu_j, \quad \ell = 1, \dots, d.$$

For definiteness suppose the former case holds.

We want to obtain the hypoellipticity of A (and hence of P) with loss of 2 derivatives at every point $(x = 0, y, \xi = 0, \eta > 0)$. A gauge for the trivial bundle $V = V_1 = V_2$ is given by $\phi = (\phi_{k^{(1)}}, \dots, \phi_{k^{(d)}})$, with

$$\phi_{k^{(\ell)}} = \phi_{k^{(\ell)}}(\eta; x) = \prod_{j=1}^{\nu}(\mu_j|\eta|)^{1/4} h_{k_j^{(\ell)}}(\sqrt{\mu_j|\eta|}\,x_j),\ \ \ell = 1, \dots, d.$$

Now

$$\begin{cases} A^{(3)}_\eta = \sum_{j=1}^{\nu}(\alpha_j D_{x_j} + \beta_j x_j \eta), \\ \\ A^{(4)}_\eta = \delta, \end{cases}$$

hence, for the corresponding $\Lambda(y, D_y)$ we have (for $\eta > 0$)

$$\begin{cases} \Lambda_1(y,\eta) = 0, \\ \\ \Lambda_{1/2}(y,\eta) = 0, \\ \\ \Lambda_0(y,\eta) = \delta I_d - \Big(\Big(E^{(-2)}_\eta A^{(3)}_\eta \phi_{k^{(\ell')}}(\eta;\cdot), (A^{(3)}_\eta)^*\phi_{k^{(\ell)}}(\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}\Big)_{1\leq \ell,\ell' \leq d}. \end{cases}$$

By computations similar to those carried out in Example 6.1 we get

$$-\Big(E^{(-2)}_\eta A^{(3)}_\eta \phi_{k^{(\ell')}}(\eta;\cdot), (A^{(3)}_\eta)^*\phi_{k^{(\ell)}}(\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} =$$

$$= -\sum_{j,j'=1}^{\nu} \sum_{k\neq k^{(1)},\dots,k^{(d)}} \frac{1}{\lambda_k(\eta)} \Big[\Big(m_j\sqrt{2(k_j^{(\ell')}+1)\mu_j|\eta|} - \frac{r_j}{2}\sqrt{2(k_j^{(\ell')}+1)}\Big)\delta_{k^{(\ell')}+e_j,k} +$$

$$-\frac{r_j}{2}\sqrt{2k_j^{(\ell')}}\delta_{k^{(\ell')}-e_j,k}\Big]\Big[\Big(m_{j'}\sqrt{2k_{j'}^{(\ell)}\mu_{j'}|\eta|} - \frac{r_{j'}}{2}\sqrt{2k_{j'}^{(\ell)}}\Big)\delta_{k^{(\ell)}-e_{j'},k} +$$

$$-\frac{r_{j'}}{2}\sqrt{2(k_{j'}^{(\ell)}+1)}\delta_{k^{(\ell)}+e_{j'},k}\Big],$$

where

$$m_j = -i\alpha_j, \quad r_j = -i\alpha_j\sqrt{\mu_j|\eta|} + \beta_j\sqrt{\frac{|\eta|}{\mu_j}},$$

and

$$\lambda_k(\eta) = 2|\eta|\sum_{j=1}^{\nu}(k_j - k_j^{(\ell)})\mu_j,$$

which is **independent of** $\ell = 1,\dots,d$.

Notice that Λ_0 is a $d\times d$ complex matrix, which is independent of y and $\eta > 0$. We get that P is hypoelliptic with loss of 2 derivatives iff Λ_0 is invertible. Notice also that when $\alpha = \beta = 0$, the above condition reduces to $\delta \neq 0$, which recaptures, in the C^∞-hypoellipticity, the well-known example by Stein [20].

6.5. **Example: Iteration of the machinery.** In this last example, we revisit Example 6.2 to show a very simple instance in which the whole machinery can be iterated by micro-microlocalizing.

Consider in $\mathbb{R}_x \times \mathbb{R}_y^{n-1}$ ($n \geq 3$) the operator

$$A = (D_x^2 + \mu^2 x^2 |D_y|^2)^2 + (1-\mu^2)D_{y_1}^2 + (\sigma^2 y_1^2 - \mu^2)|D_{y'}|^2 + aD_x^2 + \gamma, \tag{139}$$

where $a, \gamma \in \mathbb{C}$ and $\mu, \sigma \in \mathbb{R}_+$. We shall prove the following (rather surprising) result.

Proposition 6.11. *For the hypoellipticity of A we have:*

- *(i) A is hypoelliptic with loss of 2 derivatives at every point $(x = 0, y_1, y', \xi = 0, \eta_1, \eta')$ for which $(y_1, \eta_1) \neq (0, 0)$.*
- *(ii) At every point of the submanifold $\Sigma' \subset \Sigma$, $\Sigma' = \{(x = 0, y_1 = 0, y', \xi = 0, \eta_1 = 0, \eta');\ y', \eta' \in \mathbb{R}^{n-2}, \eta' \neq 0\}$, A can be hypoelliptic only with a loss of derivatives ≥ 3, and in fact is hypoelliptic with loss of 3 derivatives iff*

$$a \neq -\frac{2\sigma}{\mu}(2j+1),\ \forall j \in \mathbb{Z}_+.$$

(*iii*) *If* $a = -\dfrac{2\sigma}{\mu}(2k+1)$ *for some* $k \in \mathbb{Z}_+$, *A can be hypoelliptic at the points of* Σ' *only with a loss of derivatives* ≥ 4, *and in fact is hypoelliptic with loss of 4 derivatives iff*

$$\gamma \neq \frac{(2k+1)^2\sigma^2}{4}\left(1+\frac{1}{3\mu^2}\right).$$

(*iv*) *If* $a = -\dfrac{2\sigma}{\mu}(2k+1)$ *and* $\gamma = \dfrac{(2k+1)^2\sigma^2}{4}\left(1+\dfrac{1}{3\mu^2}\right)$, *for some* $k \in \mathbb{Z}_+$, *A can be hypoelliptic at the points of* Σ' *only with a loss of derivatives* ≥ 5, *and in fact is hypoelliptic with loss of 5 derivatives iff*

$$\mu^4 \neq \frac{2}{9}\,\frac{(2k+1)^2}{k^2+k+2}.$$

Proof. From Example 6.2 we recall what the localized operators $A^{(j)}_{(y,\eta)}$, $j \geq 4$, are. Precisely one has

$$\begin{cases} A^{(4)}_{(y,\eta)} = (D_x^2 + \mu^2 x^2|\eta|^2)^2 + (1-\mu^2)\eta_1^2 + (\sigma^2 y_1^2 - \mu^2)|\eta'|^2, \\ A^{(5)}_{(y,\eta)} = 0, \\ A^{(6)}_{(y,\eta)} = aD_x^2 \\ A^{(7)}_{(y,\eta)} = 0, \\ A^{(8)}_{(y,\eta)} = \gamma, \\ A^{(j)}_{(y,\eta)} = 0, \ \forall j \geq 9. \end{cases}$$

Moreover, one has that $A^{(4)}_{(y,\eta)} = (A^{(4)}_{(y,\eta)})^*$ has spectrum made of the eigenvalues

$$\lambda_k(y,\eta) = \Big((4k^2+4k)\mu^2+1\Big)\eta_1^2 + \Big((4k^2+4k)\mu^2+\sigma^2 y_1^2\Big)|\eta'|^2, \ k \in \mathbb{Z}_+,$$

with corresponding normalized eigenfunctions

$$\phi_k(\eta;x) = (\mu|\eta|)^{1/4} h_k(\sqrt{\mu|\eta|}\,x), \ k \in \mathbb{Z}_+,$$

with h_k given in (95).

When $\rho \notin \Sigma'$, every $\lambda_k > 0$, whence the Boutet-Grigis-Helffer Theorem 1.4 gives (i) in the statement.

Suppose now that $\rho \in \Sigma'$. In this case the lowest eigenvalue λ_0 vanishes exactly to second order at ρ. According to our Theorem 5.1, we have to construct the scalar operator $\Lambda(y, D_y) \in \mathrm{OPS}^2_{\mathrm{cl}}(\mathbb{R}^{n-1})$. In order to prove the proposition, we have to compute the terms $\Lambda_{2-j/2}(y,\eta)$, $j = 0, 1, \ldots, 6$, in the symbol of $\Lambda(y, D_y)$. We repeatedly use formulas (83) and (84) (with $\phi_{-p/2} = \phi'_{-p/2} = 0$ for every $p \geq 1$, and $\phi'_0 = \phi_0 = \psi_0 = \psi'_0$). By an elementary and rather messy computation one gets

$$\begin{cases} \Lambda_2(y,\eta) = \lambda_0(y,\eta) = \eta_1^2 + \sigma^2 y_1^2 |\eta'|^2, \\ \\ \psi'_0 = \phi_0 = (\mu|\eta|)^{1/4} h_0(\sqrt{\mu|\eta|}\, x), \end{cases}$$

$$\begin{cases} \Lambda_{3/2}(y,\eta) = 0, \\ \\ \psi'_{-1/2} = 0, \end{cases}$$

$$\begin{cases} \Lambda_1(y,\eta) = \dfrac{1}{2} a\mu|\eta|, \\ \\ \psi'_{-1} = \dfrac{\sqrt{2}}{2} \dfrac{F(y,\eta)}{\lambda_2(y,\eta)} \phi_2, \end{cases}$$

where

$$F(y,\eta) := \frac{a\mu|\eta|^3 + i\sigma^2 y_1 \eta_1 |\eta'|^2}{|\eta|^2},$$

$$\begin{cases} \Lambda_{1/2}(y,\eta) = 0, \\ \\ \psi'_{-3/2} = 0, \end{cases}$$

$$\begin{cases} \Lambda_0(y,\eta) = \dfrac{1}{4i} \dfrac{\eta_1}{|\eta|^2} \dfrac{\partial}{\partial y_1} \left(\dfrac{F}{\lambda_2} \right) \Lambda_2 - \dfrac{\sigma^2}{8} \dfrac{\eta_1^2 |\eta'|^2}{|\eta|^4} + \dfrac{6\mu^2}{i} \eta_1 \dfrac{\partial}{\partial y_1} \left(\dfrac{F}{\lambda_2} \right) - \dfrac{a\mu}{2} \dfrac{F}{\lambda_2} |\eta| + \gamma, \\ \\ \psi'_{-2} = \dfrac{1}{4i} \dfrac{\eta_1}{|\eta|^2} \dfrac{\partial}{\partial y_1} \left(\dfrac{F}{\lambda_2} \right) \phi_0 - \sqrt{2}\, H \phi_2 - \sqrt{6}\, G \phi_4, \end{cases}$$

where

$$H(y,\eta) := \frac{1}{\lambda_2(y,\eta)} \left[\frac{\sigma^2 |\eta'|^2}{4|\eta|^2} (\eta_1^2 - |\eta'|^2) - i(1 + 24\mu^2)\eta_1 \frac{\partial}{\partial y_1} \left(\frac{F}{\lambda_2} \right) + \frac{5}{4} a\mu|\eta| \frac{F}{\lambda_2} \right],$$

$$G(y,\eta) := \frac{1}{\lambda_4(y,\eta)} \left[\frac{\sigma^2}{8|\eta|^4} \eta_1^2 |\eta'|^2 - 14i\mu^2 \eta_1 \frac{\partial}{\partial y_1} \left(\frac{F}{\lambda_2} \right) - \frac{a\mu}{2} |\eta| \frac{F}{\lambda_2} \right],$$

$$\begin{cases} \Lambda_{-1/2}(y,\eta) = 0, \\ \\ \psi'_{-5/2} = 0, \end{cases}$$

and

$$\begin{aligned} \Lambda_{-1}(y,\eta) = \left[\frac{1}{8} \frac{\eta_1^2 - |\eta'|^2}{|\eta|^4} \frac{\partial^2}{\partial y_1^2} \left(\frac{F}{\lambda_2} \right) - \frac{1}{2i} \frac{\eta_1}{|\eta|^2} \frac{\partial H}{\partial y_1} \right] \Lambda_2(y,\eta) + \\ -3(\mu^2 + 2\frac{\eta_1^2}{|\eta|^2}) \frac{\partial^2}{\partial y_1^2} \left(\frac{F}{\lambda_2} \right) - \frac{1}{2} \frac{\eta_1^2}{|\eta|^2} \frac{\partial^2}{\partial y_1^2} \left(\frac{F}{\lambda_2} \right) + \\ -\frac{12}{i} \mu^2 \eta_1 \frac{\partial H}{\partial y_1} + \frac{a\mu}{8i|\eta|} \eta_1 \frac{\partial}{\partial y_1} \left(\frac{F}{\lambda_2} \right) + a\mu|\eta| H. \end{aligned}$$

In the computations it is convenient to use the formulas
(140)
$$\begin{cases} x^2\phi_k = \dfrac{1}{\mu|\eta|}\Big[\dfrac{\sqrt{k(k-1)}}{2}\,\phi_{k-2} + \dfrac{2k+1}{2}\phi_k + \dfrac{\sqrt{(k+1)(k+2)}}{2}\,\phi_{k+2}\Big] \\ D_x^2\phi_k = -\mu|\eta|\Big[\dfrac{\sqrt{k(k-1)}}{2}\,\phi_{k-2} - \dfrac{2k+1}{2}\phi_k + \dfrac{\sqrt{(k+1)(k+2)}}{2}\,\phi_{k+2}\Big] \\ \dfrac{\partial\phi_k}{\partial\eta_1} = \dfrac{\eta_1}{4|\eta|^2}\Big[\sqrt{k(k-1)}\,\phi_{k-2} - \sqrt{(k+1)(k+2)}\,\phi_{k+2}\Big] \\ \dfrac{\partial^2\phi_k}{\partial\eta_1^2} = \dfrac{\eta_1^2}{16|\eta|^4}\sqrt{(k-3)(k-2)(k-1)k}\,\phi_{k-4} - \dfrac{\eta_1^2-|\eta'|^2}{4|\eta|^4}\sqrt{k(k-1)}\,\phi_{k-2}+ \\ \qquad -\dfrac{\eta_1^2}{8|\eta|^4}(k^2+k+1)\phi_k + \dfrac{\eta_1^2-|\eta'|^2}{4|\eta|^4}\sqrt{(k+1)(k+2)}\,\phi_{k+2}+ \\ \qquad\qquad +\dfrac{\eta_1^2}{16|\eta|^4}\sqrt{(k+1)(k+2)(k+3)(k+4)}\,\phi_{k+4}. \end{cases}$$

Now observe that $\Lambda(y, D_y)$ has double characteristics and is transversally elliptic with respect to Σ'. We have hence to compute the localized operators $\Lambda^{(j)}_{(y',\eta')}$, $j \geq 2$, and in fact for j up to 6, in order to prove the proposition. Once more, a computation gives

$$\Lambda^{(2)}_{(y',\eta')} = D_{y_1}^2 + \sigma^2 y_1^2|\eta'|^2 + \frac{a\mu}{2}|\eta'|,$$

$$\Lambda^{(3)}_{(y',\eta')} = 0,$$

$$\Lambda^{(4)}_{(y',\eta')} = \frac{a\mu}{4|\eta'|}D_{y_1}^2 + \gamma - \frac{a^2}{48},$$

$$\Lambda^{(5)}_{(y',\eta')} = 0,$$

$$\Lambda^{(6)}_{(y',\eta')} = -\frac{a\mu}{16|\eta'|^3}D_{y_1}^4 + \frac{1}{8|\eta'|^2}\left(\sigma^2 + \frac{a^2}{6\cdot 24\mu^2}\right)D_{y_1}^2 + \frac{a^2\sigma^2}{2\cdot 24^2\mu^2}y_1^2 + \frac{5a^3}{4\cdot 24^2\mu|\eta'|}.$$

Now, the eigenvalues of $\Lambda^{(2)}_{(y',\eta')}$ are

$$\omega_j(y',\eta') = \Big(\frac{a\mu}{2} + (2j+1)\sigma\Big)\,|\eta'|, \quad j \in \mathbb{Z}_+,$$

with corresponding normalized eigenfunctions

$$\theta_j(\eta'; y_1) = (\sigma|\eta'|)^{1/4}h_j(\sqrt{\sigma|\eta'|}\,y_1), \ j \in \mathbb{Z}_+.$$

By the Boutet-Grigis-Helffer theorem, $\Lambda(y, D_y)$ is hypoelliptic with loss of 1 derivative at $\rho \in \Sigma'$ iff $\omega_j(\rho) \neq 0$ for all $j \in \mathbb{Z}_+$, that is to say iff

$$a \neq -\frac{2\sigma}{\mu}(2j+1), \quad \forall j \in \mathbb{Z}_+.$$

This and Theorem 5.1 give point *(ii)* of the proposition.

Suppose now that $a = -\dfrac{2\sigma}{\mu}(2k+1)$ for some $k \in \mathbb{Z}_+$. To apply our machinery once more, we have to compute the associated operator $\tilde{\Lambda}(y', D_{y'}) \in \mathrm{OPS}^1_{\mathrm{cl}}(\mathbb{R}^{n-2})$, and more precisely the terms $\tilde{\Lambda}_1(y',\eta'), \dots, \tilde{\Lambda}_{-1}(y',\eta')$ of its symbol. Once again, we use formulas (83) and (84) (with $\phi_0 = \phi_0' = \psi_0' = \psi_0' = \theta_k$ and $\phi_{-p/2} = \phi'_{-p/2} = 0$ for all $p \geq 1$), and obtain

$$\begin{cases} \tilde{\Lambda}_1(y',\eta') = 0, \\ \psi_0' = \theta_k, \end{cases}$$

$$\begin{cases} \tilde{\Lambda}_{1/2}(y',\eta') = 0, \\ \psi'_{-1/2} = 0, \end{cases}$$

$$\begin{cases} \tilde{\Lambda}_0(y',\eta') = \gamma - \dfrac{(2k+1)^2\sigma^2}{4}\left(1 + \dfrac{1}{3\mu^2}\right), \\ \psi'_{-1} = -\dfrac{\sigma}{16}\dfrac{2k+1}{|\eta'|}\left(\sqrt{(k+1)(k+2)}\,\theta_{k+2} - \sqrt{k(k-1)}\,\theta_{k-2}\right), \end{cases}$$

$$\begin{cases} \tilde{\Lambda}_{-1/2}(y',\eta') = 0, \\ \psi'_{-3/2} = 0, \end{cases}$$

$$\tilde{\Lambda}_{-1}(y',\eta') = \frac{\sigma^3(2k+1)}{8|\eta'|}\left(\frac{k^2+k+2}{2} - \frac{1}{9\mu^4}(2k+1)^2\right).$$

In the computations it is convenient to use the analogue of formulas (140) for θ_k.

As a consequece, $\Lambda(y, D_y)$ is hypoelliptic at $\rho \in \Sigma'$ with loss of 2 derivatives iff $\tilde{\Lambda}_0(\rho) \neq 0$, that is to say iff

$$\gamma \neq \frac{(2k+1)^2\sigma^2}{4}\left(1 + \frac{1}{3\mu^2}\right),$$

which yields point (*iii*) in the statement.

Finally, when $\gamma = \dfrac{(2k+1)^2\sigma^2}{4}\left(1 + \dfrac{1}{3\mu^2}\right)$, we have that $\tilde{\Lambda}(y', D_{y'})$ is hypoelliptic with loss of 2 derivatives iff $\tilde{\Lambda}_{-1} \neq 0$, so that by our Theorem 5.1 $\Lambda(y, D_y)$ is hypoelliptic at $\rho \in \Sigma'$ with loss of 3 derivatives iff

$$\mu^4 \neq \frac{2}{9}\,\frac{(2k+1)^2}{k^2+k+2}.$$

This proves point (*iv*) and concludes the proof of the proposition. □

Remark 6.12. *If* $\mu^4 = \dfrac{2}{9}\,\dfrac{(2k+1)^2}{k^2+k+2}$, *to detect hypoellipticity of* $\tilde{\Lambda}(y', D_{y'})$ *(and hence of* $\Lambda(y, D_y)$*, and hence of* A*) one should compute the terms*

$\tilde{\Lambda}_{1-j/2}$, $j \geq 5$, *but this requires the knowledge of the terms* $\Lambda_{2-j/2}$, $j \geq 7$, *a computation that we decided not to carry out!*

7. Conclusive remarks: invariance?

We finally comment on the lack of invariance of our approach.

From the very beginning we have supposed that the characteristic manifold Σ of A be flat. Now, suppose Σ is a general symplectic submanifold of $T^*\mathbb{R}^n \setminus 0$ of codimension 2ν, conditions (1) and (2) hold, and we ask for the hypoellipticity of A at some point $\rho_0 \in \Sigma$. Again, one has the Boutet-Grigis-Helffer Theorem 1.4, for the localized operator A_ρ, $\rho \in \Sigma$, is intrinsecally defined in the following way (see [6]).

One takes the Weyl-symbol of A

$$\sigma_{\mathrm{Weyl}}(A) := e^{i\langle \partial_z, \partial_\zeta \rangle/2}\sigma(A) \sim \sum_{j\geq 0} q_{m-j/2},$$

and next defines the polynomial map

$$q_\rho \colon T_\rho\Sigma^\sigma \longrightarrow \mathbb{C}, \quad \rho \in \Sigma,$$

by

$$q_\rho(v) = \sum_{j=0}^{k} \frac{1}{(k-j)!}\big(V^{k-j} q_{m-j/2}\big)(\rho),$$

where $v \in T_\rho\Sigma^\sigma$ and V is any smooth vector field on $T^*\mathbb{R}^n$ with $V(\rho) = v$ (the vanishing conditions on the $q_{m-j/2}$ ensure that the definition of $q_\rho(v)$ is independent of the extension V of v). Now take any linear symplectomorphism

$$f \colon T^*\mathbb{R}^\nu \simeq \mathbb{R}^\nu_x \times \mathbb{R}^\nu_\xi \longrightarrow T_\rho\Sigma^\sigma.$$

Put $q_{\rho,f}(x,\xi) := q_\rho(f(x,\xi))$ and

$$A_{\rho,f} := \mathrm{Op}^{\mathrm{w}}(q_{\rho,f})(x, D_x),$$

and observe that (by Thm. 18.5.9 of Hörmander [10] Vol. III) A_{ρ,f_1} and A_{ρ,f_2} are unitarily equivalent, so that the spectral properties of the $A_{\rho,f}$ are independent of the symplectomorphism f.

Remark that if we perform the above construction when Σ is flat, we get that $\mathrm{Op}^{\mathrm{w}}(q_{\rho,\mathrm{id}})(x, D_x)$ coincides with the localized operator $A_\rho^{(k)}$, that we used throughout the paper. Here id is the canonical identification of $T^*\mathbb{R}^\nu$ with $T_\rho\Sigma^\sigma$.

Theorem 1.4 reads now: *A is hypoelliptic at $\rho_0 \in \Sigma$ with loss of $k/2$ derivatives iff* $\mathrm{Ker}(A_{\rho_0,f}) = \{0\}$ *for one, and hence for all, f as above.*

Thus, the problem is: *What can one say on the hypoellipticity of A at $\rho_0 \in \Sigma$ when the injectivity condition is not satisfied?*

Our recipe goes as follows:

- take a homogeneous canonical transformation χ that brings Σ into the flat case $x = \xi = 0$;

- consider the ψdo $\tilde{A}$ obtained by conjugating A with an elliptic Fourier integral operator associated with χ;
- "compute" $\Lambda(y, D_y)$.

It is important to observe that one has

$$\tilde{A}^{(k)}_{\chi(\rho_0)} = A_{\rho_0, f},$$

with $f(x,\xi) =$ the projection onto $T_\rho\Sigma^\sigma$ of $\chi'(\rho_0)^{-1}\begin{bmatrix} x \\ 0 \\ \xi \\ 0 \end{bmatrix}$.

One of the crucial problems is therefore the following: *Is there an object (on Σ) that corresponds, via χ, to the higher order localized operator $\tilde{A}^{(k+j)}_{\chi(\rho_0)}$?*

We do not have an answer. For this reason we had to work in the flat case. An answer to this problem would lead to an "invariant" interpretation of the conditions obtained in the previously treated examples.

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