Uniqueness theorems for parabolic equations and Martin boundaries for elliptic equations in skew product form*

Minoru Murata
Department of Mathematics, Tokyo Institute of Technology,
Oh-okayama, Meguro-ku, Tokyo, 152-8551 Japan
e-mail: minoru3@math.titech.ac.jp

Abstract

We determine Martin boundaries of product domains for elliptic equations in skew product form via Widder type uniqueness theorems. We show that the fiber of the Martin boundary at infinity of the base space degenerates into one point if any nonnegative solution to the Dirichlet problem for a corresponding parabolic equation with zero initial and boundary data is identically zero.

1 Introduction

The Widder type uniqueness theorem for a parabolic equation asserts that its nonnegative solution with zero initial (and boundary) value must be identically zero; while the Martin representation theorem for an elliptic equation says that any positive solution of it is represented by an integral of the Martin kernel with respect to a finite Borel measure on the Martin boundary. During the last few decades, Widder type and related uniqueness theorems have been investigated to a satisfactory extent (cf. [9, 15–18, 21, 23, 25, 26,

*Running Title: Uniqueness theorems and Martin boundaries
2000 Mathematics Subject Classification: 31C35, 35J99, 35K15, 35K20, 58J99
Key Words and Phrases: Uniqueness of nonnegative solutions, Parabolic equation, Positive solution, Elliptic equation, Skew product, Martin boundary

28, 32, 34, 42–44, 46, 48, 57–61, 65, 69, 76, 78, 80]), and there has been a significant progress in determining explicitly Martin boundaries in many important cases (cf. [4–7, 10–13, 20, 29–31, 36, 38–40, 49–51, 53–56, 63, 64, 66, 72, 73]). Among others, Ishige and Murata [44] showed that under a general and sharp condition, any nonnegative solution to the Cauchy problem for a parabolic equation is determined uniquely by its initial value; while Murata [64] constructed Martin boundaries for a wide class of elliptic equations in skew product form.

The purpose of this paper is to determine explicitly Martin boundaries for elliptic equations in skew product form via Widder type uniquness theorems for parabolic equations by applying general results on Martin boundaries given in [64].

We consider positive solutions of an elliptic equation in skew product form

$$Lu \equiv (L_1 + W_1 L_2)u = 0 \quad \text{in } D = D_1 \times D_2 \subset M = M_1 \times M_2. \tag{1.1}$$

Here D is a non-compact domain of a product Riemannian manifold M, L_i with $i = 1$ or 2 is an elliptic operator on a domain D_i of a Riemannian manifold M_i, and W_1 is a positive measurable function on M_1. We assume that (L, D) is subcritical, i.e. there exists a minimal positive Green function of L on D. In order to determine explicitly the Martin boundary $\partial_M D$ of D with respect to L, we study uniqueness of nonnegative solutions to the initial and boundary value problem for a parabolic equation

$$(\partial_t + W_1^{-1} L_1)v = 0 \quad \text{in } D_1 \times (0, \infty), \tag{1.2}$$

$$v(x, 0) = 0 \quad \text{on } D_1, \tag{1.3}$$

$$v(x, t) = 0 \quad \text{on } \partial D_1 \times (0, \infty). \tag{1.4}$$

(It is needless to say that when $D_1 = M_1$, the condition (1.4) is redundant, and the problem reduces to the initial value problem.) We shall show from the uniqueness of nonnegative solutions that the fiber of $\partial_M(D_1 \times D_2)$ at infinity of the base space D_1 reduces into one point.

Now, in order to state our main theorem, we fix notations and recall several notions and facts. For $i = 1$ or 2, let M_i be a connected separable n_i-dimensional smooth manifold with Riemannian metric of class C^0. With $N = M_1$ or M_2, $T_x N$ and TN denote the tangent space to N at $x \in N$ and the tangent bundle, respectively. We denote by $\mathrm{End}(T_x N)$ and $\mathrm{End}(TN)$

the set of endmorphisms in T_xN and the corresponding bundle, respectively. The inner product on TN is denoted by $\langle X, Y\rangle$, where $X, Y \in TN$; and $|X| = \langle X, X\rangle^{1/2}$. The divergence and gradient with respect to the metric on N are denoted by div and ∇, respectively. Let L_1 be an elliptic differential operator on M_1 of the form

$$L_1 u = -m_1^{-1}\mathrm{div}(m_1 A_1 \nabla u - m_1 u C_1) - \langle B_1, \nabla u\rangle + V_1 u, \tag{1.5}$$

where m_1 is a positive measurable function on M_1 such that

$$m_1 \text{ and } m_1^{-1} \text{ are bounded on any compact subset of } M_1, \tag{1.6}$$

A_1 is a symmetric measurable section on M_1 of $\mathrm{End}(TM_1)$, B_1 and C_1 are measurable vector fields on M_1, and V_1 is a real-valued measurable function on M_1. We assume that L_1 is locally uniformly elliptic on M_1, i.e., for any compact set K in M_1 there exists a positive constant λ such that

$$\lambda|\xi|^2 \le \langle (A_1)_x \xi, \xi\rangle \le \lambda^{-1}|\xi|^2, \quad x \in K,\ (x, \xi) \in TM_1. \tag{1.7}$$

Denote by ν_1 the Riemannian measure on M_1, and put $d\mu_1 = m_1 d\nu_1$. For $1 \le p \le \infty$, denote by $L^p_{\mathrm{loc}}(M_1) = L^p_{\mathrm{loc}}(M_1, d\mu_1)$ the set of complex-valued functions on M_1 locally p-th integrable with respect to $d\mu_1$. We assume that

$$|B_1|^2,\ |C_1|^2,\ V_1 \in L^p_{\mathrm{loc}}(M_1, d\mu_1), \qquad \text{for some } p > \max(\frac{n_1}{2}, 1). \tag{1.8}$$

Let W_1 be a positive measurable function on M_1 such that

$$W_1, W_1^{-1} \in L^\infty_{\mathrm{loc}}(M_1, d\mu_1). \tag{1.9}$$

Let L_2 be an elliptic differential operator on D_2 of the form

$$L_2 u = -m_2^{-1}\mathrm{div}(m_2 A_2 \nabla u - m_2 u B_2) - \langle B_2, \nabla u\rangle + V_2 u, \tag{1.10}$$

where m_2, A_2, B_2, and V_2 satisfy the conditions (1.6), (1.7), and (1.8) with obvious modifications. Note the L_2 is formally selfadjoint with respect to the measure $d\mu_2$. We assume that the generalized principle eigenvalue λ_0 of L_2 on D_2 is finite, i.e., with Λ being the set of all real numbers λ such that the equation $(L_2 - \lambda)u = 0$ in D_2 has a positive solution,

$$\lambda_0 \equiv \sup \Lambda > -\infty. \tag{1.11}$$

We denote by $\mathcal{L}_2$ the Dirichlet realization of L_2 on D_2, i.e., the selfadjoint operator on $L^2(D_2, d\mu_2)$ associated with L_2 on D_2 (cf. Subsection 2.2 of [64]). We assume the hypothesis (SMI2) for (L_2, D_2), which is composed of three conditions (S), (M), and (I), i.e., semismallness, minimality and intrinsic ultracontractivity for (L_2, D_2). Let us state the conditions (S), (M) and (I). We say that the semigroup $e^{-t\mathcal{L}_2}$ generated by $-\mathcal{L}_2$ is IU (i.e., intrinsically ultracontractive) when λ_0 is the first eigenvalue of $\mathcal{L}_2$, and there exists a positive continuous decreasing function $C(t)$ on $(0, \infty)$ such that

$$p_2(x_2, y_2, t) \le C(t)e^{-\lambda_0 t}\phi_0(x_2)\phi_0(y_2), \quad x_2, y_2 \in D_2,\ t > 0, \tag{1.12}$$

where ϕ_0 is a normalized positive eigenfunction associated with λ_0, $p_2(x_2, y_2, t)$ is the integral kernel of the semigroup $e^{-t\mathcal{L}_2}$. For IU, see [22, 24, 64] and references therein. We assume the following condition (I).

(I) The semigroup $e^{-t\mathcal{L}_2}$ is IU and the function $C(t)$ in (1.12) satisfies

$$\lim_{t\to 0} t \log C(t) = 0. \tag{1.13}$$

For example, when D_2 is compact this condition is satisfied with $C(t) = \alpha t^{-n_2/2}$ for some positive constant α (cf. Example 9.2 of [64]). The condition (I) implies that the spectrum of $\mathcal{L}_2$ consists of discrete eigenvalues with finite multiplicity. Let $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $\mathcal{L}_2$ repeated according to multiplicity. Let ϕ_j be an eigenfunction associated with λ_j $(j = 0, 1, 2, \dots)$ such that $\{\phi_j\}_{j=0}^{\infty}$ is a complete orthonormal system of $L^2(D_2, d\mu_2)$. It follows from (I) that $\phi_j/\phi_0 \in L^\infty(D_2)$ for any $j \ge 1$. We assume the following condition (S).

(S) The constant function 1 is a semismall perturbation of $L_2 - \lambda$ on D_2 for some $\lambda < \lambda_0$.

This condition means that for any $\varepsilon > 0$ there exists a compact subset K of D_2 such that

$$\int_{D_2\setminus K} g(x_2^0, z)g(z, y_2)d\mu_2(z) \le \varepsilon g(x_2^0, y_2), \quad y_2 \in D_2 \setminus K, \tag{1.14}$$

where x_2^0 is a reference point in D_2, and g is the Green function of $L_2 - \lambda$ on D_2 with respect to the measure $d\mu_2$ (cf. [62]). When D_2 is compact, the condition (S) is redundant. When D_2 is non-compact, we denote by D_2^* and $\partial_M D_2$ the Martin compactification and Martin boundary of D_2 with respect to $L_2 - \lambda$, respectively (cf. [14,49,64,73,79] and references therein). We recall that for any $\eta \in \partial_M D_2$ there exists a sequence $\{y_2^j\}_j$ in D_2 such that $\{y_2^j\}_j$ has no point of accumulation in D_2 and the sequence $\{g(\cdot, y_2^j)/g(x_2^0, y_2^j)\}_j$ of functions on D_2 converges locally uniformly to the Martin kernel $h(\cdot, \eta)$, which is a positive solution of the equation $(L_2 - \lambda)u = 0$ in D_2. We also recall a positive solution u is said to be minimal if another positive solution satisfies $v \leq u$, then $v = cu$ for some constant c. When D_2 is non-compact, we assume the following condition (M).

(M) For any $\eta \in \partial_M D_2$, the Martin kernel $h(\cdot, \eta)$ for $(L_2 - \lambda, D_2)$ is minimal.

When D_2 is compact, we put $D_2^* = D_2$ and $\partial_M D_2 = \emptyset$ as convention. The condition (S) implies that for any $j = 1, 2, \ldots$, the function ϕ_j/ϕ_0 has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D_2$ (cf. Theorem 6.3 of [71] and Theorem 5.12 of [64]). The condition (M) together with (I) and (S) implies that the family $\{[\phi_j/\phi_0]; j = 0, 1, 2, \cdots\}$ separates finite Borel measures on D_2^* (cf. Proposition 9.7 of [64]). Throughout the present paper we assume the hypothesis (SMI2):

(SMI2) The conditions (S),(M) and (I) are satisfied for (L_2, D_2).

For example, (SMI2) holds when D_2 is a relatively compact Lipschitz domain and L_2 is an elliptic operator on the whole space M_2 of the form (1.10) with the coefficients satisfying the conditions (1.6), (1.7) and (1.8) with obvious modifications (cf. Example 9.3 of [64]).

We assume that either $D_1 = M_1$ and M_1 is non-compact, or D_1 is a Lipschitz domain of M_1, i.e., for any boundary point z, the domain D_1 in a coordinate neighborhood of z is the upper side of a Lipschitz continuous graph. Consider (weak) solutions of the Dirichlet problem (1.2), (1.3) and (1.4). When $\partial D_1 \neq \emptyset$, the boundary condition (1.4) means that for any $\psi \in C_0^\infty(M_1)$ and $T > 0$,

$$\psi v \in L^\infty((0,T); L^2(D_1, d\mu_1)) \cap L^2((0,T); H_0^1(D_1, d\mu_1)),$$

where $H_0^1(D_1, d\mu_1)$ is the closure of $C_0^\infty(D_1)$ in the Sobolev space $H^1(D_1, d\mu_1)$ of order 1. We introduce the following condition (U1), i.e., uniqueness for the positive Dirichlet problem for $(\partial_t + W_1^{-1}L_1, D_1)$.

(U1) Any nonnegative solution of the problem (1.2), (1.3) and (1.4) must be identically zero.

Let $L = L_1 + W_1 L_2$ and $D = D_1 \times D_2$. We assume that (L, D) is subcritical, i.e., there exists the (minimal positive) Green function G of L on D. This implies that $(L_1 + \lambda_j W_1, D_1)$ are also subcritical for any $j =$

$0, 1, \cdots$ (cf. Theorem 7.4 of [64]). Denote by H_j the Green function for $(L_1 + \lambda_j W_1, D_1)$. Fix a reference point $x^0 \in D$. Denote by

$$D^*,\ \partial_M D,\ \partial_m D,\ \text{ and }\ K(x, \xi)$$

the Martin compactification, Martin boundary, minimal Martin boundary, and Martin kernel for (L, D), respectively. Similarly,

$$D_1^*,\ \partial_M D_1,\ \partial_m D_1\ \text{ and }\ k_0(x_1, \xi_1)$$

denote those for $(L_1 + \lambda_0 W_1, D_1)$. It is known that the closure $\overline{D_1}$ of D_1 in M_1 is continuously imbedded into D_1^* and $\partial D_1 \subset \partial_m D_1$ (cf. Theorem 2.1 of [55]). We put

$$\Gamma_1 = \partial_M D_1 \setminus \partial D_1.$$

We are now ready to state our main theorem.

Theorem 1.1 Assume the conditions (SMI2) and (U1). Then the following (i)–(vi) hold true:

(i) With d_2 being an ideal point outside of D_2^*, the Martin boundary $\partial_M D$ is equal to the disjoint union of $\Gamma_1 \times \{d_2\}, \partial D_1 \times D_2^*$, and $D_1 \times \partial_M D_2$:

$$\partial_M D = \Gamma_1 \times \{d_2\} \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.15}$$

Furthermore,

$$\partial_m D = (\Gamma_1 \cap \partial_m D_1) \times \{d_2\} \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.16}$$

In particular, $\partial_m D = \partial_M D$ if and only if $\Gamma_1 \subset \partial_m D_1$, i.e., $\partial_m D_1 = \partial_M D_1$.

(ii) For $\xi_1 \in \Gamma_1$, a subset U of D^* is a neighborhood of $\tilde{\xi_1} = (\xi_1, d_2)$ if and only if there exists a neighborhood U_1 of ξ_1 in D_1^* such that

$$U \supset (U_1 \cap \Gamma_1) \times \{d_2\} \cup (U_1 \cap \overline{D_1}) \times D_2^*. \tag{1.17}$$

(iii) For $\xi \in \partial D_1 \times D_2^* \cup D_1 \times \partial_M D_2$, a subset U of D^* is a neighborhood of ξ if and only if there exist neighborhoods U_1 and U_2 of ξ_1 and ξ_2 in $\overline{D_1}$ and D_2^*, respectively, such that $U_1 \times U_2 \subset U$.

(iv) For $\xi \in \Gamma_1 \times \{d_2\}$,

$$K(x, \xi) = k_0(x_1, \xi_1)\phi_0(x_2)/\phi_0(x_2^0), \quad x \in D. \tag{1.18}$$

(v) For $\xi \in \partial D_1 \times D_2^*$,

$$K(x, \xi) = k(x, \xi)/k(x^0, \xi), \quad x \in D, \tag{1.19}$$

where $k(\cdot,\xi)$ is a positive solution of (1.1) defined by

$$k(x,\xi)=\sum_{j=0}^{\infty}k_j(x_1,\xi_1)\phi_j(x_2)[\phi_j/\phi_0](\xi_2),\quad x\in D,\tag{1.20}$$

$$k_j(x_1,\xi_1)=\lim_{D_1\ni y_1\to\xi_1}H_j(x_1,y_1)/H_0(x_1^0,y_1),\quad j=0,1,2,\cdots.\tag{1.21}$$

Here the series on the right hand side of (1.20) converges uniformly on $(F\times E)\times(\partial D_1\times D_2^*)$ for any relatively compact domains $F\subset D_1$ and $E\subset D_2$. It also converges in $L^\infty(\partial D_1\times D_2^*;L^2(F\times D_2))$ for any relatively compact domain F in D_1. Furthermore, $k(x,\xi)$ is continuous on $D\times(\partial D_1\times D_2^*)$, and $k_j(\cdot,\xi_1)$ is a positive solution of $(L_1+\lambda_jW_1)u=0$ in D_1 for any $j=0,1,2,\cdots$.

(vi) For $\xi\in D_1\times\partial_M D_2$,

$$K(x,\xi)=H(x,\xi)/H(x^0,\xi),\quad x\in D,\tag{1.22}$$

where $H(\cdot,\xi)$ is a positive solution of (1.1) determined by

$$H(x,\xi)=\sum_{j=0}^{\infty}H_j(x_1,\xi_1)\phi_j(x_2)[\phi_j/\phi_0](\xi_2),\quad x\in(D_1\setminus\{\xi_1\})\times D_2.\tag{1.23}$$

Here the series on the right hand side of (1.23) converges uniformly on any compact subset of $(D_1\setminus\{\xi_1\})\times D_2$. It also converges in $L^2(F\times D_2)$ for any relatively compact domain F in $D_1\setminus\{\xi_1\}$. Furthermore, H is continuous on $D\times(D_1\times\partial_M D_2)$.

Theorem 1.1 says that the uniqueness for a parabolic equation implies that the fiber of $\partial_M(D_1\times D_2)$ at infinity of the base space D_1 reduces into one point. This theorem will be proved in Section 5. The condition (U1) in Theorem 1.1 implies that for $\xi_1\in\Gamma_1$ the limit $k_j(x_1,\xi_1)=0$ for any $j\geq 1$ (see Lemma 5.3 in Section 5). This means that the perturbation W_1 of the operator $L_1+\lambda_0W_1$ on D_1 is big in some sense, since the Green function H_j of $L_1+\lambda_jW_1$ on D_1 becomes smaller as the positive function W_1 becomes bigger. Now, we introduce the following condition (S1), i.e., semismallness of W_1, which is complementary to the condition (U1).

(S1) W_1 is a semismall perturbation of $L_1+\lambda_0W_1$ on D_1.

This condition means that for any $\varepsilon>0$ there exists a compact subset K of D_1 such that

$$\int_{D_1\setminus K}H_0(x_1^0,z)W(z)H_0(z,y_1)d\mu_1(z)\leq\varepsilon H_0(x_1^0,y_1),\quad y_1\in D_1\setminus K,$$

where x_1^0 is a reference point in D_1. By Theorem 3.1 and Proposition 3.4 to be stated in Section 3, both the conditions (S1) and (U1) do not hold together. Interestingly, in several important cases, either (S1) or (U1) holds.

When (S1) holds, the Martin compactification $(D_1 \times D_2)^*$ of $D_1 \times D_2$ with respect to L is extremely simple. In this case, $(D_1 \times D_2)^*$ is regular: $(D_1 \times D_2)^* = D_1^* \times D_2^*$.

Theorem 1.2 Assume the conditions (SMI2) and (S1). Then the following (i)–(iii) hold true.

(i) The Martin compactification D^* of D with respect to L is homeomorphic to $D_1^* \times D_2^*$. In particular,

$$\partial_M D = \Gamma_1 \times D_2^* \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.24}$$

Furthermore,

$$\partial_m D = (\Gamma_1 \cap \partial_m D_1) \times D_2^* \sqcup \partial D_1 \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{1.25}$$

In particular, $\partial_m D = \partial_M D$ if and only if $\Gamma_1 \subset \partial_m D_1$, i.e., $\partial_m D_1 = \partial_M D_1$.

(ii) The assertion (v) of Theorem 1.1 holds with ∂D_1 replaced by $\Gamma_1 \cup \partial D_1$. In particular, the Martin kernel $K(x, \xi)$ for $\xi \in (\Gamma_1 \cup \partial D_1) \times D_2^*$ is given by (1.19).

(iii) The assertion (vi) of Theorem 1.1 holds.

Theorem 1.2 is a special case of Theorem 9.1 of [64] (see Theorem 4.2 in Section 4). This theorem says that "smallness" of W_1 implies the regularity of $(D_1 \times D_2)^*$, while Theorem 1.1 says that "bigness" of W_1 implies the degeneration of the fiber at infinity.

Here, as an application of Theorems 1.1 and 1.2, we give a simple example concerning positive harmonic functions on horn-shaped domains in $\mathbf{R}^{N+1}, N \geq 2$. Further examples will be given in Section 8.

Theorem 1.3 Let α and β be Lipschitz continuous functions on $[1, \infty)$ such that $\alpha > \beta$ and $(\alpha(r) - \beta(r))/r$ is decreasing. Let

$$D_1 = \{(r, s) \in \mathbf{R}^2;\ \alpha(r) > s > \beta(r),\ 1 < r < \infty\}.$$

Let D_2 be a Lipschitz domain in the unit sphere S^{N-1} of $\mathbf{R}^N$ or the whole space S^{N-1}, where $N \geq 2$. Let $L = -\Delta$ on $\mathbf{R}^{N+1}$ and

$$D = \{(z, s) \in \mathbf{R}^N \times \mathbf{R}^1; \alpha(|z|) > s > \beta(|z|), |z| > 1, z/|z| \in D_2\}.$$

(i) Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r))r^{-2}dr < \infty. \tag{1.26}$$

Then D^* is homeomorphic to $D_1^* \times \overline{D_2}$, where $D_1^* = \overline{D_1} \cup \{\infty\}$ is the closure of D_1 in the one-point compactification of $\mathbf{R}^2$. Furthermore, $\partial_M D = \partial_m D = \partial_M D_1 \times \overline{D_2} \cup D_1 \times \partial D_2$ and $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$.

(ii) Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r))r^{-2}dr = \infty. \tag{1.27}$$

Then D^* is homeomorphic to $(\overline{D_1} \times \overline{D_2}) \sqcup \{(\infty, d_2)\}$, where a fundamental neighborhood system of the ideal point (∞, d_2) is given by the family

$$\{(\{(r,s) \in \overline{D_1};\ \varepsilon^{-1} < r < \infty\} \times \overline{D_2}) \cup \{(\infty, d_2)\}\}_{0<\varepsilon<1}.$$

Furthermore, $\partial_M D = \partial_m D = \{(\infty, d_2)\} \cup \partial D_1 \times \overline{D_2} \cup D_1 \times \partial D_2$ and $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$.

A special case of this theorem was shown under more stringent condition by Ioffe and Pinsky [40], and related results were announced by Maz'ya [50]. The assertion (i) of Theorem 1.3 was shown by Aikawa and Murata [4] (see also Theorem 6.3 in Section 6). The assertion (ii) will be proved in Section 6.

The remainder of this paper is organized as follows. In Section 2, we recall uniqueness theorems for parabolic equations given in [44] and [65], and give an application to a concrete example related to Theorem 1.3. In Section 3, we recall criteria for non-h-bigness, and observe that the Widder type uniqueness theorem implies h-bigness. In Section 4, we recall general results on Martin boundaries for elliptic equations in skew product form given in [64]. In section 5, we prove Theorem 1.1. Theorem 1.3 is proved in Section 6. There we also give a theorem on small perturbation, and generalize the assertion (i) of Theorem 1.3. In Section 7, we give a generalization of Theorems 1.1 and 1.2. By applying it, we give several concrete examples in Section 8.

2 Uniqueness theorems for parabolic equations

In this section we recall uniqueness theorems in [42] and [65], and give an application to a simple example related to Theorem 1.3. Let N be a connected separable smooth manifold with Riemannian metric of class C^0. We assume that the Riemannian manifold N is complete. Let P be an elliptic operator on N of the form

$$Pu = -w^{-1}\text{div}(wa\nabla u - wuc) - \langle b, \nabla u\rangle + qu, \tag{2.1}$$

where w, a, b, c, q satisfy the conditions (1.6), (1.7) and (1.8) with obvious modifications. We further assume that P is uniformly elliptic on N, i.e., there exists a positive constant κ such that

$$\kappa|\xi|^2 \le \langle a_x\xi, \xi\rangle \le \kappa^{-1}|\xi|^2, \quad (x,\xi) \in TN. \tag{2.2}$$

We denote by ν the Riemannian measure on N, and put $d\lambda = wd\nu$. First, consider the Cauchy problem

$$\mathcal{P}u = 0 \quad \text{in } N \times (0,\infty), \tag{2.3}$$

$$u(x,0) = u_0(x) \quad \text{on } N, \tag{2.4}$$

where $\mathcal{P} = \partial_t + P$ and $u_0 \in L^2_{\text{loc}}(N, d\lambda)$. In order to give a Widder type uniqueness theorem, we need two conditions. Put $q^\pm = \max(\pm q, 0)$. Fix a point O in N, and let $d(x) = \text{dist}(O,x)$ be the Riemannian distance between O and $x \in N$. Put $B(O,R) = \{x \in N;\ d(x) < R\}$ for $R > 0$. Let ρ be a positive continuous increasing function on $[0,\infty)$. Then the condition [RB-ρ] (i.e., relative boundedness with scale function ρ) to be imposed on b, c, q^- is as follows.

[RB-ρ] There exist constants $\alpha_1 > 0$, $0 \le \beta_1 < 1$, $0 < \beta_2 < 1$, $0 < \beta_3 < 1$ such that $\beta_1 + \beta_2 + \beta_3 < 1$ and

$$\begin{aligned} &\int_{B(O,R)} \left[\frac{1}{4\beta_2}\langle a^{-1}b, b\rangle + \frac{1}{4\beta_3}\langle a^{-1}c, c\rangle + q^-\right]\psi^2 d\lambda \\ \le\ &\beta_1 \int_{B(O,R)} \langle a\nabla\psi, \nabla\psi\rangle d\lambda + \alpha_1\rho(R)^2 \int_{B(O,R)} \psi^2 d\lambda \end{aligned} \tag{2.5}$$

for any $R > 1$ and $\psi \in C_0^\infty(B(O,R))$.

The second condition to be imposed on P is the following condition [PHP-ρ], i.e., the parabolic Harnack principle with scale function ρ.

[PHP-ρ] There exists a positive constant α_2 such that for any

$$(x,t) \in N \times (0,\infty),\ 0 < r \leq \frac{1}{\rho(d(x))},$$

any nonnegative solution u of the equation

$$\mathcal{P}u = 0 \quad \text{in } Q = B(x,r) \times (t - r^2, t + r^2) \tag{2.6}$$

satisfies the inequality

$$\sup_{Q^-} u \leq \alpha_2 \inf_{Q^+} u, \tag{2.7}$$

where

$$\begin{aligned} Q^- &= B(x, \frac{r}{2}) \times (t - \frac{3}{4}r^2, t - \frac{1}{4}r^2), \\ Q^+ &= B(x, \frac{r}{2}) \times (t + \frac{1}{4}r^2, t + \frac{3}{4}r^2). \end{aligned}$$

For the parabolic Harnack inequality (2.7), see [15, 27, 33, 41, 44, 48, 52, 75, 77] and references therein. We are now ready to state a Widder type uniqueness theorem, which is a time independent elliptic operator case of Theorem 2.2 in [44].

Theorem 2.1 Suppose that the conditions [RB-ρ] and [PHP-ρ] hold with ρ satisfying

$$\int_1^\infty \frac{dr}{\rho(r)} = \infty. \tag{2.8}$$

Then a nonnegative solution u of the Cauchy problem (2.3) and (2.4) is determined uniquely by the initial data u_0.

Let Ω be a domain in N. We next consider the Dirichlet problem

$$\mathcal{P}u = 0 \quad \text{in } \Omega \times (0,\infty), \tag{2.9}$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega, \tag{2.10}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.11}$$

where u_0 satisfies $\eta u_0 \in L^2(\Omega, d\lambda)$ for any $\eta \in C_0^\infty(N)$. Let $\mathrm{Lip}([0,1]; \overline{\Omega})$ be the set of Lipschitz continuous curves in $\overline{\Omega}$. For $x, y \in \overline{\Omega}$, put

$$\Gamma(x,y) = \{\gamma \in \mathrm{Lip}([0,1]; \overline{\Omega});\ \gamma(0) = x,\ \gamma(1) = y,\ \gamma(s) \in \Omega \text{ for } 0 < s < 1\}.$$

Denote by $l(\gamma)$ the length of a curve $\gamma \in \mathrm{Lip}([0,1];\overline{\Omega})$, and put

$$d_\Omega(x,y) = \inf\{l(\gamma); \gamma \in \Gamma(x,y)\}. \tag{2.12}$$

Fix a point O in Ω, and put $d_\Omega(x) = d_\Omega(O,x)$ for $x \in \overline{\Omega}$. Let ρ be a positive continuous increasing function on $[0,\infty)$. For $R > 0$, put

$$\Omega_R = \{x \in \Omega;\ d_\Omega(x) < R\}, \quad \Omega_R^\wedge = \bigcup_{x \in \overline{\Omega_R}} B(x, \frac{1}{\rho(d_\Omega(x))}). \tag{2.13}$$

We impose on P and Ω the following condition [RB-$\rho\Omega$] (i.e., relative boundedness with scale function ρ and domain Ω) and [PHP-$\rho\Omega$] (i.e., the parabolic Harnack principle with scale function ρ and domain Ω).

[RB-$\rho\Omega$] The condition [RB-ρ] holds with $B(O,R)$ replaced by $\Omega_R^\wedge$.

[PHP-$\rho\Omega$] There exist a compact subset K of N and a positive constant α_2 such that for any

$$(x,t) \in (\overline{\Omega} \setminus K) \times (0,\infty), \quad 0 < r \le \frac{1}{\rho(d_\Omega(x))},$$

any nonnegative solution u of the equation (2.6) satisfies the inequality (2.7).

In order to give a Widder type uniqueness theorem for the Dirichlet problem, we further need two conditions: [PCE-ρ] (i.e., the parabolic Carleson estimate with scale function ρ) and [OBC-ρ] (i.e., off-boundary curve condition with scale function ρ). In what follows, K denotes the compact set in [PHP-$\rho\Omega$].

[PCE-ρ] There exist positive constants $\gamma_1, \gamma_2, \alpha_3$ satisfying the following: For any

$$(x,t) \in (\partial\Omega \setminus K) \times (0,\infty), \quad 0 < r \le \frac{1}{\rho(d_\Omega(x))},$$

and any connected component $\omega(x,r)$ of $\Omega \cap B(x,r)$, one can find a point $\tilde{x} \in \omega(x,r)$ such that

$$B(\tilde{x},\gamma_1 r) \Subset \omega(x,r) \cap B(x,\frac{r}{2}),\ \mathrm{dist}(B(\tilde{x},\gamma_1 r), \partial\omega(x,r)) > \gamma_2 r, \tag{2.14}$$

and any nonnegative solution u of the equation

$$\mathcal{P}u = 0 \quad \text{in } Q_\omega = \omega(x,r) \times (t - r^2, t + r^2), \tag{2.15}$$

$$u = 0 \quad \text{on } (\partial\omega(x,r) \setminus \Omega) \times (t - r^2, t + r^2) \tag{2.16}$$

satisfies the inequality

$$\sup_{Q_\omega^-} u \le \alpha_3 \inf_{Q_\omega^+} u, \tag{2.17}$$

where

$$Q_\omega^- = (\omega(x,r) \cap B(x, \frac{r}{2})) \times (t - \frac{3}{4}r^2, t - \frac{1}{4}r^2),$$
$$Q_\omega^+ = B(\tilde{x}, \gamma_1 r) \times (t + \frac{1}{4}r^2, t + \frac{3}{4}r^2).$$

[OBC-ρ] For any $\varepsilon > 0$ there exist positive constants δ and C satisfying the following: For ant $x \in \Omega \setminus K$ with $\mathrm{dist}(x, \partial\Omega) > \varepsilon/\rho(d_\Omega(x))$ there exists a curve $\gamma \in \mathrm{Lip}([0,1]; \overline{\Omega})$ such that $\gamma(0) = x, \gamma(1) = O$, and

$$l(\gamma) \le C d_\Omega(x), \quad \mathrm{dist}(\gamma(s), \partial\Omega) > \frac{\delta}{\rho(C d_\Omega(x))}, \ 0 \le s \le 1. \tag{2.18}$$

We can show that if Ω is a Lipschitz domain, then these conditions are satisfied for some ρ. For the parabolic Carleson estimate (2.18), which is also called the boundary Harnack inequality for parabolic equations, see [41] and references therein. We are now ready to state a Widder type uniqueness theorem, Theorem 4.4 of [65].

Theorem 2.2 Suppose that [RB-$\rho\Omega$], [PHP-$\rho\Omega$], [PCE-ρ] and [OBC-ρ] hold with ρ satisfying (2.8). Then a nonnegative solution u of the Dirichlet problem (2.9), (2.10) and (2.11) is determined uniquely by the initial data u_0.

Here, as an application of Theorem 2.2, we give a simple example related to Theorem 1.3.

Theorem 2.3 Let $\gamma \ge -2$. Let α and β be Lipschitz continuous functions on $[1, \infty)$ such that $\alpha > \beta$ and $(\alpha(r) - \beta(r))r^{\gamma/2}$ is decreasing. Let

$$\Omega = \{(r,s) \in \mathbf{R}^2; 1 < r < \infty, \ \alpha(r) > s > \beta(r)\}.$$

Let $P = r^{-\gamma}(\partial^2/\partial r^2 + \partial^2/\partial s^2)$. Suppose that

$$\int_1^\infty (\alpha(r) - \beta(r)) r^\gamma dr = \infty. \tag{2.19}$$

Then a nonnegative solution of the Dirichlet problem (2.9), (2.10) and (2.11) is determined uniquely by the initial data u_0.

Proof. We show the theorem along the line given in the proof of Theorem 5.6 of [65]. Let $N = \{(r, s) \in \mathbf{R}^2; r > 0\}$. Introduce a Riemannian metric $g = (f(r)\delta_{ij})$ on N, where $\delta_{11} = \delta_{22} = 1, \delta_{12} = \delta_{21} = 0$, and f is a positive smooth function on $(0, \infty)$ such that $f(r) = r^\gamma$ for $r > 1/2$ and $f(r) = r^{-2}$ for $0 < r < 1/4$. Then N becomes a complete Riemannian manifold with this metric g. Let ∇ and div be the associated gradient and divergence on N, respectively. We have

$$P = \operatorname{div} \circ \nabla \quad \text{on } \{(r, s) \in N; r > 1/2\}.$$

Thus [RB-$\rho\Omega$] obviously holds. Let us show the conditions [PHP-$\rho\Omega$], [PCE-ρ] and [OBC-ρ]. Put $h(r) = \alpha(r) - \beta(r)$. Since $h(r)/r$ is bounded, we can choose a sufficiently small positive number θ so that

$$2^{-1} r^{\gamma/2} |x - y| \le \operatorname{dist}(x, y) \le 2 r^{\gamma/2} |x - y| \tag{2.20}$$

for any $x, y \in E(r, s, \theta h(r)) = \{z \in N; |z - (r, s)| < \theta h(r)\}$ with $(r, s) \in \overline{\Omega}$. Put $\zeta(r) = (\alpha(r) + \beta(r))/2,\ O = (2, \zeta(2))$ and $\hat{r} = (r, \zeta(r))$. We have

$$d_\Omega(\hat{r}, (r, s)) = |s - \zeta(r)| r^{\gamma/2} \le 2^{-1} h(r) r^{\gamma/2} \tag{2.21}$$

for any $(r, s) \in \overline{\Omega}$. Put $F(r) = \int_2^r t^{\gamma/2} dt$ for $r \ge 2$. Since α and β are Lipschitz continuous, there exists a positive constant C such that

$$F(r) \le d_\Omega(\hat{r}) \le C F(r), \quad r \ge 2. \tag{2.22}$$

Since $h(r) r^{\gamma/2}$ is bounded, it follows from (2.21) and (2.22) that for any $(r, s) \in \overline{\Omega}$ with $r \ge 3$

$$C_1 F(r) \le d_\Omega((r, s)) \le C_2 F_2(r), \tag{2.23}$$

where C_1 and C_2 are positive constants independent of (r, s). Define a function $r(R)$ from $[0, \infty)$ to $[2, \infty)$ by $F(r) = R$. Recalling that $h(r) r^{\gamma/2}$ is decreasing, define ρ by

$$\rho(R) = 1/\kappa h(r(R)) r(R)^{\gamma/2} \tag{2.24}$$

for a sufficiently small positive constant κ. Here, in view of (2.20) and (2.23), we have chosen κ so small that for any $x = (r, s) \in \overline{\Omega}$ with $r \ge 3$

$$B(x, 1/\rho(C_1 F(r))) \subset E(r, s, \theta h(r)). \tag{2.25}$$

By (2.19), ρ satisfies (2.8). In order to show [OBC-ρ], choose a curve composed of the line segment with endpoints $x = (r, s)$ and $\hat{r}$ and the curve $\{\hat{\tau}; 2 \le \tau \le r\}$. Then [OBC-$\rho$] follows from (2.20), (2.21), (2.22) and (2.23). Let us show [PHP-$\rho\Omega$]. Let $t > 0$, $x = (r, s) \in \overline{\Omega}$, $0 < \eta < \theta$, and $E(r, s, \eta h(r))$ the Euclidean ball with center (r, s) and radius $\eta h(r)$. Put $\sigma = \eta h(r) r^{\gamma/2}$. In view of (2.20), (2.23) and (2.25), we consider the equation

$$\frac{\partial u}{\partial \tau} + y^{-\gamma}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 \quad \text{in } Q_E = E(r, s, \sigma r^{-\gamma/2}) \times (t - \sigma^2, t + \sigma^2). \tag{2.26}$$

It suffices to show that the inequality

$$\sup_{Q_E^-} u \le \alpha_2 \inf_{Q_E^+} u, \tag{2.27}$$

where

$$Q_E^- = E(r, s, \frac{1}{2}\sigma r^{-\gamma/2}) \times (t - \frac{3}{4}\sigma^2, t - \frac{1}{4}\sigma^2),$$
$$Q_E^+ = E(r, s, \frac{1}{2}\sigma r^{-\gamma/2}) \times (t + \frac{1}{4}\sigma^2, t + \frac{3}{4}\sigma^2).$$

Change the variable (y, z) to

$$(Y, Z) = (r^{\gamma/2}(y - r) + r, r^{\gamma/2}(z - s) + s).$$

Then the equation becomes

$$\frac{\partial u}{\partial \tau} + \left(\frac{r}{y}\right)^{\gamma}\left(\frac{\partial^2 u}{\partial Y^2} + \frac{\partial^2 u}{\partial Z^2}\right) = 0 \quad \text{in } \Sigma = E(r, s, \sigma) \times (t - \sigma^2, t + \sigma^2), \tag{2.28}$$

and $Q_E^{\pm}$ become

$$\Sigma^- = E(r, s, \frac{\sigma}{2}) \times (t - \frac{3}{4}\sigma^2, t - \frac{1}{4}\sigma^2),$$
$$\Sigma^+ = E(r, s, \frac{\sigma}{2}) \times (t + \frac{1}{4}\sigma^2, t + \frac{3}{4}\sigma^2).$$

Note that σ is less than some small positive number, and $(r/y)^\gamma$ is bounded from above and below by positive constants. Thus the standard parabolic Harnack inequality shows the desired inequality (2.27). It remains to show

[PCE-ρ]. We treat only a boundary point x on the lower bank. Let $x = (r, \beta(r))$ with $r \geq 3$. In view of the above argument, put

$$B(Y) = r^{\gamma/2}\beta(r^{-\gamma/2}(Y - r) + r) + (1 - r^{\gamma/2})\beta(r).$$

It suffices to consider a nonnegative solution of the equation

$$\frac{\partial u}{\partial \tau} + (r/y)^{\gamma}\left(\frac{\partial^2 u}{\partial Y^2} + \frac{\partial^2 u}{\partial Z^2}\right) = 0 \tag{2.29}$$

$$\text{in } \Sigma_B = (E(r, \beta(r), \sigma) \cap \{Z > B(y)\}) \times (t - \sigma^2, t - \sigma^2),$$

$$u = 0 \quad \text{on } (E(r, \beta(r), \sigma) \cap \{Z = B(y)\}) \times (t - \sigma^2, t - \sigma^2). \tag{2.30}$$

Since the function $B(Y)$ is Lipschitz continuous, the standard parabolic Carleson estimate yields the inequality

$$\sup_{\Sigma_B^-} u \leq \alpha_3 \inf_{\Sigma_B^+} u,$$

where

$$\Sigma^- = (E(r, \beta(r), \frac{\sigma}{2}) \cap \{Z > B(Y)\}) \times (t - \frac{3}{4}\sigma^2, t - \frac{1}{4}\sigma^2),$$
$$\Sigma^+ = E(r, \beta(r) + \frac{\sigma}{4}, \gamma_1\sigma) \times (t + \frac{1}{4}\sigma^2, t + \frac{3}{4}\sigma^2)$$

for a sufficiently small positive number γ_1. This shows [PCE-ρ]. Hence Theorem 2.3 follows from Theorem 2.2. $\square$

Remark 2.4 Actually, the condition (2.19) is also a necessary condition for the Widder type uniqueness theorem to hold. Indeed, suppose that

$$\int_1^{\infty} (\alpha(r) - \beta(r)) r^{\gamma} dr < \infty.$$

Apply Theorem 6.1 in Section 6 with $\nu_1(r) = r$ and $\Phi(t_1) = t_1^{\gamma}$. Then we obtain that r^{γ} is a small perturbation of $-\Delta$ on Ω. Thus Remark 3.5 and Theorem 3.1 in Section 3 show that there exists a positive solution of (2.9), (2.10) and (2.11) with $u_0 = 0$.

3 h-big perturbations

In this section we recall a non-uniqueness theorem in [65], and observe that the Widder type uniqueness theorem implies h-bigness.

Let N be a connected separable smooth manifold with Riemannian metric of class C^0. Let L be an elliptic operator on N of the form

$$Lu = -m^{-1}\mathrm{div}(mA\nabla u - mCu) - \langle B, \nabla u\rangle + Vu, \tag{3.1}$$

where m, A, B, C, V satisfy the conditions (1.6),(1.7) and (1.8), with obvious modifications. Let W be a positive measurable function on N such that $W, W^{-1} \in L^\infty_{\mathrm{loc}}(N, d\lambda)$, $d\lambda = md\nu$, where ν is the Riemannian measure on N. Let Ω be a domain of N. We consider the Dirichlet problem

$$(\partial_t + W^{-1}L)u = 0 \quad \text{in } \Omega \times (0, \infty), \tag{3.2}$$

$$u(x, 0) = 0 \quad \text{on } \Omega, \tag{3.3}$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{3.4}$$

It is needless to say that when $\Omega = N$, the condition (3.4) is redundant, and the problem reduces to the Cauchy problem. Suppose that (L, Ω) is subcritical, i.e., there exists the Green function G of L on Ω. Let h be a positive solution of the Dirichlet problem

$$Lv = 0 \quad \text{on } \Omega, \tag{3.5}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

Here, the boundary condition (3.6) means $v \in H^1_{0,\mathrm{loc}}(\Omega)$. Following [35], we say that W is h-big (on Ω) when any function v satisfying

$$(L + W)v = 0 \quad \text{and} \quad 0 \le v \le h \quad \text{on } \Omega \tag{3.7}$$

must be identically zero. Otherwise, W is said to be non-h-big (on Ω). Theorem 2.5 of [65] partially reads as follows.

Theorem 3.1 The following are equivalent:

(i) W is non-h-big.

(ii) There exist a non-empty domain $E \subset \Omega$ and a positive solution f of the Dirichlet problem

$$Lf = 0 \quad \text{on } E, \quad f = 0 \quad \text{on } \partial E$$

such that $0 < f \leq h$ on E and

$$\int_E G_E(x,y)W(y)f(y)d\lambda(y) < \infty, \quad x \in E, \tag{3.8}$$

where G_E is the Green function of L on E with respect to the measure $d\lambda$.

(iii) There exists a solution u of (3.2), (3.3) and (3.4) such that $0 < u(x,t) \leq h(x)$ on $\Omega \times (0,\infty)$.

We should mention here that the statement of the assertion (ii) is slightly different from that of the assertion (II) of Theorem 2.5 in [65], but they are equivalent because a nonnegative solution of an elliptic equation on a connected open set is positive or identically zero.

The following is a direct consequence of this theorem which will be used in proving Theorem 1.1.

Proposition 3.2 Suppose that the Dirichlet problem (3.2), (3.3) and (3.4) has no nonnegative solution which is not identically zero. Then W is h-big for any positive solution h of (3.5) and (3.6).

Remark 3.3 When a positive solution h satisfies an appropriate growth condition at infinity, a Täcklind type uniqueness theorem (cf. [44,65]) can be used also as a sufficient condition of h-bigness.

We conclude this section with remarks on semismall perturbations (cf. Section 5 of [64]).

Proposition 3.4 Suppose that W is a semismall perturbation of L on Ω. Then W is non-h-big for any positive solution h of (3.5).

Proof. The semismallness of W implies

$$\int_\Omega G(x,y)W(y)h(y)d\lambda(y) < \infty, \ x \in \Omega$$

(cf. Proposition 3.3 of [62]). Thus W is non-h-big by virtue of Theorem 7.19 of [35] (see also Theorem 4.1 of [35]). □

Remark 3.5 We say that W is a small perturbation of L on Ω when for any $\varepsilon > 0$ there exists a compact subset K of Ω such that

$$\int_{\Omega \setminus K} G(x,z)W(z)G(z,y)d\lambda(z) \leq \varepsilon G(x,y), \ x,y \in \Omega \setminus K.$$

It is known (cf. [62]) that if W is a small perturbation, then it is a semismall perturbation, i.e., for any $\varepsilon > 0$ there exists a compact subset K of Ω such that

$$\int_{\Omega\setminus K} G(x^0, z)W(z)G(z, y)d\lambda(z) \le \varepsilon G(x^0, y),\ y \in \Omega \setminus K,$$

where x^0 is a point fixed in Ω. Thus, if W is a small perturbation of L on Ω, then W is non-h-big for any positive solution h of (3.5).

4 Martin boundaries for elliptic equations in skew product form

In this section we recall general results in [64], from which Theorem 1.1 is derived. Consider the equation (1.1). For (L_2, D_2), we assume the same conditions as in Section 1; so L_2 is the operator (1.10) on D_2 satisfying the hypothesis (SMI2). But, in this section, we treat L_1 and D_1 under more general conditions although we use the same notations as in Section 1. Let D_1 be a non-compact domain of M_1. Let L_1 be an elliptic operator on D_1 of the form (1.5), where m_1, A_1, B_1, C_1, V_1 satisfy the conditions (1.6), (1.7) and (1.8) with M_1 replaced by D_1. Let W_1 be a positive measurable function on D_1 such that $W_1, W_1^{-1} \in L^\infty_{\mathrm{loc}}(D_1, d\mu_1)$. Let $L = L_1 + W_1 L_2$ and $D = D_1 \times D_2$. We assume that (L, D) is subcritical. We denote by $D_1^*, \partial_M D_1, \partial_m D_1$, and k_0 the Martin compactification, Martin boundary, minimal Martin boundary, and Martin kernel for $(L_1 + \lambda_0 W_1, D_1)$, respectively. For an open set $\Omega \subset D_1$, we denote by Ω^* the closure of Ω in D_1^*; while $\overline{\Omega}$ denotes the closure Ω in the relative topology of D_1. We denote by L_1^* the formal adjoint operator of L_1 with respect to $d\mu_1$. For an elliptic operator P on an open set $\Omega \subset D_1$, a subset F of Ω such that $\overline{F} \cap \Omega = F$, and a family $\mathcal{F}$ of positive solutions of $Pu = 0$ in Ω, we put $\mathcal{S} = (\mathcal{F}, P, \Omega, F)$. We say that CP (i.e., the comparison principle) holds for $\mathcal{S}$ when there exists a positive constant c such that for any u and v in $\mathcal{F}$

$$c\frac{v(x)}{v(y)} \le \frac{u(x)}{u(y)} \le c^{-1}\frac{v(x)}{v(y)}, \quad x, y \in F. \tag{4.1}$$

We impose on $\{(L_1 + \lambda_j W_1, D_1)\}_{j=0}^\infty$ the following condition (ZCS1), i.e., zero limit, comparison principle and semismallness.

(ZCS1) There exist subsets Ξ_0 and Ξ_∞ of $\partial_M D_1$ such that $\Xi_0 \cup \Xi_\infty = \partial_M D_1$

and the following conditions (ZC) and (CS) are satisfied.

(ZC) For any $\xi_1 \in \Xi_0$, there exist domains U_i $(i = 1, 2, 3, 4)$ of D_1 such that

$$\overline{U_i} \subset U_{i+1} \text{ for } i = 1, 2, 3, \quad \xi_1 \in U_1^* \cap \partial_M D_1, \quad x_1^0 \in U_3 \setminus \overline{U_1}, \tag{4.2}$$
$$\lim_{U_3 \ni y_1 \to \xi_1} h_1(x_1, y_1)/h_0(x_1^0, y_1) = 0, \quad x_1 \in U_3, \tag{4.3}$$

where h_1 (resp. h_0) is the Green function of $L_1 + \lambda_1 W_1$ (resp. $L_1 + \lambda_0 W_1$) on U_4. Furthermore, CP holds for $\mathcal{S}$ and $\mathcal{R}$, where

$$\begin{aligned} \mathcal{S} &= (\{H_0(\cdot, y_1); y_1 \in \overline{U_1} \cup (D_1 \setminus U_3)\}, L_1 + \lambda_0 W_1, U_3 \setminus \overline{U_1}, \partial U_2), \quad (4.4)\\ \mathcal{R} &= (\{H_1(x_1^0, \cdot), h_1(x_1^0, \cdot)\}, L_1^* + \lambda_1 W_1, U_4 \setminus \{x_1^0\}, \partial U_3). \end{aligned}$$

(CS) For any $\xi_1 \in \Xi_\infty$, there exist domains E_i $(i = 1, \cdots, 8)$ of D_1 such that

$$\overline{E_i} \subset E_{i+1} \text{ for } i = 1, \cdots, 7, \quad \xi_1 \in E_1^* \cap \partial_M D_1, \quad x_1^0 \in E_6 \setminus \overline{E_5}, \tag{4.5}$$
$$W_1 \text{ is a semismall perturbation of } L_1 + \lambda_0 W_1 \text{ on } E_8, \tag{4.6}$$

and CP holds for $\mathcal{S}_i$ $(i = 1, 2, 3), \mathcal{T}_j$ and $\mathcal{U}_j$ $(j = 0, 1, \cdots)$, where

$$\begin{aligned} \mathcal{S}_i &= (\{H_0(\cdot, y_1); y_1 \in \overline{E_{2i-1}} \cup (D_1 \setminus E_{2i+1})\}, \quad (4.7)\\ &\qquad L_1 + \lambda_0 W_1, E_{2i+1} \setminus \overline{E_{2i-1}}, \partial E_{2i}),\\ \mathcal{T}_j &= (\{H_j(\cdot, y_1); y_1 \in \overline{E_6}\} \cup \{h_j(\cdot, y_1); y_1 \in \overline{E_6}\},\\ &\qquad L_1 + \lambda_j W_1, E_8 \setminus \overline{E_6}, \partial E_7),\\ \mathcal{U}_j &= (\{H_j(x_1^0, \cdot), h_j(x_1^0, \cdot)\},\\ &\qquad L_1^* + \lambda_j W_1, E_8 \setminus \{x_1^0\}, \partial E_6), \ j = 0, 1, 2, \cdots. \end{aligned}$$

Here h_j is the Green function of $L_1 + \lambda_j W_1$ on E_8.

This condition (ZCS1) always holds when D_1 is one dimensional (cf. [64]). The semi-localized condition (4.3) and (4.6) are useful in treating domains having several connected components at infinity. Note that CP holds for (4.7), for example, if $\overline{E_8} \setminus E_1$ is a compact subset of D_1.

We are now ready to state Theorem 9.1 of [64] except for the case where D_1 is compact.

Theorem 4.1 Assume the conditions (SMI2) and (ZCS1). Then the following (i)–(iv) hold true:

(i) With d_2 being an ideal point outside of D_2^*, the Martin boundary $\partial_M D$ is equal to the disjoint union of $\Xi_0 \times \{d_2\}, \Xi_\infty \times D_2^*$ and $D_1 \times \partial_M D_2$:

$$\partial_M D = \Xi_0 \times \{d_2\} \sqcup \Xi_\infty \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{4.8}$$

Furthermore,

$$\partial_m D = (\Xi_0 \cap \partial_m D_1) \times \{d_2\} \sqcup (\Xi_\infty \cap \partial_m D_1) \times D_2^* \sqcup D_1 \times \partial_M D_2. \tag{4.9}$$

In particular, $\partial_m D = \partial_M D$ if and only if $\partial_m D_1 = \partial_M D_1$.

(ii) The assertions (ii) and (iv) of Theorem 1.1 hold with Γ replaced by Ξ_0.

(iii) The assertions (iii) and (v) of Theorem 1.1 hold with ∂D_1 replaced by Ξ_∞.

(iv) The assertion (vi) of Theorem 1.1 holds.

A special case of this theorem is worth stating.

Theorem 4.2 Assume (SMI2). Suppose that W_1 is a semismall perturbation of $L_1 + \lambda_0 W_1$ on D_1. Then the Martin compactification D^* of D with respect to L is homeomorphic to $D_1^* \times D_2^*$, and all the assertions of Theorem 4.1 hold with $\Xi_0 = \emptyset$ and $\Xi_\infty = \partial_M D_1$.

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying Theorem 4.1 in the last section. We use the notations in Section 1, and assume the conditions (SMI2) and (U1). We start with a lemma concerning small perturbation and the boundary Harnack principle for elliptic equations. For definition of small perturbation, see Remark 3.5 in Section 3. As for the boundary Harnack principle, see [3, 5, 19, 20, 38, 39, 55, 81].

Lemma 5.1 The condition (CS) of the hypothesis (ZCS1) holds with Ξ_∞ and E_1^* replaced by ∂D_1 and $\overline{E_1}$, respectively.

Here and in what follows we abuse notations as follows: $\overline{E_1}$ and ∂E_1 in this section mean the closure and boundary of E_1 in M_1, respectively; so $\overline{E_1} \cap D_1$ and $\partial E_1 \cap D_1$ are equal to the symbols $\overline{E_1}$ and ∂E_1 in the hypothesis (ZCS1).

Proof. Let $\xi_1 \in \partial D_1$. Since D_1 is a Lipschitz domain, we can choose a coordinate system (U, z) such that

$$\begin{aligned} U \cap D_1 &= \{z = (z', z_N) \in \mathbf{R}^N; |z'| < R,\ 0 < z_N - f(z') < R\}, & (5.1)\\ U \cap \partial D_1 &= \{z \in \mathbf{R}^N; |z'| < R,\ z_N = f(z')\}, & (5.2) \end{aligned}$$

and $\xi_1 = (0, 0)$, where f is a Lipschitz continuous function on $\mathbf{R}^{N-1}$. We denote the right hand side of (5.1) by $E(R)$. For $0 < r < R/8$ with $x_1^0 \notin \overline{E(5r)}$, choose a Lipschitz curve γ in $D_1 \setminus \overline{E(5r)}$ such that $\gamma(0) = x_1^0$ and $\gamma(1) = (0, f(0) + 6r)$. For $s > 0$, put

$$F(s) = \{p \in M_1; \mathrm{dist}(p, \gamma(t)) < s, \quad 0 \le t \le 1\}.$$

Choose s so small that $F(8s) \subset D_1 \setminus \overline{E(5r)}$. For $i = 6, 7, 8$, put $E_i = E(ir) \cup F(is)$. Modifying $F(is)$ if necessary, we may assume that E_i are relatively compact Lipschitz domain of D_1. For $i = 1, \cdots, 5$, put $E_i = E(ir)$. Then $\overline{E_i} \cap D_1 \subset E_{i+1}$ for $i = 1, \cdots, 7$, $\xi_1 \in \overline{E_1} \cap \partial D_1$, and $x_1^0 \in E_6 \setminus \overline{E_5}$. In the coordinate system (U, z) the operator $L_1 + \lambda_0 W_1$ has the form

$$w(L_1 + \lambda_0 W_1)u = -\sum_{1 \le i,j \le N} \partial_i(a_{ij}\partial_j u) - \sum_{1 \le j \le N} b_j \partial_j u + \sum_{1 \le j \le N} \partial_j(c_j u) + qu, \tag{5.3}$$

where w is a positive measurable function with $w, w^{-1} \in L^\infty(E(R))$ and $a_{i,j}, b_j, c_j, q$ satisfy the condition (1.7) and (1.8) with obvious modifications. Thus, rechoosing r and s sufficiently small if necessary, we can show by Theorem 9.1', Proposition 9.2 and the proof of Corollary 6.1 of [8] that W_1, which is bounded on E_8, is a small perturbation of $L_1 + \lambda_0 W_1$ on E_8 (see also [1, 62]). This implies (4.6). Let $i = 1, 2, 3$. By the boundary Harnack principle, there exists a positive constant c such that

$$c\frac{v(x)}{v(y)} \le \frac{u(x)}{u(y)} \le c^{-1}\frac{v(x)}{v(y)}, \quad x, y \in \partial E_{2i} \cap D_1, \tag{5.4}$$

for any positive solutions u and v of the equation $(L_1 + \lambda_0 W_1)u = 0$ in $E_{2i+1} \setminus \overline{E_{2i-1}}$ such that

$$u = v = 0 \quad \text{on } \{z \in \mathbf{R}^N; (2i-1)r < |z'| < (2i+1)r, z_n = f(z')\}$$

(cf. Theorem 1.3 of [55]). We have abused notations: ∂E_{2i} in (5.4) is the boundary of E_{2i} in M_1, and so $\partial E_{2i} \cap D_1$ is the boundary of E_{2i} in D_1 which is equal to ∂E_{2i} in (4.7). Let us give another proof of (5.4). Denote by P the operator on the right hand side of (5.3), and put

$$Qu = -\sum_{1\le i,j\le N} \partial_i(a_{ij}\partial_j u).$$

Choose a relatively compact Lipschitz domain $E \subset D_1$ such that $\overline{E} \cap D_1 \subset E_{2i+1} \setminus \overline{E_{2i-1}}$ and $E \supset U \cap D_1$ for some neighborhood U of $\overline{\partial E_{2i} \cap D_1}$. Let u and v be positive solutions of the equation $Pu = 0$ in E such that they are continuous up to the boundary and vanish on $\{z \in \partial E; z_n = f(z')\}$. Let $\hat{u}$ be a positive solution of the equation $Q\hat{u} = 0$ in E with $\hat{u} = u$ on ∂E. Denote by μ_x and ν_x, $x \in E$, the harmonic measures for P and Q, respectively. Then there exists a positive constant c_1 such that $c_1\mu_x \le \nu_x \le c^{-1}\mu_x$, $\quad x \in E$ (cf. Proposition 8.3 and the comment after Theorem 9.1' of [8]). Thus $c_1 u(x) \le \hat{u}(x) \le c_1^{-1}u(x)$, $x \in E$. Similarly, $c_1 v(x) \le \hat{v}(x) \le c_1^{-1}v(x)$, $x \in E$. By Theorem 1.4 of [19], there exists a positive constant c_2 such that

$$c_2\frac{\hat{v}(x)}{\hat{v}(y)} \le \frac{\hat{u}(x)}{\hat{u}(y)} \le c_2^{-1}\frac{\hat{v}(x)}{\hat{v}(y)},\ x, y \in \partial E_{2i} \cap D_1.$$

This implies (5.4). Now for $y_1 \in \overline{E_{2i-1}} \cup (D_1 \setminus E_{2i+1})$, $H_0(\cdot, y_1)$ is a positive solution of the equation $(L_1 + \lambda_0 W_1)u = 0$ in $E_{2i+1} \setminus \overline{E_{2i-1}}$ which vanishes on $\{z \in \partial(E_{2i+1} \setminus \overline{E_{2i-1}}); z_n = f(z')\}$. Hence CP holds for $\mathcal{S}_i$ given by (4.7). Similarly, CP holds for $\mathcal{T}_j$ and $\mathcal{U}_j$ given by (4.7). □

The following lemma is a simple observation, but plays a critical role in proving Theorem 1.1.

Lemma 5.2 Let $h(x_1) = k_0(x_1, \xi_1)$ for some $\xi_1 \in \Gamma_1 = \partial_M D_1 \setminus \partial D_1$, where k_0 is the Martin kernel for $(L_1 + \lambda_0 W_1, D_1)$. If W_1 is h-big, then

$$\lim_{D_1 \ni y_1 \to \xi_1} H_1(x_1, y_1)/H_0(x_1^0, y_1) = 0,\ x_1 \in D_1. \tag{5.5}$$

Proof. Although the lemma is essentially Lemma 5.8 of [64], we give a proof since it is simple. Suppose that (5.5) does not hold. Then there exists a sequence $\{y_1^j\}_{j=1}^\infty$ in D_1 such that $y_1^j \to \xi_1$ and

$$v(x) = \lim_{j\to\infty} H_1(x_1, y_1^j)/H_0(x_1^0, y_1^j)$$

is a positive solution of the equation $(L_1 + \lambda_1 W_1)v = 0$ in D_1 satisfying $0 < v \le h$. This is a contradiction, since $(\lambda_1 - \lambda_0)W_1$ is also h-big (cf. Propositions 7.16 and 3.7 of [35]). □

Lemma 5.3 Let $h(x_1) = k_0(x_1, \xi_1)$ for some $\xi_1 \in \Gamma_1$. Then (5.5) holds.

Proof. By the a priori estimates near boundary points, h is a positive solution in $H^1_{0,\mathrm{loc}}(D_1)$ of the equation $(L_1 + \lambda_0 W_1)h = 0$ in D_1. It follows from the assumption (U1) that any nonnegative solution of the problem

$$\begin{aligned} &(\partial_t + W_1^{-1}(L_1 + \lambda_0 W_1))v = 0 \quad \text{in } D_1 \times (0,\infty), \\ &v(x,0) = 0 \quad \text{on } D_1, \\ &v(x,t) = 0 \quad \text{on } \partial D_1 \times (0,\infty) \end{aligned}$$

must be identically zero. Thus, by Proposition 3.2 in Section 3, W_1 is h-big. Hence Lemma 5.2 implies (5.5). □

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We claim that the condition (ZC) of the hypothesis (ZCS1) holds with Ξ_0 replaced by Γ_1. Choose domains U_i $(i = 1, \cdots, 4)$ such that $D_1 \setminus U_1$ is a compact subset of D_1, $U_4 = D_1$, $\overline{U_i} \cap D_1 \subset U_{i+1}$ for $i = 1, 2, 3$, and $x_1^0 \in U_3 \setminus \overline{U_1}$. Then (4.2) holds. By Lemma 5.3, (4.3) holds. By the Harnack inequality, (CP) holds for $\mathcal{S}$ and $\mathcal{R}$ given by (4.4). This proves the claim, which together with Lemma 5.1 implies that the hypothesis (ZCS1) holds with $\Xi_0 = \Gamma_1$ and $\Xi_\infty = \partial D_1$. Hence Theorem 4.1 in the last section shows Theorem 1.1. □

6 Martin boundaries of horn-shaped domains

In this section we show the assertion (ii) of Theorem 1.3, and give a generalization of the assertion (i) of Theorem 1.3.

6.1 Small perturbations

In this subsection we give a theorem on small perturbation. By using it we also give an improvement of Theorem 4 of [4], from which the assertion (i) of Theorem 1.3 follows.

Let Ω be a domain in $\mathbf{R}^2$ such that $(-\Delta, \Omega)$ is subcritical, i.e., there exists the Green function H of $-\Delta$ on Ω (cf. Theorem 8.33 of [37]). Let $\Phi(t_1, \cdots, t_l)$ be a nonnegative Borel measurable function on $(0, \infty]^l$. Define $\Psi(t_1, \cdots, t_l)$ by

$$\Psi(t_1, \cdots, t_l) = \sup_{4^{-1} < c_1, \cdots, c_l < 4} \Phi(c_1 t_1, \cdots, c_l t_l).$$

Let ν_j $(j = 1, \cdots, l)$ be $(0, \infty]$-valued continuous superharmonic function on Ω. Put

$$W(z) = \Phi(\nu_1(z), \cdots, \nu_l(z)).$$

Then we have the following

Theorem 6.1 Suppose that

$$\int_\Omega \Psi(\nu_1(z), \cdots, \nu_l(z)) dz < \infty, \tag{6.1}$$

where dz is the Lebesgue measure on $\mathbf{R}^2$. Then W is a small perturbation of $-\Delta$ on Ω.

Proof. Let $\partial_\infty \Omega$ be the boundary of Ω in the one point compactification of $\mathbf{R}^2$. Let F be the set of points in $\partial_\infty \Omega$ which are irregular with respect to the Dirichlet problem for harmonic functions on Ω. Then there exists a positive superharmonic function v on Ω such that $\lim_{z \to x} v(z) = \infty$ for all $x \in F$ (cf. Lemmas 9.18 and 9.19 of [37]). For an interval I in $(0, \infty]$, denote by χ_I the characteristic function of I. For $\delta > 0$, put

$$\phi_\delta(z) = \chi_{(0,\delta)}(H(z, y^0)) + \chi_{(\delta^{-1}, \infty]}(v(z)),$$

where y_0 is a point fixed in Ω. Then there exists a positive constant c_l depending only on l such that

$$\begin{aligned} &\frac{1}{H(x,y)} \int_\Omega H(x,z) H(z,y) \phi_\delta(z) W(z) dz \\ \le \ & c_l \int_\Omega \phi_{4\delta}(z) \Psi(\nu_1(z), \cdots \nu_l(z)) dz, \quad x, y \in \Omega, \end{aligned} \tag{6.2}$$

(cf. [2], Theorem 1 of [4] and the remark which follows it). We have

$$\lim_{\delta \to 0} \phi_{4\delta}(z) = 0 \quad \text{for a.e. } z,$$

since $\lim_{z\to x} H(z,y^0)=0$ for any regular boundary point x in $\partial_\infty\Omega$. By the Lebesgue dominated convergence theorem, for any $\varepsilon>0$ there exists $\delta>0$ such that the right hand side of (6.2) is less than ε. Put $K=\{z\in\Omega;\phi_\delta(z)=0\}$. Since $\phi_\delta(z)\geq 1$ on a neighborhood of $\partial_\infty\Omega$, K is a relatively compact subset of Ω. Thus we have

$$\begin{aligned}&\frac{1}{H(x,y)}\int_{\Omega\setminus K} H(x,z)H(z,y)W(z)dz\\ &\leq \frac{1}{H(x,y)}\int_{\Omega} H(x,z)H(z,y)\phi_\delta(z)W(z)dz<\varepsilon\end{aligned}$$

for any $x,y\in\Omega$. Hence W is a small perturbation of $-\Delta$ on Ω. □

The following is an improvement of Theorem 4 of [4].

Theorem 6.2 Let D_1 be a domain in $\{(r,s)\in\mathbf{R}^2;r>0\}$. Let D_2 be a Lipschitz domain in S^{N-1} or the whole space S^{N-1}, where $N\geq 2$. Let $L=-\Delta$ on $\mathbf{R}^{N+1}$ and

$$D=\{(z,s)\in\mathbf{R}^N\times\mathbf{R}^1;(|z|,s)\in D_1, z/|z|\in D_2\}. \tag{6.3}$$

Suppose that

$$\int\int_{D_1}\frac{drds}{r^2}<\infty. \tag{6.4}$$

Then the Martin compactification D^* for (L,D) is homeomorphic to $D_1^*\times\overline{D_2}$, where D_1^* is the Martin compactification for $(-\Delta,D_1)$. In particular, $\partial_M D$ is homeomorphic to $(D_1\times\partial D_2)\cup(\partial_M D_1\times\overline{D_2})$. Furthermore,

$$\partial_m D=(D_1\times\partial D_2)\cup(\partial_m D_1\times\overline{D_2}).$$

Proof. We show the theorem by applying Theorem 4.2. In the polar coordinates of $\mathbf{R}^N$,

$$L=-\frac{\partial^2}{\partial r^2}-\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\Lambda}{r^2}-\frac{\partial^2}{\partial s^2}, \tag{6.5}$$

where Λ is the Laplace-Beltrami operator on the sphere S^{N-1}. With

$$L_1=-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial s^2}-\frac{N-1}{r}\frac{\partial}{\partial r},\quad W_1=\frac{1}{r^2},\quad L_2=-\Lambda, \tag{6.6}$$

we have $L = L_1 + W_1 L_2$. For (L_2, D_2), the hypothesis (SMI2) holds with $\lambda_0 \geq 0$ (cf. Examples 9.2 and 9.3 of [64]). Let us show that W_1 is a small perturbation of $L_1 + \lambda_0 W_1$ on D_1. We have

$$\begin{aligned} P &\equiv r^{-(N-1)/2} \circ (L_1 + \lambda_0 W_1) \circ r^{(N-1)/2} \\ &= -\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} + [\lambda_0 + \frac{(N-1)(N-3)}{4}]\frac{1}{r^2}. \end{aligned} \tag{6.7}$$

Apply Theorem 6.1 with $\Phi(t_1) = t_1^{-2}$ and $\nu_1(z) = z_1$. Then it follows from (6.4) that $W_1 = r^{-2}$ is a small perturbation of $-\Delta$ on D_1. Thus the Green function g of P on D_1 is comparable with the Green function H of $-\Delta$ on D_1, i.e., $cg \leq H \leq c^{-1}g$ for some positive constant c (cf. [62]). This together with Theorem 6.1 shows that W_1 is a small perturbation of P on D_1. Denote by $H_0(r, s; \tilde{r}, \tilde{s})$ and $g(r, s; \tilde{r}, \tilde{s})$ the Green functions of $L_1 + \lambda_0 W_1$ and P on D_1. Then

$$g(r, s; \tilde{r}, \tilde{s}) = (r/\tilde{r})^{(N-1)/2} H_0(r, s; \tilde{r}, \tilde{s}). \tag{6.8}$$

Thus

$$\frac{g(r, s; z_1, z_2) g(z_1, z_2; \tilde{r}, \tilde{s})}{g(r, s; \tilde{r}, \tilde{s})} = \frac{H_0(r, s; z_1, z_2) H_0(z_1, z_2; \tilde{r}, \tilde{s})}{H_0(r, s; \tilde{r}, \tilde{s})}.$$

It follows from this that W_1 is a small perturbation of $L_1 + \lambda_0 W_1$ on D_1. In view of Theorem 4.2, it remains to show that the Martin compactification $D^*_{1, L_1+\lambda_0 W_1}$ of D_1 with respect to $L_1 + \lambda_0 W_1$ is homeomorphic to Martin compactification $D^*_{1,-\Delta}$ of D_1 with respect to $-\Delta$. We have

$$\frac{H_0(r, s; \tilde{r}, \tilde{s})}{H_0(r^0, s^0; \tilde{r}, \tilde{s})} = \left(\frac{r}{r_0}\right)^{(1-N)/2} \frac{g(r, s; \tilde{r}, \tilde{s})}{g(r^0, s^0; \tilde{r}, \tilde{s})},$$

where (r^0, s^0) is a reference point in D_1. Thus $D^*_{1, L_1+\lambda_0 W_1}$ is homeomorphic to $D_{1,P}$ which is homeomorphic to $D^*_{1,-\Delta}$, since r^{-2} is a small perturbation of $-\Delta$ on D_1. Hence $D^*_{1, L_1+\lambda_0 W_1}$ is homeomorphic to $D^*_{1,-\Delta}$. □

In Theorem 4 of [4], it was assumed that every boundary point of D_1 is regular with respect to the Dirichlet problem. Theorem 6.2 removes this regularity assumption.

The following is a special case of Theorem 6.2 and a generalization of the assertion (i) of Theorem 1.3.

Theorem 6.3 Let α and β be continuous functions on $[1, \infty)$ such that $\alpha > \beta$. Let $D_1 = \{(r, s) \in \mathbf{R}^2; \alpha(r) > s > \beta(r),\ 1 < r < \infty\}$. Let D_2, D and L be as in Theorem 1.3. Then the assertion (i) of Theorem 1.3 holds.

Proof. By virtue of Theorem 6.2, it suffices to show that $D_1^* = \overline{D_1} \cup \{\infty\}$ and $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$. But this follows from the Carathéodory theorem (cf. [74]) which says that there exists a homeomorphism from $\overline{D_1} \cup \{\infty\}$ onto the closed unit disc which is conformal in D_1. □

6.2 Proof of Theorem 1.3 (ii)

In this subsection we show the assertion (ii) of Theorem 1.3 by applying Theorem 1.1.

Lemma 6.4 Let D_1 be as in Theorem 1.3. Then the Martin compactification D_1^* of D_1 with respect to $L_1 + \lambda_0 W_1$ (see (6.6)) is homeomorphic to $\overline{D_1} \cup \{\infty\}$ which is the closure of D_1 in the one point compactification of $\mathbf{R}^2$. Furthermore, $\partial_M D_1 = \partial_m D_1 = \partial D_1 \cup \{\infty\}$.

Proof. Let P be the elliptic operator given by (6.7). Let $\mathcal{F}$ be the set of all positive solution u of $Pu = 0$ in D_1 such that $u = 0$ on ∂D_1 and $u(r^0, s^0) = 1$ with $x_1^0 = (r^0, s^0)$, Along the line given in the proof of Theorem A of [55], we show that $\mathcal{F}$ consists of one element. Put $h(r) = (\alpha(r) - \beta(r))/2$. Let $R > 2 + r^0$. Noting that α and β are Lipschitz continuous on $[1, \infty)$, choose a positive number δ so small that $\delta h(R) < R/4$ and

$$|\alpha(r) - \alpha(R)| + |\beta(r) - \beta(R)| < h(R)/4 \quad \text{if } |r - R| < \delta h(R).$$

Put

$$E(R) = \{(r, s) \in \mathbf{R}^2; |r - R| < \delta h(R), \alpha(r) > s > \beta(r)\}.$$

Let $u \in \mathcal{F}$. Then, $Pu = 0$ in $E(R)$ and $u = 0$ on $\partial E(R) \cap \partial D_1$. Change the variable (r, s) to (X, Y):

$$X = (r - R)/h(R), \quad Y = [s - \gamma(R)]/h(R),$$

where $\gamma(R) = (\alpha(R) + \beta(R))/2$. Then the domain $E(R)$ becomes

$$D(R) = \{(X, Y) \in \mathbf{R}^2; |X| < \delta, a(X) > Y > b(X)\},$$

where

$$\begin{aligned} a(X) &= [\alpha(h(R)X + R) - \gamma(R)]/h(R), \\ b(X) &= [\beta(h(R)X + R) - \gamma(R)]/h(R). \end{aligned}$$

The equation $Pu = 0$ in $E(R)$ becomes

$$(-\Delta + V_R(X))\hat{u} = 0 \text{ in } D(R),$$

where $\hat{u}(X, Y) = u(r, s)$ and

$$V_R(X) = [\lambda_0 + (N-1)(N-3)/4][h(R)/R][1 + Xh(R)/R]^{-2}.$$

Furthermore $\hat{u} = 0$ on $\{(X, Y) \in \partial D(R); Y = a(X) \text{ or } b(X)\}$. We see that there exists a positive constant C independent of R such that

$$\begin{aligned} &|a(X) - a(X')| + |b(X) - b(X')| \leq C|X - X'|, \quad X, X' \in (-\delta, \delta), \\ &|V_R(X)| \leq C, \quad |X| < \delta. \end{aligned}$$

Furthermore, $|a(X) - 1| < 1/4$ and $|b(X) + 1| < 1/4$ if $|X| < \delta$. By the boundary Harnack principle, there exists a positive constant c independent of R such that for any $u, v \in \mathcal{F}$

$$c\frac{\hat{u}(0, Y)}{\hat{u}(0, 0)} \leq \frac{\hat{v}(0, Y)}{\hat{v}(0, 0)} \leq c^{-1}\frac{\hat{u}(0, Y)}{\hat{u}(0, 0)}, \quad a(0) > Y > b(0).$$

Hence

$$c\frac{u(R, s)}{u(R, \gamma(R))} \leq \frac{v(R, s)}{v(R, \gamma(R))} \leq c^{-1}\frac{u(R, s)}{u(R, \gamma(R))}, \quad \alpha(R) > s > \beta(R).$$

Since $u = v = 0$ on ∂D_1, the maximum principle shows that

$$c\frac{u(r, s)}{u(R, \gamma(R))} \leq \frac{v(r, s)}{v(R, \gamma(R))} \leq c^{-1}\frac{u(r, s)}{u(R, \gamma(R))} \tag{6.9}$$

for $(r, s) \in D_1$ with $1 < r < R$. Since $u(r^0, s^0) = v(r^0, s^0) = 1$, this implies that

$$c \leq \frac{v(R, \gamma(R))}{u(R, \gamma(R))} \leq c^{-1}.$$

Thus

$$c^2 u(r, s) \leq v(r, s) \leq c^{-2} u(r, s) \tag{6.10}$$

for $(r, s) \in D_1$ with $1 < r < R$. Since c is a constant independent of R, (6.10) holds on D_1. Put

$$\varepsilon_0 = \sup\{\varepsilon > 0; v > \varepsilon u \text{ on } D_1\}$$

and $w = v - \varepsilon_0 u$. Then either $w = 0$ or $w > 0$ on D_1. Suppose that $w > 0$. Then $w(\cdot)/w(x_1^0) \in \mathcal{F}$. By (6.10), $w \geq [w(x_1^0)c^2]u$ on D_1 This is a contradiction. Thus $v = \varepsilon_0 u$. But $v(x_1^0) = u(x_1^0) = 1$. Hence $v = u$. This shows that $\mathcal{F}$ consists of one element. Let $\mathcal{F}'$ be the set of all positive solutions u of $(L_1 + \lambda_0 W_1)u = 0$ in D_1 such that $u = 0$ on ∂D_1 and $u(x_1^0) = 1$. By (6.7), $\mathcal{F}'$ consists of one element. Choose a sequence $\{y_1^j\}_{j=1}^\infty$ in D_1 such that it has no accumulation points in $\overline{D_1}$ and there exists the limit

$$\lim_{j\to\infty} H_0(x_1, y_1^j)/H_0(x_1^0, y_1^j).$$

Denote this limit by $k_0(x_1, \eta_1)$, where η_1 is the point in $\partial_M D_1$ corresponding the sequence $\{y_1^j\}$. We claim that $\eta_1 \in \partial_M D_1 \setminus \partial D_1$. For any $\xi_1 = (R, S) \in \partial D_1$, choose R_i $(i = 1, 2, 3)$ such that $R + r^0 < R_3 < R_2 < R_1$. Put

$$\begin{aligned} U_i &= \{(r,s) \in D_1; r > R_i\}, \quad i = 1, 2, \\ U_3 &= \{(r,s) \in D_1; r > R_3\} \cup V_3, \end{aligned} \tag{6.11}$$

where V_3 is a relatively compact subset of D_1 such that $x_1^0 = (r^0, s^0) \in V_3$ and U_3 is connected. Since the boundary Harnack principle holds for positive solution of $(L_1 + \lambda_0 W_1)u = 0$ in $U_3 \setminus \overline{U_1}$ such that $u = 0$ on $\partial D_1 \cap \partial(U_3 \setminus \overline{U_1})$, we have by Lemma 1.5 of [64] that

$$(U_1^* \cap \partial_M D_1) \cap ((D_1 \setminus \overline{U_3})^* \cap \partial_M D_1) = \emptyset,$$

where U_1^* is the closure of U_1 in D_1^*. We have

$$\xi_1 \in \{(r,s) \in \partial D_1; r \leq R_3\} = (D_1 \setminus \overline{U_3})^* \cap \partial_M D_1$$

and $\eta_1 \in U_1^* \cap \partial_M D_1$. This proves the claim that $\eta_1 \in \partial_M D_1 \setminus \partial D_1$. By the a priori estimates, $k_0(\cdot, \eta_1) \in \mathcal{F}'$. But $\mathcal{F}'$ consists of one element. This implies that for any sequence $\{y_1^j\}_{j=1}^\infty$ in D_1 with no accumulation points in $\overline{D_1}$

$$k_0(x_1, \eta_1) = \lim_{j\to\infty} H_0(x_1, y_1^j)/H_0(x_1^0, y_1^j).$$

Since $\overline{D_1}$ is continuously imbedded into D_1^*, it follows from this that D_1^* is homeomorphic to $\overline{D_1} \cup \{\infty\}$ and $\eta_1 = \infty$. Since $\partial D_1 \subset \partial_m D_1$, it remains to show that $\infty \in \partial_m D_1$. Suppose that $\infty \notin \partial_m D_1$. Then the Martin representation theorem shows that there exists a finite Borel measure μ on ∂D_1 such that

$$k_0(x_1, \infty) = \int_{\partial D_1} k_0(x_1, \xi_1) d\mu(\xi_1).$$

For $R > 1$, put $F_R = \{(r,s) \in \partial D_1; r \leq R\}$ and

$$u_R(x_1) = \int_{F_R} k_0(x_1, \xi_1) d\mu(\xi_1).$$

Choose U_i ($i = 1, 2, 3$) as in (6.11). Since $u_R(\cdot) \leq k_0(\cdot, \infty)$, Lemma 1.5 of [64] together with the boundary Harnack principle shows that $\mu(\partial D_1 \cap (\partial F_R \setminus \partial U_3)) = 0$. Thus $\mu(\partial D_1 \cap \partial F_R) = 0$. Since R is arbitrary, this implies that $\mu = 0$; which is a contradiction. Hence $\infty \in \partial_M D_1$. □

We are now ready to complete the proof of Theorem 1.3 (ii).

Proof of Theorem 1.3 (ii). Recall that the hypothesis (SMI2) for (L_2, D_2) holds (cf. Examples 9.2 and 9.3 of [64]). By virtue of Lemma 6.4 and Theorem 1.1, it suffices to show that the condition (U1) holds. Consider the equation

$$(\partial_t + W_1^{-1} L_1)v = 0 \quad \text{in } D_1 \times (0, \infty)$$

(see (6.6) and (6.7)). We have

$$r^{-(N-1)/2} \circ (\partial_t + W_1^{-1} L_1) \circ r^{(N-1)/2} = \partial_t - r^2 \Delta + (N-1)(N-3)/4.$$

Thus Theorem 2.3 and the assumption (1.27) show that any nonnegative solution of (1.2), (1.3) and (1.4) must be identically zero. i.e., (U1) holds. □

7 Generalization

In this section we slightly generalize Theorems 1.1 and 1.2 for giving more concrete examples.

Let $L = L_1 + W_1 L_2$ and $D = D_1 \times D_2$ be as in Section 1. Assume (SMI2) for (L_2, D_2). Suppose that

$$D_1 = \bigcup_{j=0}^{N} E_j, \tag{7.1}$$

where N is a natural number, E_j ($j = 1, \cdots, N$) are Lipschitz domains in M_1 or the whole space M_1 such that $\overline{E_j} \cap \overline{E_k} = \emptyset$ for $j \neq k$, and E_0 is a relatively compact Lipschitz domain in M_1 or an empty set. Here $\overline{E_j}$ is the closure of E_j in M_1, while E_j^* denote the closure of E_j in D_1^*. For $j = 1, \cdots, N$, consider the Dirichlet problem

$$(\partial_t + W_1^{-1} L_1)v = 0 \quad \text{in } E_j \times (0, \infty), \tag{7.2}$$

$$v(x, 0) = 0 \quad \text{on } E_j, \tag{7.3}$$

$$v(x, t) = 0 \quad \text{on } \partial E_j \times (0, \infty). \tag{7.4}$$

We introduce the following condition.

(US1) There exists an integer l such that (i) $0 \le l \le N$, (ii) for $1 \le j \le l$, any nonnegative solution of (7.2), (7.3) and (7.4) must be identically zero, and (iii) for $l < j \le N$, W_1 is a semismall perturbation of $L_1 + \lambda_0 W_1$ on E_j.

Theorem 7.1 Assume the conditions (SMI2) and (US1). Put

$$\Xi_0 = \bigcup_{j=1}^{l} (E_j^* \cap \partial_M D_1) \setminus \partial D_1, \quad \Xi_\infty = \bigcup_{j=l+1}^{N} (E_j^* \cap \partial_M D_1) \cup \partial D_1.$$

Then all the conclusions of Theorem 4.1 hold true. Furthermore, $E_j^* \cap E_k^* = \emptyset$ for $j, k = 1, \cdots, N$ with $j \ne k$.

This theorem can be shown as Theorems 1.1 and 1.2. For proving the last assertion, use Lemma 1.5 of [64] as in the proof of Lemma 6.4.

8 Examples

In this section we give several concrete examples as applications of Theorem 7.1.

Example 8.1 Let $L = -\Delta$ on $\mathbf{R}^{2+m}$. Let D_2 be a bounded Lipschitz domain in $\mathbf{R}^m$. Let D_1 be a Lipschitz domain in $\mathbf{R}^2$ of the form

$$D_1 = \bigcup_{j=0}^{N} E_j,$$

where N is a natural number and E_j are Lipschitz domains defined as follows: For $j = 1, \cdots, N$, let f_j be a Lipschitz continuous positive function on $[1, \infty)$

such that it is decreasing and $f_j(1) < 1/2$; and let

$$\begin{aligned} E_j &= \{(r,s) \in \mathbf{R}^2; |s-j| < f_j(r),\ r > 1\},\ j = 1, \cdots, N, \\ E_0 &= \bigcup_{j=1}^{N} \{(r,s) \in \mathbf{R}^2; |s-j| < f_j(r),\ 1 \le r < 2\} \cup (0,1) \times (0, N+1). \end{aligned}$$

Let $0 \le l \le N$ be an integer. Suppose that

$$\int_1^\infty f_j(r)dr = \infty, \quad 1 \le j \le l,$$
$$\int_1^\infty f_j(r)dr < \infty, \quad l < j \le N.$$

For $j = 1, \cdots, N$, let η_j be the point at infinity of the one point compactification of $\overline{E_j}$; and set $\eta_j \ne \eta_k$ for $j \ne k$. Put

$$\Xi_0 = \{\eta_j; j = 1, \cdots, l\}, \quad \Xi_\infty = \{\eta_j; j = l+1, \cdots, N\} \cup \partial D_1.$$

Let $D = D_1 \times D_2$. Then the Martin boundary $\partial_M D$ for (L, D) is homeomorphic to

$$\Xi_0 \times \{d_2\} \cup \Xi_\infty \times \overline{D_2} \cup D_1 \times \partial D_2.$$

Furthermore, $\partial_M D = \partial_m D$. Indeed, by Theorems 2.3 and 6.1, the hypothesis (US1) holds. Furthermore, the same argument as in the proof of Lemma 6.4 shows that $\partial_M D = \partial_m D_1 = \partial D_1 \cup \{\eta_j; j = 1, \cdots, N\}$. Thus Theorem 7.1 shows the assertion.

Example 8.2 Let $D = \{x \in \mathbf{R}^n; |x| > 1\}$. let V be a locally bounded measurable real-valued function on $[1, \infty)$. Let $L = -\Delta + V(|x|)$. Suppose that (L, D) is subcritical. Then it is known that $\partial_M D = \partial_m D \supset \partial D$ and the set

$$\Gamma = \partial_M D \setminus \partial D$$

is homeomorphic to the unit sphere S^{n-1} or one point.

(i) Suppose that

$$\int_1^\infty (\sup_{1 \le s \le r} s^2 |V(s)| + 1)^{-1/2} \frac{dr}{r} = \infty. \tag{8.1}$$

Then Γ consists of one point.

(ii) Suppose that $r^2V(r)+\alpha \geq 1$ on $[1,\infty)$ for some positive constant α. Assume that

$$\int_1^\infty (r^2V(r)+\alpha)^{-1/2}\frac{dr}{r} < \infty. \tag{8.2}$$

Then Γ is homeomorphic to the unit sphere S^{n-1}.

For results related to (i) and (ii), see [45, 53, 55, 56, 67, 72], and Example 10.1 of [64].

Let us show the assertion (i) by applying Theorem 7.1. In the polar coordinates of $\mathbf{R}^n$,

$$L = -r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial}{\partial r}) + V(r) - \frac{\Lambda}{r^2}, \tag{8.3}$$

where Λ is the Laplace-Beltrami operator on the sphere S^{n-1}. Let $D_1 = (1,\infty)$, $D_2 = S^{n-1}$, $L_1 = -r^{1-n}(\partial/\partial r)(r^{n-1}\partial/\partial r)+V$, $W_1 = r^{-2}$, and $L_2 = -\Lambda$. Then $L = L_1 + W_1L_2$ on $D = D_1 \times D_2$. Put $E_0 = (2,9), E_1 = (e^2,\infty)$ and $E_2 = (1,3)$. Then $D_1 = E_0 \cup E_1 \cup E_2$. We see that W_1 is a small perturbation of L_1 on E_2 (cf. Theorem 6.3 of [64]). We claim that for $j=1$, any nonnegative solution of (7.2), (7.3) and (7.4) must be identically zero. Change the variable r to $z = \log r$. Then (7.2) becomes

$$[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} - (n-2)\frac{\partial}{\partial z} - e^{2z}V(e^z)]\hat{v} = 0 \quad \text{ in } (2,\infty)\times(0,\infty), \tag{8.4}$$

where $\hat{v}(z,t) = v(r,t)$. For $R > 0$, put

$$\phi(R) = (\sup_{0\leq z\leq R} e^{2z}|V(e^z)| + 1). \tag{8.5}$$

Choose an increasing step function ψ such that $\phi \leq \psi \leq 2\phi$, and find a positive continuous increasing function ρ such that $\psi \leq \rho \leq 2\psi$. Then $\phi \leq \rho \leq 4\phi$. By (8.5) and (8.1),

$$\int_1^\infty \frac{dR}{\rho(R)} = \infty, \qquad \sup_{0\leq z\leq R} e^{2z}|V(e^z)| \leq \rho(R)^2.$$

Let $\hat{v}$ be a nonnegative solution of (8.4) with $\hat{v}(z,0) = 0$ on $(2,\infty)$ and $\hat{v}(2,t) = 0$ on $(0,\infty)$. Then, by the scaling argument as in the proof of Theorem 6.2 of [44]. Theorem 2.2 in Section 2 shows that $\hat{v} = 0$. Thus the claim holds, and so the condition (US1) is satisfied. Hence Theorem 7.1 shows the assertion (i).

Let us show the assertion (ii). We claim that W_1 is a small perturbation of $L_1 + \alpha W_1$ on D_1. Let f and g be positive solutions of the equation $(L_1+\alpha W_1)f = 0$ in D_1 and $(L_1+(\alpha+1)W_1)g = 0$ in D_1 with $f(1) = g(1) = 0$ and $f'(1) = g'(1) = 1$, respectively. Change the variable r to $z = \log r$, and put $\hat{f}(z) = f(e^z)$ and $\hat{g}(z) = g(e^z)$. Then the equation becomes

$$P\hat{f} \equiv [\frac{d^2}{dz^2} + (n-2)\frac{d}{dz} - e^{2z}V(e^z) - \alpha]\hat{f} = 0 \quad \text{in } (0,\infty),$$
$$(P-1)\hat{g} = 0 \quad \text{in } (0,\infty),$$

with $\hat{f}(0) = \hat{g}(0) = 0$ and $\hat{f}'(0) = \hat{g}'(0) = 1$. By (8.2),

$$\int_0^\infty (e^{2z}V(e^z) + \alpha)^{-1/2} dz < \infty.$$

Then the same argument as in the proof of Lemma 2 of [58] shows that $\lim_{z\to\infty} \hat{g}(z)/\hat{f}(z) < \infty$. Thus

$$\lim_{r\to\infty} \frac{g(r)}{f(r)} < \infty.$$

We see that the Martin boundary for $(L_1+\alpha W_1, D_1)$ is $\{1,\infty\}$ and the Martin kernel $k_0(x_1,\infty)$ is a constant multiple of f (cf. Appendix of [53]). Thus, by Theorem 6.3 of [64], $W_1\chi_{(2,\infty)}$ is a small perturbation of $L_1 + \alpha W_1$ on D_1. Since $W_1\chi_{(1,2]}$ is a small perturbation of $L_1 + \alpha W_1$ on D_1, this implies that W_1 is a small perturbation of $L_1 + \alpha W_1$ on D_1, i.e., the claim holds. Thus the Green functions of $L_1+\alpha W_1$ on D_1 and L_1 on D_1 are comparable. Hence W_1 is a small perturbation of L_1 on D_1. Hence Theorem 7.1 (or Theorem 1.2) shows the assertion (ii).

Example 8.3 Let D_1 be a bounded Lipschitz domain in $\mathbf{R}^n$, and put $\delta_1(x_1) = \mathrm{dist}(x_1, \partial D_1)$. Let $L_1 = -\delta_1(x_1)^\gamma \Delta_1$, where γ is a real number and Δ_1 is the Laplacian on $\mathbf{R}^n$. Let $D_2 = M_2$ be a compact manifold. Let $L_2 = -\Delta_2$, where Δ_2 is the Laplacian on M_2. Let $L = L_1 + L_2$ and $D = D_1 \times D_2$. Let $\partial_M D$ and $\partial_m D$ be the Martin boundary and minimal Martin boundary for (L, D). Then we have the following:

(i) For $\gamma \geq 2$, $\partial_m D = \partial_M D = \partial D_1 \times \{d_2\}$.

(ii) For $\gamma < 2$, $\partial_m D = \partial_M D = \partial D_1 \times D_2$.

Let us show the assertions. We see that the Martin compactification D_1^* of D_1 with respect to L_1 is homeomorphic to $\overline{D_1}$, and $\partial_m D_1 = \partial_M D_1 = \partial D_1$. Suppose that $\gamma \geq 2$. Then, by Theorem 7.8 of [44], any nonnegative solution of the Cauchy problem

$$(\partial_t + L_1)u = 0 \quad \text{in } D_1 \times (0,\infty), \quad u(x,0) = 0 \quad \text{on } D_1$$

must be identically zero. Thus the assumption (U1) of Theorem 1.1 is satisfied with $D_1 = M_1$. Hence the assertion (i) follows from Theorem 1.1. Next, suppose that $\gamma < 2$. Then, by Theorem 9.1 of [8], 1 is a small perturbation of L_1 on D_1. Thus the assumption (S1) of Theorem 1.2 is satisfied. Hence the assertion (ii) follows from Theorem 1.2.

Example 8.4 Let $D_1 = \mathbf{R}^n$ and $L_1 = -\Delta_1 + V_1$, where Δ_1 is the Laplacian on $\mathbf{R}^n$ and V_1 is the function on $\mathbf{R}^n$ such that $V_1(z) = 1$ for $z_n > 0$ and $V_1(z) = 2$ for $z_n \leq 0$. Let $W_1(x_1) = \langle x_1 \rangle^\gamma$, where γ is a real number and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $D_2 = \mathbf{R}^m$ and $L_2 = \langle x_2 \rangle^\alpha(-\Delta_2 + 1) - \beta$, where $\alpha > 2$, Δ_2 is the Laplacian on $\mathbf{R}^m$, and β is a positive constant such that $\lambda_0 = 0$, i.e., 0 is the first eigenvalue of the selfadjoint operator $\mathcal{L}_2$ associated with L_2 on D_2. Let $L = L_1 + W_1 L_2$ and $D = D_1 \times D_2$. It is known (cf. [54]) that the Martin boundary $\partial_M D_1$ and the minimal Martin boundary $\partial_m D_1$ for (L_1, D_1) are homeomorphic to the set Σ and σ defined by

$$\begin{aligned} \sigma &= \{\omega \in \mathbf{R}^n; |\omega| = 1, \omega_n \geq 0\} \cup \{\omega \in \mathbf{R}^n; |\omega| = \sqrt{2}, \omega_n \leq -1\}, \\ \Sigma &= \sigma \cup \{(\omega', -\theta) \in \mathbf{R}^n; |\omega'| = 1, 0 < \theta < 1\}, \end{aligned}$$

i.e., $\partial_M D_1 \cong \Sigma$ and $\partial_m D_1 \cong \sigma$. Furthermore, 1 is a small perturbation of $\langle x_2 \rangle^\alpha(-\Delta_2 + 1)$ on $\mathbf{R}^m$ (cf. Theorem 5.1 of [64]); the Martin boundary $\partial_M D_2$ for $(\langle x_2 \rangle^\alpha(-\Delta_2 + 1), \mathbf{R}^m)$ is homeomorphic to the unit sphere S^{m-1} at infinity (cf. [53]), i.e., $\partial_M D_2 = S^{m-1}\infty$; $D_2^* = \mathbf{R}^m \sqcup S^{m-1}\infty$; the hypothesis (SMI2) for (L_2, D_2) is satisfied (cf. Example 9.4 of [64]); and $D_1 \times \partial_M D_2 \subset \partial_m D$. Put

$$\Gamma = \partial_M D \setminus D_1 \times \partial_M D_2.$$

Then we have the following:

(i) For $\gamma \geq -1$, $\Gamma \cong \Sigma$ and $\Gamma \cap \partial_m D \cong \sigma$.

(ii) For $\gamma < -1$, $\Gamma \cong \Sigma \times D_2^*$ and $\Gamma \cap \partial_m D \cong \sigma \times D_2^*$.

Let us show the assertions. Suppose that $\gamma < -1$. Then $W_1(x_1) = \langle x_1 \rangle^\gamma$ is a small perturbation of L_1 on $\mathbf{R}^n$ (cf. Theorem 5.1 of [64]). Thus Theorem 1.2

shows the assertion (ii). Next, suppose that $\gamma \geq -1$. Consider the Cauchy problem

$$(\partial_t + \langle z\rangle^{-\gamma} L_1)v = 0 \quad \text{in } \mathbf{R}^n \times (0,\infty), \quad v(z,0) = 0 \quad \text{on } \mathbf{R}^n. \tag{8.6}$$

In order to show that the Cauchy problem (8.5) allows no positive solution, we introduce a Riemannian metric $g = (g_{ij})$ on $\mathbf{R}^n$ by $g_{ii} = \langle z\rangle^{\gamma}$ and $g_{ij} = 0$ for $i \neq j$. Then $M_1 = \mathbf{R}^n$ becomes a complete Riemannian manifold with this metric g. The associated gradient ∇ and divergence div are written as

$$\nabla = \langle z\rangle^{-\gamma}\nabla^0, \ \mathrm{div} = \langle z\rangle^{-n\gamma/2}\mathrm{div}^0\langle z\rangle^{n\gamma/2},$$

where ∇^0 and div^0 are the standard gradient and divergence on $\mathbf{R}^n$. Put $m_1(z) = \langle z\rangle^{(1-n/2)\gamma}$. Then

$$\langle z\rangle^{-\gamma} L_1 v = -m_1^{-1}\mathrm{div}(m_1\nabla v) + \langle z\rangle^{-\gamma} V_1(z)v \tag{8.7}$$

For z with $|z| > 1$, denote by $d(z)$ the Riemannian distance from 0 to z. Then $d(z)$ is comparable with $|z|^{(\gamma/2+1)}$. Thus

$$|\langle z\rangle^{-\gamma} V_1(z)| \leq C d(z)^{-\gamma/(\gamma/2+1)} \leq C d(z)^2, \quad |z| > 1,$$

for some constant $C > 0$. We see from this that the assumption [PHP-ρ] of Theorem 2.1 is satisfied with $\rho(R) = C(R+1)$ for a sufficiently large positive constant C (cf. the proof of Theorem 6.2 of [44]). By Theorem 2.1, any nonnegative solution of (8.5) must be identically zero. Thus the assumption (U1) of Theorem 1.1 is satisfied. Hence Theorem 1.1 shows the assertion (i).

References

[1] H. Aikawa, *Norm estimate of Green operator, perturbation of Green function and integrability of superharmonic functions*, Math. Ann. **312** (1998), 289–318.

[2] H. Aikawa, *Generalized Cranston-McConnell inequalities for discontinuous superharmonic functions*, Potential Anal. **8** (1998), 127–135.

[3] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan, **53** (2001), 119–145.

[4] H. Aikawa and M. Murata, *Generalized Cranston-McConnell inequalities and Martin boundaries of unbounded domains*, J. Analyse Math. **69** (1996), 137–152.

[5] A. Ancona, *Principe de Harnack à frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Ann. Inst. Fourier (Grenoble) **28** (1978), 169–213.

[6] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. Math. **121** (1987), 429–461.

[7] A. Ancona, *Theorie du potential sur les graphes et les varietes*, Lecture Notes in Math. Vol. **1427**, Springer, Berlin (1988), 1–23.

[8] A. Ancona, *First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains*, J. Analyse Math. **72** (1997), 45–92.

[9] A. Ancona and J. C. Taylor *Some remarks on Widder's theorem and uniqueness of isolated singularities for parabolic equations*, in Partial Differential Equations with Minimal Smoothness and Applications, (Dahlberg et al., eds.) Springer, New York, 1992, 15–23.

[10] M. T. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. Math. **121** (1985), 429–461.

[11] J-P. Anker and L. Ji, *Heat kernel and Green function estimates on non-compact symmetric space*, Geom. Funct. Anal. **9** (1999), 1035–1091 .

[12] J-P. Anker and L. Ji, *Heat kernel and Green function estimates on non-compact symmetric space II*, in Topics in Probability and Lie Groups: Boundary Theory (J. C. Taylor Ed.), CRM Proc. Lecture Notes, Vol. **28** Amer. Math. Soc. 2001, 153–202.

[13] H. Arai, *Degenerate elliptic operators, Hardy spaces and diffusions on strongly pseudoconvex domains*, Tohoku Math. J. **46** (1994), 469–498.

[14] D. H. Armitage and S. J. Gardiner *Classical Potential Theory*, Springer, London, 2001.

[15] D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Sci. Norm. Sup. Pisa **22** (1968), 607–694.

[16] D. G. Aronson and P. Besala, *Uniqueness of solutions of the Cauchy problem for parabolic equations*, J. Math. Anal. Appl. **13** (1966), 516–526.

[17] D. G. Aronson and P. Besala, *Uniqueness of positive solutions of parabolic equations with unbounded coefficients*, Colloq. Math. **18** (1967), 126–135.

[18] R. Azencott, *Behavior of diffusion semi-groups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240.

[19] L. A. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), 621–640.

[20] B. E. Dahlberg, *Estimates of harmonic measure*, Arch. Rat. Mech. Anal. **65** (1977), 275–282.

[21] E. B. Davies, L^1 *properties of second order elliptic operators*, Bull. London Math. Soc. **17** (1985), 417–436.

[22] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, Cambridge, 1989.

[23] E. B. Davies, *Heat kernel bounds, conservation of probability and the Feller property*, J. Analyse Math. **58** (1992), 99–119.

[24] E. B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **59** (1984), 335–395.

[25] H. Donnelly, *Uniqueness of the positive solutions of the heat equation*, Proc. Amer. Math. Soc. **99** (1987), 353–356.

[26] D. Eidus and S. Kamin, *The filtration equation in a class of functions decreasing at infinity*, Proc. Amer. Math. Soc. **120** (1994), 825–830.

[27] E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rat. Mech. Anal. **96** (1986), 327–338.

[28] W. Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. Math. **55** (1952), 468–519.

[29] A. Freire, *On the Martin boundary of Riemannian products*, J. Diff. Geometry **33** (1991), 215–232.

[30] S. J. Gardiner, *The Martin boundary of NTA strips*, Bull. London Math. Soc. **22** (1990), 163–166.

[31] S. Giulini and W. Woes, *The Martin compactification of the Cartesian product of two hyperbolic spaces*, J. Reine Angew. Math. **444** (1993), 17–28.

[32] A. Grigor'yan, *On stochastically complete manifolds*, Soviet Math. Dokl. **34** (1987), 310–313.

[33] A. Grigor'yan, *The heat equation on non-compact Riemannian manifolds*, Math. USSR Sb. **72** (1992), 47–77.

[34] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian notion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[35] A. Grigor'yan and W. Hansen, *A Liouville property for Schrödinger operators*, Math. Ann. **312** (1998), 659–716.

[36] Y. Guivarc'h, L. Ji and J. C. Taylor, *Compactifications of Symmetric Spaces*, Birkhäuser, Boston, 1998.

[37] L. L. Helms, *Introduction to Potential Theory*, Wiley-Interscience, New York, 1969.

[38] R. A. Hunt and R. L. Wheeden, *On the boundary values of harmonic functions*, Trans. Amer. Math. Soc. **132** (1968), 307–322.

[39] R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147** (1970), 505–527.

[40] D. Ioffe and R. Pinsky, *Positive harmonic functions vanishing on the boundary for the Laplacian in unbounded horn-shaped domains*, Trans. Amer. Math. Soc. **342** (1994), 773–791.

[41] K. Ishige, *On the behavior of the solutions of degenerate parabolic equations*, Nagoya Math. J. **155** (1999), 1–26.

[42] K. Ishige, *An intrinsic metric approach to uniqueness of the positive Dirichlet problem for parabolic equations in cylinders*, J. Diff. Eq. **158** (1999), 251–290.

[43] K. Ishige and M. Murata, *An intrinsic metric approach to uniqueness of the positive Cauchy problem for parabolic equations*, Math. Z. **227** (1998), 313–335.

[44] K. Ishige and M. Murata, *Uniqueness of nonnegative solutions of the Cauchy problem for parabolic equations on manifolds or domains*, Ann. Scuola Norm. Sup. Pisa, **30** (2001), 171–223.

[45] M. Kawamura and M. Nakai, *A test of Picard principle for rotation free densities. II*, J. Math. Soc. Japan **28** (1976), 323–342.

[46] R. Z. Khas'minskii, *Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equations*, Theory of Prob. and Appl. **5** (1960), 179–196.

[47] A. Koranyi and J. C. Taylor, *Minimal solutions of the heat equations and uniqueness of the positive Cauchy problem on homogeneous spaces*, Proc. Amer. Math. Soc. **94** (1985), 273–278.

[48] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[49] R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172.

[50] V. G. Maz'ya, *On the relationship between Martin and Euclidean topologies*, Soviet Math. Dokl. **18** (1977), 283–286.

[51] R. Mazzeo and A. Vasy, *Resolvents and Martin boundaries of product spaces*, Geom. Funct. Anal. **12** (2002), 1018–1079.

[52] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134.

[53] M. Murata, *Structure of positive solutions to* $(-\triangle + V)u = 0$ *in* $\mathbb{R}^n$, Duke Math. J. **53** (1986), 869–943.

[54] M. Murata, *Martin compactification and asymptotics of Green functions for Schrödinger operators with anisotropic potentials*, Math. Ann. **288** (1990), 211–230.

[55] M. Murata, *On construction of Martin boundaries for second order elliptic equations*, Publ. RIMS, Kyoto Univ. **26** (1990), 585–627.

[56] M. Murata, *Positive harmonic functions on rotationary symmetric Riemannian manifolds*, in Potential Theory (M. Kishi ed.), Walter de Gruyter, Berlin, 1992, 251–259.

[57] M. Murata, *Uniform restricted parabolic Harnack inequality, separation principle, and ultracontractivity for parabolic equations*, in Functional Analysis and Related Topics, 1991, Lecture Notes in Math. Vol. **1540**, Springer, Berlin, 1993, 277–288.

[58] M. Murata, *Sufficient condition for nonuniqueness of the positive Cauchy problem for parabolic equations*, Adv. Stud. Pure Math **23** (1994), 275–282.

[59] M. Murata, *Non-uniqueness of the positive Cauchy problem for parabolic equations*, J. Diff. Eq. **123** (1995), 343–387.

[60] M. Murata, *Uniqueness and non-uniqueness of the positive Cauchy problem for the heat equation on Riemannian manifolds*, Proc. Amer. Math. Soc. **123** (1995), 1923–1932.

[61] M. Murata, *Non-uniqueness of the positive Dirichlet problem for parabolic equations in cylinders*, J. Func. Anal. **135** (1996), 456–487.

[62] M. Murata, *Semismall perturbations in the Martin theory for elliptic equations*, Israel J. Math. **102** (1997), 29–60.

[63] M. Murata, *Structure of positive solutions to Schrödinger equations*, Sugaku Expositions **11** (1998), 101–121.

[64] M. Murata, *Martin boundaries of elliptic skew products, semismall perturbations, and fundamental solutions of parabolic equations*, J. Funct. Anal. **194** (2002), 53–141.

[65] M. Murata, *Heat escape*, Math. Ann. (to appear).

[66] M. Murata and T. Tsuchida, *Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients*, J. Diff. Eq. (to appear).

[67] M. Nakai and T. Tada, *Monotoneity and homogeneity of Picard dimensions for signed radial densities*, Hokkaido Math. J. **26** (1997) 253–296.

[68] M. A. Perel'muter and YU. A. Semenov, *Elliptic operators preserving probability*, Theory of Probability and Its Applications, **32** (1987), 718–721.

[69] Y. Pinchover, *On existence of any λ_0-invariant positive harmonic function, a counter example to Stroock's conjecture*, Comm. P. D. E. **20** (1995), 1831–1846.

[70] Y. Pinchover, *On uniqueness and nonuniqueness of positive Cauchy problem for parabolic equations with unbounded coefficients*, Math. Z. **232** (1996), 569–586.

[71] Y. Pinchover, *Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations*, Math. Ann. **314** (1999), 555–590.

[72] R. G. Pinsky, *A new approach to the Martin boundary via diffusions conditioned to hit a compact set*, Ann. Probability **21** (1993), 453–481.

[73] R. G. Pinsky, *Positive Harmonic Functions and Diffusion*, Cambridge Univ. Press, Cambridge, 1995.

[74] C. Pommerenke *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[75] L. Saloff-Coste, *Uniform elliptic operators on Riemmanian manifolds*, J. Diff. Geom. **36** (1992), 417–450.

[76] K.-Th. Sturm, *Analysis on local Dirichlet spaces -I. Recurrence, conservativeness and L^p-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173–196.

[77] K.-Th. Sturm, *Analysis on local Dirichlet spaces -III. Poincaré and parabolic Harnack inequality*, J. Math. Pures Appl. **IX**, Ser. **75** (1996), 273–297.

[78] S. Täcklind, *Sur les classes quasianalytiques des solutions des équations aux dérivées partielles du type parabolique*, Nova Acta Regiae Soc. Scien. Upsaliensis, Ser. IX **10** (1936), 1–57 .

[79] J. C. Taylor, *The Martin compactification associated with a second order strictly elliptic partial differentail operator on a manifold M*, in Topics in Probability and Lie Groups: Boundary Theory, CRM Proc. Lecture Notes, Vol. **28**, Amer. Math. Soc., Providence, RI. 2001, 153–202.

[80] D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. **55** (1944), 85–95 .

[81] J.-M. G. Wu, *Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains*, Ann. Inst. Fourier (Grenoble) **28** (1978), 147–167.