ASYMPTOTIC CHOW SEMI-STABILITY AND INTEGRAL INVARIANTS

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Abstract. We define a family of integral invariants containing those which are closely related to asymptotic Chow semi-stability of polarized manifolds. It also contains an obstruction to the existence of Kähler-Einstein metrics and its natural extensions by the author, Calabi and Bando as Kählerian invariants and by Morita and the author as invariant polynomials of the automorphism groups of compact complex manifolds.

1. Introduction

In [6] the author defined an integral invariant for Fano manifolds, namely for compact complex manifolds with positive first Chern class, which obstructs the existence of Kähler-Einstein metrics.

Later two families of integral invariants containing this invariant as a special case were found:

(A) The first family was defined on a compact Kähler manifold of dimension \( m \) with a fixed Kähler class and parametrized by \( k \) with \( 1 \leq k \leq m \). The \( k \)-th member \( f_k \) obstructs the existence of a Kähler metric with harmonic \( k \)-th Chern form in the fixed Kähler class; this formulation is due to Bando [1]. The first Chern form is harmonic if and only if the scalar curvature is constant because of the second Bianchi identity. The case when \( k = 1 \) therefore obstructs the existence of constant scalar curvature metric; this case was also observed by the author [7] and Calabi [3]. The case when the manifold is Fano with the Kähler class \( c_1(M) \) and \( k = 1 \) reduces to the original one.

(B) The second family was studied in the author’s joint work S. Morita [11]. We began this study to understand the property of the invariant analogous to the Godbillon-Vey invariant in the foliation theory, but it turned out that the invariant could be understood as a complex analogue of the equivariant cohomology expressed by Berline-Vergne [2].

Rigorous explanations of these two families will be given in the proof of Theorem 1.1 in the next section. In this paper we present a larger family containing both of these two families.

Let \( G \) be a complex Lie group and \( P_G \rightarrow M \) a holomorphic principal bundle over a compact complex manifold \( M \) with structure group \( G \) (of right action). Suppose a complex Lie group \( H \) acts from the left on \( P_G \) covering an \( H \)-action on \( M \) and commuting with the right action of \( G \).

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Choose any type (1,0)-connection \( \theta \) of \( P_G \) and denote its curvature form by \( \Theta \). Let \( I^p(G) \) denote the set of all \( G \)-invariant polynomials of degree \( p \) on \( \mathfrak{g} \).

An element \( X \) of the Lie algebra \( \mathfrak{h} \) of \( H \) defines a \( G \)-invariant holomorphic vector field \( \tilde{X} \) on \( P_G \) and a holomorphic vector field \( X \) on \( M \). Let \( \mathfrak{h}(M) \) be the complex Lie algebra of all holomorphic vector fields on \( M \), and \( \mathfrak{h}_0(M) \) the subalgebra consisting of all \( X \in \mathfrak{h}(M) \) which has a zero. Fix any Kähler class \([\omega_0]\) of a Kähler form \( \omega \). For any Kähler form \( \omega \) in \([\omega_0]\) there exists a complex valued smooth function \( u_X \) determined up to a constant such that

\[
i(X)\omega = -\bar{\partial}u_X.
\]

We assume the normalization of \( u_X \) is so chosen that

\[
(1) \quad \int_M u_X \omega^m = 0
\]

where \( m = \dim_C M \).

For any \( \phi \in I^p(G) \) and \( X \in \mathfrak{h}(M) \) the differential form of \( \phi(\theta(\tilde{X}) + \Theta) \) on \( P_G \) is pulled back from a form on \( M \) just as in the usual Chern-Weil theory. We will denote this form on \( M \) by the same notation \( \phi(\theta(\tilde{X}) + \Theta) \). We put

\[
(2) \quad F_\phi(X) = (m - p + 1) \int_M \phi(\Theta) \wedge u_X \omega^{m-p} \\
+ \int_M \phi(\theta(\tilde{X}) + \Theta) \wedge \omega^{m-p+1}
\]

**Theorem 1.1.** \( F_\phi(X) \) is independent of the choices of the Kähler form \( \omega \in [\omega_0] \) and the type (1,0)-connection \( \theta \). The case when \( p = m + 1 \) is the same as those defined in Futaki-Morita [11], and the case when \( P_G \) is the frame bundle of the holomorphic tangent bundle of \( M \) with the natural action of the automorphism group and \( \phi = c_p \) is the same as those defined by Bando [1].

If \( M \) is a Fano manifold and \( P_G \) is the frame bundle of the holomorphic tangent bundle of \( M \) with the natural action of the automorphism group then the cases when \( \phi = c_1 \) and when \( \phi = c_{m+1} \) coincide and give the obstruction to the existence of Kähler-Einstein metrics (see [11]). If \( M \) is a general Kähler manifold then the case when \( \phi = c_1 \) is an obstruction to the existence of Kähler metric of constant scalar curvature as mentioned in (A) above. The integral invariant \( F_{c_1} \) is sometimes called the Futaki invariant. When the Kähler class is an integral class the relation of \( F_{c_1} \) with Hilbert-Mumford stability is clarified recently by Donaldson [5]. In this paper we will see that the cases when \( \phi \) is one of the Todd polynomials give obstructions to asymptotic Chow semi-stability:

**Theorem 1.2.** Let \( L \to M \) be an ample line bundle and take \( c_1(L) \) as the Kähler class. Suppose that \((M, L) \) is asymptotically semi-stable. Then for \( 1 \leq p \leq m \) we have

\[
(3) \quad F_{Td(p)}(X) = 0
\]

for \( X \) in a maximal reductive subalgebra \( \mathfrak{h}_r(M) \) of \( \mathfrak{h}_0(M) \) where \( Td(p) \) denotes the Todd polynomial of degree \( p \).

Note that, in Theorem 1.2, the case when \( p = 1 \) implies the vanishing of the Futaki invariant.
2. INTEGRAL INVARIANTS

Any Kähler form $\omega$ in the Kähler class $[\omega_0]$ is written as

\[ \omega = \omega_0 + i\partial\bar{\partial}\varphi \]

for some real smooth function $\varphi$. For the path $\omega_t = \omega_0 + it\partial\bar{\partial}\varphi$ joining $\omega$ and $\omega_0$ we have the path of Hamiltonian functions $u_{X,t}$ for $X \in h_0$. It is easy to see that they are given by

\[ u_{X,t} = u_{X,0} - itX\varphi. \]

On the other hand for a type (1,0) connection $\theta$ of $P_G$, its curvature form is $\Theta = d\theta + \frac{1}{2} [\theta, \theta]$. But the only type (1,1)-part of $\Theta$ contributes to the integration of (2), and thus we may assume $\Theta = \partial\theta$. From this

\[ i(\tilde{X})\Theta = -\partial(\theta(\tilde{X})), \tag{4} \]
\[ i(\tilde{X})\phi(\Theta) = -\partial\phi(\theta(\tilde{X}) + \Theta). \tag{5} \]

Let $p$ be the degree of $\phi$. We write $\phi(X_1, \cdots, X_p)$ for the coefficient of $t_1 \cdots t_p$ of $\phi(t_1X_1 + \cdots + t_pX_p)$. $\phi(\cdot, \cdots, \cdot)$ is called the polarization of $\phi(\cdot)$. Obviously $\phi(X) = \phi(X_1, \cdots, X)$, and an invariant polynomial $\phi$ is often identified with its polarization.

**Proof of Theorem 1.1.** Let $F_\phi(X, \omega_t)$ be the right hand side of (2) corresponding to $\omega_t$. We will show that $F_\phi(X, \omega_t)$ is independent of $t$. One computes

\[
\frac{d}{dt} F_\phi(X, \omega_t) = (m - p + 1) \int_M \phi(\Theta) \wedge (-iX\varphi)\omega^{m-p}_t + (m - p + 1) \int_M \phi(\Theta) \wedge u_{X,t} (m - p) \omega^{m-p-1}_t \wedge i\partial\bar{\partial}\varphi + \int_M \phi(\theta(\tilde{X}) + \Theta) \wedge (m - p + 1)\omega^{m-p}_t \wedge i\partial\bar{\partial}\varphi = -(m - p + 1) \int_M i(X) (\phi(\Theta) \wedge \omega^{m-p}_t \wedge i\partial\varphi)
\]

But $\phi(\Theta) \wedge \omega^{m-p}_t \wedge \partial\varphi$ is an $(2m + 1)$-form and is identically zero on $M$. Hence $F_\phi(X, \omega_t)$ is independent of $t$.

Next we consider a one parameter family $\theta_s$ of type (1,0) connections and let $F_\phi(X, \theta_s)$ be the right hand side of (2) corresponding to $\theta_s$. We will show that
\( F_{\phi}(X, \theta_s) \) is independent of \( s \). One computes
\[
\frac{d}{ds} F_{\phi}(X, \theta_s) = (m - p + 1) \int_M \phi(\tilde{\theta}_s + \Theta_s) \land u_X \omega^{m-p}
\]
\[
+ \int_M \phi(\hat{\theta}_s(\tilde{X}) + \Theta_s) \land \omega^{m-p+1}
\]
\[
+ \int_M p(p - 1) \phi(\theta_s(\tilde{X}), \tilde{\theta}_s, \Theta_s, \ldots, \Theta_s) \land \omega^{m-p+1}
\]
\[
= (m - p + 1) \int_M \phi(\hat{\theta}_s + \Theta_s) \land \tilde{\omega}_X \land \omega^{m-p}
\]
\[
+ \int_M \phi(i(\tilde{X})\hat{\theta}_s + \Theta_s) \land \omega^{m-p+1}
\]
\[
- \int_M p(p - 1) \phi(i(\tilde{\theta}_s(\tilde{X}))), \hat{\theta}_s, \Theta_s, \ldots, \Theta_s) \land \omega^{m-p+1}
\]
\[
= \int_M \phi(\hat{\theta}_s + \Theta_s) \land (-i(X)\omega^{m-p+1})
\]
\[
+ \int_M \phi(i(\tilde{X})\hat{\theta}_s + \Theta_s) \land \omega^{m-p+1}
\]
\[
- \int_M p(p - 1) \phi(\hat{\theta}_s, i(\tilde{X})\Theta_s, \Theta_s, \ldots, \Theta_s) \land \omega^{m-p+1}
\]
\[
= \int_M i(X)(\phi(\hat{\theta}_s + \Theta_s) \land \omega^{m-p+1}),
\]
where on the third terms \( \phi \) is considered as its polarization. Again \( \phi(\hat{\theta}_s + \Theta_s) \land \omega^{m-p+1} \) is an \((2m+1)\)-form on \( M \) and is identically zero on \( M \) so that \( F_{\phi}(X, \theta_s) \) is independent of \( s \).

If \( p = m + 1 \) then
\[
F_{\phi}(X) = \int_M \phi(\tilde{\theta} + \Theta).
\]

These are exactly the same as the invariant polynomials of Aut(\( M \)) considered in [11].

Let us consider the case when \( P_G \) is the associated principal bundle of the holomorphic tangent bundle of \( M \). Since \( F_{\phi} \) is independent of the connection form \( \theta \), we may assume that \( \theta \) is the Levi-Civita connection of the Kähler form \( \omega \). Let \( g \) be the Kähler metric of \( \omega \). Then with respect to local holomorphic coordinates \( z^1, \ldots, z^m \) the connection matrix \( \theta_0 \) on the base is given by \( \theta_0 = g^{-1} \partial g \).

The general frame is expressed as \( e = e_0 A \) where \( e_0 = (\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m}) \) is the standard frame and \( A \in GL(m, \mathbb{C}) \). At \( e \) the connection form \( \theta \) and the curvature form \( \Theta \) are expressed as in the usual formulae
\[
\theta = A^{-1} dA + A^{-1} \theta_0 A, \quad \Theta = A^{-1} \Theta_0 A,
\]
where \( \Theta_0 \) denotes the curvature matrix with respect to the standard frame \( e_0 \). The flow of \( X \) lifts to \( P_G \) and its infinitesimal action for the fiber direction is minus the Lie derivative. Since
\[
-L_X \frac{\partial}{\partial z^4} = \frac{\partial X}{\partial z^4}
\]
the vector field $\tilde{X}$ is of the form
\[ \tilde{X} = \sum_{i,j,k} \frac{\partial X}{\partial a_j^k} a_j^k \frac{\partial}{\partial a_i^j} + X \]
at $eA$ where $A = (a_{ij})$ and $a_{ij}$ are regarded as fiber coordinates of $P_G$. Thus we get
\[
\theta(\tilde{X}) = A^{-1}(-L_X + \nabla_X)A, \\
\phi(\theta(\tilde{X}) + \Theta) = \phi(L(X) + \Theta_0)
\]
where we have set $L(X) = \nabla_X - L_X$ which is a section of the endomorphism bundle $\text{End}(T' M)$ of the holomorphic tangent bundle $T' M$. But then from the torsion freeness of the Levi-Civita connection
\[ L(X) = \nabla_X - L_X = \nabla_X. \]
If we identify the holomorphic tangent bundle $T' M$ with the cotangent bundle $T'' M^* \otimes T'' M^*$, and through this correspondence $\nabla_X$ corresponds to $i\partial\bar{\partial}u_X$. Let us take $\phi = c_p$ and see in this case $F_p$ is the same as that Bando defined.

Let $H_{c_p}(\omega)$ be the harmonic part of the $p$-th Chern form $c_p(\omega)$ with respect to $\omega$. Then there is a real $(p - 1, p - 1)$-form $F_p$ uniquely up to harmonic forms such that
\[ c_p(\omega) - H_{c_p}(\omega) = i\partial\bar{\partial}F_p. \]
Bando defined $f_p : \mathfrak{h}(M) \to \mathbb{C}$ by
\[
f_p(X) = \int_M L_X F_p \wedge \omega^{m-p+1},
\]
and proved that $f_p$ is independent of the choice of the Kähler form $\omega$ in the fixed Kähler class $[\omega_0]$. Note that $f_p$ is independent of the choice of $F_p$. Obviously $f_p$ is an obstruction to the existence of the Kähler metric with harmonic $p$-th Chern form in the fixed Kähler class. Since $H_{c_p}(\omega) \wedge \omega^{m-p} = \text{const} \ \omega^m$ we have
\[ \int_M H_{c_p}(\omega) \wedge u_X \omega^{m-p} = 0 \]
because of our normalization (1) of $u_X$. Hence
\[
\begin{align*}
(m - p + 1) \int_M c_p(\omega) \wedge u_X \omega^{m-p} & = (m - p + 1) \int_M (c_p(\omega) - H_{c_p}(\omega)) \wedge u_X \omega^{m-p} \\
& = (m - p + 1) \int_M i\partial\bar{\partial}F_p \wedge u_X \omega^{m-p} = \int_M i\partial F_p \wedge (i(X) \omega^{m-p+1}) \\
& = \int_M i(X) \partial F_p \wedge \omega^{m-p+1} = \int_M L_X F_p \wedge \omega^{m-p+1}.
\end{align*}
\]
Thus it is sufficient to show that the first term of the right hand side in (2) vanishes. But

$$\int_M c_p(\theta(\bar{X}), \Theta, \cdots, \Theta) \wedge \omega^{m-p+1} = \int_M c_m(\omega \otimes I, \cdots, \omega \otimes I, \omega \otimes L(X), \Theta, \cdots, \Theta).$$

Here the left hand side of $\otimes$ stands for the differential form part while the right hand side stands for the endomorphism part. As explained before $\text{End}(T'M) \cong T'M^* \wedge T''M^*$ by using the metric, and through this isomorphism $L(X)$ corresponds to $i\partial\bar{\partial}u_X$. Furthermore both of form part and endomorphism part of $c_m$ are determinant. In this sense the form part and the endomorphism part of $c_m$ are symmetric. Thus we have

$$\int_M c_m(\omega \otimes I, \cdots, \omega \otimes I, i\partial\bar{\partial}u \otimes I, \Theta, \cdots, \Theta)$$

$$= - \int_M \bar{\partial}c_m(\omega \otimes I, \cdots, \omega \otimes I, i\partial u \otimes I, \Theta, \cdots, \Theta)$$

$$= 0.$$

This completes the proof of Theorem 1.1. □

3. **Geometric invariant theory**

Geometric invariant theory is an idea that, to have a moduli space with good properties such as Hausdorff property and compactness, one may collect semi-stable ones ([14]). The definitions of stability and semi-stability are stated as follows. Let $V$ be a vector space and $G$ a subgroup of $\text{SL}(V)$. $x \in V$ is said to be stable with respect to the $G$-action if the orbit $Gx$ of $x$ is closed in $V$ and the stabilizer of $x$ is a finite subgroup of $G$, and $x \in V$ is said to be semi-stable if the closure of the orbit $Gx$ does not contain zero of $V$.

Let $L \rightarrow M$ be an ample line bundle. Put $V_k := H^0(M, L^k)^*$, and let $\Phi_{|L^k|} : M \rightarrow \text{P}(V_k)$ be the Kodaira embedding defined by using the sections of $L^k$. Let $d$ be the degree of $M$ in $\text{P}(V_k)$. An element of the product $\text{P}(V_k^*) \times \cdots \times \text{P}(V_k^*)$ of $m+1$ copies of $\text{P}(V_k^*)$ defines $m+1$ hyperplanes $H_1, \cdots, H_{m+1}$ in $\text{P}(V_k)$. The set of all $m+1$ hyperplanes such that $H_1 \cap \cdots \cap H_{m+1} \cap M$ is non-empty defines a divisor in $\text{P}(V_k^*) \times \cdots \times \text{P}(V_k^*)$. Since the degree of $M$ is $d$ this divisor is defined by some $\tilde{M}_k \in (\text{Sym}^d(V_k))^\otimes(m+1)$. Of course $\tilde{M}_k$ is determined up to constant. The point $[\tilde{M}_k] \in \text{P}((\text{Sym}^d(V_k))^\otimes(m+1))$ is called the Chow point. $M$ is said to be Chow stable with respect to $L^k$ if $M$ is stable in $(\text{Sym}^d(V_k))^\otimes(m+1)$ with respect to the action of $\text{SL}(V_k)$. $M$ is said to be asymptotically Chow stable with respect to $L$ if there exists a $k_0 > 0$ such that $\tilde{M}_k$ is stable for all $k \geq k_0$. Asymptotic Chow semi-stability is defined similarly.

The stabilizer $\tilde{G}_k \subset \text{SL}(V_k)$ of $\tilde{M}_k$ has a finite covering $p_k : \tilde{G}_k \rightarrow G_k$ onto a subgroup $G_k$ of the automorphism group $\text{Aut}(M)$ of $M$. $\tilde{G}_k$ includes the connected subgroup of $\text{Aut}(M)$ generated by the vector fields in $h_0(M)$. We take an $X \in h_0(M)$ such that the real part $2\text{Re} X$ of $2X$ generates an $S^1$-action $M$, and denote

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1 This argument is due to Shigetoshi Bando in his unpublished paper [1]
the complexification of this one parameter subgroup by \( H = \{ \sigma_t \mid t \in \mathbb{C} \} \cong \mathbb{C}^* \). Further put \( \tilde{H}_k := p^{-1}_k(H) \). Then since \( L \to L^k \) is a covering ramified over the zero section there is a subgroup \( H_k \) of the group of bundle automorphisms of \( L \) such that \( \tilde{H}_k \)-action on \( L^k \) is induced from the \( H_k \)-action on \( L \). Obviously \( H_k \)-action on \( L \) induces a finite covering of \( H_k \) onto \( H \), and by contragradient representation \( H_k \) acts on \( H^0(M, L^k) \). The following result is a slight modification of Theorem A in Mabuchi’s paper [12].

**Proposition 3.1.** Suppose that \( M \) is asymptotically Chow semi-stable, and let the notations be as above.

1. \( H_k \)-actions coincide for all \( k \geq k_0 \). We put \( \hat{H} := H_k \). Thus the natural actions on \( L^k \) induced by \( \hat{H} \)-action on \( L \) are \( \text{SL} \)-actions for all \( k \geq k_0 \).
2. If we regard \( X \) as an infinitesimal generator of \( \hat{H} \) then for the left action of \( \hat{H} \) on the principal \( \mathbb{C}^* \)-bundle associated with the line bundle \( L \) we have
   \[
   \mathcal{F}_{c^{n+1}}(X) = 0.
   \]

To prove this we recall some known facts.

**Lemma 3.2.** Let \( L \to M \) be an ample line bundle, \( \omega \) a Kähler form which represents \( c_1(L) \). Let \( \theta \) be a type \((1,0)\)-connection form of the associated principal \( \mathbb{C}^* \)-bundle \( P_{\mathbb{C}^*} \) with its curvature equal to \(-2\pi \omega\). If we choose a lifting \( \tilde{X} \) to \( P_{\mathbb{C}^*} \) of a holomorphic vector field \( X \) on \( M \), then \( \frac{i}{2\pi} \theta(\tilde{X}) \) is a Hamiltonian function in the sense that it satisfies
   \[
   i(X)\omega = -\bar{\partial} \left( \frac{i}{2\pi} \theta(\tilde{X}) \right).
   \]

A different choice of lifting reflects a different choice of additive constant of the Hamiltonian functions.

**Proof.** The equality (11) follows from the definition of the curvature \( \Theta = \bar{\partial} \theta \). The difference of two choices of lifts \( \tilde{X} \) come from the difference in the fiber direction, and if the two \( \tilde{X} \)'s are different then the two \( \theta(\tilde{X}) \)'s have to be different. Conversely, suppose that one has a Hamiltonian function \( u_X \):
   \[
   i(X)\omega = -\bar{\partial} u_X.
   \]

Let \( X^\sharp \) denote the horizontal lift of \( X \) to the principal bundle \( P_{\mathbb{C}^*} \) associated with \( L \) with respect to the connection \( \theta \), and put
   \[
   \tilde{X} = -2\pi i u_X z \frac{\partial}{\partial z} + X^\sharp
   \]

where \( z \) denotes the fiber coordinate of \( L \). Then \( u_X = \frac{1}{2\pi} \theta(\tilde{X}) \). But this implies that \( \tilde{X} \) is a holomorphic vector field because
   \[
   -i(X)\omega = \bar{\partial} \theta(\tilde{X}) = -i(X)\omega + \frac{i}{2\pi} \theta(\tilde{X}).
   \]

Thus, given a Hamiltonian function with arbitrary normalization, one has a holomorphic lift \( \tilde{X} \) of \( X \). This completes the proof. \( \square \)

**Theorem 3.3 ([18]).** Let \( M \) be a nonsingular algebraic variety of dimension \( m \) imbedded in \( \mathbb{P}(V) \). We identify \( V \) with \( \mathbb{C}^{N+1} \) with standard Hermitian metric.
Then there is a norm, called Chow norm and denoted by $\| \cdot \|_C$, on $(\text{Sym}^d V)^{(m+1)}$ such that, for any one parameter subgroup $\sigma_t$ of $SL(V)$, we have

\begin{equation}
\frac{d}{dt} \log \| \sigma_t(M) \|_C = \int_M \dot{\varphi}_t \sigma_t^* \omega^m_{FS},
\end{equation}

where $\varphi_t = \log \| \sigma_t z \|/\| z \|$ for $z \in V - \{ 0 \}$ with $[z] \in M \subset \mathbb{P}(V)$, and $\omega_{FS}$ denotes the Fubini-Study Kähler form restricted to $M$.

This result was proved by Zhang using DeLigne pairing (c.f. Theorem 3.4 in [18]). Direct proofs are given also by Phong and Sturm [15] and Y.Sano [16].

**Lemma 3.4.** Suppose that $M$ is Chow semi-stable in $\mathbb{P}(V)$. If $\sigma_t$ in Theorem 3.3 preserves $M$ and induces an action on $M$ generated by a holomorphic vector field $X$ on $M$, then for the natural action of $\sigma_t$ on $\mathcal{O}_M(1)$ we have

\begin{equation}
\mathcal{F}_{c^m+1}(X) = 0.
\end{equation}

Here of course the relevant principal bundle of $\mathcal{F}_{c^m+1}$ is the associated $C^*$-bundle of $\mathcal{O}_M(1)$ with the left action of the one parameter group.

**Proof.** Since $M$ is Chow semi-stable then (14) must be zero, because otherwise $\| \sigma_t(M) \|_C$ tends to zero as $t$ tends to $\infty$ or $-\infty$. On the other hand if we write down the left hand side of (15) in terms of the Hermitian connection of the canonical metric of $\mathcal{O}_M(1)$ which is induced from $\mathcal{O}_{\mathbb{P}^n}(1)$ and the first Chern form of which is $\omega_{FS}$, then we get the right hand side of (14). Hence (15) holds. This completes the proof. \hfill \Box

**Proof of Proposition 3.1** Let $\pi_k : L^k \to M$ and $\pi : L \to M$ be the projections respectively. Let $X_k$ and $X$ respectively be the vector fields on $L^k$ and $L$ generating $\tilde{H}_k$ and $H_k$ actions on $L^k$ and $L$ such that

$$\pi_k(\tilde{X}_k) = \pi(X_k) = X.$$  

By our assumption $M$ is Chow semi-stable. Thus by Lemma 3.4, $\mathcal{F}_{c^m+1}(X) = 0$ for the principal $C^*$-bundle associated with $\mathcal{O}_{\mathbb{P}(V_k)}(1)|_M \cong L^k$ with the left $\tilde{H}_k$-action.

But this implies that the Hamiltonian function $\mu_X^{(k)}$ determined by $\tilde{X}_k$ on $L^k$ with respect to $k\omega$ is normalized so that the average over $M$ is zero. But this also implies that the Hamiltonian function $\mu_X^{(0)}$ determined by $X_k$ on $L$ with respect to $\omega$ has average 0. For different $k$’s the Hamiltonian functions $\mu_X^{(k)}$ differ by constants. So they must all coincide. By the last statement of Lemma 3.2 all the $X_k$’s must coincide. This proves (3.1.1). The fact that the average of the Hamiltonian function $\mu_X^{(k)}$ is zero implies (3.1.2). This completes the proof of Proposition 3.1. \hfill \Box

**4. Obstructions to asymptotic Chow semi-stability**

In this section we prove Theorem 1.2, and remark that the conclusion of Theorem 1.2 is equivalent to the stability of isotropic actions for $(M, L)$ in the sense of T. Mabuchi [12], [13].

**Proof of Theorem 1.2.** The reductive part of $\mathfrak{h}_0(M)$ is a complexification of a Lie algebra of a compact Lie group. Since any element of the compact Lie group belongs to a maximal torus, we may assume that the real part $2\mathbb{R}X$ of $2X$ is in the Lie algebra of a maximal torus. Since $\mathcal{F}_{T(d\phi)}$ is linear in $X$ we may also assume
that $2\text{Re}X$ generates $S^1$. Hence we are in the situation of Proposition 3.1 and $X$ generates $H$ in the notation there. A covering $\tilde{H}$ of $H$ acts on $L$ and the induced action on $L^k$ gives an SL-action. We shall apply the equivariant Riemann-Roch theorem to the $S^1 \subset \tilde{H}$.

Put $d_k = H^0(M, L^k)$. Then the weight of $\hat{H}_k$-action on $L^k$ is $0$ since $\hat{H}_k$ gives an SL-action on $H^0(M, L^k)$. By the equivariant Riemann-Roch theorem this weight is given by the coefficient of $t$ ([5]) of the following:

$$e^{k(\omega + t\mu_X)}Td(tL(X) + \Theta) = \sum_{p=0}^{\infty} \frac{k^p}{p!} (\omega + t\mu_X)^p \sum_{q=0}^{\infty} Td^q(tL(X) + \Theta).$$

where $\mu_X$ is the Hamiltonian function defined by $\hat{H}$-action on $L$, namely $\mu_X = \mu^{|k|}_X$ in Proof of Proposition 3.1, and where $Td^q$ is the $q$-th Todd polynomial. From the property of $\mu^{|k|}_X$:

$$\int_M \mu_X \omega^m = 0.$$  

(16)

Note that this is equivalent to (3.1.2) in Proposition 3.1. Thus $\mu_X = u_X$ (c.f. (1)).

By writing the coefficient of $t$ explicitly we have

$$0 = \sum_{p=0}^{m+1} \frac{k^p}{p!} \int_M (\omega^p \wedge Td^{m-p+1}(L(X) + \Theta) + \mu \omega^{p-1} \wedge u_X Td^{m-p+1}(\Theta))$$

for all $k \geq k_0$. But from a result in [10] (see also Theorem 5.3.10 in [8])

$$\int_M Td^{m+1}(L(X) + \Theta) = 0$$

(18)

which implies that the term $p = 0$ in (17) vanishes. The term $p = m + 1$ also vanishes because of (16). Thus the vanishing of the terms for $p = 1, \cdots, m$ in (17) gives the desired result. This completes the proof of Theorem 2.1.

Note that $\hat{H}_k$ and $H_k$ in the Proposition 3.1 can be defined for any subgroup $H$ of the identity component of Aut($M$). Let Aut$_r(M)$ denote the Lie subgroup of the identity component of Aut($M$) corresponding to $\mathfrak{h}_r(M)$ (see the statement of Theorem 1.2).

**Proposition 4.1.** The following three conditions are equivalent.

(a) The conclusions (3.1.1) and (3.1.2) of Proposition 3.1 hold when $H =$ Aut$_r(M)$.

(b) The conclusions (3.1.1) and (3.1.2) of Proposition 3.1 hold when $H$ is the center of Aut$_r(M)$.

(c) The conclusion of Theorem 1.2 holds.

**Proof.** That (a) implies (b) is obvious.

By Theorem 1.1, $\mathcal{F}_\phi$ is independent of the choice of the Kähler form in the fixed Kähler class and the choice of the type (1,0)-connection. Therefore it is invariant under the automorphisms preserving the Kähler class, in particular under the identity component of Aut($M$). This implies that $\mathcal{F}_\phi$ is a Lie algebra character and vanishes on the semisimple part of the Lie algebra $\mathfrak{h}_r(M)$. If the assumption (b) holds then one can show as in the proof of Theorem 1.2 that $\mathcal{F}_{Td^{(p)}}$ vanishes for the center of $\mathfrak{h}_r(M)$. Thus (b) implies (c).
Now assume (c). To prove (a) it is enough to consider holomorphic vector fields \( X \) such that \( 2\text{Re}X \) generates an \( S^1 \). Let \( \tilde{X} \) be as in (13) with \( \int_M u_X \omega^m = 0 \). Then \( \tilde{X} \) is a holomorphic lift of \( X \) as we saw in the proof of Lemma 3.2, and \( \int_M u_X \omega^m = \int_M \frac{1}{2\pi} \theta(\tilde{X}) \omega^m = 0 \). By a result of the author and Mabuchi [9] the image of \( u_X \) is an interval of rational vertices and thus the orbit of the flow generated by \( 2\text{Re}\tilde{X} \) is periodic. With the \( S^1 \)-action that \( 2\text{Re}\tilde{X} \) generates we apply the equivariant Riemann-Roch theorem as in the proof of Theorem 1.2, and then see that the \( \mathbb{C}^* \)-action that \( \tilde{X} \) generates induces \( \text{SL} \)-actions on \( H^0(M, L^k) \) for all large \( k \). This completes the proof. \( \square \)

We say that the isotropy action of \( (M, L) \) is stable if one of the equivalent conditions in Proposition 4.1 holds. This definition goes back to T. Mabuchi who defined it using the condition (b) and proved that if \( M \) further has a Kähler metric in \( c_1(L) \) with constant scalar curvature then the orbit of the Chow point \( \hat{M}_k \) under the action of \( SL(V_k) \) is closed for all large \( k \). Hence we have the following.

**Corollary 4.2.** Let \( L \to M \) be an ample line bundle and suppose that we have Kähler metric in \( c_1(L) \) of constant scalar curvature. Suppose also that for \( 1 \leq \ell \leq m \) we have \( F_{C\ell}(X) = 0 \) for \( X \in h_p(M) \). Then the orbit of the Chow point \( \hat{M}_k \) under the action of \( SL(V_k) \) is closed for all large \( k \).

5. **Variants of the integral invariants**

In this section we take up various variants of the integral invariants \( F_\phi \). The first one is a symplectic analogue as in [2]. Let \( H \) be a compact Lie group acting on a compact symplectic manifold \( (M, \omega_0) \) in the Hamiltonian fashion. Choose any symplectic form \( \omega \in [\omega_0] \). For an element \( X \in h \) of the Lie algebra of \( H \) we take the Hamiltonian function \( u_X \) given by

\[
i(X)\omega = -du_X, \quad \int_M u_X \omega^m = 0
\]

where \( \text{dim} M = 2m \). Let \( G \) be a compact Lie group and \( P_G \to M \) a principal bundle over \( M \) with structure group \( G \) (of right action). Suppose \( H \) acts from the left on \( P_G \) covering the \( H \)-action on \( M \) and commuting with the right action of \( G \).

Choose any \( H \)-invariant connection \( \theta \) of \( P_G \) and denote its curvature form by \( \Theta \). Let \( p^p(G) \) denote the set of all \( G \)-invariant polynomials of degree \( p \) on \( g \).

An element \( X \) of the Lie algebra \( h \) of \( H \) defines a \( G \)-invariant vector field \( \tilde{X} \) on \( P_G \) which descends to the Hamiltonian vector field \( X \) on \( M \).

For any \( \phi \in p^p(G) \) and \( X \in h(M) \) the differential form of \( \phi(\theta(\tilde{X}) + \Theta) \) on \( P_G \) is pulled back from a form on \( M \) just as in the usual Chern-Weil theory. We will denote this form on \( M \) by the same notation \( \phi(\theta(\tilde{X}) + \Theta) \). We put

\[
F_\phi(X) = (m - p + 1) \int_M \phi(\Theta) \wedge u_X \omega^{m-p} \\
+ \int_M \phi(\theta(\tilde{X}) + \Theta) \wedge \omega^{m-p+1}
\]

**Theorem 5.1.** \( F_\phi(X) \) is independent of the choices of the symplectic form \( \omega \in [\omega_0] \) and the \( H \)-invariant connection \( \theta \).
The proof is left to the reader.

If we simply consider $F_{c^m+p}(X)$ for a line bundle $L \rightarrow M$ with connection whose Chern form equal to the symplectic form. Then we have

\begin{equation}
F_{c^m+p}(X) = \left( \frac{m+p}{p} \right) \int_M \mu_X^p \omega^m
\end{equation}

where $\omega$ is the first Chern form of $L$. If we take

\[ X = t_1X_1 + \cdots + t_pX_p \]

and plug it into (20), then as the coefficient of $t_1 \cdots t_p$ we get $\int_M \mu_X \cdots \mu_X \omega^m$ which must be independent of the choice of the symplectic form in a fixed de Rham class. This is a simple way of deriving the multi-linear form considered in [9].

Let us return to the complex situation. Let $L \rightarrow M$ be an ample line bundle and suppose that a subgroup $H$ of automorphism group acts from the left on the principal bundle $P_{\mathbb{C}^*}$ associated with $L^p \otimes K^{-q}_M$. Let $I_{pq}$ be $F_{c^m+1}(X)$ in this situation. X.X.Chen and G.Tian [4] used the coefficients of $I_{1q}$ effectively to show that on the complex projective spaces the Ricci flow with initial metric of non-negative bisectional curvature will converge to a Kähler-Einstein metric of constant bisectional curvature.

Next let us consider (20) in the complex situation. Then the right hand side of (20) is regarded as an invariant of the Kähler class. Since $p$ is arbitrary

\[ f(X) := \int_M e^\mu_X \omega^m \]

is also an invariant of the Kähler class. If we take derivative of $f(X + tY)$ with respect to $t$ at $t = 1$ we get

\[ \frac{d}{dt}f(X + tY)|_{t=1} = \int_M \mu_Y e^\mu_X \omega^m. \]

This is also an invariant of the Kähler class. When $M$ is a Fano manifold Tian and Zhu [17] used it to show the uniqueness of Kähler-Ricci solitons.

We hope the new family of integral invariants in Theorem 1.1 and their variants will be useful in various situations.

References


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