JULIA SETS OF PERMUTABLE ENTIRE FUNCTIONS

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Abstract. Let $C(f)$ be the set of entire functions permutable with a given entire function f. In this paper we show that $C(f) = \{f^n : n \geq 0\}$ holds for almost all transcendental entire function f. For such a function f, it follows that every transcendental entire function in $C(f)$ has the same Julia set as that of f.

1. Introduction and statement of results

Let f be an entire or a rational function. We denote by f^n the nth iterate of f. The Fatou set $F(f)$ and the Julia set $J(f)$ of f are defined by $F(f) = \{z \in \widehat{\mathbf{C}} : \{f^n\}_{n\in\mathbf{N}} \text{ is defined and normal in some neighborhood of } z\}$ and $J(f) = \widehat{\mathbf{C}} \setminus F(f)$. Julia [6] and Fatou [5] proved that rational functions $f,\ g$ of degree at least two satisfying $f \circ g = g \circ f$ have the same Julia set. It is natural to ask whether this is valid for entire functions. Baker [1] showed that a polynomial of degree at least two cannot be permutable with any transcendental entire function. In this paper we consider the case that $f,\ g$ are transcendental. We denote by $C(f)$ the set of entire functions permutable with f. Baker [2] proved that, if $g = cf^n + d$ $(c \neq 0)$ and $f\circ g = g\circ f$, then $J(f) = J(g)$. Therefore, if $C(f) \subset \{cf^n + d : n \geq 0,\ c \neq 0\}$ holds, then $f \circ g = g \circ f$ implies that $J(f) = J(g)$.

We say that f is prime (left-prime, pseudo-prime), if every factorization $f = g\circ h$ with a meromorphic function g and an entire function h implies that either g or h is a linear transformation (either g is a linear transformation or h is a polynomial, either g is a rational function or h is a polynomial). When factors $g,\ h$ are restricted to entire functions, f is said to be prime (left-prime, pseudo-prime) in entire sense. The author proved that almost all functions are prime.

Theorem A ([9])**.** *Let f be a transcendental entire function, h be a holomorphic function in $\mathbf{C}^*$ with essential singularities at $0,\ \infty$ and m be a non-zero integer. Then there exist countable sets $E_f,\ E_h \subset \mathbf{C}$ such that*

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$f(z)+az,\ (z-a)f(z)$ *are prime for each* $a \notin E_f$*, and that* $h(e^z)+ae^{mz}$ *is prime for each* $a \notin E_h$*.*

Theorem B ([9]). *Let* f *be a transcendental entire function. Put* $A_c = \{z \in \mathbf{C} : f'(z) = 0,\ f(z) = c\}$*.*

(i) *If* $\sharp A_c < \infty$ *for every* $c \in \mathbf{C}$ *and* $N(r, 1/f') > kT(r, f')$ $(k > 0,\ r \in E,\ E \subset \mathbf{R}_+,\ |E| = \infty)$*, then* f *is left-prime in entire sense.*

(ii) *If* $\sharp A_c < \infty$ *for every* $c \in \mathbf{C} \setminus \{0\}$ *and* $N(r, f/f') > kT(r, f'/f)$ $(k > 0,\ r \in E,\ E \subset \mathbf{R}_+,\ |E| = \infty)$*, then* f *is left-prime in entire sense.*

(iii) *If* $\sharp A_c < \infty$ *for every* $c \in \{|z| > R\}$ $(R > 0)$ *and* $T(r, f) < r^k$ $(k > 0)$*, then* f *is pseudo-prime.*

(For the definitions of $T(r, f),\ N(r, f)$*, see §5.)*

Recently Ng [8] gives a sufficient condition which yields $C(f) \subset \{cf^n + d : n \geq 0,\ c \neq 0\}$. By making use of Ng's method, we obtain the following theorems.

Theorem 1. *Let* f *be a transcendental entire function and* m *be a positive integer. Put* $f_a(z) = f(z) + az^m$*. Then there exists a countable set* E_f *such that if* $f(z) = z\sum_{k=0}^{\infty} a_k z^{pk}$ $(a_k \in \mathbf{C},\ p \in \mathbf{N},\ p \geq 2)$ *and* $m \in \{jp + 1 : j \in \mathbf{N}\}$*, then* $C(f_a) = \{e^{2\pi ki/p} {f_a}^n : n \geq 0,\ k = 1, \ldots, p\}$ *for each* $a \notin E_f$*, and that if otherwise,* $C(f_a) = \{{f_a}^n : n \geq 0\}$ *for each* $a \notin E_f$*.*

The case of $m = 1$ was proved by Ng [8]. He showed that for each $a \notin E_f$, any element g of $C(f_a)$ is of the form $c{f_a}^n + d$ $(n \geq 0)$, where c is a kth root of unity and $d \in \mathbf{C}$.

Theorem 2. *Let* f *be a transcendental entire function. Put* $f_a(z) = (z - a)f(z)$*. Then there exists a countable set* E_f *such that* $C(f_a) = \{{f_a}^n : n \geq 0\}$ *for each* $a \notin E_f$*.*

Theorem 3. *Let* h *be a holomorphic function in* $\mathbf{C}^*$ *with essential singularities at* $0,\ \infty$ *and* m *be a non-zero integer. Put* $f_a(z) = h(e^z) + ae^{mz}$*. Then there exists a countable set* E_f *such that* $C(f_a) = \{{f_a}^n : n \geq 0\}$ *for each* $a \notin E_f$*.*

2. Common right factor theorem

Let $f,\ g$ be holomorphic functions in a domain D. Let S be a Riemann surface and $h : D \to S$ be a holomorphic surjective map. We say that (h, S) is a common right factor of f and g, if there exist holomorphic functions $\widehat{f},\ \widehat{g}$ on S such that $f = \widehat{f} \circ h,\ g = \widehat{g} \circ h$. Let $(\widetilde{h}, \widetilde{S})$ denote any common right factor. We say that (h, S) is a greatest common right factor of f and g, if for any $(\widetilde{h}, \widetilde{S})$ there exists a holomorphic map $k : \widetilde{S} \to S$ such that $h = k \circ \widetilde{h}$. If $(h, S),\ (\widetilde{h}, \widetilde{S})$ are two greatest common right factors, then the above holomorphic map $k : \widetilde{S} \to S$ is a conformal map. The existence of a greatest common right factor in the case of $D = \mathbf{C}$ was proved in [4]. In this section we shall give another proof for a general domain D.

We define an equivalence relation $\sim_{(f,g)}$ in $\{z \in D : f'(z)g'(z) \neq 0\}$ as follows: $z_1 \sim_{(f,g)} z_2$ if and only if $f(z_1) = f(z_2)$, $g(z_1) = g(z_2)$ and there exist neighborhoods U_j of z_j $(j = 1, 2)$ such that $f(U_1) = f(U_2)$, $g(U_1) = g(U_2)$ and $(f|_{U_2})^{-1} \circ (f|_{U_1}) \equiv (g|_{U_2})^{-1} \circ (g|_{U_1})$ in U_1.

Lemma 2.1. *Let f, g be holomorphic functions in a domain D and $c \in D$. Then there exist a neighborhood U of c, a holomorphic function h in U and holomorphic functions $\widehat{f}$, $\widehat{g}$ in $h(U)$ satisfying the following conditions.*

1) $f'(z) \neq 0$, $g'(z) \neq 0$, $h'(z) \neq 0$ $(z \in U \setminus \{c\})$.
2) $z \sim_{(f,g)} w$ *if and only if* $h(z) = h(w)$ $(z, w \in U \setminus \{c\})$.
3) $f = \widehat{f} \circ h$, $g = \widehat{g} \circ h$.

Proof. We may assume that $c = 0$, $f(0) = g(0) = 0$. Then there exist a neighborhood U of 0 and a univalent function f_0 in U such that $f_0(0) = 0$, $f(z) = (f_0(z))^m$ $(z \in U, m \in \mathbf{N})$, $f_0(U) = \{|z| < \epsilon\}$ $(\epsilon > 0)$ and $g'(z) \neq 0$ $(z \in U \setminus \{0\})$. Put $f_1 = f \circ {f_0}^{-1}$, $g_1 = g \circ {f_0}^{-1}$. Then $f_1(z) = z^m$.

Put $\lambda_k = e^{2\pi i k/m}$ $(k = 1, \dots, m)$. Let x, $y \in \{0 < |z| < \epsilon\}$. Then $x \sim_{(f_1,g_1)} y$ if and only if $y = \lambda_k x$, $g_1(z) \equiv g_1(\lambda_k z)$ for some k. Put $k_0 = \min\{k : g_1(z) \equiv g_1(\lambda_k z),\ k = 1, \dots, m\}$. Then each element of this set is a multiple of k_0. Put $p = m/k_0$. Then $x \sim_{(f_1,g_1)} y$ if and only if $x^p = y^p$. Since $f_1(z) \equiv f_1(e^{2\pi i/p} z)$, $g_1(z) \equiv g_1(e^{2\pi i/p} z)$, there exist holomorphic functions $\widehat{f}$, $\widehat{g}$ in $\{|z| < \epsilon^p\}$ such that $f_1(z) = \widehat{f}(z^p)$, $g_1(z) = \widehat{g}(z^p)$. Put $h = (f_0)^p$. Then we have $f = \widehat{f} \circ h$, $g = \widehat{g} \circ h$. Further $z \sim_{(f,g)} w$ holds if and only if $h(z) = h(w)$ $(z, w \in U \setminus \{0\})$. □

In the above lemma, $z \sim_{(\widehat{f},\widehat{g})} w$ holds if and only if $z = w$ for $z, w \in h(U)$. In the case of $f'(c)g'(c) \neq 0$, we see that $\widehat{f} \equiv f$, $\widehat{g} \equiv g$ and $h \equiv$ id. satisfy the conclusion of Lemma 2.1.

Lemma 2.2. *Let f, g be holomorphic functions in a domain D and $c \in D$. Let U_j $(j = 1, 2)$ be neighborhoods of c and h_j $(j = 1, 2)$ be holomorphic functions in U_j which satisfy the conclusion of Lemma 2.1 with U, h replaced by U_j, h_j. Then there exists a conformal map φ defined in $h_1(U_1 \cap U_2)$ such that $\varphi \circ h_1 = h_2$ in $U_1 \cap U_2$.*

Proof. Put $[z] = \{w \in U_1 \cap U_2 \setminus \{c\} : z \sim_{(f,g)} w\}$ $(z \in U_1 \cap U_2 \setminus \{c\})$. Then there exists one to one correspondence between $\{[z] : z \in U_1 \cap U_2 \setminus \{c\}\}$ and $h_j(U_1 \cap U_2 \setminus \{c\})$ $(j = 1, 2)$. Therefore there exists a bijection $\varphi : h_1(U_1 \cap U_2 \setminus \{c\}) \to h_2(U_1 \cap U_2 \setminus \{c\})$. φ is represented by $\varphi = h_2 \circ {h_1}^{-1}$ and $h_j'(z) \neq 0$ $(z \in U_1 \cap U_2 \setminus \{c\},\ j = 1, 2)$. Therefore φ is conformal in $h_1(U_1 \cap U_2) \setminus \{h_1(c)\}$. Further $h_1(c)$ is a removable singularity of φ. Therefore φ is conformal in $h_1(U_1 \cap U_2)$. □

We extend $\sim_{(f,g)}$ to D as follows: $z_1 \sim_{(f,g)} z_2$ if and only if $f(z_1) = f(z_2)$, $g(z_1) = g(z_2)$ and there exists a conformal map φ defined in some neighborhood of $h_1(z_1)$ such that $\varphi(h_1(z_1)) = h_2(z_2)$, $\widehat{f}_1 = \widehat{f}_2 \circ \varphi$, $\widehat{g}_1 = \widehat{g}_2 \circ \varphi$, where z_j, h_j, $\widehat{f}_j$, $\widehat{g}_j$ $(j = 1, 2)$ satisfy the conclusion of Lemma 2.1. By Lemma 2.2, this definition does not depend on the choice of h_j, $\widehat{f}_j$, $\widehat{g}_j$. We simply write $z_1 \sim z_2$, if there is no confusion.

Lemma 2.3. *Let z_1, z_2 be distinct points in D and $\{z_k^{(j)}\}_k$ $(j = 1, 2)$ be sequences of complex numbers such that $z_k^{(j)} \to z_j$ $(k \to \infty,\ j = 1, 2)$ and that $z_k^{(1)} \sim z_k^{(2)}$. Then we have $z_1 \sim z_2$.*

Proof. By the assumption, we have $f(z_1) = f(z_2)$, $g(z_1) = g(z_2)$. Let U_j $(j = 1, 2)$ be neighborhoods of z_j $(j = 1, 2)$ and h_j $(j = 1, 2)$ be holomorphic functions in U_j $(j = 1, 2)$ and $\widehat{f}_j$, $\widehat{g}_j$ $(j = 1, 2)$ be holomorphic functions in $h_j(U_j)$ satisfying the conclusion of Lemma 2.1 with c replaced by z_j. Further we assume that $\widehat{f}_j'(z)\widehat{g}_j'(z) \neq 0$ in $h_j(U_j) \setminus \{h_j(z_j)\}$ and $\{z_k^{(j)}\}_k \subset U_j$ for $j = 1, 2$.

By Lemma 2.2, there exists a conformal map φ defined in some neighborhood V_k of $h_1(z_k^{(1)})$ such that $\varphi(h_1(z_k^{(1)})) = h_2(z_k^{(2)})$, $\widehat{f}_1 = \widehat{f}_2 \circ \varphi$, $\widehat{g}_1 = \widehat{g}_2 \circ \varphi$. Since $f(z_1) = f(z_2)$, $g(z_1) = g(z_2)$, we have $\widehat{f}_1(h_1(z_1)) = \widehat{f}_2(h_2(z_2))$. Therefore, there exists a neighborhood W of $h_1(z_1)$ such that $\varphi = \widehat{f}_2^{-1} \circ \widehat{f}_1 = \widehat{g}_2^{-1} \circ \widehat{g}_1$ has analytic continuation along any path in $W \setminus \{h_1(z_1)\}$. We may assume that $V_k \subset W$.

If φ has an analytic continuation which is multiple-valued in $W \setminus \{h_1(z_1)\}$, then there exist a point $c \in W$ and holomorphic functions α, β defined in some neighborhood of c satisfying $\alpha(c) \neq \beta(c)$ and $\widehat{f}_1 = \widehat{f}_2 \circ \alpha$, $\widehat{g}_1 = \widehat{g}_2 \circ \alpha$, $\widehat{f}_1 = \widehat{f}_2 \circ \beta$, $\widehat{g}_1 = \widehat{g}_2 \circ \beta$. We may assume that α, β are injections. Then we have $\widehat{f}_2 = \widehat{f}_1 \circ \alpha^{-1}$, $\widehat{f}_1 = \widehat{f}_2 \circ \beta$. Therefore $\widehat{f}_2 = \widehat{f}_2 \circ (\beta \circ \alpha^{-1})$. Similarly $\widehat{g}_2 = \widehat{g}_2 \circ (\beta \circ \alpha^{-1})$. Thus we have $\alpha(c) \sim_{(\widehat{f}_2, \widehat{g}_2)} \beta(c)$. This contradicts $\alpha(c) \neq \beta(c)$. Therefore φ has an analytic continuation which is single-valued in $W \setminus \{h_1(z_1)\}$. Since $h_1(z_1)$ is a removable singularity, φ is holomorphic in W. Similarly, we see that φ^{-1} is single-valued in some neighborhood of $h_2(z_2)$. Therefore φ is conformal in some neighborhood of $h_1(z_1)$. Thus we have $z_1 \sim z_2$. □

Lemma 2.4. *Let f, g be holomorphic functions in a domain D and $c \in D$. Let U, h, $\widehat{f}$, $\widehat{g}$ be the same as in Lemma 2.1, and k be a holomorphic function in some neighborhood U' of $c' \in \mathbf{C}$ such that $k(U') \subset U$, $k(c') = c$, $k'(z) \neq 0$ $(z \in U' \setminus \{c'\})$. Put $F = f \circ k$, $G = g \circ k$, $H = h \circ k$. Then U', H, $\widehat{f}$, $\widehat{g}$ satisfy the conclusion of Lemma 2.1 with c, f, g replaced by c', F, G.*

Proof. 1) and 3) are obvious. Let $x \sim_{(F,G)} y$ $(x,\ y \in U' \setminus \{c'\})$. Put

$a = k(x)$, $b = k(y)$. Since $k'(x)k'(y) \neq 0$, we have $a \sim_{(f,g)} b$. Hence $h(a) = h(b)$. Therefore $H(x) = H(y)$. Conversely, if $H(x) = H(y)$, then we have $h(k(x)) = h(k(y))$. Hence $k(x) \sim_{(f,g)} k(y)$. Since $k'(x)k'(y) \neq 0$, we have $x \sim_{(F,G)} y$. Thus 2) holds. □

Let f, g be holomorphic functions in a domain D. Let S be the set of equivalence classes and $F(c)$ be the class of the point $c \in D$. We put $N(c, r) = \{F(z) : |z - c| < r,\ z \in D\}$ $(r > 0)$. If $p = F(c) \in S$, we call $N(c, r)$ a basic neighborhood of p. These basic neighborhoods define a topology on S. Further, by Lemma 2.3, S is a Hausdorff space.

Let $p = F(c) \in S$, U be a neighborhood of c and h be a holomorphic function defined in U such that U, h satisfy the conclusion of Lemma 2.1. Then we have $F(x) = F(y)$ if and only if $h(x) = h(y)$ $(x, y \in U)$. We define a local coordinate in $F(U)$ by $h \circ (F|_U)^{-1} : F(U) \to h(U)$. We see that S is a Riemann surface and F is holomorphic in D. Further there exist holomorphic functions $\widehat{f}$, $\widehat{g}$ on S such that $f = \widehat{f} \circ F$, $g = \widehat{g} \circ F$. By Lemma 2.1, 2.2 and 2.4, we see that (F, S) is a greatest common right factor of f and g.

If $D = \mathbf{C}$, then S is one of $\mathbf{C}$, $\mathbf{C}^*$, $\widehat{\mathbf{C}}$ or a torus. Since f, g are entire, S is either $\mathbf{C}$ or $\mathbf{C}^*$.

The following lemma is essentially the same as in [8]. For the sake of completeness, we describe its proof.

Lemma 2.5. *Let f, g be analytic self mappings of a domain D satisfying $f \circ g = g \circ f$ and (F, S) be a greatest common right factor of f, g. Assume that there exists a subset $\mathcal{A} \subset D$ such that $\sharp f(\mathcal{A}) = 1$, $\sharp g(\mathcal{A}) = 1$. Let q be the order of f at $g(\mathcal{A})$. Then there exists a subset $\mathcal{A}' \subset \mathcal{A}$ such that $\sharp F(\mathcal{A}') = 1$, $\sharp \mathcal{A}' \geq \sharp \mathcal{A}/q$.*

Proof. We may assume that $q < \sharp \mathcal{A}$. Let $\sharp \mathcal{A} = K$ and $\mathcal{A} = \{z_j\}_{j=1}^K$. Put $a = g(z_1)$. Let U be a neighborhood of a such that $f(z) = f(a) + (\psi(z))^q$ $(z \in U)$, where ψ is a conformal map in U with $\psi(a) = 0$, $\psi(U) = \{|w| < \epsilon\}$ $(\epsilon > 0)$. Let x_j $(j = 1, \ldots, K)$ be distinct points near z_j satisfying $f(x_1) = \cdots = f(x_K)$. Further let U_j $(j = 1, \ldots, K)$ be neighborhoods of x_j such that $f|_{U_j}$ is an injection and that $f(U_1) = \cdots = f(U_K)$, $g(U_j) \subset U$ $(j = 1, \ldots, K)$.

Put $V = f(U_1)$, $\varphi_j = (f|_{U_j})^{-1}$ $(j = 1, \ldots, K)$. Then we have $g = g \circ f \circ \varphi_1 = \cdots = g \circ f \circ \varphi_K$ in V. Hence $f \circ g \circ \varphi_1 = \cdots = f \circ g \circ \varphi_K$. From $q < K$ there exists a subset $\mathcal{N} \subset \{1, 2, \ldots K\}$ with $\sharp \mathcal{N} \geq K/q$ such that $g \circ \varphi_\mu \equiv g \circ \varphi_\nu$ in V for every μ, $\nu \in \mathcal{N}$. On the other hand $f \circ \varphi_\mu \equiv f \circ \varphi_\nu \equiv \mathrm{id.}$ $(\mu, \nu \in \mathcal{N})$. Therefore $x_\mu \sim x_\nu$ $(\mu, \nu \in \mathcal{N})$.

Thus there exist sequences of complex numbers $\{x_k^{(\nu)}\}_k$ $(\nu \in \mathcal{N})$ such that $x_k^{(\nu)} \to z_\nu$ $(k \to \infty)$, $x_k^{(\mu)} \sim x_k^{(\nu)}$ $(k = 1, 2, \ldots)$ for every μ, $\nu \in \mathcal{N}$. Let $\mathcal{A}' = \{z_\nu : \nu \in \mathcal{N}\}$. By Lemma 2.3, we have $z \sim w$ for every z, $w \in \mathcal{A}'$. Hence $\sharp F(\mathcal{A}') = 1$. □

3. Proof of Theorem 1

We shall make use of the following Baker's result about permutable entire functions.

Lemma A ([1, Satz 6])**.** *Let f be a transcendental entire function which is permutable with a polynomial g. Then $g(z) = \alpha z + \beta$ ($\alpha = e^{2\pi ki/p}$, $k, p \in \mathbf{N}$, $(k, p) = 1$, $\beta \in \mathbf{C}$). Further, if $\alpha \neq 1$, then $f(z) = c + (z - c)F((z - c)^p)$ ($c = \beta/(1 - \alpha)$), where F is an entire function.*

In what follows we write $M(r, f) = \max_{|z|=r} |f(z)|$.

Lemma B ([1, Satz 7])**.** *Let f, g be permutable transcendental entire functions. Then there exist a positive integer n and $R_0 > 0$ such that $M(r, g) < M(r, f^n)$ for all $r > R_0$.*

We also use the following results of Clunie.

Lemma C ([3, Theorem 1])**.** *Let $f(z)$, $g(z)$ be entire and transcendental. Then $\limsup_{r\to\infty} \log M(r, f \circ g)/\log M(r, g) = \infty$.*

Lemma D ([3, Lemma 1])**.** *Let $g(z)$ be entire and transcendental. Given $K > 0$ there is a number $R_0 > 0$ and an increasing sequence $\{R_n\}_1^\infty$ with $R_1 > R_0$ and $R_n \to \infty$ ($n \to \infty$) such that for $n \geq 1$ and all r in $R_n \leq r \leq {R_n}^2$ and all w satisfying $R_0 \leq |w| \leq r$ we have $\sharp(g^{-1}(\{w\}) \cap \{|z| \leq r\}) > K$.*

From Lemma D, we obtain the following lemma, which is essentially the same as in [8].

Lemma 3.1. *Let f be a transcendental entire function. Let A be a subset of $\mathbf{C}$ satisfying that $\sharp f^{-1}(A) = \infty$ and $\sharp(A \cap \{|z| < r\}) < \infty$ for every $r > 0$. Then $\sup_{w\in A} \sharp(f^{-1}(\{w\}) \cap A^C) = \infty$. (By Picard's theorem, $\sharp A \geq 2$ implies that $\sharp f^{-1}(A) = \infty$.)*

Proof. If $\sharp A < \infty$, it is obvious. Assume that $\sharp A = \infty$. Let K, $\{R_n\}_1^\infty$ satisfy the conclusion of Lemma D. Put $N_n = \sharp(A \cap \{|z| \leq R_n\})$ ($n = 0, 1, \dots$). By Lemma D, $\sharp(f^{-1}(\{w\}) \cap \{|z| \leq R_n\}) > K$ for all w with $R_0 \leq |w| \leq R_n$. Therefore, if $2(N_n - N_0) > N_n$, then

$$\begin{aligned}&\sharp(f^{-1}(A \cap \{R_0 < |z| \leq R_n\}) \cap A^C \cap \{|z| \leq R_n\})\\ &> K(N_n - N_0) - N_n > (K - 2)(N_n - N_0).\end{aligned}$$

This implies that $\sharp(f^{-1}(\{w\}) \cap A^C \cap \{|z| \leq R_n\}) > K - 2$ for some $w \in A \cap \{R_0 < |z| \leq R_n\}$. By choosing K arbitrarily large, we have the desired result. □

We shall show the following result, which is an extension of Theorem 2 of [9].

Lemma 3.2. *Let f be a transcendental entire function and m be a positive integer. Put $f_a(z) = f(z) + az^m$. Then there exists a countable set $E_f \subset \mathbf{C}$ such that f_a satisfies the following conditions for each $a \notin E_f$.*

1) *f_a is left-prime in entire sense.*
2) *$\sharp\{z \in \mathbf{C} : f_a'(z) = 0\} = \infty$.*
3) *$\sharp\{z^m : f_a(z) = c,\ f_a'(z) = 0,\ z \in \mathbf{C}\} \le 1$ for all $c \in \mathbf{C}$.*
4) *If $f_a = g \circ Q$ with an entire function g and a polynomial Q, then g is not periodic and $Q(z) = \alpha z^p + \beta$ $(\alpha \neq 0)$, where p is a divisor of m.*
5) *f_a' has only simple zeros in $\mathbf{C}^*$.*
6) *If $\alpha z + \beta$ $(\alpha \neq 0)$ is permutable with f_a, then either $\alpha = 1,\ \beta = 0$ or $f(z) = z\sum_{k=0}^{\infty} a_k z^{pk}$ $(a_k \in \mathbf{C},\ p \in \mathbf{N},\ p \ge 2),\ m \in \{jp+1 : j \in \mathbf{N}\},\ \alpha = e^{2\pi ki/p}$ $(k \in \{1, \dots, p\}),\ \beta = 0$.*

Proof. 3) follows from Lemma 3 of [9, p.488].

Put $F(z) = -f'(z)/(mz^{m-1})$. Then $F(z) = a$ if and only if $f_a'(z) = 0$ in $\mathbf{C} \setminus \{0\}$. Therefore, by the second fundamental theorem ([7, p.246]), there exists a countable set $E \subset \mathbf{C}$ and a positive number k such that $|\{r : N(r, 1/(F-a)) > kT(r, F)\}| = \infty$ for every $a \notin E$. Hence $|\{r : N(r, 1/f_a') > kT(r, f_a')\}| = \infty$ for every $a \notin E$. Therefore 1) follows from Theorem A of [9, p.480] and 3). Further we see that 2) holds.

4) If $f_a = g \circ Q$, then $f_a' = g' \circ QQ'$. By 2) and 3), we may assume that g' has infinitely many zeros $\{w_j\}_{j \in \mathbf{N}}$, and that $\sharp\{z^m : Q(z) = w_j,\ z \in \mathbf{C}\} = 1$ for all $j \in \mathbf{N}$. Therefore the degree p of Q is a divisor of m. Let z_j be a root of $Q(z) - w_j$. Then $Q(z_j) = Q(e^{2\pi i/p} z_j)$ $(j = 1, 2, \dots)$. Thus $Q(z) \equiv Q(e^{2\pi i/p} z)$. Therefore $Q(z)$ is represented by $Q(z) = \alpha z^p + \beta$ $(\alpha \neq 0)$. Put $\widehat{f}(z) = f(z^{1/p})$. Then $\widehat{f}$ is entire and $\widehat{f}(z) + az^{m/p} = g(\alpha z + \beta)$. If $\widehat{f}(z) + az^{m/p}$, $\widehat{f}(z) + bz^{m/p}$ are periodic functions with periods λ, μ respectively, then $\widehat{f}^{(m/p)}$ is a periodic function with periods $\lambda,\ \mu$. Hence $\lambda/\mu \in \mathbf{R}$. Therefore $(\widehat{f}(z) + az^{m/p}) - (\widehat{f}(z) + bz^{m/p}) = (a-b)z^{m/p}$ is bounded on $\lambda\mathbf{R}$. Hence $a = b$. Thus there exists a finite set $E \subset \mathbf{C}$ such that if $a \notin E$, then g is not periodic.

5) If $f_a'(z) = 0,\ f_a''(z) = 0$ $(z \neq 0)$, then $zf''(z) - (m-1)f'(z) = 0,\ a = -f'(z)/(mz^{m-1})$. It is easy to see that $zf''(z) - (m-1)f'(z) \not\equiv 0$. Put $E' = \{-f'(z)/(mz^{m-1}) : zf''(z) - (m-1)f'(z) = 0,\ z \neq 0\}$. If $a \notin E'$, then we have $\{z \in \mathbf{C}^* : f_a'(z) = 0,\ f_a''(z) = 0\} = \emptyset$. Therefore f_a' has only simple zeros in $\mathbf{C}^*$ for each $a \notin E'$.

6) Assume that $f_a(z + \alpha) = f_a(z) + \alpha,\ f_b(z + \beta) = f_b(z) + \beta$ $(a \neq b,\ \alpha \neq 0,\ \beta \neq 0)$. Then $f^{(m)}$ is a periodic function with periods $\alpha,\ \beta$. Hence $\alpha/\beta \in \mathbf{R}$. Further $f_a(z) - z,\ f_b(z) - z$ are periodic functions with periods $\alpha,\ \beta$ respectively. Therefore $(f_a(z) - z) - (f_b(z) - z) = (a-b)z^m$ is bounded on $\alpha\mathbf{R}$. Hence $a = b$. Thus any linear polynomial $z + \alpha$ $(\alpha \neq 0)$

is not permutable with f_a for all $a \in \mathbf{C}$ with at most one exception.

Next we assume that $f_a(\alpha z + \beta) = \alpha f_a(z) + \beta$ $(\alpha \neq 1, 0,\ \beta \neq 0)$. From Lemma A we have $f_a(z) = c + (z - c)F((z - c)^p)$ $(p \geq 2,\ c = \beta/(1 - \alpha))$. Then $f_a(c) = c,\ f_a^{(k)}(c) = 0$ $(k \notin \{jp + 1 : j = 0, 1, \dots\})$. Therefore $c \neq 0,\ a = (c - f(c))/c^m,\ f^{(m+1)}(c) f^{(m+2)}(c) = 0$. Put $E'' = \{(z - f(z))/z^m : f^{(m+1)}(z) f^{(m+2)}(z) = 0,\ z \neq 0\}$. Then any linear polynomial $\alpha z + \beta$ $(\alpha \neq 0, 1,\ \beta \neq 0)$ is not permutable with f_a for each $a \notin E''$.

Finally we assume that $f_a(\alpha z) = \alpha f_a(z)$ $(\alpha \neq 1, 0)$. From Lemma A we have $f_a(z) = zF(z^p)$ and $\alpha = e^{2\pi k i/p}$ $(k = 1, \dots, p)$. If $a \neq 0$, then $m \in \{jp + 1 : j \in \mathbf{N}\}$. Thus we have the desired result. □

Proof of Theorem 1. The following proof is essentially the same as in [8]. For the sake of completeness, we describe it. We assume that f satisfies the conclusion of Lemma 3.2 with f_a replaced by f. Let g be a transcendental entire function which is permutable with f. Assume that $\sharp\{z : g'(z)(g' \circ f(z)) = 0\} < \infty$. Since $f' \circ g(z) g'(z) = g' \circ f(z) f'(z)$, we see that $f' \circ g(z) = 0$ if and only if $f'(z) = 0$ in $\mathbf{C} \setminus \{z : g'(z)(g' \circ f(z)) = 0\}$. By $\sharp\{z : f'(z) = 0\} = \infty$ and Lemma 3.1, this is a contradiction. Hence $\sharp\{z : g'(z)(g' \circ f(z)) = 0\} = \infty$. This implies that $\sharp f^{-1}(\{z : g'(z) = 0\}) = \infty$. From Lemma 3.1, we see that for every $N \in \mathbf{N}$, there exists $c \in \{z : g'(z) = 0\}$ such that $\sharp\{z : f(z) = c,\ g'(z) \neq 0\} \geq N$. By $f' \circ g g' = g' \circ f f'$ we have $\sharp\{z : f(z) = c,\ f' \circ g(z) = 0\} \geq N$.

Put $\mathcal{A} = \{z : f(z) = c,\ f' \circ g(z) = 0\}$. Since $f \circ g(\mathcal{A}) = g \circ f(\mathcal{A}) = \{g(c)\}$, we have $g(\mathcal{A}) \subset \{z : f(z) = g(c),\ f'(z) = 0\}$. By 3) of Lemma 3.2, we have $\sharp g(\mathcal{A})^m = 1$. Therefore there exists a subset $\mathcal{B} \subset \mathcal{A}$ such that $\sharp g(\mathcal{B}) = 1,\ \sharp \mathcal{B} \geq N/m$. On the other hand $f(\mathcal{B}) = \{c\}$. By Lemma 2.5 and 5) of Lemma 3.2, there exist an entire function F and holomorphic functions $\widehat{f},\ \widehat{g}$ on $F(\mathbf{C})$ such that $f = \widehat{f} \circ F,\ g = \widehat{g} \circ F$. Further there exists a subset $\mathcal{B}' \subset \mathcal{B}$ such that $\sharp F(\mathcal{B}') = 1,\ \sharp \mathcal{B}' \geq N/(mq)$, where $q = \max\{\mathrm{ord}_0 f, 2\}$. Since we can choose N arbitrarily large, F is transcendental.

Assume that $F = a + e^Q$ with a constant a and an entire function Q. Then $f = \widehat{f}(a + e^w) \circ Q$. Since f is left-prime, Q is a polynomial. Since $\widehat{f}(a + e^w)$ is periodic, we have a contradiction from 4) of Lemma 3.2. Thus $F(\mathbf{C}) = \mathbf{C}$ and $\widehat{f},\ \widehat{g}$ are entire. Since F is transcendental and f is left-prime, we see that $\widehat{f}$ is linear. Hence $g = \widehat{g} \circ \widehat{f}^{-1} \circ f$. Put $g_1 = \widehat{g} \circ \widehat{f}^{-1}$. Then $g = g_1 \circ f$. Note that $f \circ g_1 = g_1 \circ f$. If g_1 is transcendental, then by the same reasoning, there exists an entire function g_2 such that $g = g_1 \circ f = g_2 \circ f^2$. Similarly we have $g = g_n \circ f^n$ $(n = 1, 2, \dots)$, whenever all g_n $(n = 1, 2, \dots)$ are transcendental. By Lemma B and Lemma C, this is a contradiction. Thus g_n is a polynomial for some n. By Lemma A we have $g_n(z) = \alpha z + \beta$ $(\alpha \neq 0)$. Hence $g = \alpha f^n + \beta$. By 6) of Lemma 3.2, we complete the proof of Theorem 1.

4. Proof of Theorem 2

First we prove the following lemma.

Lemma 4.1. *Let f be a transcendental entire function. Put $f_a(z) = (z-a)f(z)$. Then there exists a countable set $E \subset \mathbf{C}$ such that, if f_a is permutable with $\alpha z + \beta$ $(\alpha \neq 0)$, then $\alpha = 1,\ \beta = 0$ for each $a \notin E$.*

Proof. 1) Assume that $f_a(z+\alpha) = f_a(z) + \alpha,\ f_b(z+\beta) = f_b(z) + \beta$ $(a \neq b,\ \alpha \neq 0,\ \beta \neq 0)$. Put $h_a(z) = f_a(z) - z,\ h_b(z) = f_b(z) - z$. Then $h_a,\ h_b$ are periodic functions with periods $\alpha,\ \beta$ respectively. We may assume that $\alpha,\ \beta$ are their fundamental periods. Since $(z-a)f(z) = h_a(z) + z,\ (z-b)f(z) = h_b(z) + z$, we have $f(z) = (h_a - h_b)/(b-a)$. Therefore $(h_a - h_b)/(b-a) = h_a/(z-a) + 1 + a/(z-a)$. Hence $h_b = h_a + (a-b) + (a-b)(h_a + a)/(z-a)$. This relation implies that h_b is bounded in $\{\alpha z : |\Im z| < K\} \cup \{\beta z : |\Im z| < K\}$, where K is an arbitrarily fixed positive number. Therefore, if $\alpha/\beta \notin \mathbf{R}$, then h_b is bounded in $\mathbf{C}$. This is a contradiction. Hence $\alpha/\beta \in \mathbf{R}$.

We shall show that $\alpha = \pm\beta$. Assume that $\alpha \neq \pm\beta$. Further assume $|\beta| > |\alpha|$. Let $0 < \epsilon < |\alpha|$ and $|w| < \epsilon$. Then

$$
\begin{aligned}
(1)\qquad & h_b(n\alpha + w) - h_b((n+1)\alpha + w) \\
&= (a-b)(h_a(w) + a)\Big(\frac{1}{n\alpha + w - a} - \frac{1}{(n+1)\alpha + w - a}\Big) \\
&= O(1/n) \quad (n \to \infty).
\end{aligned}
$$

Put $M(c) = \max\{|h_b(z) - h_b(z+\alpha)| : |z-c| \leq \epsilon\}$. Since $\alpha \neq \pm\beta$, we have $M(c) > 0$ for all $c \in \mathbf{C}$. Further $M(c)$ is a continuous function. Therefore $\inf\{M(x\beta) : x \in \mathbf{R}\} = \inf\{M(x\beta) : 0 \leq x \leq 1\} > 0$. This contradicts (1). Hence $\alpha = \pm\beta$. Thus $h_a,\ h_b$ have a period α.

Since $(b-a)f = h_a - h_b$, f and f' have a period α. From $f_a' = h_a' + 1 = f + (z-a)f'$, we see that $(z-a) = (h_a' + 1 - f)/f'$ has a period α. This is a contradiction. Thus any linear polynomial $z + \alpha$ $(\alpha \neq 0)$ is not permutable with f_a for all $a \in \mathbf{C}$ with at most one exception.

2) Next we assume that $f_a(\alpha z + \beta) = \alpha f_a(z) + \beta$ $(\alpha \neq 1, 0,\ \beta \neq 0)$. From Lemma A we have $f_a(z) = c + (z-c)F((z-c)^p)$ $(p \geq 2,\ c \in \mathbf{C})$. Then $f_a(c) = c,\ f_a''(c) = 0$. Therefore $a = c - c/f(c),\ cf''(c) + 2f(c)f'(c) = 0,\ f(c) \neq 0$. Assume that $zf''(z) + 2f(z)f'(z) \equiv 0$. Then we have $(zf'(z))' - f'(z) + (f(z)^2)' = 0$. Hence $zf' = -f^2 + f + k$ $(k \in \mathbf{C})$. Therefore either $f(z) = (\log z + d)^{-1} + 1/2$ $(k = -1/4,\ d \in \mathbf{C})$ or $f(z) = d(d'z^d - 1)^{-1} + (d+1)/2$ $(k \neq -1/4,\ d, d' \in \mathbf{C} \setminus \{0\})$. This is a contradiction. Thus $zf''(z) + 2f(z)f'(z) \not\equiv 0$. Put $E' = \{z - z/f(z) : zf''(z) + 2f(z)f'(z) = 0,\ f(z) \neq 0\}$. Then any linear polynomial $\alpha z + \beta$ $(\alpha \neq 0, 1,\ \beta \neq 0)$ is not permutable with f_a for each $a \notin E'$.

3) Finally we assume that $f_a(\alpha z) = \alpha f_a(z)$ $(\alpha \neq 1, 0)$. From Lemma A we have $f_a(z) = zF(z^p)$ $(p \geq 2)$. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$. Assume that $a \neq 0$.

Then $c_0 = 0$ and $\sum_{k=0}^{\infty}(c_k - ac_{k+1})z^k = F(z^p)$. Therefore $c_k - ac_{k+1} = 0$ $(k \notin p\mathbf{N})$. Since $(z-a)f(z) = zF(z^p)$, $f(z) \not\equiv zG(z^p)$ for any entire function G. Therefore $c_{K+1} \neq 0$ for some $K \notin p\mathbf{N}$. Hence $c_K - ac_{K+1} = 0$. Thus we have $a = c_K/c_{K+1}$. Put $E'' = \{0\} \cup \{kf^{(k-1)}(0)/f^{(k)}(0) : f^{(k)}(0) \neq 0,\ k \in \mathbf{N}\}$. Then any linear polynomial αz $(\alpha \neq 0, 1)$ is not permutable with f_a for each $a \in \mathbf{C} \setminus E''$. □

Lemma 4.2. *Let f be a transcendental entire function. Put $f_a(z) = (z - a)f(z)$. Then there exists a countable set $E_f \subset \mathbf{C}$ such that f_a satisfies the following conditions for each $a \notin E_f$.*

1) *f_a is prime.*
2) *$\sharp\{z \in D_a : f_a(z) = c,\ f_a'(z) = 0\} \leq 1$ for all $c \in \mathbf{C}^*$, where $D_a = \{z \in \mathbf{C} : f_a(z) \neq 0\}$.*
3) *f_a is not periodic.*
4) *f_a' has only simple zeros in D_a.*
5) *If $\alpha z + \beta$ $(\alpha \neq 0)$ is permutable with f_a, then $\alpha = 1,\ \beta = 0$.*
6) *There exists a positive integer K such that $\{f_a(z) : f_a'(z) = 0,\ z \in D_a\} \cap \{f_a{}^n(0) : n \geq K\} = \emptyset$.*

Proof. 1) and 2) follow from Theorem 3 and Lemma 2 in [9] respectively.

3) It is easy to see that f_a is not periodic for every $a \in \mathbf{C}$ with at most one exception.

4) If $z \in \mathbf{C}$ satisfies $f_a(z) \neq 0,\ f_a'(z) = 0,\ f_a''(z) = 0$, then we have $f'(z) \neq 0,\ f(z)f''(z) - 2f'(z)^2 = 0,\ a = z + f(z)/f'(z)$. Assume that $f(z)f''(z) - 2f'(z)^2 \equiv 0$. Then $f''/f' \equiv 2f'/f$. Hence $(\log f')' = 2(\log f)'$, $f' = cf^2$ $(c \in \mathbf{C} \setminus \{0\})$. Therefore $f = -(cz + d)^{-1}$ $(d \in \mathbf{C})$. This is a contradiction. Thus $f(z)f''(z) - 2f'(z)^2 \not\equiv 0$. Put $E = \{z + f(z)/f'(z) : f(z)f''(z) - 2f'(z)^2 = 0,\ f'(z) \neq 0\}$. Then, for every $a \notin E$, we have $\{z \in D_a : f_a'(z) = 0,\ f_a''(z) = 0\} = \emptyset$. Therefore f_a' has only simple zeros in D_a.

5) follows from Lemma 4.1.

6) Assume that $f(0) \neq 0, -1$. Put $F(a) = (f(0) + 1)f(-af(0))$. Then $f_a(0) = -af(0),\ f_a{}^2(0) = -aF(a),\ f_a{}^3(0) = -a(F(a) + 1)f(-aF(a)), \ldots$. Put $\varphi_n(a) = f_a{}^n(0)$ $(n = 1, 2, \ldots)$. If $z \in \mathbf{C}$ satisfies $f_a(z) \neq 0,\ f_a'(z) = 0,\ f_a(z) = f_a{}^n(0)$ for some $n \in \mathbf{N}$, then $(z - a)f(z) = \varphi_n(a) \neq 0,\ (z - a)f'(z) + f(z) = 0$. Therefore we have $f'(z) \neq 0,\ a = z + f(z)/f'(z)$ and $f(z)^2/f'(z) + \varphi_n(z + f(z)/f'(z)) = 0$. Put

$$\Phi_n(z) = f(z)^2/f'(z) + \varphi_n(z + f(z)/f'(z)).$$

If $\Phi_n(z) \equiv 0,\ \Phi_m(z) \equiv 0$ for $n \neq m$, then we have $\varphi_n(z) \equiv \varphi_m(z)$. By Lemma C, this is a contradiction. Therefore there exists a positive integer K such that $\Phi_n \not\equiv 0$ $(n \geq K)$. Put $X = \cup_{n=K}^{\infty}\{z : \Phi_n(z) = 0,\ f'(z) \neq$

$0\}$, $E' = \{z + f(z)/f'(z) : z \in X\}$. Then, for every $a \notin E'$, we have $\{f_a(z) : f_a'(z) = 0,\ z \in D_a\} \cap \{f_a{}^n(0) : n \geq K\} = \emptyset$.

If $f(0) = 0$, then $f_a{}^n(0) = 0$ $(n = 1, 2, \dots)$. Therefore $f_a(z) = f_a{}^n(0)$ implies $z \notin D_a$.

If $f(0) = -1$, then $f_a{}^{2n}(0) = 0,\ f_a{}^{2n-1}(0) = a$ $(n = 1, 2, \dots)$. If $z \in \mathbf{C}$ satisfies $f_a(z) \neq 0,\ f_a'(z) = 0,\ f_a(z) = f_a{}^n(0)$ for some $n \in \mathbf{N}$, then we have $a \neq 0,\ f'(z) \neq 0,\ a = z + f(z)/f'(z),\ zf'(z) + (f(z) + 1)f(z) = 0$. Assume that $zf' \equiv -(f + 1)f$. Then $f = (cz - 1)^{-1}$ $(c \in \mathbf{C})$. This is a contradiction. Thus $zf' \not\equiv -(f + 1)f$. Put $E'' = \{z + f(z)/f'(z) : zf'(z) + (f(z) + 1)f(z) = 0,\ f'(z) \neq 0\}$. Then, for every $a \notin E''$, we have $\{f_a(z) : f_a'(z) = 0,\ z \in D_a\} \cap \{f_a{}^n(0) : n \geq 1\} = \emptyset$. □

Lemma 4.3. *Let f, g be transcendental entire functions satisfying $f \circ g = g \circ f$. Assume that $\sharp\{z : f'(z) = 0,\ f(z) \neq 0\} = \infty$, $\sharp\{z : f'(z) = 0,\ f(z) = c\} < \infty$ for every $c \in \mathbf{C}^*$ and that there exists a positive integer K such that $\{f(z) : f'(z) = 0,\ f(z) \neq 0\} \cap \{f^n(0) : n \geq K\} = \emptyset$. Then, for every $N \in \mathbf{N}$, there exists $c \in \{z : g'(z) = 0,\ g(z) \neq 0\}$ such that $\sharp\{z : f(z) = c,\ f'(g(z)) = 0\} \geq N$.*

Proof. Put $\mathcal{E} = \{g(z) : g'(z) = 0\} \setminus \{f^n(0) : n \geq 0\}$. First we shall prove that $\sharp\mathcal{E} = \infty$. Assume that $\sharp\mathcal{E} < \infty$. Put

$$\begin{aligned}
\mathcal{A} &= \{z : f'(z) = 0,\ f(z) \neq 0\} \cap \mathcal{E}^C, \\
\mathcal{B} &= \{z : f'(z) = 0,\ f(z) \neq 0\} \cap \mathcal{E}, \\
\mathcal{C} &= \{z : f'(z) = 0,\ f(z) = 0\}, \\
\mathcal{D} &= \{z : g'(z) = 0\}.
\end{aligned}$$

By the assumption, $\sharp\mathcal{A} = \infty$. If $z \in g^{-1}(\mathcal{A})$, then $f'(g(z)) = 0$. From $f'(g(z))g'(z) = g'(f(z))f'(z)$, we have $g'(f(z)) = 0$ or $f'(z) = 0$. Therefore

$$g^{-1}(\mathcal{A}) \subset \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup f^{-1}(\mathcal{D}).$$

Put

$$\mathcal{A}_0 = \{a \in \mathcal{A} : g^{-1}(\{a\}) \cap \mathcal{A}^C \neq \emptyset\}.$$

By Lemma 3.1, $\mathcal{A}_0 \neq \emptyset$. We shall show that $\sharp\mathcal{A}_0 < \infty$. Let $a \in \mathcal{A}_0$, $x \in g^{-1}(\{a\}) \cap \mathcal{A}^C$. Then, by $x \in \mathcal{A}^C$ and $x \in g^{-1}(\mathcal{A})$, we have $x \in \mathcal{B} \cup \mathcal{C} \cup f^{-1}(\mathcal{D})$. Therefore

$$\begin{aligned}
f(a) &= f \circ g(x) = g \circ f(x) \\
&\in g \circ f(\mathcal{B} \cup \mathcal{C} \cup f^{-1}(\mathcal{D})) \\
&= g \circ f(\mathcal{B}) \cup g \circ f(\mathcal{C}) \cup g \circ f(f^{-1}(\mathcal{D})) \\
&\subset g \circ f(\mathcal{E}) \cup \{g(0)\} \cup g(\mathcal{D})
\end{aligned}$$

$$\subset g \circ f(\mathcal{E}) \cup \{g(0)\} \cup \mathcal{E} \cup \{f^n(0) : n \geq 0\}.$$

By the definition of $\mathcal{A}$, we have $f(a) \neq 0,\ f'(a) = 0$. Therefore, by the assumption, $f(a) \notin \{f^n(0) : n \geq K\}$. Thus

$$f(a) \in g \circ f(\mathcal{E}) \cup \{g(0)\} \cup \mathcal{E} \cup \{f^n(0) : 0 \leq n < K\}.$$

The right-hand side is a finite set. Therefore, by the assumption $\sharp\{z : f'(z) = 0,\ f(z) = c\} < \infty$ ($c \in \mathbf{C}^*$), we have $\sharp\mathcal{A}_0 < \infty$.

Put $\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{A}_0$. Then we have $\sharp\mathcal{A}_1 = \infty$. By Lemma 3.1, for every $N \in \mathbf{N}$, there exists $c \in \mathcal{A}_1$ such that

$$\sharp(g^{-1}(\{c\}) \cap {\mathcal{A}_1}^C) \geq N.$$

If we choose $N > \sharp\mathcal{A}_0$, then $g^{-1}(\{c\}) \cap \mathcal{A}^C \neq \emptyset$. This contradicts $c \notin \mathcal{A}_0$. Thus we have $\sharp\mathcal{E} = \infty$.

Put $\mathcal{F} = \{z : g'(z) = 0,\ g(z) \notin \{f^n(0) : n \geq 0\}\}$. From $\sharp\mathcal{E} = \infty$, we have $\sharp\mathcal{F} = \infty$. By Lemma 3.1, for every $N \in \mathbf{N}$, there exists $c \in \mathcal{F}$ such that

$$\sharp(f^{-1}(\{c\}) \cap \mathcal{F}^C) \geq N.$$

Since $c \in \mathcal{F}$, we have $g'(c) = 0,\ g(c) \notin \{f^n(0) : n \geq 0\}$. If $x \in f^{-1}(\{c\}) \cap \mathcal{F}^C$, then $f' \circ g(x) g'(x) = g' \circ f(x) f'(x) = g'(c) f'(x) = 0$. Therefore $f' \circ g(x) =$ or $g'(x) = 0$. If $g'(x) = 0$, then, from $x \notin \mathcal{F}$, we have $g(x) = f^n(0)$ for some $n \geq 0$. Therefore $f^{n+1}(0) = f \circ g(x) = g \circ f(x) = g(c)$. This contradicts $c \in \mathcal{F}$. Hence $f' \circ g(x) = 0$. Thus $\sharp\{x : f(x) = c,\ f' \circ g(x) = 0\} \geq \sharp(f^{-1}(\{c\}) \cap \mathcal{F}^C) \geq N$. □

Lemma 4.4. *Let $f = Pe^Q$ with polynomials $P,\ Q$ ($P \not\equiv 0,\ Q \not\equiv$ const.) and $f_a = (z-a)f(z)$. Then there exists a finite set $E \subset \mathbf{C}$ such that $\sharp {f_a}^{-1}(\{z : g'(z) = 0\}) = \infty$ holds for every $a \notin E$ and every transcendental entire function g which is permutable with f_a.*

Proof. First we assume that either $\deg P \geq 1$ or $\deg Q \geq 2$. Then we have $f_a' = (P + (z-a)(P' + PQ'))e^Q$, $\deg(P + (z-a)(P' + PQ')) \geq \max\{\deg P, 1\}$. Let $R,\ S,\ T$ be polynomials such that $P + (z-a)(P' + PQ') = R(S + (z-a)T)$, $\{z : S(z) = T(z) = 0\} = \emptyset$. Assume that $\{z : f_a'(z) = 0\} = \{c\}$ ($c \in \mathbf{C}$). Then $\deg T \geq \max\{\deg S, 1\}$, $S(c) + (c-a)T(c) = 0$, $S'(c) + T(c) + (c-a)T'(c) = 0$. Therefore $T(c) \neq 0$, $a = c + S(c)/T(c)$, $S(c)T'(c) - S'(c)T(c) - T(c)^2 = 0$. Put $E = \{z + S(z)/T(z) : S(z)T'(z) - S'(z)T(z) - T(z)^2 = 0,\ T(z) \neq 0\}$. Then f_a' has at least two distinct zero points for each $a \notin E$. Therefore $\sharp\{z : f_a' \circ g(z) = 0,\ f_a'(z) \neq 0\} = \infty$. From $f_a' \circ g g' = g' \circ f_a f_a'$, we have $\sharp {f_a}^{-1}(\{z : g'(z) = 0\}) = \infty$ ($a \notin E$).

Next we assume that $P \equiv$ const. and $\deg Q = 1$. Let $f(z) = e^{cz+d}$ ($c \neq 0$). Then we have ${f_a}^{-1}(\{0\}) = \{a\}$, $(f_a')^{-1}(\{0\}) = \{(ca-1)/c\}$. Put

$p = (ca-1)/c$. Assume that $\{z : g'(z) = 0\} = \emptyset$. Since $f_a' \circ gg' = g' \circ f_a f_a'$, we get $g^{-1}(\{p\}) = \{p\}$. If $a \neq 1/c$, then $p \neq 0$. By Picard's theorem, we have $\sharp\{z : f_a(z) = p\} = \infty$. From $f_a \circ g = g \circ f_a$, we see that $(f_a \circ g)^{-1}(\{p\}) = g^{-1}(\{z : f_a(z) = p\}) = \{z : f_a(z) = p\}$. By Lemma 3.1, this is a contradiction. Thus $\{z : g'(z) = 0\} \neq \emptyset$.

If $\sharp {f_a}^{-1}(\{z : g'(z) = 0\}) < \infty$, then by Picard's theorem $\sharp\{z : g'(z) = 0\} = 1$ and the only element of $\{z : g'(z) = 0\}$ is a Picard exceptional value of f_a. Therefore $\{z : g'(z) = 0\} = \{0\}$. Hence ${f_a}^{-1}(\{z : g'(z) = 0\}) = \{a\}$. Further $g'(0) = 0$. From $0 = f_a' \circ g(0)g'(0) = g' \circ f_a(0)f_a'(0)$ we have either $g' \circ f_a(0) = 0$ or $f_a'(0) = 0$. Thus $a = 0$ or $p = (ca-1)/c = 0$. Therefore we have $\sharp {f_a}^{-1}(\{z : g'(z) = 0\}) = \infty$ for all $a \in \mathbf{C} \setminus \{0, 1/c\}$. □

Proof of Theorem 2. We assume that f_a satisfies the conclusion of Lemma 4.2. Let g be a transcendental entire function which is permutable with f_a. First we assume that f is not of the form $f = Pe^Q$ with polynomials P, Q. In this case f'/f is not a rational function. We see that $f_a'(z) = 0$ if and only if $z + f(z)/f'(z) = a$ in D_a. Therefore, by Picard's theorem, $\sharp\{z \in D_a : f_a'(z) = 0\} = \infty$ holds for every $a \in \mathbf{C}$ with at most two exceptions. By Lemma 4.3 and 6) of Lemma 4.2, for every $N \in \mathbf{N}$, there exists $c \in \{z : g'(z) = 0,\ g(z) \neq 0\}$ such that $\sharp\{z : f_a(z) = c,\ f_a' \circ g(z) = 0\} \geq N$. Put $\mathcal{A} = \{z : f_a(z) = c,\ f_a' \circ g(z) = 0\}$. Since $f_a \circ g(\mathcal{A}) = g \circ f_a(\mathcal{A}) = \{g(c)\}$, we have $g(\mathcal{A}) \subset \{z : f_a(z) = g(c),\ f_a'(z) = 0\}$, $g(c) \neq 0$. By 2) of Lemma 4.2, we have $\sharp g(\mathcal{A}) = 1$. On the other hand $f_a(\mathcal{A}) = \{c\}$ and $g(\mathcal{A}) \subset D_a$. Thus, by 4) of Lemma 4.2, we see that f_a, g satisfy the assumption of Lemma 2.5. From the same reasoning as in §3, we have $g = \alpha {f_a}^n + \beta$ $(\alpha \neq 0)$. From 5) of Lemma 4.2, we deduce the desired result.

Next we assume that $f = Pe^Q$ with polynomials P, Q. By Lemma 4.4, we may assume that $\sharp {f_a}^{-1}(\{z : g'(z) = 0\}) = \infty$. From Lemma 3.1, we see that for every $N \in \mathbf{N}$, there exists $c \in \{z : g'(z) = 0\}$ such that $\sharp\{z : f_a(z) = c,\ g'(z) \neq 0\} \geq N$. By $f_a' \circ gg' = g' \circ f_a f_a'$, we have $\sharp\{z : f_a(z) = c,\ f_a' \circ g(z) = 0\} \geq N$. On the other hand $\sharp\{z : f_a(z) = 0\} \leq (\deg P) + 1$. Therefore, from 2) of Lemma 4.2, we see that $\sharp\{z \in \mathbf{C} : f_a(z) = c,\ f_a'(z) = 0\} \leq (\deg P) + 1$ for all $c \in \mathbf{C}$. Therefore, by the same reasoning as in §3, we get $g = \alpha {f_a}^n + \beta$ $(\alpha \neq 0)$. From 5) of Lemma 4.2, we have the desired result.

5. Proof of Theorem 3

We need an extension of Lemma 2.5 as follows.

Lemma 5.1. *Let f be a periodic entire function with a period $2\pi i$ and g be an entire function satisfying $f \circ g = g \circ f$ and (F, S) be a greatest common right factor of f, g. Assume that there exists a subset $\mathcal{A} \subset \mathbf{C}$ such that $\sharp f(\mathcal{A}) = 1$, $\sharp \exp \circ g(\mathcal{A}) = 1$. Let q be the order of f at some point of $g(\mathcal{A})$. Then there exists a subset $\mathcal{A}' \subset \mathcal{A}$ such that $\sharp F(\mathcal{A}') = 1$, $\sharp \mathcal{A}' \geq \sharp \mathcal{A}/q$.*

Proof. We may assume that $q < \sharp\mathcal{A}$. Let $\sharp\mathcal{A} = K$ and $\mathcal{A} = \{z_j\}_{j=1}^K$. Further let n_j $(j = 1, \dots, K)$ be integers satisfying $n_1 = 0,\ g(z_1) = g(z_j) + 2n_j\pi i$ $(j = 2, \dots, K)$. Put $a = g(z_1)$. Assume that the order of f at a is q $(q < K)$. Let U be a neighborhood of a such that $f(z) = f(a) + (\psi(z))^q$ $(z \in U)$, where ψ is a conformal map in U with $\psi(a) = 0,\ \psi(U) = \{|w| < \epsilon\}$ $(\epsilon > 0)$. Let x_j $(j = 1, \dots, K)$ be distinct points near z_j satisfying $f(x_1) = \cdots = f(x_K)$. Further let U_j $(j = 1, \dots, K)$ be neighborhoods of x_j such that $f|_{U_j}$ is an injection and that $f(U_1) = \cdots = f(U_K)$, $(g + 2n_j\pi i)(U_j) \subset U$ $(j = 1, \dots, K)$.

Put $V = f(U_1),\ \varphi_j = (f|_{U_j})^{-1}$ $(j = 1, \dots, K)$. Then we have $g = g \circ f \circ \varphi_1 = \cdots = g \circ f \circ \varphi_K$ in V. Hence $f \circ g \circ \varphi_1 = \cdots = f \circ g \circ \varphi_K$. Therefore $f \circ (g + 2n_1\pi i) \circ \varphi_1 = \cdots = f \circ (g + 2n_K\pi i) \circ \varphi_K$. From $q < K$ there exists a subset $\mathcal{N} \subset \{1, 2, \dots K\}$ with $\sharp\mathcal{N} \geq K/q$ such that $g \circ \varphi_\mu + 2n_\mu\pi i \equiv g \circ \varphi_\nu + 2n_\nu\pi i$ in V for every $\mu,\ \nu \in \mathcal{N}$. Hence $(\exp \circ g) \circ \varphi_\mu \equiv (\exp \circ g) \circ \varphi_\nu$ $(\mu, \nu \in \mathcal{N})$. On the other hand $f \circ \varphi_\mu \equiv f \circ \varphi_\nu \equiv \mathrm{id.}$ $(\mu, \nu \in \mathcal{N})$. Therefore $x_\mu \sim x_\nu$ $(\mu, \nu \in \mathcal{N})$. Let $\mathcal{A}' = \{z_\nu : \nu \in \mathcal{N}\}$. By the same reasoning as in the proof of Lemma 2.5, we have $\sharp F(\mathcal{A}') = 1$. □

Let f be a periodic entire function with a fundamental period λ. Put $z^* = \{z + n\lambda : n \in \mathbf{Z}\}$, $(A)^* = \{z^* : z \in A\}$ $(A \subset \mathbf{C},\ A \neq \mathbf{C})$ and

$$\nu(r, 1/(f - a)) = \sharp\{z^* : f(z) = a,\ |z| \leq r\} \quad (a \in \mathbf{C}).$$

The following result is an extension of Lemma D for periodic entire functions. We define $T(r, f),\ N(r, f),\ m(r, f)$ for a meromorphic function f as follows. We assume that $f(0) \neq \infty$. For the details, see [7].

$$A(r, f) = \frac{1}{\pi} \iint_{|z|<r} \frac{|f'(z)|^2 dxdy}{(1 + |f(z)|^2)^2}, \quad T(r, f) = \int_0^r A(t, f) t^{-1} dt,$$

$$n(r, f) = \sum_{z \in f^{-1}(\{\infty\}) \cap \{|z| \leq r\}} \mathrm{ord}_z f, \quad N(r, f) = \int_0^r n(t, f) t^{-1} dt,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(z)|^2)^{1/2} d\theta.$$

Lemma 5.2. *Let $g(z)$ be a periodic entire function. Given $K > 0$ there is a number R_0 and an increasing sequence $\{R_n\}_{n \in \mathbf{N}}$ with $R_1 > R_0$ and $R_n \to \infty$ $(n \to \infty)$ such that for $n \geq 1$ and all ω satisfying $R_0 \leq |\omega| \leq R_n$ we have $\nu(R_n, 1/(g - \omega)) > K$.*

Proof. First we assume that there exists a positive number R_0 such that for every $\omega \in \mathbf{C}$ satisfying $|\omega| \geq R_0$ we have $\nu(|\omega|, 1/(g - \omega)) > K$. If $R_0 \leq R$ and $R_0 \leq \omega \leq R$, then $\nu(R, 1/(g - \omega)) \geq \nu(|\omega|, 1/(g - \omega)) > K$. Thus we have the desired result.

Next we assume that there exists a sequence $\{\omega_n\}_{n=1}^{\infty}$ of complex numbers such that $\{|\omega_n|\}_{n=1}^{\infty}$ is a strictly increasing sequence satisfying $|\omega_n| \to \infty$ $(n \to \infty)$ and $\nu(|\omega_n|, 1/(g-\omega_n)) \le K$ $(n = 1, 2, \dots)$. Put $R_n = |\omega_n|/2$ $(n = 1, 2, \dots)$. Then

$$\nu(2R_n, 1/(g-\omega_n)) \le K \quad (n = 1, 2, \dots). \tag{2}$$

We choose β so that $|\beta| > |g(0)|$ and $N(r, 1/(g-\beta)) \sim T(r, g)$ $(r \to \infty)$. We put $R_0 = |g(0)| + |\beta| + 1$ and assume that $R_1 > R_0$.

Now we assume that for every n there exists Ω_n such that $R_0 \le |\Omega_n| \le R_n$ and

$$\nu(R_n, 1/(g-\Omega_n)) \le K \quad (n = 1, 2, \dots). \tag{3}$$

Let α be a non-zero complex number. From (3.3) of [7, p.245], we have

$$m(r, g'/(g-\alpha)) \le 4\log^+|\alpha| + 8\log T(r, g)$$
$$(r \notin E,\ E \subset \mathbf{R}_+,\ |E| < \infty),$$

where E depends only on g. (In (3.3) of [7, p.245], set $w = g'/(g-\alpha)$ and $\rho = r + 1/\log T(r, g)$, and use Lemma 2 of [7, p.245].) Therefore we can choose ρ_n with $R_n/2 \le \rho_n \le R_n$ such that

$$m(\rho_n, g'/(g-\omega_n)) = o(T(\rho_n, g)) \quad (n \to \infty),$$
$$m(\rho_n, g'/(g-\Omega_n)) = o(T(\rho_n, g)) \quad (n \to \infty).$$

By (2) and (3), we have

$$n(r, g'/(g-\omega_n)) \le K'r, \quad n(r, g'/(g-\Omega_n)) \le K'r \quad (0 < r \le R_n).$$

(Note that $g'/(g-\omega_n),\ g'/(g-\Omega_n)$ have only simple poles.) Therefore

$$N(r, g'/(g-\omega_n)) \le K'r, \quad N(r, g'/(g-\Omega_n)) \le K'r \quad (0 < r \le R_n).$$

Since $T(r, g)/r \to \infty$ $(r \to \infty)$, we have

$$\begin{aligned} T(\rho_n, g'/(g-\omega_n)) &= o(T(\rho_n, g)) \quad (n \to \infty), \\ T(\rho_n, g'/(g-\Omega_n)) &= o(T(\rho_n, g)) \quad (n \to \infty). \end{aligned} \tag{4}$$

By the same reasoning as in Clunie [3], we have a contradiction. For the sake of completeness, we describe the proof. Put $\alpha_n = -(\beta-\omega_n)/(\beta-\Omega_n)$ and $h_n = g'/(g-\omega) + \alpha_n g'/(g-\Omega_n)$. Then $h_n = (\omega_n - \Omega_n)(g -$

$\beta)g'/[(g-\omega_n)(g-\Omega_n)]$ and $\log|\alpha_n| = O(\log \rho_n)$ $(n \to \infty)$. From (4) we have $T(\rho_n, h_n) = o(T(\rho_n, g))$. We note that

$$\begin{aligned}\log|1/h_n(0)| &= O(\log(|g(0)-\omega_n||g(0)-\Omega_n|/|\omega_n-\Omega_n|)) \\ &= O(\log \rho_n) \quad (n \to \infty).\end{aligned}$$

Therefore

$$\begin{aligned}&N(\rho_n, 1/h_n) \le T(\rho_n, 1/h_n) = T(\rho_n, h_n) + \log|1/h_n(0)| \\ &= T(\rho_n, h_n) + O(\log \rho_n) = o(T(\rho_n, g)) + O(\log \rho_n) \\ &= o(T(\rho_n, g)) \quad (n \to \infty).\end{aligned}$$

On the other hand $N(\rho_n, 1/h_n) \ge N(\rho_n, 1/(g-\beta)) = (1+o(1))T(\rho_n, g)$. This is a contradiction. Thus we have the desired result. □

Lemma 5.3. *Let f be a periodic entire function. Let A be a subset of $\mathbf{C}$ satisfying that $\sharp(f^{-1}(A))^* = \infty$ and $\sharp(A \cap \{|z| < r\}) < \infty$ for every $r > 0$. Then $\sup_{w \in A} \sharp(f^{-1}(\{w\}) \cap A^C)^* = \infty$.*

Proof. If $\sharp A < \infty$, it is obvious. Assume that $\sharp A = \infty$. Let K, $\{R_n\}_1^\infty$ be the same as in Lemma 5.2. Put $N_n = \sharp(A \cap \{|z| \le R_n\})$ $(n = 0, 1, \dots)$. By Lemma 5.2, $\sharp(f^{-1}(\{w\}) \cap \{|z| \le R_n\})^* > K$ for all w with $R_0 \le |w| \le R_n$. Therefore, if $2(N_n - N_0) > N_n$, then

$$\begin{aligned}&\sharp(f^{-1}(A \cap \{R_0 < |z| \le R_n\}) \cap A^C \cap \{|z| \le R_n\})^* \\ &> K(N_n - N_0) - N_n > (K-2)(N_n - N_0).\end{aligned}$$

This implies that $\sharp(f^{-1}(\{w\}) \cap A^C \cap \{|z| \le R_n\})^* > K - 2$ for some $w \in A \cap \{R_0 < |z| \le R_n\}$. By choosing K arbitrarily large, we have the desired result. □

Lemma 5.4. *Let $h(w)$ be a holomorphic function in $\mathbf{C} \setminus \{0\}$ satisfying that $\sharp\{w : h'(w) = 0\} = \infty$. Put $f(z) = h(e^z)$. Let g be a transcendental entire function permutable with f. Then, for each $N \in \mathbf{N}$, there exists c such that $g'(c) = 0$, $\sharp\{z^* : f(z) = c,\ f'(g(z)) = 0\} \ge N$.*

Proof. Put $A = \{z : g'(z) = 0\}$ and $B = \{z : f'(z) = 0\}$. Assume that $A = \emptyset$. From $f' \circ g g' = g' \circ f f'$, we have $\{z : f'(z) = 0\} = \{z : f' \circ g(z) = 0\}$. By Lemma 3.1 and $\sharp B = \infty$, this is a contradiction. Therefore $A \ne \emptyset$. We shall show that $\sharp(f^{-1}(A))^* = \infty$. By Picard's theorem, $\sharp A \ge 2$ implies $\sharp h^{-1}(A) = \infty$. Hence $\sharp(f^{-1}(A))^* = \infty$. Therefore we assume that $\sharp A = 1$ and $\sharp h^{-1}(A) < \infty$.

Put $X = \{z : g'(z)(g' \circ f(z)) = 0\}$. Then we have $\sharp(X \cap \{|z| \le r\}) = O(r)$ $(r \to \infty)$. By Lemma D, we have $\sharp(g^{-1}(B \cap \{R_0 \le |z| \le R_n\}) \cap \{|z| \le$

$R_n\}) > K(N_n - N_0)$, where K is an arbitrarily fixed positive number, $\{R_n\}_0^\infty$ is an increasing sequence with $R_n \to \infty$ $(n \to \infty)$ and $N_n = \sharp(B \cap \{|z| \le R_n\})$ $(n = 0, 1, \dots)$. From $f' \circ gg' = g' \circ ff'$, we see that $f' \circ g = 0$ if and only if $f' = 0$ in $\mathbf{C} \backslash X$. Hence $\sharp(g^{-1}(B) \cap \{|z| \le R_n\} \cap X^C) = \sharp(B \cap \{|z| \le R_n\} \cap X^C)$. Since $\sharp\{w : h'(w) = 0\} = \infty$, we have $R_n/N_n \to 0$ $(n \to \infty)$. Therefore

$$\begin{aligned}&\sharp(g^{-1}(B) \cap \{|z| \le R_n\}) \le \sharp(B \cap \{|z| \le R_n\}) + O(R_n)\\ &= N_n + O(R_n) = N_n(1 + o(1)) \quad (n \to \infty).\end{aligned}$$

On the other hand we see that $\sharp(g^{-1}(B) \cap \{|z| \le R_n\}) \ge \sharp(g^{-1}(B \cap \{R_0 \le |z| \le R_n\}) \cap \{|z| \le R_n\}) > K(N_n - N_0)$. Thus $N_n(1 + o(R_n)) > K(N_n - N_0)$. When $K > 1$, we have a contradiction. Therefore $\sharp h^{-1}(A) = \infty$. Thus we have $\sharp(f^{-1}(A))^* = \infty$.

By Lemma 5.3, $\sup_{w \in A} \sharp(f^{-1}(\{w\}) \cap A^C)^* = \infty$. Thus, for each $N \in \mathbf{N}$, there exists c such that $g'(c) = 0$ and $\sharp\{z^* : f(z) = c,\ g'(z) \ne 0\} \ge N$. From $f' \circ gg' = g' \circ ff'$, we have $\sharp\{z^* : f(z) = c,\ f'(g(z)) = 0\} \ge N$. □

Lemma 5.5. *Let h be a holomorphic function in $\mathbf{C} \setminus \{0\}$ with essential singularities at 0, ∞ and m be a non-zero integer. Put $h_a(w) = h(w) + aw^m$, $f_a(z) = h_a(e^z)$ Then there exists a countable set $E_h \subset \mathbf{C}$ such that h_a satisfies the following conditions for each $a \notin E_h$.*

1) *f_a is prime.*
2) $\sharp\{w \in \mathbf{C} \setminus \{0\} : h_a'(w) = 0\} = \infty$.
3) $\sharp\{w^m : h_a(w) = c,\ h_a'(w) = 0,\ w \in \mathbf{C} \setminus \{0\}\} \le 1$ *for all* $c \in \mathbf{C}$.
4) *h_a' has only simple zeros in $\mathbf{C} \setminus \{0\}$.*
5) *If $\alpha z + \beta$ $(\alpha \ne 0)$ is permutable with f_a, then $\alpha = 1$, $\beta = 0$.*

Proof. 1) and 3) follow from Theorem 4 and Lemma 3 in [9] respectively.

2) Put $H(w) \equiv h'(w)/mw^{m-1}$. Then $h_a'(w) = 0$ if and only if $H(w) = -a$. From Picard's theorem, we see that $\sharp\{w \in \mathbf{C} \setminus \{0\} : h_a'(w) = 0\} = \infty$ for every $a \in \mathbf{C}$ with at most one exception.

4) follows from 5) of Lemma 3.2.

5) Suppose that $f_a(z) = h_a(e^z)$ has a fundamental period $2\pi i/p$ $(p \in \mathbf{N})$. Then $h_a(e^{\alpha z + \beta})$ has a fundamental period $2\pi i/p\alpha$. Therefore $h_a(e^{\alpha z + \beta}) = \alpha h_a(e^z) + \beta$ implies that $\alpha = \pm 1$.

Suppose that $\alpha = 1$, $\beta \ne 0$. Then $g(z) \equiv f_a(z) - z$ has a period β. Therefore $g' = f_a' - 1$ has a period β. Hence $\beta = 2\pi ik/(pq)$ $(k \in \mathbf{Z},\ q \in \mathbf{N})$. ($2\pi i/(pq)$ is a fundamental period of f_a'.) Therefore $z = f_a(z) - g(z)$ has a period $q\beta = 2\pi ik/p$. This is a contradiction.

Suppose that $\alpha = -1$. Then $-h_a(e^z) + \beta = h_a(e^{-z+\beta})$. Therefore $-h(w) - aw^m + \beta = h(e^\beta/w) + ae^{m\beta}/w^m$. Let $h(w) = \sum_{j=-\infty}^{\infty} c_j w^j$. Then we have $\beta = 2c_0$. Hence $h(w) + h(\exp(2c_0)/w) - 2c_0 = -a(w^m +$

$\exp(2mc_0)/w^m$). Therefore $a = -(c_m + c_{-m}\exp(-2mc_0))$. Thus $f_a(-z+\beta) \not\equiv -f_a(z)+\beta$ for every $a \in \mathbf{C}$ with at most one exception. □

Proof of Theorem 3. Let h be a holomorphic function in $\mathbf{C}\setminus\{0\}$ with essential singularities at 0, ∞ and $f(z) = h(e^z)$ satisfies the conclusion of Lemma 5.5. Let g is a transcendental entire function which is permutable with f. By Lemma 5.4 and 2) of Lemma 5.5, there exists c such that $g'(c) = 0$ and $\sharp\{z^* : f(z) = c,\ f'(g(z)) = 0\} \geq N$ for every positive integer N. Let $\mathcal{A}$ be a subset of $\mathbf{C}$ such that $z^* \neq w^*$ $(z, w \in \mathcal{A},\ z \neq w)$ and that $(\mathcal{A})^* = \{z^* : f(z) = c,\ f'(g(z)) = 0\}$. Since $f \circ g(\mathcal{A}) = g \circ f(\mathcal{A}) = \{g(c)\}$, we have $g(\mathcal{A}) \subset \{z : f(z) = g(c),\ f'(z) = 0\}$. Hence $g(\mathcal{A}) \subset \{z : h(e^z) = g(c),\ h'(e^z) = 0\}$. Therefore $\exp(g(\mathcal{A})) \subset \{w : h(w) = g(c),\ h'(w) = 0\}$. By 3) of Lemma 5.5, we have $\sharp\exp(mg(\mathcal{A})) = 1$. Therefore there exists a subset $\mathcal{B} \subset \mathcal{A}$ such that $\sharp\exp(g(\mathcal{B})) = 1$, $\sharp\mathcal{B} \geq N/m$. On the other hand, $f(\mathcal{B}) = \{c\}$. By Lemma 5.1 and 4) of Lemma 5.5, there exist an entire function F, holomorphic functions $\widehat{f}$, $\widehat{g}$ on $F(\mathbf{C})$ and a subset $\mathcal{B}' \subset \mathcal{B}$ such that $f = \widehat{f} \circ F$, $\exp\circ g = \widehat{g} \circ F$, $\sharp\mathcal{B}' \geq N/(2m)$, $\sharp F(\mathcal{B}') = 1$.

Assume that $F = C + e^Q$ with a constant C and an entire function Q. Then $f = \widehat{f}(C + e^w) \circ Q$. Since f is prime, we have $Q(z) = \alpha z + \beta$ $(\alpha \neq 0)$. Hence f has a period $2\pi i/\alpha$. Let $2\pi i/p$ $(p \in \mathbf{N})$ be a fundamental period of f. Then $2\pi i/\alpha = 2\pi ik/p$ for some integer k. Thus $\alpha = p/k$, $F(z) = C + \exp((zp/k) + \beta)$. From $\sharp F(\mathcal{B}') = 1$ we have $(z - w)p/k \in 2\pi i\mathbf{Z}$ for all $z,\ w \in \mathcal{B}'$. Hence $\sharp(\mathcal{B}')^* = 1$. This is a contradiction. Thus we have $F(\mathbf{C}) = \mathbf{C}$. Therefore $\widehat{f}$, $\widehat{g}$ are entire functions.

Since $\exp\circ g = \widehat{g} \circ F$, we have $\widehat{g}(z) \neq 0$ $(z \in \mathbf{C})$. Therefore $\widehat{g} = \exp\circ G$ with an entire function G. Hence $\exp\circ g = \exp\circ G \circ F$. This relation yields $g = G \circ F + 2\pi i\nu$ for some $\nu \in \mathbf{Z}$. Put $H \equiv G + 2\pi i\nu$. Then we have $g = H \circ F$. Since F is transcendental and f is prime, we see that $\widehat{f}$ is linear. Therefore $g = H \circ \widehat{f}^{-1} \circ f$. Put $g_1 = H \circ \widehat{f}^{-1}$. Then $g = g_1 \circ f$. By the same reasoning as in §3, we get $g = \alpha {f_a}^n + \beta$ $(\alpha \neq 0)$. From 5) of Lemma 5.5, we have the desired result.

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