NON-MOISHEZON TWISTOR SPACES OF $4\mathbf{CP}^2$ WITH NON-TRIVIAL AUTOMORPHISM GROUP

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Abstract. We show that a twistor space of a self-dual metric on $4\mathbf{CP}^2$ with $U(1)$-isometry is not Moishezon iff there is a $\mathbf{C}^*$-orbit biholomorphic to a smooth elliptic curve, where the $\mathbf{C}^*$-action is the complexification of the $U(1)$-action on the twistor space. It follows that the $U(1)$-isometry has a two-sphere whose isotropy group is $\mathbf{Z}_2$. We also prove the existence of such twistor spaces in a strong form to show that a problem of Campana and Kreußler is affirmative even though a twistor space is required to have a non-trivial automorphism group.

1. Introduction

Let (M, g) be a compact self-dual four-manifold and Z the associated twistor space, which is a compact complex threefold. In 1981 N. Hitchin [10] showed that Z does not admit Kähler metric except two well-known examples. Five years later, Y. S. Poon [26] showed that the connected sum of two complex projective planes admits self-dual metrics whose twistor spaces are not projective algebraic, but Moishezon. (A compact complex manifold is called Moishezon if the transcendental degree of its meromorphic function field over $\mathbf{C}$ is equal to the complex dimension.) Since then many works have been done about Moishezon twistor spaces of $n\mathbf{CP}^2$, the connected sum of n copies of the complex projective planes, for arbitrary n.

In contrast, non-Moishezon twistor spaces of $n\mathbf{CP}^2$ are not seriously investigated so far, to the best of the author's knowledge. In the present paper, we study complex geometric properties of non-Moishezon twistor spaces of $4\mathbf{CP}^2$, having a holomorphic $\mathbf{C}^*$-action. We note that these are equivalent to studying non-Moishezon twistor spaces whose identity component of the holomorphic automorphism group is non-trivial. Here and in what follows, every $\mathbf{C}^*$-action on twistor spaces are assumed to come from non-trivial $U(1)$-isometry. Namely, any $\mathbf{C}^*$-action we consider is a holomorphic one that is the complexification of $U(1)$-action, where this $U(1)$-action is the canonical lift of a non-trivial circle isometry of the corresponding self-dual metric on $4\mathbf{CP}^2$. In particular, $\mathbf{C}^*$-actions are

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assumed to be non-trivial. We note that for $n \le 3$ any twistor space of $n\mathbf{CP}^2$ is known to be Moishezon, and $n = 4$ is the first case in which non-Moishezon twistor spaces actually appear.

Our first main result is a geometric characterization of such twistor spaces of $4\mathbf{CP}^2$. More precisely, we prove that *every non-Moishezon twistor space of* $4\mathbf{CP}^2$ *with* $\mathbf{C}^*$*-action has a unique real* $\mathbf{C}^*$*-orbit which is a smooth elliptic curve* (Theorem 3.6). This contrasts to the Moishezon case, because for Moishezon manifold with $\mathbf{C}^*$-action with non-empty fixed locus, the closure of any orbit is either a point or $\mathbf{CP}^1$, and the isotropy group is either finite or $\mathbf{C}^*$, whereas the isotropy group of the above orbit of elliptic curve is a infinite cyclic group. But we remark that a spacial class of Moishezon twistor spaces of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action can be equivariantly deformed into non-Moishezon twistor space, as a small deformation [13]. We also mention that Theorem 3.6 has a corollary that the corresponding $U(1)$-isometry on $4\mathbf{CP}^2$ has a two-sphere along which the isotropy group is $\mathbf{Z}_2$ (Proposition 3.7).

In our proof of the above result we see that the normal bundle of such a $\mathbf{C}^*$-orbit is of the form $F \oplus F$, where F is a line bundle of degree zero over the elliptic curve. Then applying a work of F. Campana and B. Kreußler [5], the algebraic dimension of a twistor space Z, denoted by $a(Z)$, is determined by F: $a(Z) = 2$ if and only if F is of finite order in the Picard group, and otherwise $a(Z) = 1$. Further, in case $a(Z) = 2$ the algebraic reduction of Z is induced by its anticanonical system multiplied by the half of the order of F [5]. Thus a natural question arises as to which $F \to C$ can be realized as above by a non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action. In this direction we first obtain a necessary condition for a line bundle F to be a direct summand of the normal bundle (of the $\mathbf{C}^*$-orbit in a non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action). Namely we show that such F is real (i.e. $\sigma^* F \simeq \overline{F}$; σ is the real structure of Z) and can be continuously deformed into the trivial line bundle preserving the reality. (This condition is not necessarily satisfied for real $F \in \mathrm{Pic}^0 C$ in general. See Definition 4.3 for precisely.) Next we show our main result stating that *if* C *is an elliptic curve without non-trivial automorphism, and if* $F \to C$ *is a line bundle of degree zero which can be continuously deformed into the trivial line bundle preserving the real structure, then there exists a non-Moishezon twistor space of* $4\mathbf{CP}^2$ *with* $\mathbf{C}^*$*-action such that the* $\mathbf{C}^*$*-orbit of elliptic curve is biholomorphic to* C *and its normal bundle is isomorphic to* $F \oplus F$ *as a holomophic vector bundle* (Theorem 4.5). In particular, any positive integer can be realized as the order of F, and a question of Campana-Kreußler [5, Open Problem] turns out to be affirmative, even if the twistor space is required to have a non-trivial $\mathbf{C}^*$-action.

This paper is organized as follows. In Section 2 we study twistor spaces of $n\mathbf{CP}^2$ whose fundamental systems are two-dimensional and show that for $n \ge 5$ such a twistor space is always Moishezon (Corollary 2.5). This reproves a result of Kreußler [18]. Next in Section 3 we prove Theorem 3.6 by using results in the previous section. In Section 4 we prove Theorem 4.5. Our proof is a refinement of the construction in our previous paper [15], where we showed, by using so-called a Kummer type construction [22], the existence of non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action whose F is trivial. The key point for this case was the existence of some $\mathbf{C}^*$-invariant smooth divisor on the twistor space. When F is non-trivial, such a divisor does not exist, and the proof of [15] does not work. Here is a rough sketch of the present proof. For given data (an elliptic curve C and a line bundle $F \to C$), we first construct a simply connected orbifold with a conformally flat metric, whose twistor space Z_0 has a $\mathbf{C}^*$-action with an orbit biholomorphic to C whose normal bundle is $F \oplus F$. Then applying a theory of [6, 22], we construct some normal crossing variety Z', one of whose irreducible component is a resolution of Z_0. The ingredient of the proof of Theorem 4.5 is to show that *Z' can be $\mathbf{C}^*$-equivariantly deformed keeping the complex structure of C and its normal bundle $F \oplus F$ fixed, to get a desired twistor space of* $4\mathbf{CP}^2$. We also see in the course of our proof that the obstruction space for deformation theory of twistor spaces of primary Hopf surfaces vanishes (Proposition 4.11), generalizing a result of Pontecorvo [25]. This is done by taking the universal cover and calculating a Čech cohomology using an explicit Stein open covering.

Notations and Conventions. (i) Let X be a complex manifold and $\mathcal{S}$ a coherent sheaf on X. The structure sheaf and the tangent of X are denoted by $\mathcal{O}_X$ and Θ_X respectively. If a Lie group G acts on X holomorphically, and if the action lifts on $\mathcal{S}$ as $\mathcal{O}_X$-homomorphism, then G naturally acts on the cohomology groups $H^i(X, \mathcal{S})$. We denotes by $H^i(X, \mathcal{S})^G$ the space of G-fixed elements, which is a vector subspace of $H^i(X, \mathcal{S})$. (ii) Let Z be a twistor space associated to a self-dual four-manifold. Then the anticanonical bundle of Z has a canonical square root, which is called *the fundamental line bundle*. The associated complete linear system is called *the fundamental system* and denoted by $|-(1/2)K_Z|$. A member of the fundamental system is called *a fundamental divisor*. (iii) An anti-holomorphic involution on a complex manifold is often called *a real structure*. If A is the subset of the complex manifold, its image by the real structure is denoted by $\overline{A}$. A is said to be *real* if $A = \overline{A}$.

2. Twistor spaces with two-dimensional fundamental system

In this section we study twistor spaces whose fundamental system is two-dimensional. Some of the results will be needed in the next section to prove Theorem 3.6. First we recall a basic fact and a definition concerning fundamental divisors.

Proposition 2.1. [23] *Let S be a real irreducible fundamental divisor of a twistor space associated to a self-dual metric on $n\mathbf{CP}^2$. Then there exists a birational morphism $\nu : S \to \mathbf{CP}^1 \times \mathbf{CP}^1$ such that (i) ν preserves the real structure, (ii) the induced real structure of $\mathbf{CP}^1 \times \mathbf{CP}^1$ is (anti-podal)$\times$ (complex conjugation), (iii) $\nu(E) \cap p_2^{-1}(S^1)$ is empty, where E denotes the exceptional divisor of ν,*

$p_2 : \mathbf{CP}^1 \times \mathbf{CP}^1 \to \mathbf{CP}^1$ *the projection to the second factor, and* $S^1 \subset \mathbf{CP}^1$ *the real locus of the complex conjugation. (iv)* $c_1^2(S) = 8 - 2n$.

Definition 2.2. [11] Let $\nu : S \to \mathbf{CP}^1 \times \mathbf{CP}^1$ and $E \subset S$ be as in Proposition 2.1. Then S is called of *type* (a, b) if $\nu(E)$ is contained in a real irreducible curve of bidegree (a, b).

Lemma 2.3. *Let* Z *be a twistor space of* $n\mathbf{CP}^2$ *with* $n \geq 4$ *and* S *a real irreducible fundamental divisor of* Z*. Assume that* $\dim |-K_S| = 1$*. Then (i) if* $|-K_S|$ *is free from the base locus,* $n = 4$ *and* $|-K_S|$ *induces an elliptic fibration* $S \to \mathbf{CP}^1$*, (ii) if* $|-K_S|$ *is not free,* S *is of type* $(2, 1)$.

Proof. (i) If the anticanonical system of S is free and is positive dimension, $(-K_S)^2 = c_1^2(S) = 0$. On the other hand, we have $c_1^2(S) = 8 - 2n$. Therefore it follows $n = 4$. In this case, a standard argument shows that general anticanonical curve of S is a smooth elliptic curve. Thus we obtain (i).

So we assume that the anticanonical system is one-dimensional and that its base locus is non-empty, and show that S is of type $(2, 1)$. Since $c_1^2(S) = 8-2n \leq 0$, the fixed locus of the anticanonical system is not isolated and therefore has a fixed divisor. Let C_0 be the fixed component of the anticanonical system. C_0 is real. Let $\nu : S \to \mathbf{CP}^1 \times \mathbf{CP}^1$ be a blowing-down fulfilling the conditions of Proposition 2.1. Suppose that $\nu(C_0)$ is zero-dimensional. Then since the induced real structure on $\mathbf{CP}^1 \times \mathbf{CP}^1$ has no fixed points and since $\nu(C_0)$ is real, the number of points of $\nu(C_0)$ must be even (≥ 2). Hence C_0 is disconnected. Let $C_1 \in |-K_S - C_0|$ be a member of the movable part. Cohomology exact sequence of the sequence $0 \to \mathcal{O}_S(-C_0) \to \mathcal{O}_S \to \mathcal{O}_{C_0} \to 0$ and the disconnectedness of C_0 imply that $H^1(\mathcal{O}_S(-C_0))$ is of positive dimension. Further, since $C_0 + C_1$ is an anticanonical curve, $H^1(\mathcal{O}_S(-C_0))$ and $H^1(\mathcal{O}_S(-C_1))$ are dual of each other by Serre duality. It follows from the sequence $0 \to \mathcal{O}_S(-C_1) \to \mathcal{O}_S \to \mathcal{O}_{C_1} \to 0$ that C_1 is also disconnected. Since S is rational and since C_1 is a member of the movable part, this implies that $\dim |-K_S| \geq 2$, contradicting our assumption. Therefore $\dim \nu(C_0) \neq 0$ and $\nu(C_0)$ must be a curve.

Let C be any anticanonical curve of S. Then since $C_0 \subset C$, and since $\nu(C)$ is an anticanonical curve of $\mathbf{CP}^1 \times \mathbf{CP}^1$, the bidegree (a, b) of $\nu(C_0)$ satisfies $a \leq 2$ and $b \leq 2$. Further, because $\dim |-K_S| \geq 1$, $(a, b) \neq (2, 2)$. Moreover, the reality of $\nu(C_0)$ and properties (ii) and (iii) of Proposition 2.1 imply $(a, b) \neq (1, 0), (0, 1), (1, 1)$ or $(1, 2)$. Suppose that $(a, b) = (2, 0)$ or $(a, b) = (0, 2)$. In these cases we readily deduce $\dim |-K_S| = 2$, and again a contradiction. Therefore, we get $(a, b) = (2, 1)$. Moreover, our assumption $\dim |-K_S| = 1$ implies that all of the blowing-up points of ν are on (the strict transform of) $\nu(C_0)$, because otherwise we have $\dim |-K_S| = 0$. Finally we show that $\nu(C_0)$ is irreducible. First, $\nu(C_0)$ cannot contain an irreducible component whose bidegree is $(1, 1)$, since there exists no real curve of bidegree $(1, 1)$. Next, a curve of bidegree $(0, 1)$ cannot be an irreducible component of $\nu(C_0)$, since in such a case the blown-up points on ν is on $p_2^{-1}(S^1)$ (because we have assumed $\dim |-K_S| = 1$), contradicting (iii) of Proposition 2.1. Combining these, it easily follows that $\nu(C_0)$ does not contain an irreducible component of bidegree $(1,0)$. Thus $\nu(C_0)$ is irreducible. This implies that S is of type $(2,1)$. $\square$

We use this lemma to get the following

Proposition 2.4. *Let Z be a twistor space of $n\mathbf{CP}^2$ and assume that its fundamental system is two-dimensional. Then we have: (i) $n \geq 4$, (ii) if the fundamental system is free, we have $n = 4$ and the system induces a surjective morphism $Z \to \mathbf{CP}^2$, which is an algebraic reduction of Z. In particular, the algebraic dimension of Z is two, (iii) if the fundamental system is not free, any real irreducible fundamental divisor is of type $(2,1)$. In particular, Z is Moishezon. (iv) Z does not have an effective divisor of degree one (= a divisor whose intersection number with a twistor line is one).*

Proof. (i) is well-known. Indeed, if $n = 1$, the fundamental system is very ample. If $n = 2$ or $n = 3$, the system is five-dimensional [26], or three-dimensional [19, 27] respectively.

So we assume that Z is a twistor space of $n\mathbf{CP}^2$ with $n \geq 4$ and that its fundamental system is two-dimensional. Let S be a real irreducible fundamental divisor of Z. Then the anticanonical system of S is one-dimensional. Further, the base locus of the fundamental system of Z and the anticanonical system of S coincide. Hence if the fundamental system is free, so is the anticanonical system of S. Therefore by Lemma 2.3, we get $n = 4$, and the anticanonical system induces an elliptic fibration on S. The anti-Kodaira dimension $\kappa^{-1}(S)$ of rational elliptic surface is one. On the other hand we have the inequality $a(Z) \leq 1+\kappa^{-1}(S)$ [3]. Hence $a(Z) \leq 2$. Further, the morphism $f : Z \to \mathbf{CP}^2$ induced by the fundamental system is surjective, for otherwise general fundamental divisor would be reducible, which implies that Z is a LeBrun twistor space [27], whose fundamental system is three-dimensional [20]. Thus we get $a(Z) = 2$. It is easily seen from the existence of four-dimensional family of rational curves on Z that the meromorphic function field of Z is purely transcendental extension of $\mathbf{C}$. Therefore f must be an algebraic reduction of Z. Thus we obtain (ii). If the fundamental system is not free, the anticanonical system of any real irreducible fundamental divisor S is also not free. Hence again by Lemma 2.3, S must be of type $(2,1)$. The existence of such a fundamental divisor implies that Z is Moishezon [16]. Thus we get (iii).

In order to show (iv) we recall that in [11, Prop. 1.2] it was shown that if Z has a real irreducible fundamental divisor of type $(2,1)$ or $(2,2)$, then Z does not have an effective divisor of degree one. It is obvious from the above argument that any real irreducible fundamental divisor of Z is of type $(2,1)$ (resp. $(2,2)$), if the fundamental system of Z is not free (resp. free). Therefore we can conclude that in both cases the twistor spaces do not have an effective divisor of degree one. $\square$

Thus we have reproved a result of Kreußler [18].

Corollary 2.5. *Let Z be a twistor space of $n\mathbf{CP}^2$ and assume that the fundamental system is two dimensional. If Z is non-Moishezon, $n = 4$.*

Next we make a brief remark on the case that the twistor space has a $\mathbf{C}^*$-action.

Proposition 2.6. *Let Z be a twistor space of $n\mathbf{CP}^2$ and assume that its fundamental system is two-dimensional. If Z admits a non-trivial $\mathbf{C}^*$-action, there exists a real $\mathbf{C}^*$-invariant fundamental divisor. Further, such a divisor is unique and irreducible (hence smooth).*

Proof. Since a holomorphic action naturally lifts on the fundamental line bundle, $\mathbf{C}^*$ naturally acts on $\mathbf{CP}^2$, the dual projective plane of the space of fundamental divisors of Z. Let ρ be this $\mathbf{C}^*$-action on $\mathbf{CP}^2$.

Suppose that there exists a line $l \subset \mathbf{CP}^2$ which is pointwise fixed by ρ. We may assume that l is real, since otherwise ρ is the trivial action and we can replace l with any real line. Let S be the $\mathbf{C}^*$-invariant real fundamental divisor corresponding to l. S is irreducible by Proposition 2.4 (iv). If the anticanonical system of S is free, it follows from Lemma 2.3 that ($n = 4$ and) the anticanonical system induces an elliptic fibration on S. This is a contradiction, since in this situation $\mathbf{C}^*$ does not preserve general fiber of the elliptic fibration, which is not compatible with our assumption on ρ. Hence suppose that the anticanonincal system is not free. Let C_0 be the base curve of the system, which is automatically $\mathbf{C}^*$-invariant. Let $\nu : S \to \mathbf{CP}^1 \times \mathbf{CP}^1$ be a birational morphism as in the proof of Lemma 2.3. Then as seen in the proof of the lemma, $\nu(C_0)$ is a real irreducible curve of bidegree $(2, 1)$, and is clearly $\mathbf{C}^*$-invariant. Since our $\mathbf{C}^*$-action on S is non-trivial, it follows that $\nu(C_0)$ is the closure of an orbit. Therefore, because we know that $\nu(C_0)$ is irreducible, the induced $\mathbf{C}^*$-actions on both factors of $\mathbf{CP}^1 \times \mathbf{CP}^1$ are non-trivial. This is again a contradiction, because the morphism $S \to \mathbf{CP}^1$ associated to the anticanonical system is induced by $|\nu^*\mathcal{O}(0, 1)|$, and because we have assumed that ρ is trivial on l. Therefore ρ fixes no lines on $\mathbf{CP}^2$.

It then follows that the set of ρ-invariant lines forms a triangle of $\mathbf{CP}^2$, and that each of these three lines is not pointwise fixed by ρ. This triangle is clearly real. To prove the proposition, it suffices to show that there exists just one real line among these three lines. Since the real structure is involutive, at least one of these lines, again denoted by l, is real. Let $\{0, \infty\}$ be the set of ρ-fixed points on l. Then the real structure interchanges 0 and ∞. In fact, if both of 0 and ∞ are real, 0 and ∞ belong to the real circle of l. But the real circle must be an orbit of the $U(1)$-action which is the restriction of ρ, because ρ is non-trivial on l. This is a contradiction and $\overline{0} = \infty$ must hold. Therefore the remaining two $\mathbf{C}^*$-invariant lines (among the triangle) are conjugate of each other. This completes a proof of the proposition. □

3. Non-Moishezon twistor spaces of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action and an elliptic curve

Proposition 3.1. *Let Z be a twistor space of $n\mathbf{CP}^2$ with $\mathbf{C}^*$-action and assume that the fundamental system is of one-dimensional. If the induced $\mathbf{C}^*$-action on the parameter space of the pencil ($\simeq \mathbf{CP}^1$) is non-trivial, then Z is non-Moishezon and the base locus of the pencil is a smooth elliptic curve which is an orbit of the $\mathbf{C}^*$-action.*

Proof. Let Z be as in the proposition and C the base locus of the pencil of fundamental divisors. Then by [17, Prop. 3.7], C is either a smooth elliptic curve or a cycle of rational curves. Suppose that C is a cycle of rational curves, and let $k \geq 2$ be the number of the irreducible components of C. Then again by [17, Prop. 3.7], k is equal to the number of degree one divisors on Z. On the other hand, since Z has at most finite number of degree one divisors, every degree one divisor is $\mathbf{C}^*$-invariant. Therefore, together with its conjugation, each degree one divisor determines a $\mathbf{C}^*$-fixed point on $\mathbf{CP}^1$ (= the parameter space of the pencil). Since we have assumed that the induced $\mathbf{C}^*$-action on $\mathbf{CP}^1$ is non-trivial, there exists just two $\mathbf{C}^*$-invariant fundamental divisor, which we denote S_0 and S_∞. Then we have $\overline{S}_0 = S_\infty$ (cf. an argument in the proof of Prop. 2.6). Therefore both of S_0 and S_∞ must be reducible. So we may write $S_0 = D_0 + D_0'$ and $S_\infty = \overline{D}_0 + \overline{D}_0'$. On the other hand, $D_0 + \overline{D}_0$ is also a (reducible) fundamental divisor which is not contained in the pencil. This is a contradiction and C must be a smooth elliptic curve.

C is obviously real and $\mathbf{C}^*$-invariant. Further, since the connected component of the fixed locus of the original $U(1)$-action on $n\mathbf{CP}^2$ is either a point or a two-dimensional sphere [9], C cannot be pointwise fixed. Hence C must be an orbit of the $\mathbf{C}^*$-action. It follows immediately from the existence of such an orbit that Z is non-Moishezon, because the closure of any orbit of $\mathbf{C}^*$-action with fixed points on any Moishezon manifold is a point or a rational curve. □

Proposition 3.2. *Let Z be a twistor space of $n\mathbf{CP}^2$ with $\mathbf{C}^*$-action and assume that the fundamental system is of one-dimensional. If each fundamental divisor is $\mathbf{C}^*$-invariant, Z is Moishezon.*

To prove this proposition, we show the following

Lemma 3.3. *Let Z be a twistor space as in the above proposition and S a real irreducible $\mathbf{C}^*$-invariant fundamental divisor of Z. Then the anti-Kodaira dimension $\kappa^{-1}(S)$ of S is two.*

Proof. Let C be the base locus of the pencil of fundamental divisors. C is an anticanonical curve of S. As in the proof of Proposition 3.1, C is either a smooth elliptic curve or a cycle of rational curves. But since S is this time $\mathbf{C}^*$-invariant, C cannot be a smooth elliptic curve and must be a cycle of rational curves.

First suppose that the fixed locus of the $\mathbf{C}^*$-action on S is isolated and show that $\kappa^{-1}(S) = 2$. Let E be any (-1)-curve on S. Since C is an anticanonical curve of S, we have $C \cdot E = 1$ by adjunction formula. Hence E is not disjoint from C. Suppose that $E \not\subset C$. Then C and E must intersect transversally at just one point. Because both C and E are $\mathbf{C}^*$-invariant, the intersection is $\mathbf{C}^*$-fixed point. On the other hand we have supposed that all of the fixed locus of the $\mathbf{C}^*$-action on S is isolated. Therefore $C \cap E$ must be the double point of C. This contradicts to $C \cdot E = 1$. Therefore E must be one of the irreducible components of C. Let $\nu : S \to S'$ be the contraction of E. Then $\nu(C)$ is again a $\mathbf{C}^*$-invariant anticanonical curve, and the induced $\mathbf{C}^*$-action still has no fixed curve. Moreover, $\nu(E)$ is clearly a double point of $\nu(C)$. Therefore we have $\kappa^{-1}(S) = \kappa^{-1}(S')$. It is obvious that we can repeat this blowing-down process keeping the anti-Kodaira dimensions, to get a minimal surface. Therefore we get $\kappa^{-1}(S) = 2$, as claimed.

Next assume that the fixed locus of the $\mathbf{C}^*$-action on S is not isolated. Let C_0 be a one-dimensional connected component of the fixed locus. Since S is rational, C_0 is a (smooth) rational curve. Because the $U(1)$-action commutes with the real structure, $\overline{C}_0$ is also pointwise fixed. Further, $C_0 \neq \overline{C}_0$, since otherwise $C_0(= \overline{C}_0)$ becomes source and sink, which is impossible. Let $g : S \to \mathbf{CP}^1$ be the quotient map of the $\mathbf{C}^*$-action. Because our $\mathbf{C}^*$-action has two fixed curves, g is a holomorphic map, which is biholomorphic on C_0 and $\overline{C}_0$. Now we see that C_0 and $\overline{C}_0$ are irreducible components of C. General fiber of g being a smooth rational curve, we easily see by adjunction formula that the intersection number of C with a general fiber of g is two. Therefore $g(C) = \mathbf{CP}^1$. Hence just two of the irreducible components of C must be $\mathbf{C}^*$-invariant sections of g, which must be pointwise-fixed. On the other hand there is no pointwise fixed curve, other than C_0 and $\overline{C}_0$. Therefore the two sections must be C_0 and $\overline{C}_0$. Thus we get $C_0 \cup \overline{C}_0 \subset C$. Let E be a (-1)-curve on S such that $(C_0 \cup \overline{C}_0) \cap E$ is empty, if any. Then by the same argument (based on $C \cdot E = 1$) as above, we can see that such E is an irreducible component of C. Further, E is not real, since otherwise the surface obtained by blowing-down E has a real structure such that the contacted curve is a real point, and since the real structure on the tangent space of this real point must be the usual complex conjugation (because the tangent space is a vector space), which yields a (-1)-curve with the real circle. Further, $E \cap \overline{E}$ is empty. In fact, E and $\overline{E}$ are also contained in fibers of g, since otherwise E and $\overline{E}$ will be $\mathbf{C}^*$-fixed curves which contradicts our choice of E and $\overline{E}$. Hence if $E \cap \overline{E} \neq \phi$, it follows that $E \cap \overline{E}$ is a point. And this point is real, which is impossible. Therefore we can simultaneously blow down E and $\overline{E}$. Let S' be the resulting surface. Then we have $\kappa^{-1}(S) = \kappa^{-1}(S')$, because E and $\overline{E}$ will again be contracted to double points of an anticanonical curve. Further, since E and $\overline{E}$ are contained in fibers of g as seen above, g naturally induces a morphism $g' : S' \to \mathbf{CP}^1$ which is still the $\mathbf{C}^*$-quotient map. Repeating this blow-down process as far as possible, we get a surface S_1 with $\mathbf{C}^*$-action and a morphism

$g_1 : S_1 \to \mathbf{CP}^1$, satisfying the following two properties: (i) any (-1)-curve of S_1 is not disjoint from $C_0 \cup \overline{C}_0$, (ii) $\kappa^{-1}(S_1) = \kappa^{-1}(S)$. Thus in order to finish a proof of the lemma it suffices to show that $\kappa^{-1}(S_1) = 2$. Let $\nu_1 : S_1 \to S_0$ be a birational morphism to a minimal surface S_0 preserving the real structure, which is a succession of blowing-downs of (-1)-curves contained in the fiber of g_1. By reality, S_0 is biholomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$. Then it is easily seen from the condition (i) above that any of the (-1)-curves contracted in each steps of ν_1 always intersect just one of (the image of) C_0 and $\overline{C}_0$. Therefore setting $k := 8 - c_1^2(S_1)$, the self-intersection numbers of C_0 and $\overline{C}_0$ in S_1 are both equal to $-k$. From this, it is readily seen that the degree of S_1 is $4/k > 0$. (Here the degree means $(P \cdot P)_{S_1}$, where $-K = P + N$ is the Zariski decomposition of the anticanonical class of S_1.) Therefore we get $\kappa^{-1}(S_1) = 2$. □

Proof of Proposition 3.2. Let Z be a twistor space satisfying the properties of the proposition. We show that any smooth fundamental divisor has anti-Kodaira dimension two. By Lemma 3.3 this is true for any real irreducible members S. Let C be as above and $C = P + N$ the Zariski decomposition of the anticanonical class of S, as a divisor on S. Then since $\kappa^{-1}(S) = 2$ by Lemma 3.3, we have $(P^2)_S > 0$. Let S' be any smooth fundamental divisor. Then C is also an anticanonical curve of S'. Further, since the Zariski decomposition of a divisor is determined by the self-intersection numbers of the irreducible components, and since the self-intersection numbers are independent of the choice of smooth S', $C = P + N$ also gives the Zariski decomposition of the anticanonical class of S', and $(P^2)_{S'} = (P^2)_S > 0$. Therefore we have $\kappa^{-1}(S') = 2$ for any smooth fundamental divisor S'. Then [28, Prop. 12.2] readily implies the proposition. □

Propositions 3.1 and 3.2 imply

Corollary 3.4. *Let Z be a twistor space of $n\mathbf{CP}^2$ with $\mathbf{C}^*$-action and assume that the fundamental system is of one-dimensional. Then Z is Moishezon if and only if each fundamental divisor is $\mathbf{C}^*$-invariant.*

Remark 3.5. The conclusion of the Proposition 3.1 (and hence Corollary 3.4) does not hold if we drop the assumption $\dim |-(1/2)K_Z| = 1$ and only assume the existence of real $\mathbf{C}^*$-invariant pencil of fundamental divisors. In fact, as shown in [14], there exists a Moishezon twistor space of $n\mathbf{CP}^2$ with $\mathbf{C}^*$-action such that the fundamental system contains a real irreducible $\mathbf{C}^*$-invariant member of type $(2, 1)$. Such a twistor space contains a real $\mathbf{C}^*$-invariant pencil whose general member is not $\mathbf{C}^*$-invariant. This pencil is not complete. (The fundamental system is two dimensional.)

From now on we restrict our attention to the case $n = 4$. The following result gives an account for non-meromorphicity of $\mathbf{C}^*$-action on non-Moishezon twistor spaces of $4\mathbf{CP}^2$.

Theorem 3.6. *Let Z be a non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action. Then there exists a $\mathbf{C}^*$-orbit C satisfying the following two properties: (i) C is a smooth elliptic curve, (ii) C is real. Further, such an orbit is unique and $\mathbf{R}$-homologous to zero.*

Proof. It is easily seen from the Riemann-Roch formula and the vanishing theorem of Hitchin that the fundamental system of a twistor space of $4\mathbf{CP}^2$ satisfies $\dim|-(1/2)K_Z| \geq 1$. First assume that the system is one-dimensional. Then since Z is assumed to be non-Moishezon, it follows from Proposition 3.2 that the induced $\mathbf{C}^*$-action on the parameter space of the pencil is non-trivial. Hence by Proposition 3.1 the base curve C of the pencil is a smooth real elliptic curve which is an orbit of the $\mathbf{C}^*$-action.

Next suppose that the fundamental system is two-dimensional. Then Proposition 2.4 implies the system is free, because Z is assumed to be non-Moishezon. Let $f : Z \to \mathbf{CP}^2$ be the associated morphism (which is an algebraic reduction of Z). As in the proof of Proposition 2.6, the set of $\mathbf{C}^*$-invariant lines of the induced $\mathbf{C}^*$-action ρ on $\mathbf{CP}^2$ forms a triangle and just one of the three lines is real. Let l be this real invariant line and y the intersection of the other two lines, which is clearly real $\mathbf{C}^*$-fixed point. Set $S := f^{-1}(l)$ and $C := f^{-1}(y)$. Both S and C is real and $\mathbf{C}^*$-invariant. Further, since C can be regarded as an anticanonical curve of a smooth rational surface, C is a connected curve. Since S is rational with $c_1^2(S) = 0$, the Euler number of S is easily seen to be 12. On the other hand the Euler number of Z is also 12. Therefore every fixed points of $\mathbf{C}^*$-action is contained in S, because the connected component of the $\mathbf{C}^*$-fixed points on Z is either a rational curve or a point, whose Euler numbers are positive. It follows that no point of C is fixed. Therefore C cannot be singular and hence is a smooth elliptic curve.

Next we show the uniqueness of real $\mathbf{C}^*$-orbit of an elliptic curve. Let $C' \subset Z$ be such an orbit other than C. Since both are orbits, $C \cap C'$ is empty. Let $\tilde{Z} \to Z$ be the blowing-up along C, $\tilde{C}' \subset \tilde{Z}$ the inverse image of C', and $f : \tilde{Z} \to \mathbf{CP}^1$ the morphism associated to the pencil of fundamental divisors whose base locus is C. f is $\mathbf{C}^*$-equivariant and $\tilde{C}'$ is $\mathbf{C}^*$-invariant. Let 0 and $\infty \in \mathbf{CP}^1$ be the $\mathbf{C}^*$-fixed points, which are conjugate of each other (cf. an argument in the proof of Proposition 2.6). Therefore $\tilde{C}'$ is not contained in $f^{-1}(0)$ or $f^{-1}(\infty)$. It follows that $f(\tilde{C}') = \mathbf{CP}^1$. This implies that $\tilde{C}' \cap f^{-1}(0) \neq \phi$, which contradicts to the fact that there exists no fixed point on $\tilde{C}'$. Therefore such an orbit C' does not exist.

Finally we show that C is $\mathbf{R}$-homologous to zero, by showing that the intersection numbers of C with generators of $H^2(Z, \mathbf{R})$ are all zero. Let $[C] \in H^4(Z, \mathbf{R})$ be the cohomology class of C. As in the above proof, C is the intersection of fundamental divisors. Hence $[C] = \mathcal{F} \cdot \mathcal{F}$, where we set $\mathcal{F} = -\frac{1}{2}K_Z$. On the other hand $H^2(Z, \mathbf{R})$ is generated by $\mathcal{F}$ and $\xi_1, \xi_2, \xi_3, \xi_4$, where $\{\xi_i\}_{i=1}^4$ is the lifted orthonormal bases of $H^2(4\mathbf{CP}^2, \mathbf{Z})$. Further we have [10]

$$(\mathcal{F}\cdot\mathcal{F})\cdot\mathcal{F} = -\frac{1}{8}K_Z^3 = 2(4-n) = 0$$

and since $c_1(Z)^2$ is the pull-back of a cohomology class of $H^4(4\mathbf{CP}^2, \mathbf{Z})$ [10], we have

$$(\mathcal{F}\cdot\mathcal{F})\cdot\xi_i = 0$$

for any ξ_i. Therefore we have $[C] = 0 \in H^4(Z, \mathbf{R})$, and all of the claims of the theorem are proved. □

Next we study $U(1)$-action on the base four-manifold $4\mathbf{CP}^2$. A $U(1)$-action on a four-manifold is said to be *semi-free* if the isotropy group of any point is either $\{\mathrm{id}\}$ or $U(1)$. Let g be a self-dual metric on $4\mathbf{CP}^2$ and assume that g has a non-trivial isometric $U(1)$ action ρ. C. LeBrun [21] has shown that if ρ is semi-free, g is a LeBrun's metric, whose twistor space is Moishezon [20]. Therefore, if the twistor space of g is non-Moishezon, ρ is not semi-free. That is, there is a point on $4\mathbf{CP}^2$ whose isotropy group is neither $\{\mathrm{id}\}$ nor $U(1)$. In this direction Theorem 3.6 implies the following.

Proposition 3.7. *Let g be a self-dual metric on $4\mathbf{CP}^2$ of positive scalar curvature, and assume that g has a non-trivial isometric $U(1)$-action, and that the associated twistor space is non-Moishezon. Then there is a $U(1)$-invariant two-sphere in $4\mathbf{CP}^2$ whose isotropy group is $\mathbf{Z}_2$.*

Proof. Let Z be the twistor space of g and σ the real structure. Then by Theorem 3.6 there is a real smooth elliptic curve C in Z which is $U(1)$-invariant. It can be readily seen that while $U(1)$ acts freely on C, there are just two orbits which are σ-invariant. Let $K \subset 4\mathbf{CP}^2$ be the image of C. $K = C/\langle\sigma\rangle$ and is a Klein bottle which is evidently $U(1)$-invariant. Then the two orbits corresponding to the above two σ-invariant orbits on C have $\mathbf{Z}_2$ as the isotropy group. On the other hand, by a result of Fintushel [9], the set of points on a four-manifold which have a non-trivial finite isotropy group forms a set of disjoint $S^2\backslash\{0,\infty\}$, where 0 and ∞ are fixed points. Thus the claim follows. □

Proposition 3.8. *Let Z be a non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action, σ the real structure, and C the orbit as in Theorem 3.6. Then the normal bundle of C in Z is of the form $F\oplus F$, where F is a line bundle of degree zero satisfying $\sigma^*F \simeq \overline{F}$.*

Proof. As seen in the proof of Theorem 3.6, C is a base curve of a $\mathbf{C}^*$-invariant real pencil of fundamental divisors, whose general member is smooth and not $\mathbf{C}^*$-invariant. Take a real (irreducible) member S among the pencil, and put $F := N_{C/S}$. Then the claim is obvious. □

The following is a direct consequence of a result of Campana and Kreußler [5, Theorem 3.4] (where they do not assume the existence of $\mathbf{C}^*$-action and instead suppose the existence of C).

Proposition 3.9. *Let Z, C and F be as in Proposition 3.8. Then $a(Z) = 2$ if and only if the order of F in $\mathrm{Pic}^0 C$ is finite. (Otherwise $a(Z) = 1$.)*

4. Existence

As shown in the last section, non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action has a unique real orbit which is a smooth elliptic curve whose normal bundle is of the form $F \oplus F$ with $\deg F = 0$. In this section we show the existence of such twistor spaces over $4\mathbf{CP}^2$.

4.1 Some lemmas on elliptic curves and a statement of the main result. In order to state our main result precisely, we need the following lemmas.

Lemma 4.1. *Let C be a smooth elliptic curve and σ a real structure without real point. Then there exists $0 < \lambda \leq 1/e^{2\pi}$ such that C is biholomorphic to $\mathbf{C}^*/\langle z \mapsto \lambda z\rangle$, and σ is induced by $z \mapsto -1/\overline{z}$. Further, such λ is uniquely determined by the biholomorphic class of C.*

Proof. Let $\Gamma = \mathbf{Z} + \mathbf{Z}\omega \subset \mathbf{C}$ be a lattice such that $C \simeq \mathbf{C}/\Gamma$, where ω satisfies $|\omega| \geq 1$ and $-1/2 < \mathrm{Re}\omega \leq 1/2$, and $\mathrm{Im}\omega > 0$, under which ω is uniquely determined. Then by a standard argument we can show that, if C admits a real structure, then ω satisfies $|\omega| = 1$, $\mathrm{Re}\omega = 1/2$, or $\mathrm{Re}\omega = 0$. But if $\mathrm{Re}\omega \neq 0$, then the real locus of any real structures on C turns out to be non-empty, which is always a (connected) circle. Hence if the real structure does not have real point, $\mathrm{Re}\omega = 0$ must hold.

Assume $\mathrm{Re}\omega = 0$. Then we can show that any real structure on C is given by $w \mapsto \overline{w} + b\omega$, $w \mapsto \overline{w} + 1/2 + b\omega$, $w \mapsto -\overline{w} + a$, or $w \mapsto -\overline{w} + a + \omega/2$ on the universal cover, where $a, b \in \mathbf{R}$. For the first and third cases, the real locus on C consists of two disjoint circle. Hence our real structure must be the second or fourth ones. But these two cases represent the same real structure, as is seen by changing a coordinate by $w' = \omega w$. Thus we may assume that the real structure is given by $w \mapsto \overline{w} + 1/2 + b\omega$. But we can suppose $b = 0$, as is seen by setting $w' = w - (b/2)\omega$. Descending on $\mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^*$ by $z = e^{2\pi\sqrt{-1}w}$, we get the claim of the lemma. □

Lemma 4.2. *Let C and σ be as in Lemma 4.1, and $F \to C$ a holomorphic line bundle of degree zero satisfying $\sigma^* F \simeq \overline{F}$. Then F is obtained from the trivial line bundle over $\mathbf{C}^*$ as a quotient bundle by the $\mathbf{Z}$-action defined by $(z, \xi) \mapsto (\lambda z, \zeta\xi)$, where $0 < \lambda \leq 1/e^{2\pi}$ is as in Lemma 4.1, $\zeta \in U(1) \cup \sqrt{\lambda} \cdot U(1)$, and ξ is a fiber coordinate on the trivial bundle. Further, such ζ is uniquely determined by the isomorphic class of F.*

Proof. Let $\Gamma = \mathbf{Z} + \mathbf{Z}\omega \subset \mathbf{C}$ be a lattice such that $C \simeq \mathbf{C}/\Gamma$ as in the proof of the previous lemma. Then, as is well known, $\mathrm{Pic}^0 C$ is naturally identified with the set of characters $\{\chi : \Gamma \to U(1)\}$ of Γ; Concretely, any $F \in \mathrm{Pic}^0 C$ is obtained from the trivial bundle over $\mathbf{C}$, as a quotient line bundle by the Γ-action defined by

$$\Gamma \ni m + n\omega : (w, \xi) \mapsto (w + m + n\omega, \chi(1)^m \chi(\omega)^n \xi).$$

Here, ξ denotes a fiber coordinate of the trivial line bundle over $\mathbf{C}$. If we change the fiber coordinate by setting $\xi' = e^{2\pi\sqrt{-1}az}\xi$ for $a \in \mathbf{C}^*$, then $\chi(1)$ and $\chi(\omega)$ are multiplied by $e^{2\pi\sqrt{-1}a}$ and $e^{2\pi\sqrt{-1}a\omega}$ respectively. Hence by setting $e^{-2\pi\sqrt{-1}a} = \chi(1)$ (i.e. by setting $a = -(\log\chi(1))/(2\pi\sqrt{-1})$), we see that $\chi(1)$ can be assumed to be 1 and correspondingly $\chi(\omega)$ is multiplied by $e^{2\pi\sqrt{-1}a\omega} \in \mathbf{C}^*$. Further, the ambiguity of the choice of such $a \in \mathbf{C}^*$ (i.e. the indeterminacy of values of log) implies that, when fixing $\chi(1)$ to be 1, $\zeta := \chi(\omega)$ has ambiguity of multiplications by $e^{2\pi\sqrt{-1}n\omega}$, where n moves in $\mathbf{Z}$. Descending again to $\mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^*$, and noting that $\lambda = e^{2\pi\sqrt{-1}\omega}$, we can conclude that $\mathrm{Pic}^0 C$ is identified with $\mathbf{C}^*/\langle \zeta \mapsto \lambda\zeta\rangle$ (which is C itself, of course). Hence $\zeta \in \mathbf{C}^*$ can be uniquely chosen from the fundamental domain $\{\zeta \in \mathbf{C}^* \,|\, 1/\sqrt{\lambda} < |\zeta| \le \sqrt{\lambda}\}$.

Next we assume that C has a real structure σ without real points, so that $0 < \lambda \le 1/e^{2\pi}$ by the previous lemma. Let $F \in \mathrm{Pic}^0 C$ be any element and $\zeta \in \mathbf{C}^*$, $1/\sqrt{\lambda} < |\zeta| \le \sqrt{\lambda}$ be the uniquely determined number as above. We denote by F_z the fiber of F over z. Then we have natural isomorphisms $(\sigma^*F)_1 \simeq F_{\sigma(1)} = F_{-1}$ and $(\sigma^*F)_\lambda \simeq F_{\sigma(\lambda)} = F_{-1/\lambda}$. Therefore, the identification $(\overline{\sigma^*F})_1 \to (\overline{\sigma^*F})_\lambda$ is given by the multiplication of $\overline{\zeta}^{-1}$. Namely the real structure on the above fundamental domain is given by $\zeta \mapsto \overline{\zeta}^{-1}$. Thus the set of real line bundles just corresponds to the set $\{|\zeta| = 1 \text{ or } \sqrt{\lambda}\}$. This proves the claim of the lemma. □

Definition 4.3. Let C, σ and $F \to C$ be as in Lemma 4.2. Then we say F *can be continuously deformed into the trivial line bundle preserving the real structure* if $\zeta \in U(1)$ (i.e. $\zeta \notin \sqrt{\lambda}\cdot U(1)$).

Let Z be a non-Moishezon twistor space of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action and C the unique real orbit which is an elliptic curve (see Theorem 3.6). Then by Proposition 3.8 the normal bundle, $N_{C/Z}$, is of the form $F \oplus F$, where F is a line bundle of degree zero satisfying $\sigma^*F \simeq \overline{F}$. Then we have the following

Proposition 4.4. *F can be continuously deformed into the trivial line bundle preserving the real structure.*

Proof. As in the proof of Proposition 3.8, let S be a (general) real member of the $\mathbf{C}^*$-invariant pencil of fundamental divisors having C as a base locus. Then C is a real anticanonical curve of S and $F = N_{C/S}$. (S does not have $\mathbf{C}^*$-action in general.) Let $\nu : S \to S_0 = \mathbf{CP}^1 \times \mathbf{CP}^1$ be a birational morphism as in Proposition 2.1. Then $C_0 := \nu(C)$ is an anticanonical curve of S_0 which is a smooth elliptic curve. Put $N_0 = N_{C_0/S_0}$. N_0 is a real line bundle of degree eight. Let $\{p_i, \overline{p_i} \in C_0 \,|\, i = 1, 2, 3, 4\}$ be a zero locus of a real section of N_0. Let $\{q_i, \overline{q}_i \in C_0 \,|\, i = 1, 2, 3, 4\}$ be the blown-up points of ν. Then we have

$F = N_{C/S} \simeq \mathcal{O}_{C_0}(\sum_{i=1}^4 (p_i + \overline{p_i})) \otimes \mathcal{O}_{C_0}(-\sum_{i=1}^4 (q_i + \overline{q_i})) \simeq \mathcal{O}_{C_0}(\sum_{i=1}^4 (p_i - q_i) + \sum_{i=1}^4 (\overline{p}_i - \overline{q}_i))$. Thus by taking continuous paths connecting p_i and q_i in C_0 for $1 \le i \le 4$, we get the conclusion. □

The following is the main result in this section. The rest of this paper is devoted to proving it.

Theorem 4.5. *Let C be a non-singular elliptic curve and σ a real structure without real points. Suppose that C has no non-trivial automorphism. Let F be any holomorphic line bundle over C of degree zero satisfying $\sigma^* F \simeq \overline{F}$. Further assume F can be continuously deformed into the trivial line bundle preserving the real structure. Then there exists a non-Moishezon twistor space Z of $4\mathbf{CP}^2$ with $\mathbf{C}^*$-action such that the orbit appears in Theorem 3.6 is biholomorphic to C and such that its normal bundle in Z is isomorphic to $F \oplus F$.*

Remark 4.6. An elliptic curve $C = \mathbf{C}^*/\langle z \mapsto \lambda z \rangle$ with $0 < \lambda \le 1/e^{2\pi}$ has no non-trivial automorphism iff $\lambda = 1/e^{2\pi}$. The order of a holomorphic line bundle $F = (\mathbf{C}^* \times \mathbf{C})/\langle (z, \xi) \mapsto (\lambda z, \zeta \xi) \rangle$ in $\mathrm{Pic}^0 C$ coincides with the order of ζ in $\mathbf{C}^*$.

Theorem 4.5 answers a question of Campana and Kreußler [5, Open Problem] affirmatively in a stronger form. For related work, see [12].

4.2. Construction of a family of primary Hopf surfaces with involution. Let $0 < \lambda \le 1/e^{2\pi}$ be a real number and $\zeta \in U(1)$ a complex number of unit length. Let $g = g(\lambda, \zeta)$ be the matrix

$$g = \begin{pmatrix} \zeta^{-\frac{1}{2}} & 0 \\ 0 & \lambda \zeta^{\frac{1}{2}} \end{pmatrix}.$$

We regard g as an automorphism of $\mathbf{HP}^1$, the projective space of quaternionic lines in $\mathbf{H}^2$. (Because g acts $\mathbf{H}^2$ by the multiplication from the left, $\mathbf{H}^\times$ acts on $\mathbf{H}^2 \backslash (0,0)$ by the multiplication from *the right*). Then there is no distinction between two choices of the square root in the definition of g. We next define a primary Hopf surface $H_0 = H_0(\lambda, \zeta)$ by

$$H_0 := (\mathbf{HP}^1 \backslash \{0, \infty\})/\langle g \rangle = \mathbf{H}^\times / \langle g \rangle,$$

where $0 = {}^t(1:0)$ and $\infty = {}^t(0:1)$. If we use the complex coordinate (z, w) with $z + jw = q \in \mathbf{H}^\times$, $g(z, w) = (\lambda \zeta z, \lambda w)$. (Since $\mathbf{H}^\times$ acts on $\mathbf{H}^2$ from the right, non-homogeneous coordinate on the set $\{(q_0 : q_1) \in \mathbf{HP}^1 \,|\, q_0 \neq 0\}$ should be $q = q_1 \cdot q_0^{-1}$.)

Next we define a $U(1)$-action ρ on $\mathbf{HP}^1$. For each $t \in U(1)$, define ρ_t to be the matrix

$$\rho_t := \begin{pmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix}.$$

We also regard ρ_t as an automorphism of $\mathbf{HP}^1$. Since $\mathbf{H}$ is non-commutative, this defines a non-trivial automorphism on $\mathbf{HP}^1$, as far as $t \neq 1$. We have

$\rho_t(z,w) = (z, tw)$ for the complex coordinate (z,w) as above. Since g and ρ_t commute, ρ defines a $U(1)$-action on H_0, which will be still denoted by ρ.

Further we define an involution of $\mathbf{HP}^1$ by $\tau(q_0 : q_1) = (q_1 : q_0)$. Since $\tau g \tau^{-1} = g^{-1}$ as an automorphism of $\mathbf{HP}^1$, τ maps g-orbits to g-orbits and defines a (non-holomorphic) involution on H_0, which is also denoted by τ. The set of fixed points of $\tau : H_0 \to H_0$ is four-points: the g-orbits of $(1 : \pm 1)$ and of $(1 : \pm(\lambda\zeta)^{\frac{1}{2}})$. These are contained in the fixed set of ρ ($= (\mathbf{CP}^1\backslash\{0,\infty\})/\langle g\rangle$, a two-torus).

We then define an orbifold $M_0 = M_0(\lambda, \zeta)$ to be $M_0 := H_0/\langle\tau\rangle = H_0(\lambda,\zeta)/\langle\tau\rangle$. Since τ interchanges orientation of a generator of $\pi_1(H_0) \simeq \mathbf{Z}$, M_0 is a simply connected orbifold, and has four isolated singularities. Further M_0 has a $U(1)$-action induced by ρ, which we still denoted by ρ.

Finally we define a conformally flat metric on these spaces. The Riemmanian metric $|dq|^2/|q|^2$, where $q = q_1 q_0^{-1} \in \mathbf{H}^\times$ as above and $|\cdot|$ is the usual norm on the quaternions, defines a conformally flat metric on $\mathbf{H}^\times$, H_0, and M_0. All of these will be denoted by h_0.

Thus we have obtained, a connected family of simply connected orbifolds $\{M_0 = M_0(\lambda,\zeta)\}$ parametrized by $0 < \lambda \le 1/e^{2\pi}$ and $\zeta \in U(1)$ equipped with a conformally flat metric h_0 (or more precisely, a family of conformally flat metrics on the orbifold M_0).

Remark 4.7. Our choice of g comes from the following fact: *let $g \in GL(2,\mathbf{H})$ be a matrix such that (i) g fixes $(1:0)$ and $(0:1)$, (ii) g commutes with every ρ_t, (iii) τ maps every g-orbits to g-orbits. Then there exist λ and ζ such that $g = g(\lambda,\zeta)$.* This can be shown elementally, and in the following we do not need this fact. So we omit a proof.

4.3. Associated twistor spaces. In this subsection we describe the twistor spaces of H_0 and M_0. As is well known, the twistor space of $\mathbf{HP}^1$ is $\mathbf{CP}^3$, and the twistor fibration is explicitly given by $(z_0 : z_1 : z_2 : z_3) \mapsto (z_0 + jz_1 : z_2 + jz_3)$, and the real structure is given by $(z_0 : z_1 : z_2 : z_3) \mapsto (\overline{z}_0 : -\overline{z}_1 : \overline{z}_2 : -\overline{z}_3)$, where $(z_0 : z_1 : z_2 : z_3)$ is a homogeneous coordinate on $\mathbf{CP}^3$. The twistor space of $\mathbf{H}^\times$ is $\mathbf{CP}^3\backslash(L_0 \cup L_\infty)$, where L_0 and L_∞ are twistor lines $z_2 = z_3 = 0$ and $z_0 = z_1 = 0$ respectively. It is easily verified that the lift of $g, \tau, \rho_t : \mathbf{H}^\times \to \mathbf{H}^\times$ (see §4.2) upto the twistor space are explicitly given by (using the same symbols)

$$g : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : \zeta z_1 : \lambda\zeta z_2 : \lambda z_3), \tag{1}$$

$$\tau : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3 : z_0 : z_1), \tag{2}$$

$$\rho_t : (z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : tz_1 : z_2 : tz_3). \tag{3}$$

In the following we regard ρ as a $\mathbf{C}^*$-action which is the complexification of the original ρ. That is, t is allowed to move in $\mathbf{C}^*$ in (3).

Let $W_0 = (\mathbf{CP}^3\backslash(L_0 \cup L_\infty))/\langle g\rangle$ be the twistor space of (H_0, h_0) and $Z_0 = W_0/\langle\tau\rangle$ the twistor space of (M_0, h_0). Needless to say, these depend on λ and

ζ. Since τ and ρ_t commute, W_0 has a holomorphic action of $\mathbf{C}^* \times \mathbf{Z}_2$. Let l^* and $\overline{l}^* \subset \mathbf{CP}^3$ be non-real disjoint (two point punctured) lines defined by $l^* = \{z_1 = z_2 = 0 \mid z_0 z_3 \neq 0\}$ and $\overline{l}^* = \{z_0 = z_3 = 0 \mid z_1 z_2 \neq 0\}$. By (1), both l^* and $\overline{l}^*$ are g-invariant. Hence $C_0 := l^*/\langle g\rangle$ and $\overline{C}_0 := \overline{l}^*/\langle g\rangle$ can be regarded as holomorphic curves in W_0. Again by (1), C_0 and $\overline{C}_0$ are biholomorphic to $\mathbf{C}^*/\langle z \mapsto \lambda z\rangle$. Further we have $\tau(l^*) = \overline{l}^*$ by (2). Therefore C_0 and $\overline{C}_0$ are mapped biholomorphically, by the quotient map $W_0 \to Z_0$, to the same curve, which we still denote by $C_0 \subset Z_0$. That is, we define the curve $C_0 \subset Z_0$ to be the image of l^* (and $\overline{l}^*$) by the composition of the quotient maps $\mathbf{CP}^3\backslash(L_0 \cup L_\infty) \to W_0 \to Z_0$. C_0 is real in Z_0. Further, since both l^* and $\overline{l}^*$ are $\mathbf{C}^*$-orbits with respect to the complexified ρ, and since the quotient maps $\mathbf{CP}^3\backslash(L_0 \cup L_\infty) \to W_0 \to Z_0$ are $\mathbf{C}^*$-equivariant, C_0 is an orbit of the $\mathbf{C}^*$-action.

Next we see that $C_0 \subset Z_0$ is disjoint from the singular twistor lines of Z_0. The image of l^* by the twistor fibration $\mathbf{CP}^3 \to \mathbf{HP}^1$ is $\{(1 : jz) \mid z \in \mathbf{C}^*\}$. On the other hand, all of the τ-fixed points (on $\mathbf{C}^2\backslash(0,0) \simeq \mathbf{H}^\times$) lie on the set $\{(1 : z) \mid z \in \mathbf{C}^*\}$. These two sets are disjoint and g-invariant. Therefore $C_0 \subset Z_0$ is disjoint from the singular twistor lines, and Z_0 is smooth on a neighborhood of C_0. We look at the normal bundle of C_0 in Z_0. Since the quotient map $W_0 \to Z_0$ is biholomorphic on a neighborhood of $C_0 \subset W_0$, N_{C_0/Z_0} is biholomorphic to N_{C_0/W_0}. If we use $(z_1/z_0, z_2/z_0)$ as a fiber coordinate, the latter normal bundle is seen to be of the form $F \oplus F'$, where F is given by $(\mathbf{C}^* \times \mathbf{C})/\langle (z,\xi) \mapsto (\lambda z, \zeta\xi)\rangle$ and F' is given by $(\mathbf{C}^* \times \mathbf{C})/\langle (z,\xi') \mapsto (\lambda z, \lambda\zeta\xi')\rangle$. But F and F' are biholomorphic as line bundles over C_0, as is seen by setting $\xi' = z\xi$. (In general ζ and $\lambda^n\zeta$ determines the same line bundle for any $n \in \mathbf{Z}$. See the proof of Lemma 4.2) Thus we have obtained the following

Lemma 4.8. *Let C, σ and $F \in \mathrm{Pic}^0 C$ be as in Theorem 4.5. Let $0 < \lambda \le 1/e^{2\pi}$ and $\zeta \in U(1)$ be the numbers uniquely determined by C and F as in Lemma 4.2, and Z_0 be the twistor space of the conformally flat orbifold $M_0 = M_0(\lambda, \zeta)$ constructed as above. Then there exists a smooth elliptic curve $C_0 \subset Z_0$ with the following properties: (i) C_0 is real with respect to the real structure of Z_0, (ii) C_0 is biholomorphic to C, (iii) C_0 is disjoint from the singular twistor lines of Z_0, (iv) N_{C_0/Z_0} is isomorphic to $F^{\oplus 2}$, (v) C_0 is an orbit of the $\mathbf{C}^*$-action.*

4.4. The Kuranishi family of the twistor spaces. In this subsection we show that our family $\{W_0 = W_0(\lambda, \zeta)\}$ of twistor spaces (constructed in the last subsection) is the real part of the versal family of $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0. This will be a key step in proving Theorem 4.5.

For convenience we put $V := \mathbf{CP}^3\backslash(L_0 \cup L_\infty)$, which is the universal cover of W_0. Let $\pi : V \to W_0 = V/\langle g\rangle$ be the covering map. Then by Douady [7] there is the following long exact sequence of cohomology groups:

$$(4) \quad 0 \to H^0(W_0, \Theta_{W_0}) \to H^0(V, \Theta_V) \xrightarrow{1-g_*} H^0(V, \Theta_V) \to H^1(W_0, \Theta_{W_0}) \to \cdots .$$

Lemma 4.9. $H^2(V, \Theta_V) = 0$.

Proof. Put $U_0 := \mathbf{CP}^3 \backslash L_0$ and $U_\infty := \mathbf{CP}^3 \backslash L_\infty$. We consider the following standard long exact sequence

$$(5) \qquad 0 \to H^1_{L_0 \cup L_\infty}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3}) \to H^1(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3}) \to H^1(V, \Theta_V) \to \cdots .$$

Here, since the codimensions of L_0 and L_∞ are two, the sequence starts from the H^1-terms. Since $H^i(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3}) = 0$ for any $i \geq 1$ we get by (5) $H^i(V, \Theta_V) \simeq H^{i+1}_{L_0 \cup L_\infty}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3})$ for any $i \geq 1$. The same argument shows $H^i(U_0, \Theta_{U_0}) \simeq H^{i+1}_{L_0}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3})$ and $H^i(U_\infty, \Theta_{U_\infty}) \simeq H^{i+1}_{L_\infty}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3})$ for any $i \geq 1$. On the other hand, since L_0 and L_∞ are disjoint, Mayer Vietoris sequence implies $H^{i+1}_{L_0 \cup L_\infty}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3}) \simeq H^{i+1}_{L_0}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3}) \oplus H^{i+1}_{L_\infty}(\mathbf{CP}^3, \Theta_{\mathbf{CP}^3})$ for any i. Combining these, we get, for $i \geq 1$, a natural isomorphism

$$(6) \qquad H^i(V, \Theta_V) \simeq H^i(U_0, \Theta_{U_0}) \oplus H^i(U_\infty, \Theta_{U_\infty}).$$

Put $V_i = \mathbf{CP}^3 \backslash \{z_i = 0\}$ for $0 \leq i \leq 3$. These are open subset of U_0 or U_∞, and we have $U_0 = V_2 \cup V_3$ and $U_\infty = V_0 \cup V_1$. Moreover, $V_0 \cap V_1$ and $V_2 \cap V_3$ are easily seen to be biholomorphic to $\mathbf{C}^2 \times \mathbf{C}^*$ and hence Stein. Thus U_0 and U_∞ admit a Stein covering consisting of just two open subset. Therefore by a theorem of Leray we get $H^2(U_0, \Theta_{U_0}) = H^2(U_\infty, \Theta_{U_\infty}) = 0$. Hence by (6) we get $H^2(V, \Theta_V) = 0$. □

Lemma 4.10. *$1 - g_* : H^1(V, \Theta_V) \to H^1(V, \Theta_V)$ is an isomorphism.*

Proof. We continue to use the same symbols in the previous lemma. By (6) it suffices to show that $1 - g_* : H^1(U_\infty, \Theta_{U_\infty}) \to H^1(U_\infty, \Theta_{U_\infty})$ is an isomorphism, because by symmetry it then follows that $1 - g_* : H^1(U_0, \Theta_{U_0}) \to H^1(U_0, \Theta_{U_0})$ is also an isomorphism

Let $U_\infty = V_0 \cup V_1$ be the Stein covering defined in the previous lemma. Then again by a theorem of Leray we have $H^1(U_\infty, \Theta) \simeq \Gamma(V_{01}, \Theta)/\mathrm{Im}\,(\delta : \Gamma(V_0, \Theta) \oplus \Gamma(V_1, \Theta) \to \Gamma(V_{01}, \Theta))$, where Γ denotes the space of sections and δ is the derivation (i.e. taking the difference on V_{01}), and $V_{01} = V_0 \cap V_1$. Let $v_i = z_i/z_0\,(i = 1, 2, 3)$ and $u_j = z_j/z_1\,(j = 0, 2, 3)$ be non-homogeneous coordinates on V_0 and V_1 respectively, and put $\partial_i := \partial/\partial v_i\,(i = 1, 2, 3)$. Then by GAGA we may have $\Gamma(V_0, \Theta) = \oplus_{i=1}^3 \mathbf{C}[v_1, v_2, v_3]\partial_i$, $\Gamma(V_1, \Theta) = \oplus_{j=0,2,3}\, \mathbf{C}[u_0, u_2, u_3]\partial/\partial u_j$, and $\Gamma(V_{01}, \Theta) = \oplus_{i=1}^3 \mathbf{C}[v_1, v_1^{-1}, v_2, v_3]\partial_i$.

First by taking modulo about $\Gamma(V_0, \Theta)$ into account, it is obvious that $H^1(\Theta_{U_\infty})$ is generated by $\oplus_{i=1}^3 \mathbf{C}[v_1^{-1}, v_2, v_3]\partial_i$. On the other hand, we have relations $v_1 = 1/u_0, v_2 = u_2/u_0, v_3 = u_3/u_0$, from which it easily follows $\partial/\partial u_0 = -v_1(v_1\partial_1 + v_2\partial_2 + v_3\partial_3)$, $\partial/\partial u_2 = v_1\partial_2$, and $\partial/\partial u_3 = v_1\partial_3$. Thus $\Gamma(V_1, \Theta)$ is a vector space over $\mathbf{C}$ whose basis is the set

$$\left\{ \frac{v_2^b v_3^c}{v_1^{a+b+c-1}}(v_1\partial_1 + v_2\partial_2 + v_3\partial_3),\ \frac{v_2^b v_3^c}{v_1^{a+b+c-1}}\partial_2,\ \frac{v_2^b v_3^c}{v_1^{a+b+c-1}}\partial_3 \;\middle|\; a, b, c \geq 0 \right\}.$$

But we have

$$\frac{v_2^b v_3^c}{v_1^{a+b+c-1}}(v_1\partial_1 + v_2\partial_2 + v_3\partial_3) = \frac{v_2^b v_3^c}{v_1^{a+b+c-2}}\partial_1 - \frac{v_2^{b+1} v_3^c}{v_1^{a+b+c-1}}\partial_2 - \frac{v_2^b v_3^{c+1}}{v_1^{a+b+c-1}}\partial_3,$$

and if $a \geq 1$ the second and third terms of the right hand side are elements of the above set. Therefore $\Gamma(V_1, \Theta)$ is a vector space over $\mathbf{C}$ whose basis is the set

$$\left\{ \frac{v_2^b v_3^c}{v_1^{b+c-1}}(v_1\partial_1 + v_2\partial_2 + v_3\partial_3),\ \frac{v_2^b v_3^c}{v_1^{a+b+c-1}}\partial_i \ \middle|\ a, b, c \geq 0,\ i = 1, 2, 3 \right\}.$$

Let $\mathcal{V}$ be a vector space over $\mathbf{C}$ whose basis is the set $\{(v_2^b v_3^c / v_1^a)\partial_i \mid a, b, c \geq 0, a - b - c \leq -2, i = 1, 2, 3\}$. Let $\mathcal{W}$ be a vector subspace of $\mathcal{V}$ generated by $\{(v_2^b v_3^c / v_1^{b+c-1})(v_1\partial_1 + v_2\partial_2 + v_3\partial_3) \,|\, b, c \geq 0, b + c \geq 2\}$. Then by the above consideration we have $H^1(\Theta_{U_\infty}) \simeq \mathcal{V}/\mathcal{W}$. Thus in order to prove that $1 - g_*$ is an isomorphism on $H^1(\Theta_{U_\infty})$, it suffices to show that $1 - g_* : \mathcal{V} \to \mathcal{V}$ is an isomorphism preserving $\mathcal{W}$ invariant. It is readily seen that, by g_*, each $(v_2^b v_3^c / v_1^a)\partial_i$ is simply multiplied by $\zeta^{1+a-b}\lambda^{-b-c}$ (for $i = 1$), $\zeta^{1+a-b}\lambda^{1-b-c}$ (for $i = 2$), and $\zeta^{a-b}\lambda^{1-b-c}$ (for $i = 3$), respectively. Since $\lambda \leq 1/e^{2\pi} < 1$ and since $b + c \geq a + 2 \geq 2$, these three numbers cannot be zero. Therefore $1 - g_*$ is an isomorphism on $\mathcal{V}$. Further it can be easily checked that $\mathcal{W}$ is invariant by g_* (or this is rather obvious if one recalls that each generator of $\mathcal{W}$ was originally $(u_1^a u_2^b u_3^c)\partial/\partial u_0$.) This completes a proof of the lemma. □

The sequence (4) and Lemmas 4.9 and 4.10 immediately imply the following

Proposition 4.11. $H^2(W_0, \Theta_{W_0}) = 0$.

Remark 4.12. If $\zeta = 1$ this is a result of Pontecorvo [25]. Obviously his proof does not work for $\zeta \neq 1$.

The Kuranishi family of W_0 can be explicitly written down:

Proposition 4.13. *Let $\Delta \subset \mathbf{C}$ be a sufficiently small neighborhood of 0 in $\mathbf{C}$, and put $g(x_1, x_2, x_3) := \mathrm{diag}(1, e^{x_1}\zeta, e^{x_2}\lambda\zeta, e^{x_3}\lambda) \in GL(4, \mathbf{C})$ for $x_i \in \Delta$ ($i = 1, 2, 3$), where $\mathrm{diag}(\cdot)$ denotes the diagonal matrix whose entries are $(\cdot)$. Consider a family $\mathcal{W} = \{W(x_1, x_2, x_3) := V/\langle g(x_1, x_2, x_3)\rangle \,|\, x_i \in \Delta\} \to \Delta^3 = \Delta \times \Delta \times \Delta$, regarding as a deformation of $W_0 = W(0, 0, 0)$. Then $\mathcal{W} \to \Delta^3$ is the Kuranishi family of $\mathbf{C}^*$-equivariant deformations of W_0. Further, if $\zeta \neq 1$, $\mathcal{W} \to \Delta^3$ is the usual Kuranishi family of W_0.*

Proof. By Lemma 4.10 and (4) there exists an exact sequence

$$(7) \qquad 0 \to H^0(W_0, \Theta_{W_0}) \to H^0(V, \Theta_V) \xrightarrow{1-g_*} H^0(V, \Theta_V) \xrightarrow{\delta} H^1(W_0, \Theta_{W_0}) \to 0.$$

A geometric meaning of this sequence is as follows. We identify $H^0(\Theta_V) = H^0(\Theta_{\mathbf{CP}^3})$ with $pgl(4, \mathbf{C}) = gl(4, \mathbf{C})/\mathbf{C}I_4$. The exactness of the former three terms of (7) implies that a matrix $A \in gl(4, \mathbf{C})$ (or $e^{tA} \in GL(4, \mathbf{C})$, precisely) induces an automorphism of W_0 iff $A - gAg^{-1} = cI$ for some $c \in \mathbf{C}$. (Notice that $g_*A = gAg^{-1}$.) The exactness of the latter three terms of (7) implies that, for $B \in gl(4, \mathbf{C})$, $g_t = e^{tB}g$ defines a trivial deformation of W_0 iff $B = A - gAg^{-1} + cI$ for some $A \in gl(4, \mathbf{C})$ and for some $c \in \mathbf{C}$.

For the case $\zeta \neq 1$, it suffices to prove that the Kodaira-Spencer map of our family $\mathcal{W} \to \Delta^3$ is isomorphic, since $\mathbf{C}^*$-equivariance is obvious from our explicit construction. It is easy to see that if $\zeta \neq 1$, $\{A - gAg^{-1} \mid A \in gl(4, \mathbf{C})\}$ is the set of matrices whose diagonal entries are all zero. Therefore, if $\zeta \neq 1$, δ (in (7)) induces an isomorphism

$$\{\mathrm{diag}(x_0, x_1, x_2, x_3) \in gl(4, \mathbf{C}) \mid x_i \in \mathbf{C}\}/\mathbf{C}I \quad \simeq H^1(W_0, \Theta_{W_0}).$$

Since $\delta(A) \in H^1(\Theta_{W_0})$ for $A \in gl(4, \mathbf{C})$ is nothing but the Kodaira-Spencer class of a one-parameter family defined by $g_t = e^{tA}g$ [7], and since Δ^3 corresponds to the subspace $\{\mathrm{diag}(x_0, x_1, x_2, x_3) \in gl(4, \mathbf{C})\}/\mathbf{C}I$, the Kodaira-Spencer map of $\mathcal{W} \to \Delta^3$ is an isomorphism (onto $H^1(\Theta_{W_0})$). Hence $\mathcal{W} \to \Delta^3$ is the Kuranishi family of W_0 (provided $\zeta \neq 1$).

If $\zeta = 1$, $\{A - gAg^{-1} \mid A \in gl(4, \mathbf{C})\}$ is the set of matrices

$$\left\{ \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} ; A_1, A_2 \in gl(2, \mathbf{C}) \right\}.$$

Therefore δ induces an isomorphism

$$\left\{ \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} ; B_1, B_2 \in gl(2, \mathbf{C}) \right\} \Big/ \mathbf{C}I \quad \simeq H^1(W_0, \Theta_{W_0}).$$

On the other hand, the $\mathbf{C}^*$-invariance (i.e. commutativity with the matrices $\mathrm{diag}(1, t, 1, t), t \in \mathbf{C}^*$) eliminates the off-diagonal entries of B_1 and B_2. Therefore $H^1(\Theta_{W_0})^{\mathbf{C}^*}$ is identified with $\{\mathrm{diag}(x_0, x_1, x_2, x_3) \in gl(4, \mathbf{C})\}/\mathbf{C}I$ by the above isomorphism. Thus by the same reason for the case $\zeta \neq 1$, the Kodaira-Spencer map of $\mathcal{W} \to \Delta^3$ is injective whose image is $H^1(\Theta_{W_0})^{\mathbf{C}^*}$, and the claim for the case $\zeta = 1$ also follows. □

We also need to determine which deformation preserves the involution τ.

Proposition 4.14. *Let $\mathcal{W} \to \Delta^3$ be as in Proposition 4.13 and $\mathcal{W}' \to \Delta^2$ be the restriction of $\mathcal{W} \to \Delta^3$ onto the smooth subspace $\Delta^2 = \{x_2 = x_1 + x_3\}$. Then $\mathcal{W}' \to \Delta^2$ is the Kuranishi family of $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0, and further if $\zeta \neq 1$, $\mathcal{W}' \to \Delta^2$ is the Kuranishi family of $\mathbf{Z}_2$-equivariant deformations of W_0*

Proof. It suffices to see that $W(x_1, x_2, x_3)(\subset \mathcal{W})$ is τ-invariant if and only if $x_2 = x_1 + x_3$. Namely, it suffices to show that $\tau : V \to V$ maps each $g(x_1, x_2, x_3)$-orbit to a $g(x_1, x_2, x_3)$-orbit iff $x_2 = x_1 + x_3$. This is equivalent to the conditions $\tau g(x_1, x_2, x_3)\tau^{-1} = g(x_1, x_2, x_3)$ or $\tau g(x_1, x_2, x_3)\tau^{-1} = g(x_1, x_2, x_3)^{-1}$ as elements of $PGL(4, \mathbf{C})$. But the former cannot happen for small x_i, and the latter must holds. From this, the conclusion follows readily. □

Concerning the automorphism group, it is easy to see the following:

Proposition 4.15. *The Lie algebra of the holomorphic automorphism group of $W_0 = W_0(\lambda, \zeta)$ is given by*

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ; A, B \in gl(2, \mathbf{C}) \right\} \Big/ \mathbf{C}I$$

for the case $\zeta = 1$, and $\{\mathrm{diag}(x_0, x_1, x_2, x_3) \,|\, x_i \in \mathbf{C}\}/\mathbf{C}I$ for the case $\zeta \neq 1$. The Lie algebra of the holomorphic automorphism group of $Z_0 = W_0(\lambda, \zeta)/\langle \tau \rangle$ is given by

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} ; A \in gl(2, \mathbf{C}) \right\} \Big/ \mathbf{C}I$$

for the case $\zeta = 1$, and $\{\mathrm{diag}(x_0, x_1, x_0, x_1) \,|\, x_i \in \mathbf{C}\}/\mathbf{C}I$ for the case $\zeta \neq 1$.

Note that if $(x_1, x_3) \in \Delta^2$ moves in $\sqrt{-1}\mathbf{R} \times \mathbf{R}$ the family coincides with the family of twistor spaces constructed in §4.3. Namely, the family $\{W_0 = W_0(\lambda, \zeta)\}$ of twistor spaces constructed in §4.3 is the real part of $\mathcal{W}' \to \Delta^2$. Let C_0 (and $\overline{C}_0$) be the curve in W_0 which is an orbit of the $\mathbf{C}^*$-action. As seen in §4.3 its normal bundle is of the form $F \oplus F$ with $F \in \mathrm{Pic}^0 C_0$.

If $\zeta \neq 1$, F is non-trivial. Therefore $H^1(N_{C_0/W_0}) = 0$ and C is a stable submanifold (in the sense of Kodaira). Further, since $H^0(N_{C_0/W_0}) = 0$, C extends to nearby fibers in a unique way. Let $\mathcal{C}' \subset \mathcal{W}'$ be such extension. $\mathcal{C}'$ is explicitly defined by the equations $z_1 = z_2 = 0$. From this, by tracing our calculation in §4.3, the normal bundle of each fiber of $\mathcal{C}' \to \Delta^2$ turns out to be the direct sum of the same line bundle (although the complex structure of the base curves and the line bundles deform, of course).

If $\zeta = 1$, F is trivial: $N_{C_0/W_0} \simeq \mathcal{O}^{\oplus 2}$. In this case we have $H^1(N_{C_0/W_0})^{\mathbf{C}^*} = 0$. Therefore C_0 does not disappear under $\mathbf{C}^*$-equivariant deformations of W_0. Further since $H^0(N_{C_0/W_0})^{\mathbf{C}^*} = 0$, C extends to nearby fibers $\mathbf{C}^*$-invariantly in a unique way. Let $\mathcal{C}' \subset \mathcal{W}'$ be such extension. $\mathcal{C}'$ is also explicitly defined by the equations $z_1 = z_2 = 0$. From this we again see that the normal bundle of each $\mathbf{C}^*$-invariant fiber of $\mathcal{C}' \to \Delta^2$ is the direct sum of the same line bundle

Thus we get the following, which will be needed to prove Theorem 4.5.

Proposition 4.16. *Let $\zeta \in U(1)$, $0 < \lambda \leq 1/e^{2\pi}$ and $W_0 = W_0(\lambda, \zeta)$ be as in §4.3, $\mathcal{W}' \to \Delta^2$ the Kuranishi family of $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0 explicitly constructed in Propositions 4.13 and 4.14. Let $\mathcal{C}' \subset \mathcal{W}'$ be the unique extension of $C_0 \subset W_0$, which is a family of $\mathbf{C}^*$-orbits of elliptic curves. Then for any $x \in \Delta^2$, N_{C_x/W_x} is of the form $F_x \oplus F_x$ with $\deg F_x = 0$, where $C_x \subset W_x$ denote the fibers over x.*

4.5. Existence of smoothing keeping the normal bundle fixed. In this section, we apply an orbifold version of the construction of Donaldson-Friedman [6, 22] to our orbifold M_0 and complete a proof of Theorem 4.5.

Let $C, F, \lambda, \zeta, M_0 = M_0(\lambda, \zeta), Z_0$ and $C_0(\simeq C)$ be as in Lemma 4.8. Further we assume $\lambda \neq 1/e^{2\pi}$ ($\Leftrightarrow$ C does not have non-trivial automorphisms). Z_0

is a twistor space of the orbifold M_0 and can be written $W_0/\langle\tau\rangle$. Let $\tilde{L}_{0i} \subset W_0\,(1 \le i \le 4)$ be the twistor lines which are fixed (pointwisely) by τ, and L_{0i} the corresponding twistor lines in Z_0. Z_0 has A_1-singularities along L_{0i} and is smooth on the outside. In what follows the subscript i always moves from 1 to 4, and we use the notation $\cup X_i$ for $\cup_{i=1}^4 X_i$. Let $\mu : Z_0' \to Z_0$ be blowing-up along $\cup L_{0i}$, which is a resolution of singularities. The $\mathbf{C}^*$-action we have considered naturally lifts on Z_0'. Let Q_{0i} be the exceptional divisor over L_{0i} and put $Q_0 = \cup Q_{0i}$. Correspondingly, let $\tilde{\mu} : W_0' \to W_0$ be the blowing-up along $\cup\tilde{L}_{0i}$ and $\tilde{Q}_0 = \cup\tilde{Q}_{0i}$ the exceptional divisor.

On the other hand let Z_{EH} be the twistor space of compactified Eguchi-Hanson space $M_{EH} = \mathcal{O}(-2) \cup \{\infty\}$, and L_∞ the twistor lines over the unique orbifold point ∞. Z_{EH} has A_1-singularities along L_∞. Let $Z_{EH}' \to Z_{EH}$ be the blowing-up long L_∞, Q_{EH} the exceptional divisor. Next we put $Z_i' = Z_{EH}', Q_i = Q_{EH}$ for $1 \le i \le 4$. Further, we give $Z_i' = Z_{EH}'$ the $\mathbf{C}^*$-action described in [15, §3.1].

Then just as in [15, §4.1] we choose a $\mathbf{C}^*$-equivariant isomorphism $\phi_i : Q_{0i} \to Q_i$ commuting with the real structures, and set $Z' = Z_0' \cup_{\phi_i} (\cup Z_i')$, which is a simple normal crossing variety with $\mathbf{C}^*$-action. Note by Lemma 4.8 (iii) C_0 is disjoint from the singular locus $Q = \cup Q_{0i} = \cup Q_i$ of Z'.

Proposition 4.17. $T^2_{Z'} = \mathrm{Ext}^2(\Omega_{Z'}, \mathcal{O}_{Z'}) = 0$.

Proof. As in [22] it suffices to show $H^2(\Theta_{Z_0',Q_0}) = 0$. Again by [22] this follows from $H^2(\Theta_{W_0',\tilde{Q}_0}) = 0$. From the cohomology exact sequences of $0 \to \Theta_{W_0',\tilde{Q}_0} \to \Theta_{W_0'} \to N_{\tilde{Q}_0/W_0'} \to 0$ and $0 \to \Theta_{W_0'} \to \tilde{\mu}^*\Theta_{W_0} \to \Theta_{\tilde{Q}_0/\cup\tilde{L}_{0i}} \otimes \mathcal{O}_{\tilde{Q}_0}(-1) \to 0$, together with isomorphisms $\Theta_{\tilde{Q}_0/\cup\tilde{L}_{0i}} \simeq \oplus\mathcal{O}_{\tilde{Q}_{0i}}(2,0)$ and $\mathcal{O}_{\tilde{Q}_0}(-1) \simeq \oplus\mathcal{O}_{\tilde{Q}_{0i}}(-1,-1)$, we obtain $H^2(\Theta_{W_0',\tilde{Q}_0}) \simeq H^2(\Theta_{W_0})$. But we have $H^2(\Theta_{W_0}) = 0$ by Proposition 4.11. □

Lemma 4.18. $H^1(Z_{EH}', \Theta_{Z_{EH}',Q_{EH}}) = 0$.

Proof. In this proof we use an explicit description of Z_{EH}' explained in [15, §3.1] without recalling the constructions and notations there. First, because the center of the blowing up $\mu_2 : Y \to Z_{EH}'$ is disjoint from Q_{EH}, we have an exact sequence $0 \to \Theta_{Y,Q_Y} \to \mu_2^*\Theta_{Z_{EH}',Q_{EH}} \to \mathcal{O}_{\tilde{X}_0}(1,-1) \oplus \mathcal{O}_{\tilde{X}_\infty}(1,-1) \to 0$. From this we get an isomorphism $H^1(\Theta_{Z_{EH}',Q_{EH}}) \simeq H^1(\Theta_{Y,Q_Y})$. Further we have a similar exact sequence $0 \to \Theta_{Y,Q_Y} \to \mu_1^*\Theta_{X,Q_X} \to \mathcal{O}_{Z_0'} \oplus \mathcal{O}_{Z_\infty'} \to 0$. (Here, Z_0' and Z_∞' are the fibers of $f : Z_{EH}' \to \mathbf{CP}^1$ over 0 and $\infty \in \mathbf{CP}^1$ respectively, or equivalently, the exceptional divisor of $\mu_1 : Y \to X = \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$. One of the last non-trivial term of this sequence is originally $\Theta_{Z_0'/\Delta_0} \otimes \mathcal{O}_{Z_0'}(-1) \otimes \mathcal{O}_{Z_0'}(-\Delta_0)$, which can be seen to be trivial.) As is easily verified $H^1(\mu_1^*\Theta_{X,Q_X}) \simeq H^1(\Theta_{X,Q_X}) = 0$. Hence we get an exact sequence $0 \to H^0(\Theta_{Y,Q_Y}) \to H^0(\Theta_{X,Q_X}) \to H^0(\mathcal{O}_{Z_0'}) \oplus H^0(\mathcal{O}_{Z_\infty'}) \to H^1(\Theta_{Y,Q_Y}) \to 0$. Furthermore we can explicitly give two linearly independent vector fields on (X, Q_X) which cannot be lifted to Y. Hence the map $H^0(\Theta_{X,Q_X})$

$\to H^0(\mathcal{O}_{Z'_0\cup Z'_\infty})$ is surjective and and we get $H^1(\Theta_{Y,Q_Y})=0$, finishing a proof. $\square$

To investigate the Kuranishi family of $\mathbf{C}^*$-equivariant deformations of Z', we consider the following fundamental exact sequence derived from the spectral sequence for the Ext groups [6, 22]:

$$0\to H^1(Z',\Theta_{Z'})\to T^1_{Z'}\to H^0(Q,\mathcal{O}_Q)\to 0. \tag{8}$$

Here we denote $\Theta_{Z'}=\mathcal{H}om(\Omega^1_{Z'},\mathcal{O}_{Z'})$. The last zero is a consequence of $H^2(\Theta_{W'_0,\tilde{Q}_0})=0$, which was proved in Proposition 4.17, and $H^2(\Theta_{Z'_{EH},Q_{EH}})=0$, which was proved in [22, Lemma 2]. Because we are interested in $\mathbf{C}^*$-equivariant deformations of Z', we consider the $\mathbf{C}^*$-fixed part of (8).

Lemma 4.19. *$H^1(Z',\Theta_{Z'})^{\mathbf{C}^*}$ is naturally identified with $H^1(Z'_0,\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}$ and is two-dimensional.*

Proof. As in [22, p.172] there is an exact sequence (coming from the normalization sequence)

$$H^0(\Theta_{Z'_0,Q_0})\oplus H^0(\Theta_{Z'_{EH},Q_{EH}})^{\oplus 4}\to H^0(\Theta_Q)\to H^1(\Theta_{Z'})\to H^1(\Theta_{Z'_0,Q_0})\to 0. \tag{9}$$

Here we have used Lemma 4.18 for the last non-trivial term. From the explicit form of our $\mathbf{C}^*$-action on Z'_{EH}, it can be readily seen that every $\mathbf{C}^*$-invariant vector fields on Q_{EH} (i.e. elements of $H^0(\Theta_Q)^{\mathbf{C}^*}$) come from that on Z'_{EH}. Therefore (9) induces a natural isomorphism $H^1(\Theta_{Z'})^{\mathbf{C}^*}\simeq H^1(\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}$.

Next we show that $H^1(\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}$ is two-dimensional As in [22, p.173], $H^1(\Theta_{Z'_0,Q_0})$ is naturally identified with $H^1(\Theta_{W'_0,\tilde{Q}_0})^{\mathbf{Z}_2}$. It is easy to see that there are natural isomorphisms $H^1(\Theta_{W'_0,\tilde{Q}_0})\simeq H^1(\Theta_{W'_0})$ and $H^1(\Theta_{W'_0})^{\mathbf{Z}_2}\simeq H^1(\Theta_{W_0})^{\mathbf{Z}_2}$. (The latter isomorphism follows from the fact that τ acts on each $N_{\tilde{L}_{0i}/W_0}\simeq\mathcal{O}(1)^{\oplus 2}$ as the fiberwise scalar multiplication by (-1).) Hence we get a natural isomorphism $H^1(\Theta_{W'_0,\tilde{Q}_0})^{\mathbf{Z}_2}\simeq H^1(\Theta_{W_0})^{\mathbf{Z}_2}$. Thus we get an isomorphism $H^1(\Theta_{W'_0,\tilde{Q}_0})^{\mathbf{C}^*\times\mathbf{Z}_2}\simeq H^1(\Theta_{W_0})^{\mathbf{C}^*\times\mathbf{Z}_2}$.

As in the proof of Propositions 4.13 and 4.14, $H^1(\Theta_{W_0})^{\mathbf{C}^*\times\mathbf{Z}_2}$ is naturally identified with the space $\{\mathrm{diag}(0,x_1,x_2,x_3)\in gl(4,\mathbf{C})\,|\,x_2=x_1+x_3\}$, and this is two-dimensional. Hence we have $H^1(\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}\simeq H^1(\Theta_{W_0})^{\mathbf{C}^*\times\mathbf{Z}_2}$ is two-dimensional. $\square$

On the other hand $\mathbf{C}^*$ acts trivially on $H^0(Q,\mathcal{O}_Q)$ in (8) ([24]). Therefore the $\mathbf{C}^*$-fixed part of (8) is

$$0\to H^1(Z'_0,\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}\to (T^1_{Z'})^{\mathbf{C}^*}\to H^0(Q,\mathcal{O}_Q)\to 0, \tag{10}$$

and $(T^1_{Z'})^{\mathbf{C}^*}$ is $(2+4=)$ 6-dimensional. By Proposition 4.17, deformation theory of Z' is unobstructed. Therefore the base space of the Kuranishi family of deformations of Z' is naturally identified with a neighborhood of 0 in $T^1_{Z'}$. The Kuranishi family of $\mathbf{C}^*$-equivariant deformations of Z' is then the restriction onto the $\mathbf{C}^*$-fixed locus of the base space. Let $p : \mathcal{Z} \to B, 0 \in B, p^{-1}(0) \simeq Z'$ be the Kuranishi family of $\mathbf{C}^*$-equivariant deformations of Z', where B can be naturally identified with a neighborhood of 0 in $(T^1_{Z'})^{\mathbf{C}^*}$ as above. Just as in the case of $C_0 \hookrightarrow W_0$, C_0 has a natural and unique $\mathbf{C}^*$-invariant extension (after possible restriction of B), which is denoted by $\mathcal{C} \hookrightarrow \mathcal{Z}$. Then the following result is a key to prove Theorem 4.5.

Lemma 4.20. *Let $p : \mathcal{Z} \to B$ and $\mathcal{C} \hookrightarrow \mathcal{Z}$ be as above. Then after a possible restriction of B, the normal bundle $N_t = N_{C_t/Z_t}$ is isomorphic to $F_t^{\oplus 2}$ for some $F_t \in \mathrm{Pic}^0 C_t$ for every $t \in B$, where $Z_t = p^{-1}(t)$ and $C_t = Z_t \cap \mathcal{C}$.*

Proof. First we show the statement for a locally trivial (i.e. smooth) deformations of Z'. Regarding B as an open neighborhood of 0 in $(T^1_{Z'})^{\mathbf{C}^*}$ as above, and using the injection in (10), define $B' := B \cap H^1(\Theta_{Z'_0,Q_0})^{\mathbf{C}^*}$. Let $\mathcal{Z}' \to B'$ be the restriction of $\mathcal{Z} \to B$. From this, we can obtain a $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformation of W_0 in the following way. As explained in [6, p.217] $\mathcal{Z}' \to B'$ is a locally trivial family. Therefore, by looking an irreducible component, we get a $\mathbf{C}^*$-equivariant deformation of the pair (Z'_0, Q_0). Taking the double cover, we get a $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of the pair $(W'_0, \tilde{Q}_0)$. Then we can simultaneously blow-down the exceptional divisors, and consequently we get a $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0, which we denote by $\mathcal{W}'' \to B'$. It is obvious that this procedure is nothing but a realization of the isomorphism $H^1(\Theta_{Z'_0,Q_0})^{\mathbf{C}^*} \simeq H^1(\Theta_{W_0})^{\mathbf{C}^* \times \mathbf{Z}_2}$ in the proof of Lemma 4.19, in the level of actual deformations. Therefore there is an isomorphism $B' \simeq \Delta^2$ (on some neighborhoods of the origins) inducing an isomorphism $\mathcal{W}'' \simeq \mathcal{W}'$. (Here $\mathcal{W}' \to \Delta^2$ is the Kuranishi family of $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0 explicitly given in Proposition 4.16.) Thus $\mathcal{Z}' \to B'$ is explicitly obtained as the $\mathbf{Z}_2$-quotient of $\mathcal{W}' \to \Delta^2$. Hence we can regard $\mathcal{C}' \to \Delta^2$ in Proposition 4.16 as a subfamily of $\mathcal{Z}' \to B'$. Then it is clear by the uniqueness of extension that $\mathcal{C}' = \mathcal{C} \cap \mathcal{Z}'$. By Proposition 4.16, the normal bundle of C_t in W_t is of the form $F_t^{\oplus 2}$ with $F_t \in \mathrm{Pic}^0 C_t$ for any $t \in B'$. Therefore so is the normal bundle of C_t in Z_t for any $t \in B'$.

Next we show the statement of the lemma for $t \in B^\sigma$(=the real part of B) whose image in $H^0(\mathcal{O}_Q)$ is non-zero. By [6] and [22], $Z_t = p^{-1}(t)$ is a twistor space of $M_0 \# 4M_{EH} = 4\mathbf{CP}^2$ for such t after a possible restriction of B. Further, since such Z_t contains an elliptic curve $C_t = \mathcal{C} \cap Z_t$ which is a $\mathbf{C}^*$-orbit, Z_t is not Moishezon. Therefore, by Proposition 3.8, N_{C_t/Z_t} is of the form $F_t^{\oplus 2}$ for some $F_t \in \mathrm{Pic}^0 C_t$.

Now we fix a splitting of (10) compatible with the real structures: $(T^1_{Z'})^{\mathbf{C}^*} \simeq (H^1)^{\mathbf{C}^*} \oplus H^0$. Then what we have shown so far is that, the conclusion of the lemma holds true for $t \in (H^1 \oplus 0) \cap B$ and for $t \in ((H^1)^\sigma \oplus (H^0)^\sigma) \cap B$, where $(H^i)^\sigma$ denotes the real part of H^i. Let $f : B \to \mathbf{Z}$ be the function defined by $f(t) = \dim_{\mathbf{C}} H^0(C_t, \mathrm{End}(N_{C_t/Z_t}))$. By the upper-seimicontinuity theorem of Grauert [8], the set $A := \{t \in B \,|\, f(t) \geq 4\}$ is a complex analytic subset of B. Further, since $\dim H^0(\mathrm{End}(L^{\oplus 2})) = 4$ for any $L \in \mathrm{Pic} C_t$, A contains $\{(H^1 \oplus 0) \cup ((H^1)^\sigma \oplus (H^0)^\sigma)\} \cap B$. In particular, A is an analytic subset of B containing the real part B^σ of B. From this it follows that $A = B$. More generally we claim that if A is an analytic subset of $\mathbf{C}^n$ ($n \geq 2$) with $0 \in A$, and if A contains $\mathbf{R}^n$ in some neighborhood of 0 in $\mathbf{C}^n$, then A is a neighborhood of 0 in $\mathbf{C}^n$. First we note that if A is non-singular at 0, the claim is obvious since $T_0\mathbf{R}^n + T_0(J(\mathbf{R}^n)) = T_0\mathbf{C}^n$, where J denotes the complex structure of the real tangent space $T_0\mathbf{C}^n$. So we assume that A is singular at 0 and prove the claim by induction on n. Let $\dim_0 A$ denote the (analytic) dimension of A at 0 and consider the case $n = 2$. Since $\mathbf{R}^2$ is assumed to be contained in A in some neighborhood of 0, and since the topological dimension coincides with the analytic dimension, we have $\dim_0 A \geq 1$. If $\dim_0 A = 1$, A is smooth outside the origin, since the singular locus of a complex analytic curve is isolated. Therefore A is smooth on $\mathbf{R}^2 \backslash \{0\}$ (near 0). Then by the above remark for the smooth case, it follows that $\dim_x A = 2$ for any $x \in A \cap \mathbf{R}^2 \backslash \{0\}$. Hence by the upper-semicontinuity of the analytic dimension [8], we get $\dim_0 A \geq 2$. This is a contradiction and hence we have $\dim_0 A = 2$ and so A is a neighborhood of 0 in $\mathbf{C}^2$. Thus the claim is proved for $n = 2$. Suppose the claim holds for $n - 1 \geq 2$ and assume that A is an analytic subset of $\mathbf{C}^n$ containing $\mathbf{R}^n$ in a neighborhood of the origin. Let $\mathbf{C}^{n-1}$ be a hyperplane in $\mathbf{C}^n$ which is invariant by the complex conjugation. Then by the assumption of induction we have $A \cap \mathbf{C}^{n-1} = \mathbf{C}^{n-1}$ in some neighborhood of the origin. Therefore $\dim_0 A \geq n - 1$. Moreover we can move such a hyperplane continuously. This implies that $\dim_0 A > n - 1$ and so $\dim_0 A = n$. Hence A is a neighborhood of 0 in $\mathbf{C}^n$ and we have proved the claim. Thus in our situation A is a neighborhood of 0 in B, which implies that $\dim H^0(\mathrm{End} N_{C_t/Z_t}) = 4$ for any $t \in B$.

Because the degree of vector bundles is a topological invariant, $\deg N_{C_t/Z_t} =$ ($\deg N_{C_0/Z_0} =$) 0 for any $t \in B$. Hence to prove the lemma it suffices to show that, if $E \to C$ is a rank-two vector bundle over an elliptic curve with the properties: (a) $\deg E = 0$, (b) $\dim H^0(\mathrm{End} E) = 4$, (c) $E \to C$ is obtained by a small deformation of $F \oplus F \to C$ for some $F \in \mathrm{Pic}^0 C$, then $E \simeq F_t^{\oplus 2}$ for some $F_t \in \mathrm{Pic}^0 C$. By Atiyah [1] rank-two vector bundle E of degree zero is isomorphic to either (i) $F \oplus F$ ($F \in \mathrm{Pic}^0 C$), (ii) $F_1 \oplus F_2$ with $F_1 \not\simeq F_2 \in \mathrm{Pic} C$ and $\deg F_1 + \deg F_2 = 0$, (iii) an indecomposable bundle obtained as a non-trivial extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$. For the last case, it is well known that E is a stable vector bundle, and hence simple, i.e. $H^0(\mathrm{End} E) \simeq \mathbf{C}$, contradicting (b). For the case (ii) we easily obtain by Riemann-Roch $\dim H^0(\mathrm{End}(F_1 \oplus F_2)) = 2 + 2d$, where $d = |\deg F_1| (= |\deg F_2|)$. Hence $d = 1$ follows. But such $F_1 \oplus F_2$ cannot be obtained as a small deformation of the bundle $F^{\oplus 2}$ with $F \in \mathrm{Pic}^0 C$ by upper semicontinuity, and not compatible with the condition (c). Therefore E falls into the case (i), as desired. □

We complete our proof of Theorem 4.5 by the following

Lemma 4.21. *Let $p : \mathcal{Z} \to B$ and $\mathcal{C} \hookrightarrow \mathcal{Z}$ be as in Lemma 4.20. Then there exists a real four-dimensional submanifold D of B with $0 \in D$ satisfying the following properties: (i) D is real with respect to the natural real structure of B, (ii) for any $t \in D$ with $t \neq 0$, $Z_t = p^{-1}(t)$ is a twistor space of $4\mathbf{CP}^2$, (iii) for any $t \in D$, $C_t = \mathcal{C} \cap Z_t$ is biholomorphic to C and N_{C_t/Z_t} is isomorphic to $F \oplus F$. (Recall that $C_0 \simeq C$ and $N_{C_0/Z'_0} \simeq F^{\oplus 2}$ by construction.)*

Proof. Let $\mathcal{M}$ be the total space of the Kuranishi family of deformations of C: if $C = \mathbf{C}^*/\langle z \to \lambda z\rangle$ as in our case, $\mathcal{M}$ is explicitly constructed as the quotient space $(U \times \mathbf{C}^*)/\langle(\omega, z) \mapsto (\omega, \omega z)\rangle$, where U is a neighborhood of λ in $\mathbf{C}^*$. Since we have assumed that C does not have non-trivial automorphism, $\mathcal{M} \to U$ is universal (with respect to deformations of C). Let $r_1 : \mathcal{M} \to U$ be the natural projection and $r_2 : \mathcal{M}^\vee \to U$ the identity component of the relative Picard variety of $\mathcal{M} \to U$. Since $\mathrm{Pic}^0 C$ is biholomorphic to C itself, $\mathcal{M}^\vee$ is isomorphic to $\mathcal{M}$. Let $\mathcal{F} \to \mathcal{M} \times_U \mathcal{M}^\vee$ be the tautological line bundle over the fiber product. Let $q_1 : \mathcal{M} \times_U \mathcal{M}^\vee \to \mathcal{M}$ and $q_2 : \mathcal{M} \times_U \mathcal{M}^\vee \to \mathcal{M}^\vee$ be the natural projections.

Because we have assumed C has no non-trivial automorphism, it is obvious that the total space of $\mathcal{M}^\vee$ is the universal family of deformations of the pair (C, F). By Lemma 4.20, N_{C_t/Z_t} is of the form $F_t^{\oplus 2}$ with $F_t \in \mathrm{Pic}^0 C_t$ for any $t \in B$. Since $\mathcal{M}^\vee$ is universal, there exists a unique holomorphic map $\alpha : B \to \mathcal{M}^\vee$ such that C_t is biholomorphic to $r_1^{-1}(r_2(\alpha(t)))$ and F_t is isomorphic to $\mathcal{F}|_{q_2^{-1}(\alpha(t))}$ as holomorphic line bundles over $q_2^{-1}(\alpha(t)) \simeq r_1^{-1}(r_2(\alpha(t))) \simeq C_t$.

Let $B' \subset B$ be the subspace defined in the proof of Lemma 4.20. As in the proof, B' parametrizes $\mathbf{C}^*$-equivariant smooth deformations of Z', and such deformations canonically come from $\mathbf{C}^* \times \mathbf{Z}_2$-equivariant deformations of W_0. Let $\mathcal{W}' \to \Delta^2$ and $\mathcal{C}' \subset \mathcal{W}'$ be as in the proof of Lemma 4.20 (explicitly constructed in Proposition 4.16. Then it can be easily seen from the explicit constructions that the map $t \mapsto (C_t, F_t) \in \mathcal{M}^\vee$ for $t \in \Delta^2$ is an isomorphism. Therefore α is isomorphic on B'. Hence the differential $d\alpha : T_0 B \to T_{(C,F)}\mathcal{M}^\vee$ is surjective. Therefore by implicit function theorem the fiber $\alpha^{-1}(\alpha(0))$ is a complex submanifold of complex codimension two (= the complex dimension of $\mathcal{M}^\vee$), and $\alpha^{-1}(\alpha(0))$ intersects B' transversally.

Therefore (after shrinking B if necessary) the image in $H^0(\mathcal{O}_Q)$ is non-zero for any $t \in \alpha^{-1}(\alpha(0)), t \neq 0$. Further, $\alpha^{-1}(\alpha(0))$ is real in B, because α is obviously real. Then we define D to be the real part of $\alpha^{-1}(\alpha(0))$. If $t \in D$ is non-zero, $Z_t = p^{-1}(t)$ is smooth and is a twistor space of $4\mathbf{CP}^2$ by [6, 22]. Further, as $\alpha(t) = \alpha(0)$, we have $C_t \simeq r_1^{-1}(r_2(\alpha(t))) \simeq r_1^{-1}(r_2(\alpha(0))) \simeq C_0 \simeq C$ and $F_t \simeq \mathcal{F}|_{q_2^{-1}(\alpha(t))} \simeq \mathcal{F}|_{q_2^{-1}(\alpha(0))} \simeq F_0 \simeq F$. Namely, we have $C_t \simeq C$ and $N_{C_t/Z_t} \simeq F^{\oplus 2}$ for any $t \in D$.

As for the dimension of D, we have already shown $\dim(T^1_{Z'})^{\mathbf{C}^*} = 6$ in Lemma 4.19. Therefore $\alpha^{-1}(\alpha(0))$ is four-dimensional over $\mathbf{C}$. Hence D, the real part of

$\alpha^{-1}(\alpha(0))$, is a four-dimensional real submanifold. Thus we have proved all of the claim of the lemma, and Theorem 4.5 is now proved. □

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