ON THE PLANCHEREL FORMULA FOR THE (DISCRETE) LAPLACIAN IN A WEYL CHAMBER WITH REPULSIVE BOUNDARY CONDITIONS AT THE WALLS

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Abstract. It is known from early work of Gaudin that the quantum system of n Bosonic particles on the line with a pairwise delta-potential interaction admits a natural generalization in terms of the root systems of simple Lie algebras. The corresponding quantum eigenvalue problem amounts to that of a Laplacian in a convex cone, the Weyl chamber, with linear homogeneous boundary conditions at the walls. In this paper we study a discretization of this eigenvalue problem, which is characterized by a discrete Laplacian on the dominant cone of the weight lattice endowed with suitable linear homogeneous conditions at the boundary. The eigenfunctions of this discrete model are computed by the Bethe Ansatz method. The orthogonality and completeness of the resulting Bethe wave functions (i.e. the Plancherel formula) turn out to follow from an elementary computation performed by Macdonald in his study of the zonal spherical functions on p-adic simple Lie groups. Through a continuum limit, the Plancherel formula for the ordinary Laplacian in the Weyl chamber with linear homogeneous boundary conditions is recovered. Throughout this paper we restrict ourselves to the case of repulsive boundary conditions.

Contents

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1. Introduction

It is well known that the quantum eigenvalue problem for n Bosons on the line that interact pairwise through a delta-potential can be solved by the Bethe Ansatz method [LL, M, BZ, Y1, Y2, G1, G2, O]. From a physical point of view, this many-body system describes the n-particle sector of the quantized nonlinear Schrödinger field theory (i.e. the quantum NLS). For an overview of the literature concerning both the mathematical and physical aspects of this model we refer to the collections [ML, G4, KBI].

The Hamiltonian of the n-particle system in question is given by the Schrödinger operator

$$H = -\Delta + g \sum_{1\leq j\neq k\leq n} \boldsymbol{\delta}(x_j - x_k), \tag{1.1}$$

where $x_1, \ldots, x_n$ denote the position variables, $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$, $\boldsymbol{\delta}(\cdot)$ refers to the delta distribution, and g represents a real coupling parameter determining the strength of the interaction. For $g > 0$ the pairwise interaction is *repulsive* and for $g < 0$ it is *attractive*. Mathematically, the eigenvalue problem for H (1.1) amounts to that of a free Laplacian $-\Delta$ with jump conditions on the normal derivative of the (continuous) wave function at the hyperplanes $x_j = x_k$, $1 \leq j < k \leq n$. (Specifically, the jump of the normal derivative of the wave functions at the hyperplanes should be $2g$ times the value of the wave function.) By exploiting the permutation- and translational symmetry, the eigenvalue problem at issue can be reduced to the form

$$-\Delta\psi = \|\boldsymbol{\xi}\|^2\psi \qquad (\text{where } \|\boldsymbol{\xi}\|^2 := \xi_1^2 + \cdots + \xi_n^2), \tag{1.2a}$$

for a domain of wave functions $\psi = \psi(\mathbf{x};\boldsymbol{\xi}) := \psi(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ supported in the closure of the fundamental convex cone

$$\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n,\ x_1 + \cdots + x_n = 0\}, \tag{1.2b}$$

and subject to linear homogeneous boundary conditions at the walls given by

$$(\partial_{x_j} - \partial_{x_{j+1}} - g)\psi\big|_{x_j = x_{j+1}} = 0, \quad j = 1, \ldots, n-1. \tag{1.2c}$$

(Here the variable $\boldsymbol{\xi} \in \mathbb{R}^N$ plays the role of the spectral parameter.)

The idea of the Bethe Ansatz method is now to construct the solution of this eigenvalue problem as a permutation-invariant linear combination of plane waves, with suitable coefficients such that the boundary conditions at the walls are satisfied. An important problem is the question of the orthogonality and completeness of the Bethe eigenfunctions in a Hilbert space setting. This problem is commonly referred to in the mathematically oriented literature as the *Plancherel Problem*. For the repulsive regime $g > 0$, the spectrum of the Hamiltonian is absolutely continuous; the corresponding Plancherel formula was demonstrated formally by Gaudin [G1, G2, G4]. For the attractive regime $g < 0$, one has both discrete and continuous spectrum; in this case the Plancherel problem was solved by Oxford [O] by building on work of Yang [Y1] and exploiting ideas from an analysis of a related Plancherel problem for the infinite volume XXX isotropic Heisenberg spin chain by Babbitt and Thomas [T, BT].

Thanks to a fundamental observation by Gaudin, it is known that the n-Boson system with delta-potential interaction admits natural generalization in terms of the root systems of simple Lie algebras [G3, G4]. From this perspective, the original n-particle model with pairwise interaction corresponds to a root system of type A_{n-1} (i.e. the Lie algebra $\mathrm{sl}(n;\mathbb{C})$). Other classical root systems appear when restricting the particles to a half-line or by distributing them symmetrically around the origin. It turns out that the eigenfunctions of these generalized delta-potential models related to root systems can again be constructed with the Bethe Ansatz method [G3, GS, G, G4]. The corresponding Plancherel formula was proven recently by Heckman and Opdam, who considered both the repulsive and the attractive regime [HO].

The aim of the present paper is to study a discrete version of the spectral problem for the Laplace operator with a delta-potential on root systems. Throughout the paper, we restrict ourselves to the repulsive case. More specifically, we consider a discrete Laplacian acting on lattice functions with support in the dominant cone of the weight lattice of the root system, subject to suitable repulsive boundary conditions. We construct the eigenfunctions of this discrete Laplacian through the Bethe Ansatz method. The resulting eigenfunctions turn out to correspond to (the parameter deformations of) the zonal spherical functions on p-adic Lie groups studied by Macdonald [M1, M3]. In particular, the Plancherel problem reduces in this discrete setting to an elementary calculation already carried out by Macdonald to prove the orthogonality of the spherical functions in question with respect to the Plancherel measure. Finally, we perform a continuum limit as the lattice spacing tends to zero and recover the repulsive case of the Plancherel formula for the Laplace operator with a delta-potential on root systems from [HO]. In this limit the discrete Laplacian converges in the strong resolvent topology to the Laplacian of the continuous model. To give rigorous meaning to our continuum limit in a Hilbert space sense, we employ techniques developed by Ruijsenaars in his study of the continuum limit of the infinite isotropic Heisenberg spin chain [R].

The material is organized as follows. Section 2 serves to prepare some basic definitions and notations from the theory of root systems that are needed to formulate the results. Section 3 recalls the eigenfunctions and exhibits the Plancherel formula for the Laplacian in the Weyl chamber with repulsive boundary conditions at the walls. Section 4 is devoted to the discretization of this Laplacian. Specifically, we introduce our discrete Laplacian on the dominant cone of the weight lattice endowed with linear homogeneous conditions at the boundary. The eigenfunctions of the discrete Laplacian are constructed with the Bethe Ansatz method and the Plancherel problem for the repulsive case is resolved by connecting to Macdonald's theory of zonal spherical functions on p-adic Lie groups. In Section 5 it is shown how—by passing to the continuum limit—the eigenfunctions and the Plancherel formula for the (continuous) Laplacian in Section 3 can be recovered from the eigenfunctions and the Plancherel formula for the discrete Laplacian in Section 4. A few technical points concerning the proof of the Plancherel inversion formula in the continuous situation have been isolated in Appendix A. Furthermore, some crucial results due to Macdonald—which constitute the backbone of the proof for the Plancherel formula in the discrete situation—have been outlined in Appendix B at the end of the paper.

2. Preliminaries on Root Systems

Throughout the paper we will make extensive use of the language of root systems. For a thorough treatment of the concepts and theory surrounding root systems the reader is referred to the standard texts [B, H1, H2, K]. Here we restrict ourselves to recalling just the bare minimum of definitions, notations, and properties needed for our purposes. This section is probably best skipped at first reading and referred back to as needed.

2.1. **Roots.** Let $\mathbf{E}$ be a real finite-dimensional Euclidean vector space with the inner product denoted by $\langle\cdot,\cdot\rangle$. For a nonzero vector $\alpha\in\mathbf{E}$, the action of the orthogonal reflection $r_\alpha:\mathbf{E}\to\mathbf{E}$ in the hyperplane through the origin perpendicular to it is given explicitly by

$$r_\alpha(\mathbf{x})=\mathbf{x}-\langle\mathbf{x},\alpha^\vee\rangle\alpha \qquad (\mathbf{x}\in\mathbf{E}), \tag{2.1}$$

where $\alpha^\vee:=2\alpha/\langle\alpha,\alpha\rangle$. By definition, a (crystallographic) *root system* is a nonempty subset $\boldsymbol{R}\subset\mathbf{E}\setminus\{\mathbf{0}\}$ satisfying the properties

$$\begin{array}{llll}\text{(i)} & r_\alpha(\boldsymbol{R})=\boldsymbol{R}, & \forall\alpha\in\boldsymbol{R} & \text{(reflection invariance)},\\ \text{(ii)} & \langle\alpha,\beta^\vee\rangle\in\mathbb{Z}, & \forall\alpha,\beta\in\boldsymbol{R} & \text{(integrality)}.\end{array} \tag{2.2}$$

A vector in $\boldsymbol{R}$ is referred to as a *root*. The roots generate an abelian group $\mathcal{Q}:=\mathrm{Span}_{\mathbb{Z}}(\boldsymbol{R})$ called the *root lattice* of $\boldsymbol{R}$. The dimension of $\mathcal{Q}$ is called the *rank* of the root system. Here we will always assume that the ambient Euclidean space $\mathbf{E}$ is chosen minimal in the sense that $\dim(\mathbf{E})$ is equal to the rank of the root system (i.e. $\mathrm{Span}_{\mathbb{R}}(\boldsymbol{R})=\mathbf{E}$).

If one fixes a choice of normal vector generically, in the sense that the hyperplane through the origin perpendicular to it does not intersect $\boldsymbol{R}$, then the hyperplane in question divides the root system in two subsets of equal size called the *positive-* and *negative roots*:

$$\boldsymbol{R}=\boldsymbol{R}^+\cup\boldsymbol{R}^- \qquad \text{with} \qquad \boldsymbol{R}^-=-\boldsymbol{R}^+. \tag{2.3}$$

The positive roots determine a nonnegative semigroup $\mathcal{Q}^+:=\mathrm{Span}_{\mathbb{N}}(\boldsymbol{R}^+)$ of the root lattice. A positive root α is called *simple* if $\alpha-\beta\not\in\boldsymbol{R}^+$ for any $\beta\in\boldsymbol{R}^+$. Let us denote the simple roots by $\alpha_1,\ldots,\alpha_N$. These simple roots form a basis for $\mathcal{Q}$ and $\mathcal{Q}^+$, i.e.

$$\mathcal{Q}=\mathbb{Z}\,\alpha_1\oplus\cdots\oplus\mathbb{Z}\,\alpha_N \quad \text{and} \quad \mathcal{Q}^+=\mathbb{N}\,\alpha_1\oplus\cdots\oplus\mathbb{N}\,\alpha_N. \tag{2.4}$$

(Hence, the number of simple roots N is equal to the rank of the root system). It means that starting from the origin we can reach any vector in the root lattice $\mathcal{Q}$ by successive addition or subtraction of simple roots. One defines the *height* of a vector $\kappa\in\mathcal{Q}$ as

$$\mathrm{ht}(\kappa):=\langle\kappa,\rho^\vee\rangle, \quad \text{with} \quad \rho^\vee:=\textstyle\sum_{\alpha\in\boldsymbol{R}^+}\alpha^\vee/2. \tag{2.5}$$

In the basis of simple roots the height reads $\mathrm{ht}(\kappa)=\mathrm{ht}(k_1\alpha_1+\cdots+k_N\alpha_N)=k_1+\cdots+k_N$. In particular, for $\kappa\in\mathcal{Q}^+$ the height function $\mathrm{ht}(\cdot)$ counts the number of simple roots in κ. The (unique) positive root α_0 such that $\mathrm{ht}(\alpha)\leq\mathrm{ht}(\alpha_0)$ for all $\alpha\in\boldsymbol{R}^+$ is called the *maximal root* of $\boldsymbol{R}$.

A root system is said to be *irreducible* if it cannot be decomposed as a direct orthogonal sum of two (smaller) root systems. Furthermore, a root system is called *reduced* if any half-line starting from the origin contains at most *one* single root

$\alpha \in \boldsymbol{R}$. (This amounts to the condition that for any $\alpha \in \boldsymbol{R}$ the multiple $k\alpha$ is a root if and only if $k = 1$ or $k = -1$.)

2.2. **The Weyl group.** The group $W \subset \mathrm{O}(\mathbf{E};\mathbb{R})$ generated by all reflections r_α, $\alpha \in \boldsymbol{R}$ is called the *Weyl group* of $\boldsymbol{R}$. The first defining property (i) of a root system states that $\boldsymbol{R}$ is invariant with respect to the action of the Weyl group; the second defining property (ii) guarantees moreover that the root lattice $\mathcal{Q}$ is also invariant with respect to this action. In the case of an irreducible reduced root system, the action of the Weyl group splits up $\boldsymbol{R}$ in at most two orbits. More specifically, there are two possible situations: (i) either all roots have the same length, in which case the action of W on $\boldsymbol{R}$ is transitive, or (ii) the roots come in two different sizes, in which case $\boldsymbol{R}$ splits up in an orbit $\boldsymbol{R}_s$ consisting of the *short roots* and an orbit $\boldsymbol{R}_l$ consisting of the *long roots*.

The reflections in the simple roots $r_j := r_{\alpha_j}$, $j = 1, \ldots, N$ are referred to as the *simple reflections*. They form a minimal set of generators for the Weyl group W. In other words, any Weyl group element $w \in W$ can be decomposed (non-uniquely) in terms of simple reflections

$$w = r_{j_1} r_{j_2} \cdots r_{j_\ell} \tag{2.6}$$

(with the indices $j_1, \ldots, j_\ell \in \{1, \ldots, N\}$ not necessarily distinct). The number ℓ is referred to as the *length* of the decomposition. If, for given $w \in W$, the length ℓ is minimal then the corresponding decomposition is called *reduced*. An important property of Weyl groups (used frequently in our analysis below) is that a group element $w \in W$ admits a reduced decomposition ending in the simple reflection r_j (i.e. with r_{j_ℓ} in (2.6) equal to r_j) if and only if $w(\alpha_j) \in \boldsymbol{R}^-$.

Let us—for $\boldsymbol{R}$ both irreducible and reduced—define the following (length) functions on W

$$\ell(w) \;:=\; |\{\alpha \in \boldsymbol{R}^+ \mid w(\alpha) \in \boldsymbol{R}^-\}|, \tag{2.7a}$$
$$\ell_s(w) \;:=\; |\{\alpha \in \boldsymbol{R}_s^+ \mid w(\alpha) \in \boldsymbol{R}_s^-\}|, \tag{2.7b}$$
$$\ell_l(w) \;:=\; |\{\alpha \in \boldsymbol{R}_l^+ \mid w(\alpha) \in \boldsymbol{R}_l^-\}|, \tag{2.7c}$$

where $\boldsymbol{R}_s^\pm := \boldsymbol{R}_s \cap \boldsymbol{R}^\pm$, $\boldsymbol{R}_l^\pm := \boldsymbol{R}_l \cap \boldsymbol{R}^\pm$, and $|\cdot|$ refers to the cardinality of the set in question. Clearly $\ell(w) = \ell_s(w) + \ell_l(w)$. (If all roots have the same length, then by convention $\boldsymbol{R}_s := \boldsymbol{R}$ and $\boldsymbol{R}_l := \emptyset$, so $\ell_s(w) = \ell(w)$ and $\ell_l(w) = 0$.) It turns out that the numbers $\ell(w)$, $\ell_s(w)$ and $\ell_l(w)$ count, respectively, the number of simple reflections, the number of short simple reflections and the number of long simple reflections that appear in a reduced decomposition (2.6) of w into simple reflections.

For later use, it will be convenient to split up the height function $\mathrm{ht}(\cdot)$ (2.5) as a sum of partial height functions as well

$$\mathrm{ht}_s(\kappa) := \langle \kappa, \rho_s^\vee \rangle, \quad \text{with} \quad \rho_s^\vee := \textstyle\sum_{\alpha \in R_s^+} \alpha^\vee/2, \tag{2.8a}$$
$$\mathrm{ht}_l(\kappa) := \langle \kappa, \rho_l^\vee \rangle, \quad \text{with} \quad \rho_l^\vee := \textstyle\sum_{\alpha \in R_l^+} \alpha^\vee/2. \tag{2.8b}$$

For $\kappa = k_1\alpha_1 + \cdots + k_N\alpha_N$, this gives

$$\mathrm{ht}_s(\kappa) = \sum_{\substack{1 \le j \le N \\ \alpha_j \text{ short}}} k_j, \quad \mathrm{ht}_l(\kappa) = \sum_{\substack{1 \le j \le N, \\ \alpha_j \text{ long}}} k_j, \tag{2.9}$$

which for $\kappa \in \mathcal{Q}^+$ amounts to a count of, respectively, the number of short and long simple roots in κ.

2.3. **Weights.** The *weight lattice* $\mathcal{P}$ and its nonnegative *dominant cone* $\mathcal{P}^+$ are the duals of the root lattice $\mathcal{Q}$ and its nonnegative semigroup $\mathcal{Q}^+$, i.e.

$$\mathcal{P} := \{\lambda \in \mathbf{E} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \forall \alpha \in \boldsymbol{R}\}, \tag{2.10a}$$

$$\mathcal{P}^+ := \{\lambda \in \mathbf{E} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{N},\ \forall \alpha \in \boldsymbol{R}^+\}. \tag{2.10b}$$

One has that $\mathcal{Q} \subset \mathcal{P}$ but $\mathcal{Q}^+ \not\subset \mathcal{P}^+$ (unless $N = 1$). A vector in $\mathcal{P}$ is called a *weight*. Furthermore, a weight in $\mathcal{P}^+$ is called a *dominant weight*. The special dominant weights $\omega_1, \ldots, \omega_N$ that are related to the simple roots via the duality $\langle \omega_j, \alpha_k^\vee \rangle = \delta_{j,k}$ are referred to as the *fundamental weights*. These fundamental weights form a basis for $\mathcal{P}$ and $\mathcal{P}^+$, i.e.

$$\mathcal{P} = \mathbb{Z}\,\omega_1 \oplus \cdots \oplus \mathbb{Z}\,\omega_N \quad \text{and} \quad \mathcal{P}^+ = \mathbb{N}\,\omega_1 \oplus \cdots \oplus \mathbb{N}\,\omega_N. \tag{2.11}$$

The following definition

$$\forall \lambda, \mu \in \mathcal{P}: \qquad \lambda \succeq \mu \Longleftrightarrow \lambda - \mu \in \mathcal{Q}^+ \tag{2.12}$$

endows the weight lattice with a natural partial order. This partial order is usually referred to as the *dominance order*.

The cone of dominant weights $\mathcal{P}^+$ constitutes a fundamental domain for $\mathcal{P}$ with respect to the action of the Weyl group, in the sense that for any $\mu \in \mathcal{P}$ the Weyl orbit

$$W(\mu) := \{w(\mu) \mid w \in W\} \tag{2.13}$$

intersects the dominant cone $\mathcal{P}^+$ precisely *once*. For $\mu \in \mathcal{P}$, one defines $w_\mu \in W$ as the unique shortest Weyl group element such that

$$w_\mu(\mu) \in \mathcal{P}^+. \tag{2.14}$$

The group element w_μ admits a reduced decomposition ending in r_j if and only if $\langle \mu, \alpha_j^\vee \rangle < 0$ (i.e., if and only if the hyperplane perpendicular to α_j separates μ and $w_\mu(\mu)$). It is instructive to reformulate this criterion in terms of the partial order $\succeq$ in Eq. (2.12): the group element w_μ admits a reduced decomposition ending in r_j if and only if $r_j(\mu) \succeq \mu$. In particular, it means that any dominant weight λ is maximal in its Weyl orbit $W(\lambda)$, i.e.

$$\forall \lambda \in \mathcal{P}^+: \qquad \lambda \succeq w(\lambda), \quad \forall w \in W. \tag{2.15}$$

The stabilizer of a weight $\lambda \in \mathcal{P}$ is defined as

$$W_\lambda := \{w \in W \mid w(\lambda) = \lambda\}. \tag{2.16}$$

The stabilizer W_λ is a subgroup of the Weyl group W that is generated by the simple reflections r_j such that $r_j(\lambda) = \lambda$.

3. Laplacian on the Weyl Chamber

In this section we review the solution of the spectral problem for the Laplacian in a Weyl chamber with repulsive boundary conditions at the walls and formulate the associated Plancherel theorem.

Note. From now on it will always be assumed that our root system $\boldsymbol{R}$ is both *irreducible* and *reduced*. A helpful list of all irreducible root systems and their concrete properties can be found in Bourbaki's tables [B].

3.1. **Eigenvalue problem.** The *Weyl chamber* is the open convex cone

$$\mathbf{C} = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \alpha \rangle > 0,\ \forall \alpha \in \boldsymbol{R}^+\}. \tag{3.1}$$

It is bounded by the walls

$$\mathbf{C}_j = \{\mathbf{x} \in \mathbf{E} \mid \langle \mathbf{x}, \alpha_j \rangle = 0 \text{ and } \langle \mathbf{x}, \alpha \rangle > 0,\ \forall \alpha \in \boldsymbol{R}^+ \setminus \{\alpha_j\}\ \} \tag{3.2}$$

perpendicular to the simple roots α_j, $j = 1, \ldots, N$. Let g_s, g_l be two generic (possibly complex) parameters and let us set

$$g_\alpha := \begin{cases} g_s & \text{if } \alpha \in \boldsymbol{R}_s, \\ g_l & \text{if } \alpha \in \boldsymbol{R}_l. \end{cases} \tag{3.3}$$

The generalization of the eigenvalue problem in Eqs. (1.2a)–(1.2c) to the case of an arbitrary root system $\boldsymbol{R}$ is given by

$$-\nabla^2_{\mathbf{x}} \psi(\mathbf{x}; \boldsymbol{\xi}) = \|\boldsymbol{\xi}\|^2 \psi(\mathbf{x}; \boldsymbol{\xi}), \qquad \mathbf{x}, \boldsymbol{\xi} \in \mathbf{C}, \tag{3.4a}$$

with linear homogeneous boundary conditions at the walls of the form

$$\Big(\langle \nabla_{\mathbf{x}} \psi, \alpha_j \rangle - g_{\alpha_j} \psi \Big)\Big|_{\mathbf{x} \in \mathbf{C}_j} = 0, \qquad j = 1, \ldots, N. \tag{3.4b}$$

Here $\nabla^2_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}$ denote the Laplacian and gradient on $\mathbf{E}$, respectively, and $\|\boldsymbol{\xi}\| := \sqrt{\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle}$.

Theorem 3.1 (Eigenfunction)**.** *The wave function*

$$\Psi_0(\mathbf{x}; \boldsymbol{\xi}) = \sum_{w \in W} \Big(\prod_{\alpha \in \boldsymbol{R}^+} \frac{\langle \alpha, \boldsymbol{\xi}_w \rangle - i g_\alpha}{\langle \alpha, \boldsymbol{\xi}_w \rangle} \Big) e^{i \langle \mathbf{x}, \boldsymbol{\xi}_w \rangle}, \tag{3.5}$$

with $\boldsymbol{\xi}_w := w(\boldsymbol{\xi})$, solves the eigenvalue problem in Eqs. (3.4a)*,* (3.4b)*.*

Theorem 3.1 is due to Gaudin, who constructed the wave function in question by means of the Bethe Ansatz Method [G3, G4]. It is clear that the linear combination of plane waves $\Psi_0(\mathbf{x}; \boldsymbol{\xi})$ (3.5) solves the eigenvalue equation in Eq. (3.4a), since $-\nabla^2_{\mathbf{x}} e^{i\langle \mathbf{x}, \boldsymbol{\xi}_w \rangle} = \langle \boldsymbol{\xi}_w, \boldsymbol{\xi}_w \rangle e^{i\langle \mathbf{x}, \boldsymbol{\xi}_w \rangle} = \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle e^{i\langle \mathbf{x}, \boldsymbol{\xi}_w \rangle}$. To infer that the boundary conditions in Eq. (3.4b) are also satisfied it suffices to perform a small computation based on the action of the directional derivative on plane waves: $\langle \nabla_{\mathbf{x}} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle}, \alpha_j \rangle = i \langle \alpha_j, \boldsymbol{\xi} \rangle e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle}$. Specifically, the following sequence of elementary manipulations reveals that for $\mathbf{x}\in\mathbf{C}_j$

$$
\begin{aligned}
&\langle \nabla_{\mathbf{x}}\Psi_0,\alpha_j\rangle\\
&= \sum_{w\in W}\Bigl(\prod_{\alpha\in\boldsymbol{R}^+}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr) i\langle\alpha_j,\boldsymbol{\xi}_w\rangle e^{i\langle\mathbf{x},\boldsymbol{\xi}_w\rangle}\\
&= \sum_{w\in W}(g_{\alpha_j}+i\langle\alpha_j,\boldsymbol{\xi}_w\rangle)\Bigl(\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \alpha\neq\alpha_j}}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr) e^{i\langle\mathbf{x},\boldsymbol{\xi}_w\rangle}\\
&\overset{(i)}{=} g_{\alpha_j}\sum_{w\in W}\Bigl(\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \alpha\neq\alpha_j}}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr) e^{i\langle\mathbf{x},\boldsymbol{\xi}_w\rangle}\\
&\overset{(ii)}{=} g_{\alpha_j}\sum_{w\in W}\Bigl(1-\frac{ig_{\alpha_j}}{\langle\alpha_j,\boldsymbol{\xi}_w\rangle}\Bigr)\Bigl(\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \alpha\neq\alpha_j}}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr) e^{i\langle\mathbf{x},\boldsymbol{\xi}_w\rangle}\\
&= g_{\alpha_j}\sum_{w\in W}\Bigl(\prod_{\alpha\in\boldsymbol{R}^+}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr) e^{i\langle\mathbf{x},\boldsymbol{\xi}_w\rangle}\\
&= g_{\alpha_j}\Psi_0 .
\end{aligned}
$$

In Steps (i) and (ii) one exploits the fact the expressions under consideration are symmetrized with respect to the action of the Weyl group W. Notice in this connection that the relevant terms on the third and fifth line are built of factors that are (skew-)symmetric with respect to the simple reflection r_j. Indeed, we have the skew-symmetry $\langle\alpha_j, r_j(\boldsymbol{\xi}_w)\rangle = -\langle\alpha_j,\boldsymbol{\xi}_w\rangle$ (as $r_j(\alpha_j)=-\alpha_j$) and the symmetries

$$
r_j\Bigl(\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \alpha\neq\alpha_j}}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}\Bigr)=\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \alpha\neq\alpha_j}}\frac{\langle\alpha,\boldsymbol{\xi}_w\rangle-ig_\alpha}{\langle\alpha,\boldsymbol{\xi}_w\rangle}
$$

(as the simple reflection r_j permutes the positive roots other than α_j) and $\langle\mathbf{x}, r_j(\boldsymbol{\xi}_w)\rangle = \langle\mathbf{x},\boldsymbol{\xi}_w\rangle$ (as $\mathbf{x}\in\mathbf{C}_j$ so $r_j(\mathbf{x})=\mathbf{x}$). When symmetrizing with respect to the action of the Weyl group the skew-symmetric parts involving $\langle\alpha_j,\boldsymbol{\xi}_w\rangle$ thus drop out.

3.2. **Continuous Plancherel formula.**
Note. From now on we will restrict attention to the *repulsive* case of *nonnegative* parameters g_s, g_l (and hence g_α).

Let $\mathcal{H}_0 = L^2(\mathbf{C},\mathrm{d}\mathbf{x})$ be the Hilbert space of square-integrable functions on the Weyl chamber equipped with the standard inner product

$$
(f,g)_{\mathcal{H}_0}=\int_{\mathbf{C}} f(\mathbf{x})\overline{g(\mathbf{x})}\mathrm{d}\mathbf{x}\qquad (\forall f,g\in\mathcal{H}_0), \tag{3.6}
$$

and let $\hat{\mathcal{H}}_0 = L^2(\mathbf{C},(2\pi)^{-N}\hat{\Delta}_0(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi})$ be the Hilbert space of square-integrable functions on the Weyl chamber with respect to the positive weight function

$$
\hat{\Delta}_0(\boldsymbol{\xi})=\prod_{\alpha\in\boldsymbol{R}}\Bigl(1+\frac{ig_\alpha}{\langle\alpha,\boldsymbol{\xi}\rangle}\Bigr)^{-1}, \tag{3.7}
$$

equipped with the normalized inner product

$$
(\hat{f},\hat{g})_{\hat{\mathcal{H}}_0}=\frac{1}{(2\pi)^N}\int_{\mathbf{C}}\hat{f}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})}\,\hat{\Delta}_0(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}\qquad(\forall \hat{f},\hat{g}\in\hat{\mathcal{H}}_0). \tag{3.8}
$$

For $f \in \mathcal{H}_0$, we now define the eigenfunction transform $\hat{f}_0 = \mathcal{F}_0 f$ by means of the pairing

$$\hat{f}_0(\boldsymbol{\xi}) = (\mathcal{F}_0 f)(\boldsymbol{\xi}) := \int_{\mathbf{C}} f(\mathbf{x})\overline{\Psi_0(\mathbf{x};\boldsymbol{\xi})}\mathrm{d}\mathbf{x}, \tag{3.9a}$$

with $\Psi_0(\mathbf{x};\boldsymbol{\xi})$ given by Eq. (3.5). Reversely, for $\hat{f} \in \hat{\mathcal{H}}_0$ we define the adjoint eigenfunction transform $f_0 = \hat{\mathcal{F}}_0 \hat{f}$ as

$$f_0(\mathbf{x}) = (\hat{\mathcal{F}}_0 \hat{f})(\mathbf{x}) := \frac{1}{(2\pi)^N} \int_{\mathbf{C}} \hat{f}(\boldsymbol{\xi})\Psi_0(\mathbf{x};\boldsymbol{\xi})\, \hat{\Delta}_0(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{3.9b}$$

(So formally: $\hat{f}_0(\boldsymbol{\xi}) = (f, \Psi_0(\boldsymbol{\xi}))_{\mathcal{H}_0}$ and $f_0(\mathbf{x}) = (\hat{f}, \overline{\Psi_0(\mathbf{x})})_{\hat{\mathcal{H}}_0}$.) For $g_s, g_l = 0$, the transformations $\mathcal{F}_0$ and $\hat{\mathcal{F}}_0$ amount to the Fourier- and inverse-Fourier transformation on $\mathbf{C}$, respectively. The following theorem generalizes this state of affairs to the case of general nonnegative parameter values g_s, g_l.

Theorem 3.2 (Continuous Plancherel Formula)**.** *The eigenfunction transform* $\mathcal{F}_0$ (3.9a) *constitutes a unitary Hilbert space isomorphism between* $\mathcal{H}_0$ *and* $\hat{\mathcal{H}}_0$*, with the inverse transformation given by* $\hat{\mathcal{F}}_0$ (3.9b)*, i.e.*

$$\mathcal{H}_0 \overset{\mathcal{F}_0,\hat{\mathcal{F}}_0}{\longleftrightarrow} \hat{\mathcal{H}}_0, \qquad \hat{\mathcal{F}}_0\mathcal{F}_0 = \mathbf{I}_{\mathcal{H}_0}, \quad \mathcal{F}_0\hat{\mathcal{F}}_0 = \mathbf{I}_{\hat{\mathcal{H}}_0}. \tag{3.10}$$

Below we will show that Theorem 3.2 arises as a degeneration of a more elementary "polynomial" Plancherel formula for a discretization of the eigenvalue problem in Eqs. (3.4a), (3.4b).

The Plancherel formula of Theorem 3.2 is in agreement with the previous results due to Gaudin [G1, G2, G4] (for root systems of type A) and Heckman-Opdam [HO] (for arbitrary root systems), who showed that the transformation $\mathcal{F}_0$ (3.9a) constitutes an isometry of $\mathcal{H}_0$ into $\hat{\mathcal{H}}_0$ with left-inverse $\hat{\mathcal{F}}_0$ (3.9b). The idea of the proof for this inversion formula outlined by Heckman and Opdam [HO] is far from elementary: it hinges on a deep result due to Peetre concerning the abstract characterization of differential operators as support preserving linear operators acting on spaces of smooth functions [P1, P2]. For the reader's convenience, we have included a completely elementary proof of this inversion formula in Appendix A at the end of the paper.

It follows from Theorems 3.1 and 3.2 that the operator $-\nabla_{\mathbf{x}}^2$ in the Weyl chamber, with repulsive boundary conditions at the walls of the form in Eq. (3.4b), determines a unique self-adjoint extension in $\mathcal{H}_0$ given by the pullback of the multiplication operator $\hat{f}(\boldsymbol{\xi}) \mapsto \|\boldsymbol{\xi}\|^2 \hat{f}(\boldsymbol{\xi})$ in $\hat{\mathcal{H}}_0$ with respect to the eigenfunction transformation $\mathcal{F}_0$. From this observation the following corollary is immediate.

Corollary 3.3 (Spectrum and Self-adjointness)**.** *The operator* $-\nabla_{\mathbf{x}}^2$ *in the Weyl chamber* $\mathbf{C}$ (3.1)*, with repulsive boundary conditions of the form in Eq.* (3.4b) *at the walls, is essentially self-adjoint in* $\mathcal{H}_0$ *and (its closure) has a purely absolutely continuous spectrum filling the nonnegative real axis.*

4. Discrete Laplacian on the Cone of Dominant Weights

In this section we introduce a discrete Laplacian with repulsive boundary conditions on the cone of dominant weights and solve the associated spectral problem.

4.1. **Action of the discrete Laplacian and boundary conditions.** A nonzero dominant weight σ is called *minuscule* if $\langle \sigma, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \boldsymbol{R}^+$ and it is called *quasi-minuscule* if $\langle \sigma, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \boldsymbol{R}^+ \setminus \{\sigma\}$ (without it being minuscule). The number of minuscule weights is one less than the index $|\mathcal{P}/\mathcal{Q}|$, which means that there are no minuscule weights iff the root lattice $\mathcal{Q}$ fills the whole weight lattice $\mathcal{P}$. A quasi-minuscule weight, on the other hand, always exists and it is moreover unique. Specifically, it is given by the dominant weight σ such that $\sigma^\vee$ is the maximal root of the dual root system $\boldsymbol{R}^\vee := \{\alpha^\vee \mid \alpha \in \boldsymbol{R}\}$.

We will now associate to a (quasi-)minuscule weight σ a discrete Laplace operator L_σ acting on the space $C(\mathcal{P}^+)$ of complex functions over the cone of dominant weights $\mathcal{P}^+$ (2.10b).

Definition (Discrete Laplacian)**.** Let $\sigma \in \mathcal{P}^+$ be (quasi-)minuscule and let t_s, t_l denote two generic complex parameters. The action of the *discrete Laplace operator* $L_\sigma : C(\mathcal{P}^+) \longrightarrow C(\mathcal{P}^+)$ on an arbitrary lattice function $\psi \in C(\mathcal{P}^+)$ is defined as

$$L_\sigma \psi_\lambda = \sum_{\nu \in W(\sigma)} \psi_{\lambda+\nu} \qquad (\lambda \in \mathcal{P}^+), \tag{4.1a}$$

where for $\lambda+\nu \in \mathcal{P}\setminus\mathcal{P}^+$ the value of $\psi_{\lambda+\nu}$ is determined by the boundary condition

$$\begin{aligned} \psi_{\lambda+\nu} &= t_s^{\ell_s(w_{\lambda+\nu})} t_l^{\ell_l(w_{\lambda+\nu})} \psi_{w_{\lambda+\nu}(\lambda+\nu)} \\ &\quad + \theta_{\lambda+\nu} t_s^{-\mathrm{ht}_s(\nu)} t_l^{-\mathrm{ht}_l(\nu)} (1 - t_s^{-1}) \psi_\lambda, \end{aligned} \tag{4.1b}$$

with

$$\theta_\mu := \mathrm{ht}(w_\mu(\mu) - \mu) - \ell(w_\mu). \tag{4.1c}$$

To appreciate the structure of the above boundary conditions the following proposition is helpful. It exploits the decomposition of Weyl group elements in terms of simple reflections to disentangle the boundary conditions completely in terms of simple reflection relations. In this alternative characterization it turns out to be convenient to work with W invariant parameters t_α, $\alpha \in \boldsymbol{R}$ upon setting (cf. Eq. (3.3))

$$t_\alpha := \begin{cases} t_s & \text{if } \alpha \in \boldsymbol{R}_s, \\ t_l & \text{if } \alpha \in \boldsymbol{R}_l. \end{cases} \tag{4.2}$$

Proposition 4.1 (Boundary Reflection Relations)**.** *Let λ be a dominant weight and let $\sigma \in \mathcal{P}^+$ be (quasi-)minuscule. Then the boundary conditions in Eqs.* (4.1b), (4.1c) *are equivalent to the requirement that $\forall \nu \in W(\sigma)$ such that $\lambda + \nu \in \mathcal{P} \setminus \mathcal{P}^+$, and for all simple roots α_j such that $\langle \lambda+\nu, \alpha_j^\vee \rangle < 0$, the following reflection relations are satisfied*

$$\psi_{\lambda+\nu} = \begin{cases} t_{\alpha_j} \psi_{r_j(\lambda+\nu)} & \text{if } ht(r_j(\lambda+\nu) - \lambda - \nu) = 1, \quad (I) \\ t_{\alpha_j} \psi_{r_j(\lambda+\nu)} + (t_{\alpha_j} - 1)\psi_\lambda & \text{if } ht(r_j(\lambda+\nu) - \lambda - \nu) = 2, \quad (II) \end{cases}$$

or equivalently

$$\psi_{\lambda+\nu} = \begin{cases} t_{\alpha_j} \psi_{\lambda+\nu+\alpha_j} & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 0 \text{ and } \langle \nu, \alpha_j^\vee \rangle = -1, \quad (Ia') \\ t_{\alpha_j} \psi_\lambda & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 1 \text{ and } \langle \nu, \alpha_j^\vee \rangle = -2, \quad (Ib') \\ t_{\alpha_j} \psi_{\lambda+\alpha_j} + (t_{\alpha_j} - 1)\psi_\lambda & \text{if } \langle \lambda, \alpha_j^\vee \rangle = 0 \text{ and } \langle \nu, \alpha_j^\vee \rangle = -2. \quad (II') \end{cases}$$

Proof. Let us first check that the reflection relations in (I), (II) and in (Ia'), (Ib'), (II') are indeed equivalent. Since $\langle \lambda, \alpha_j^\vee \rangle \geq 0$ (as λ is dominant) and $\langle \nu, \alpha_j^\vee \rangle \geq -2$ with equality holding only when $\nu = -\alpha_j$ (as $\nu \in W(\sigma)$ with σ (quasi-)minuscule), the condition $\langle \lambda + \nu, \alpha_j^\vee \rangle < 0$ breaks up in the three cases (Ia'), (Ib') or (II'). It is readily verified that Cases (Ia') and (Ib') correspond to (I) and Case (II') corresponds to (II). Indeed, we have: $r_j(\lambda+\nu) = \lambda+\nu+\alpha_j$ in Case (Ia'), $\nu = -\alpha_j$ and $r_j(\lambda+\nu) = \lambda$ in Case (Ib'), and $\nu = -\alpha_j$ and $r_j(\lambda+\nu) = \lambda+\alpha_j$ in Case (II'). Hence, the corresponding reflection relations match in each case. (Notice also that for σ minuscule we are always in Case (Ia') (i.e. (I)); the Cases (Ib') or (II') (i.e. (II)) can only occur when σ is quasi-minuscule.)

Next we verify that the conditions in the proposition amount to the boundary conditions in Eqs. (4.1b), (4.1c). To this end we exploit the decomposition in simple reflections to perform induction on the length of $w_{\lambda+\nu}$, starting from the trivial induction base $\ell(w_{\lambda+\nu}) = 0$. (Notice in this connection that $\ell(w_{\lambda+\nu}) = 0$ implies that $\lambda+\nu$ is dominant, which agrees with the fact that formally the r.h.s. of Eq. (4.1b) reduces in this situation to $\psi_{\lambda+\nu}$.) For $\ell(w_{\lambda+\nu}) > 0$, there exists a simple reflection r_j such that $w_{\lambda+\nu} = w_{r_j(\lambda+\nu)} r_j$ with $\ell(w_{r_j(\lambda+\nu)}) = \ell(w_{\lambda+\nu}) - 1$. One furthermore has that $r_j(\lambda+\nu) \succ \lambda+\nu$, i.e. $\langle \lambda+\nu, \alpha_j^\vee \rangle < 0$. We thus fall in either one of the three cases (Ia'), (Ib') or (II'), which are to be analyzed separately below.

- -(Ia') In this situation $r_j(\lambda+\nu) = \lambda + r_j(\nu)$, which implies that $w_{\lambda+\nu} = w_{\lambda+r_j(\nu)} r_j$. By applying first the reflection relation in (Ia') and then the induction hypothesis we get

$$
\begin{aligned}
\psi_{\lambda+\nu} &= t_{\alpha_j} \psi_{\lambda+r_j(\nu)} \\
&= t_{\alpha_j} t_s^{\ell_s(w_{\lambda+r_j(\nu)})} t_l^{\ell_l(w_{\lambda+r_j(\nu)})} \psi_{w_{\lambda+r_j(\nu)}(\lambda+r_j(\nu))} \\
&\quad + t_{\alpha_j} t_s^{-\mathrm{ht}_s(r_j(\nu))} t_l^{-\mathrm{ht}_l(r_j(\nu))} \theta_{\lambda+r_j(\nu)} (1-t_s^{-1}) \psi_\lambda \\
&= t_s^{\ell_s(w_{\lambda+\nu})} t_l^{\ell_l(w_{\lambda+\nu})} \psi_{w_{\lambda+\nu}(\lambda+\nu)} \\
&\quad + t_s^{-\mathrm{ht}_s(\nu)} t_l^{-\mathrm{ht}_l(\nu)} \theta_{\lambda+\nu} (1-t_s^{-1}) \psi_\lambda,
\end{aligned}
$$

  which coincides with the expression on the r.h.s. of Eq. (4.1b).
- -(Ib') In this situation $r_j(\lambda+\nu) = \lambda$, which implies that $w_{\lambda+\nu} = r_j$ and $\nu = -\alpha_j \in \boldsymbol{R}_s$. We get from the reflection relation in (Ib')

$$\psi_{\lambda+\nu} = t_{\alpha_j} \psi_\lambda = t_s \psi_\lambda,$$

  which corresponds to Eq. (4.1b) with $\ell_s(w_{\lambda+\nu}) = \ell_s(r_j) = 1$, $\ell_l(w_{\lambda+\nu}) = \ell_l(r_j) = 0$, $\mathrm{ht}_s(\nu) = \mathrm{ht}_s(-\alpha_j) = -1$, $\mathrm{ht}_l(\nu) = \mathrm{ht}_l(-\alpha_j) = 0$, and $\theta_{\lambda+\nu} = \theta_{\lambda-\alpha_j} = \mathrm{ht}(\alpha_j) - \ell(r_j) = 0$.
- -(II') In this situation $r_j(\lambda+\nu) = \lambda + r_j(\nu) = \lambda + \alpha_j$, which implies that $w_{\lambda+\nu} = w_{\lambda+\alpha_j} r_j$ and $\nu = -\alpha_j \in \boldsymbol{R}_s$. By applying first the reflection relation in (II') and then the induction hypothesis we get

$$
\begin{aligned}
\psi_{\lambda+\nu} &= t_{\alpha_j} \psi_{\lambda+\alpha_j} + (t_{\alpha_j} - 1) \psi_\lambda \\
&= t_{\alpha_j} t_s^{\ell_s(w_{\lambda+\alpha_j})} t_l^{\ell_l(w_{\lambda+\alpha_j})} \psi_{w_{\lambda+\alpha_j}(\lambda+\alpha_j)} + (t_{\alpha_j} - 1) \psi_\lambda \\
&= t_s^{\ell_s(w_{\lambda+\nu})} t_l^{\ell_l(w_{\lambda+\nu})} \psi_{w_{\lambda+\nu}(\lambda+\nu)} \\
&\quad + t_s^{-\mathrm{ht}_s(\nu)} t_l^{-\mathrm{ht}_l(\nu)} \theta_{\lambda+\nu} (1-t_s^{-1}) \psi_\lambda,
\end{aligned}
$$

which coincides with the expression on the r.h.s. of Eq. (4.1b).

Since all three cases lead to the boundary condition in Eqs. (4.1b), (4.1c), this completes the induction step (and therewith the proof of the proposition). □

It is clear from the proof of the proposition that for σ minuscule $\theta_{\lambda+\nu} = \mathrm{ht}(w_{\lambda+\nu}(\lambda+\nu)-\lambda-\nu)-\ell(w_{\lambda+\nu}) = 0$ (as we are always in Case (Ia')). Hence, in this situation the boundary condition in Eq. (4.1b) reduces to

$$\psi_{\lambda+\nu} = t_s^{\ell_s(w_{\lambda+\nu})} t_l^{\ell_l(w_{\lambda+\nu})} \psi_{w_{\lambda+\nu}(\lambda+\nu)}. \tag{4.4}$$

The parameters t_s and t_l play the role of coupling parameters that determine the strength of the boundary conditions. There are two special extremal situations worth singling out. For $t_s, t_l \to 1$ the action of L_σ reduces to that of a free Laplacian $L_\sigma^{(n)} : C(\mathcal{P}^+) \to C(\mathcal{P}^+)$ with Neumann type boundary conditions:

$$L_\sigma^{(n)} \psi_\lambda = \sum_{\nu \in W(\sigma)} \psi_{w_{\lambda+\nu}(\lambda+\nu)}. \tag{4.5a}$$

For $t_s, t_l \to 0$ the action of L_σ reduces to that of a free Laplacian $L_\sigma^{(d)} : C(\mathcal{P}^+) \to C(\mathcal{P}^+)$ with Dirichlet type boundary conditions:

$$L_\sigma^{(d)} \psi_\lambda = -N_\sigma(\lambda)\psi_\lambda + \sum_{\substack{\nu \in W(\sigma) \\ \lambda+\nu \in \mathcal{P}^+}} \psi_{\lambda+\nu}, \tag{4.5b}$$

where $N_\sigma(\lambda) = 0$ if σ is minuscule and $N_\sigma(\lambda)$ is equal to the number of short simple roots perpendicular to λ if σ is quasi-minuscule.

Let $L_\sigma^{(0)} : C(\mathcal{P}) \longrightarrow C(\mathcal{P})$ denote the free Laplacian on the (full) weight lattice characterized by the action

$$L_\sigma^{(0)} \psi_\lambda = \sum_{\nu \in W(\sigma)} \psi_{\lambda+\nu} \qquad (\lambda \in \mathcal{P}). \tag{4.6}$$

The operators $L_\sigma^{(n)}$ (4.5a) and $L_\sigma^{(d)}$ (4.5b) can be seen as the reduction of $L_\sigma^{(0)}$ (4.6) to the space of W invariant functions and W skew-invariant functions on $\mathcal{P}$, respectively (upon restriction to the fundamental domain $\mathcal{P}^+$).

4.2. **Bethe Ansatz solution.** Let $\mathcal{Q}^\vee$ denote the dual root lattice $\mathrm{Span}_{\mathbb{Z}}(\boldsymbol{R}^\vee)$ and let us write $\mathbb{T}_R$ for the torus $\mathbf{E}/(2\pi\mathcal{Q}^\vee)$. It is evident that the plane waves $\psi_\lambda(\boldsymbol{\xi}) = \exp(i\langle\lambda, \boldsymbol{\xi}\rangle)$, $\boldsymbol{\xi} \in \mathbb{T}_R$ constitute a (Fourier) basis of eigenfunctions for the free Laplacian $L_\sigma^{(0)} : C(\mathcal{P}) \to C(\mathcal{P})$ in Eq. (4.6). The corresponding eigenvalues are given by $E_\sigma(\boldsymbol{\xi}) = \sum_{\nu \in W(\sigma)} \exp(i\langle\nu, \xi\rangle)$, $\boldsymbol{\xi} \in \mathbb{T}_R$. Following the Bethe Ansatz method, we will now construct suitable linear combination of plane waves that satisfies the boundary conditions in Eqs. (4.1b), (4.1c). By construction, the resulting wave function will thus constitute an eigenfunction of our Laplacian L_σ (4.1a)–(4.1c).

Specifically, as Bethe Ansatz wave function we take an arbitrary Weyl-group invariant linear combination of plane waves of the form

$$\Psi_\lambda(\boldsymbol{\xi}) = \frac{1}{\delta(\boldsymbol{\xi})} \sum_{w \in W} (-1)^w \mathcal{C}(\boldsymbol{\xi}_w) e^{i\langle\rho+\lambda, \boldsymbol{\xi}_w\rangle}, \tag{4.7a}$$

with $(-1)^w := \det(w) = (-1)^{\ell(w)}$, and

$$\delta(\boldsymbol{\xi}) = \prod_{\alpha\in\boldsymbol{R}^+} (e^{i\langle\alpha,\boldsymbol{\xi}\rangle/2} - e^{-i\langle\alpha,\boldsymbol{\xi}\rangle/2}), \tag{4.7b}$$

$$\rho = \frac{1}{2}\sum_{\alpha\in\boldsymbol{R}^+} \alpha. \tag{4.7c}$$

(This wave function is W invariant in the sense that $\Psi_\lambda(\boldsymbol{\xi}_w) = \Psi_\lambda(\boldsymbol{\xi})$.) The following theorem matches the coefficients so as to meet the boundary conditions (4.1b), (4.1c).

Theorem 4.2 (Bethe Wave Function)**.** *Let $L_\sigma : C(\mathcal{P}^+) \to C(\mathcal{P}^+)$ be the discrete Laplacian with boundary conditions defined in Eqs.* (4.1a)–(4.1c)*. Then the Bethe Ansatz wave function $\Psi_\lambda(\boldsymbol{\xi})$* (4.7a)–(4.7c) *solves the eigenvalue equation*

$$L_\sigma\psi(\boldsymbol{\xi}) = E_\sigma(\boldsymbol{\xi})\psi(\boldsymbol{\xi}) \quad \textit{with} \quad E_\sigma(\boldsymbol{\xi}) = \sum_{\nu\in W(\sigma)} \exp(i\langle\nu,\xi\rangle), \tag{4.8}$$

provided that

$$\mathcal{C}(\boldsymbol{\xi}) = \prod_{\alpha\in\boldsymbol{R}^+} (1 - t_\alpha e^{-i\langle\alpha,\boldsymbol{\xi}\rangle}) \tag{4.9}$$

(or a scalar multiple thereof).

Proof. It suffices to check that the Bethe Ansatz wave function $\Psi_\lambda(\boldsymbol{\xi})$ (4.7a)–(4.7c) satisfies the boundary conditions (4.1b), (4.1c), provided that $\mathcal{C}(\boldsymbol{\xi})$ is of the form stated by the theorem. To this end we compute $\mathcal{C}(\boldsymbol{\xi})$ from the boundary reflection relations of Proposition 4.1. Indeed, upon assuming the technical conditions detailed in the proposition, substitution of the Bethe Ansatz wave function in the boundary reflection relations readily leads to the stated expression for the coefficients $\mathcal{C}(\boldsymbol{\xi})$. Specifically, we find in Case (I) that equating

$$\begin{aligned}\Psi_{\lambda+\nu}(\boldsymbol{\xi}) &= \frac{1}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\rho+\lambda+\nu,\boldsymbol{\xi}_w\rangle} \\ &= (-1)^{w_{\rho+\lambda+\nu}}\delta^{-1}(\boldsymbol{\xi})\sum_{\mu\in W(\rho+\lambda+\nu)}(-1)^{w_\mu}e^{i\langle\mu,\boldsymbol{\xi}\rangle}\sum_{w\in W_{\rho+\lambda+\nu}}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)\end{aligned}$$

to

$$\begin{aligned}t_{\alpha_j}\Psi_{r_j(\lambda+\nu)}(\boldsymbol{\xi}) &= t_{\alpha_j}\Psi_{\lambda+\nu+\alpha_j}(\boldsymbol{\xi}) \\ &= \frac{t_{\alpha_j}}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}e^{i\langle\rho+\lambda+\nu,\boldsymbol{\xi}_w\rangle} \\ &= t_{\alpha_j}(-1)^{w_{\rho+\lambda+\nu}}\delta^{-1}(\boldsymbol{\xi})\sum_{\mu\in W(\rho+\lambda+\nu)}(-1)^{w_\mu}e^{i\langle\mu,\boldsymbol{\xi}\rangle} \\ &\quad\times\sum_{w\in W_{\rho+\lambda+\nu}}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}\end{aligned}$$

leads to the relation

$$\sum_{w\in W_{\rho+\lambda+\nu}}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w) = t_{\alpha_j}\sum_{w\in W_{\rho+\lambda+\nu}}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}.$$

Because r_j stabilizes $\rho+\lambda+\nu$ (i.e. $r_j \in W_{\rho+\lambda+\nu}$), the latter relation can be rewritten as

$$\sum_{\substack{w\in W_{\rho+\lambda+\nu}\\ w^{-1}(\alpha_j)\in \boldsymbol{R}^+}} (-1)^w[\mathcal{C}(\boldsymbol{\xi}_w)-\mathcal{C}(r_j(\boldsymbol{\xi}_w))]$$
$$= t_{\alpha_j} \sum_{\substack{w\in W_{\rho+\lambda+\nu}\\ w^{-1}(\alpha_j)\in \boldsymbol{R}^+}} (-1)^w[\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi}_w))e^{-i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}].$$

By induction on the cardinality of the stabilizer $W_{\rho+\lambda+\nu}$, starting from the smallest value $|W_{\rho+\lambda+\nu}|=2$ (as it contains as subgroup the cyclic group of order 2 generated by r_j), one concludes that

$$\mathcal{C}(\boldsymbol{\xi})-\mathcal{C}(r_j(\boldsymbol{\xi}))=t_{\alpha_j}[\mathcal{C}(\boldsymbol{\xi})e^{i\langle\alpha_j,\boldsymbol{\xi}\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi}))e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle}],$$

or equivalenty (assuming $\mathcal{C}(\boldsymbol{\xi})$ is nontrivial in the sense that it does not vanish identically)

$$\frac{\mathcal{C}(\boldsymbol{\xi})}{\mathcal{C}(r_j(\boldsymbol{\xi}))}=\frac{1-t_{\alpha_j}e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle}}{1-t_{\alpha_j}e^{i\langle\alpha_j,\boldsymbol{\xi}\rangle}}. \tag{4.10}$$

From varying λ and ν, it is clear that the reflection relation in Eq. (4.10) should hold for all simple reflections r_j, $j=1,\ldots,N$. We thus conclude that $\mathcal{C}(\boldsymbol{\xi})$ must in fact be of the form

$$\mathcal{C}(\boldsymbol{\xi})=c_0(\boldsymbol{\xi})\prod_{\alpha\in\boldsymbol{R}^+}(1-t_\alpha e^{-i\langle\alpha,\boldsymbol{\xi}\rangle}),$$

where $c_0(\boldsymbol{\xi})$ denotes an arbitrary W invariant overall factor (i.e. $c_0(\boldsymbol{\xi}_w)=c_0(\boldsymbol{\xi})$, $\forall w\in W$).

It remains to check that this choice for the coefficient $\mathcal{C}(\boldsymbol{\xi})$ is also compatible with the boundary conditions of Case (II). This follows from an analysis similar to that of Case (I). Indeed, we get by equating

$$\begin{aligned}\Psi_{\lambda+\nu}(\boldsymbol{\xi}) &= \Psi_{\lambda-\alpha_j}(\boldsymbol{\xi})\\ &= \frac{1}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\rho+\lambda-\alpha_j,\boldsymbol{\xi}_w\rangle}\\ &= \frac{1}{\delta(\boldsymbol{\xi})}\sum_{\substack{w\in W\\ w^{-1}(\alpha_j)\in\boldsymbol{R}^+}}(-1)^w[\mathcal{C}(\boldsymbol{\xi}_w)e^{-i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi}_w))]e^{i\langle\rho+\lambda,\boldsymbol{\xi}_w\rangle}\end{aligned}$$

to the sum of

$$\begin{aligned}&t_{\alpha_j}\Psi_{r_j(\lambda+\nu)}(\boldsymbol{\xi})=t_{\alpha_j}\Psi_{\lambda+\alpha_j}\\ &= \frac{t_{\alpha_j}}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\rho+\lambda+\alpha_j,\boldsymbol{\xi}_w\rangle}\\ &= \frac{t_{\alpha_j}}{\delta(\boldsymbol{\xi})}\sum_{\substack{w\in W\\ w^{-1}(\alpha_j)\in\boldsymbol{R}^+}}(-1)^w[\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi}_w))e^{-2i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}]e^{i\langle\rho+\lambda,\boldsymbol{\xi}_w\rangle}\end{aligned}$$

and

$$
\begin{aligned}
&(t_{\alpha_j}-1)\Psi_\lambda(\boldsymbol{\xi})\\
&= \frac{(t_{\alpha_j}-1)}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\mathcal{C}(\boldsymbol{\xi}_w)e^{i\langle\rho+\lambda,\boldsymbol{\xi}_w\rangle}\\
&= \frac{(t_{\alpha_j}-1)}{\delta(\boldsymbol{\xi})}\sum_{\substack{w\in W\\ w^{-1}(\alpha_j)\in\boldsymbol{R}^+}}(-1)^w[\mathcal{C}(\boldsymbol{\xi}_w)-\mathcal{C}(r_j(\boldsymbol{\xi}_w))e^{-i\langle\alpha_j,\boldsymbol{\xi}_w\rangle}]e^{i\langle\rho+\lambda,\boldsymbol{\xi}_w\rangle},
\end{aligned}
$$

that it is sufficient to require that

$$
\begin{aligned}
\mathcal{C}(\boldsymbol{\xi})e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi})) &= t_{\alpha_j}[\mathcal{C}(\boldsymbol{\xi})e^{i\langle\alpha_j,\boldsymbol{\xi}\rangle}-\mathcal{C}(r_j(\boldsymbol{\xi}))e^{-2i\langle\alpha_j,\boldsymbol{\xi}\rangle}]\\
&\quad +(t_{\alpha_j}-1)[\mathcal{C}(\boldsymbol{\xi})-\mathcal{C}(r_j(\boldsymbol{\xi}))e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle}].
\end{aligned}
$$

The latter relation can be rewritten as

$$
\mathcal{C}(\boldsymbol{\xi})(1-t_{\alpha_j}e^{i\langle\alpha_j,\boldsymbol{\xi}\rangle})(1-e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle})=\mathcal{C}(r_j(\boldsymbol{\xi}))(1-t_{\alpha_j}e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle})(1-e^{-i\langle\alpha_j,\boldsymbol{\xi}\rangle}),
$$

which leads us back to the reflection relation in Eq. (4.10). □

4.3. **Discrete Plancherel formula.** Next we will address the question of the orthogonality and completeness of the Bethe wave functions given by Theorem 4.2.

Note. From now on it will always be assumed that the parameters lie in the (repulsive) domain $0<t_s,t_l<1$ (unless explicitly stated otherwise).

It is straightforward to rewrite the Bethe wave function of Theorem 4.2 as

$$
\Psi_\lambda(\boldsymbol{\xi})=\sum_{w\in W}\Bigl(\prod_{\alpha\in\boldsymbol{R}^+}\frac{1-t_\alpha e^{-i\langle\alpha,\boldsymbol{\xi}_w\rangle}}{1-e^{-i\langle\alpha,\boldsymbol{\xi}_w\rangle}}\Bigr)e^{i\langle\lambda,\boldsymbol{\xi}_w\rangle}. \tag{4.11}
$$

From this expression it is clear that the functions $\Psi_\lambda(\boldsymbol{\xi})$ amount in essence to (a parameter deformation of) the zonal spherical functions on p-adic Lie groups computed by Macdonald [M1, M3]. The solution of Plancherel problem is now a direct consequence of Macdonald's orthogonality relations for these (deformed) spherical functions. To describe the result, some notation is needed. Let $\mathcal{H}=\ell^2(\mathcal{P}^+,\Delta_\lambda)$ denote the Hilbert space of complex functions on the cone of dominant weights $\mathcal{P}^+$ (2.10b) that are square-summable with respect to the positive weight function

$$
\Delta_\lambda=\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \langle\lambda,\alpha^\vee\rangle=0}}\frac{1-t_s^{\mathrm{ht}_s(\alpha)}t_l^{\mathrm{ht}_l(\alpha)}}{1-t_\alpha t_s^{\mathrm{ht}_s(\alpha)}t_l^{\mathrm{ht}_l(\alpha)}} \tag{4.12}
$$

($\lambda\in\mathcal{P}^+$). The standard inner product on $\mathcal{H}$ is given by

$$
(f,g)_{\mathcal{H}}=\sum_{\lambda\in\mathcal{P}^+}f_\lambda g_\lambda\Delta_\lambda\qquad(\forall f,g\in\mathcal{H}). \tag{4.13}
$$

Furthermore, let $\hat{\mathcal{H}}=L^2(\mathbf{A},|W|^{-1}\mathrm{Vol}(\mathbf{A})^{-1}\hat{\Delta}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi})$ denote the Hilbert space of complex functions on the Weyl alcove

$$
\mathbf{A}=\{\boldsymbol{\xi}\in\mathbf{E}\mid 0<\langle\boldsymbol{\xi},\alpha\rangle<2\pi,\ \forall\alpha\in\boldsymbol{R}^+\} \tag{4.14}
$$

that are square-integrable with respect to the positive weight function

$$\hat{\Delta}(\boldsymbol{\xi}) = \frac{|\delta(\boldsymbol{\xi})|^2}{\mathcal{C}(\boldsymbol{\xi})\mathcal{C}(-\boldsymbol{\xi})} \tag{4.15a}$$

$$= \prod_{\alpha\in\boldsymbol{R}} \frac{1-e^{i\langle\alpha,\boldsymbol{\xi}\rangle}}{1-t_\alpha e^{i\langle\alpha,\boldsymbol{\xi}\rangle}} \tag{4.15b}$$

($\boldsymbol{\xi}\in\mathbf{A}$). The normalized inner product on $\hat{\mathcal{H}}$ reads

$$(\hat{f},\hat{g})_{\hat{\mathcal{H}}} = \frac{1}{|W|\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}} \hat{f}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})}\hat{\Delta}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \qquad (\forall \hat{f},\hat{g}\in\hat{\mathcal{H}}). \tag{4.16}$$

To the Bethe wave function $\Psi_\lambda(\boldsymbol{\xi})$ in Theorem 4.2 we associate the integral transformation $\mathcal{F}:\mathcal{H}\to\hat{\mathcal{H}}$ given by the Fourier pairing

$$\hat{f}(\boldsymbol{\xi}) = (\mathcal{F}f)(\boldsymbol{\xi}) := (f,\Psi(\boldsymbol{\xi}))_{\mathcal{H}} \tag{4.17a}$$

$$= \sum_{\lambda\in\mathcal{P}^+} f_\lambda \overline{\Psi_\lambda(\boldsymbol{\xi})}\Delta_\lambda \tag{4.17b}$$

($\forall f\in\mathcal{H}$), and the adjoint integral transformation $\hat{\mathcal{F}}:\hat{\mathcal{H}}\to\mathcal{H}$ given by the Fourier pairing

$$f_\lambda = (\hat{\mathcal{F}}\hat{f})_\lambda := (\hat{f},\overline{\Psi_\lambda})_{\hat{\mathcal{H}}} \tag{4.18a}$$

$$= \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})}\int_{\mathbf{A}} \hat{f}(\boldsymbol{\xi})\Psi_\lambda(\boldsymbol{\xi})\hat{\Delta}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \tag{4.18b}$$

($\forall\hat{f}\in\hat{\mathcal{H}}$).

Theorem 4.3 (Discrete Plancherel Formula)**.** *The eigenfunction transformation $\mathcal{F}:\mathcal{H}\to\hat{\mathcal{H}}$ in Eqs.* (4.17a), (4.17b) *constitutes a unitary Hilbert space isomorphism with the inverse transformation $\mathcal{F}^{-1}$ given by the adjoint eigenfunction transformation $\hat{\mathcal{F}}:\hat{\mathcal{H}}\to\mathcal{H}$ in Eqs.* (4.18a), (4.18b)*.*

Proof. The theorem is a direct consequence of the fact that the zonal spherical functions $\Psi_\lambda(\boldsymbol{\xi})$, $\lambda\in\mathcal{P}^+$ form an orthogonal basis of $\hat{\mathcal{H}}$ satisfying the orthogonality relations [M1, M3]

$$(\Psi_\lambda,\Psi_\mu)_{\hat{\mathcal{H}}} = \begin{cases}\Delta_\lambda^{-1} & \text{if } \lambda=\mu,\\ 0 & \text{if } \lambda\neq\mu.\end{cases} \tag{4.19}$$

To keep our treatment self-contained, a brief outline of Macdonald's proof of these orthogonality relations is isolated in Appendix B at the end of the paper. □

Let $\hat{E}_\sigma:\hat{\mathcal{H}}\to\hat{\mathcal{H}}$ be the multiplication operator

$$(\hat{E}_\sigma\hat{f})(\boldsymbol{\xi}) := E_\sigma(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}), \tag{4.20}$$

where $E_\sigma(\boldsymbol{\xi})$ stands for the eigenvalue in Eq. (4.8). It is a straightforward consequence of Theorem 4.2 and Theorem 4.3 that the discrete Laplace operator L_σ (4.1a)–(4.1c) is the pullback of the multiplication operator $\hat{E}_\sigma$ with respect to the transformation $\mathcal{F}$. From this observation the following two corollaries are immediate.

Corollary 4.4 (Spectrum)**.** *The discrete Laplace operator L_σ* (4.1a)–(4.1c) *has a purely absolutely continuous spectrum in the Hilbert space $\mathcal{H}$ given by the compact set Spec$(L_\sigma)=\{E_\sigma(\boldsymbol{\xi})\mid\boldsymbol{\xi}\in\bar{\mathbf{A}}\}\subset\mathbb{C}$.*

The complex conjugate of the function $E_\sigma(\boldsymbol{\xi})$ is given by $E_{-w_0(\sigma)}(\boldsymbol{\xi})$, where w_0 denotes the longest element in the Weyl group W (i.e., the unique element $w_0 \in W$ such that $w_0(\mathbf{A}) = -\mathbf{A}$).

Corollary 4.5 (Adjoint)**.** *The adjoint of L_σ in $\mathcal{H}$ is given by $L_{-w_0(\sigma)}$.*

In particular, this means that L_σ is self-adjoint if and only if $w_0(\sigma) = -\sigma$. This is for instance the case when σ is quasi-minuscule or when $w_0 = -\mathbf{Id}$. If $w_0(\sigma) \neq -\sigma$, then one can make the eigenvalue problem self-adjoint by passing to the operator $(L_\sigma + L_{-w_0(\sigma)})$.

For $t_s, t_l \to 1$, the Bethe wave function $\Psi_\lambda(\boldsymbol{\xi})$ (4.11) reduces to the monomial symmetric function

$$\Psi_\lambda^{(n)}(\boldsymbol{\xi}) = |W_\lambda|\, m_\lambda(\boldsymbol{\xi}), \quad \text{with} \quad m_\lambda(\boldsymbol{\xi}) = \sum_{\mu \in W(\lambda)} e^{i\langle \mu, \boldsymbol{\xi}\rangle}. \tag{4.21}$$

The eigenfunction transform $\mathcal{F}$ amounts in this situation to the W invariant part of the Fourier transformation on $\ell^2(\mathcal{P})$:

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\lambda \in \mathcal{P}^+} f_\lambda \overline{m_\lambda(\boldsymbol{\xi})}, \tag{4.22a}$$

with the inversion formula

$$f_\lambda = \frac{1}{|W(\lambda)|\,\mathrm{Vol}(\mathbf{A})} \int_{\mathbf{A}} \hat{f}(\boldsymbol{\xi}) m_\lambda(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}. \tag{4.22b}$$

For $t_s, t_l \to 0$ the Bethe wave function $\Psi_\lambda(\boldsymbol{\xi})$ (4.11) reduces to the Weyl character

$$\Psi_\lambda^{(d)}(\boldsymbol{\xi}) = \chi_\lambda(\boldsymbol{\xi}), \quad \text{with} \quad \chi_\lambda(\boldsymbol{\xi}) = \frac{1}{\delta(\boldsymbol{\xi})} \sum_{w \in W} (-1)^w e^{i\langle \rho+\lambda, \boldsymbol{\xi}\rangle}. \tag{4.23}$$

The eigenfunction transform $\mathcal{F}$ amounts in this situation to the W skew-invariant part of the Fourier transformation on $\ell^2(\mathcal{P})$:

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\lambda \in \mathcal{P}^+} f_\lambda \overline{\chi_\lambda(\boldsymbol{\xi})}, \tag{4.24a}$$

with the inversion formula

$$f_\lambda = \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})} \int_{\mathbf{A}} \hat{f}(\boldsymbol{\xi}) \chi_\lambda(\boldsymbol{\xi}) |\delta(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi}. \tag{4.24b}$$

5. The Continuum Limit

In this section it is shown that the discrete Plancherel formula of Theorem 4.3 degenerates to continuous Plancherel formula of Theorem 3.2 in the continuum limit as the lattice distance tends to zero. The discrete Laplacian from Eqs. (4.1a)–(4.1c) degenerates in this limit—upon symmetrization and rescaling—in the strong resolvent sense to the continuous Laplacian from Eqs. (3.4a)–(3.4b). The approach in this section is inspired by Ruijsenaars' proof of the fact that the ground-state representation of the infinite isotropic Heisenberg spin chain converges in the continuum limit to a free Boson gas [R].

Note. Throughout this section we will employ the parametrization $t_\alpha = e^{-\epsilon g_\alpha}$ with $\epsilon > 0$ and with g_α positive (and W invariant, cf Eq. (3.3)).

5.1. **Embedding.** To perform the continuum limit, we embed the Hilbert space $\mathcal{H} = \ell^2(\mathcal{P}^+, \Delta_\lambda)$ from Section 4 isometrically in the Hilbert space $\mathrm{H}_0 = L^2(\mathbf{C}, \mathrm{d}\mathbf{x})$ with standard inner product $(f,g)_{\mathrm{H}_0} = \int_{\mathbf{C}} f(\mathbf{x})\overline{g(\mathbf{x})}\mathrm{d}\mathbf{x}$. This is done via the one-parameter family of embeddings $J_\epsilon : \mathcal{H} \to \mathrm{H}_0$, $\epsilon > 0$, which associate to a lattice function $f \in \mathcal{H}$ the staircase function $f_\epsilon \in \mathrm{H}_0$ defined by

$$f_\epsilon(\mathbf{x}) = (J_\epsilon f)(\mathbf{x}) := \frac{\epsilon^{-N/2}}{\sqrt{\det(\mathcal{P})}}\, \Delta^{1/2}_{[\epsilon^{-1}\mathbf{x}]} f_{[\epsilon^{-1}\mathbf{x}]}. \tag{5.1a}$$

Here $\det(\mathcal{P}) := \det(\omega_1, \ldots, \omega_N)$ and for $\mathbf{x} \in \overline{\mathbf{C}}$

$$[\mathbf{x}] := [\langle \mathbf{x}, \alpha_1^\vee\rangle]\,\omega_1 + \cdots + [\langle \mathbf{x}, \alpha_N^\vee\rangle]\,\omega_N \in \mathcal{P}^+$$

(where $[x]$ denotes the integral part of a nonnegative real number x obtained via truncation). Similarly, the dual Hilbert space $\hat{\mathcal{H}} = L^2(\mathbf{A}, |W|^{-1}\mathrm{Vol}(\mathbf{A})^{-1}\hat{\Delta}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}) = L^2(\mathbf{A}, (2\pi)^{-N}\det(Q^\vee)^{-1}\hat{\Delta}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi})$ from Section 4 is embedded isometrically in the Hilbert space $\hat{\mathrm{H}}_0 = L^2(\mathbf{C}, (2\pi)^{-N}\mathrm{d}\boldsymbol{\xi})$ with normalized inner product $(\hat{f}, \hat{g})_{\hat{\mathrm{H}}_0} = \frac{1}{(2\pi)^N}\int_{\mathbf{C}} \hat{f}(\boldsymbol{\xi})\overline{\hat{g}(\boldsymbol{\xi})}\mathrm{d}\boldsymbol{\xi}$. This is done via the one-parameter family of embeddings $\hat{J}_\epsilon : \hat{\mathcal{H}} \to \hat{\mathrm{H}}_0$, $\epsilon > 0$, which associate to a function $\hat{f} \in \hat{\mathcal{H}}$ the rescaled function $\hat{f}_\epsilon \in \hat{\mathrm{H}}_0$ defined by

$$\hat{f}_\epsilon(\boldsymbol{\xi}) = (\hat{J}_\epsilon \hat{f})(\boldsymbol{\xi}) := \frac{\epsilon^{N/2}}{\sqrt{\det(\mathcal{Q}^\vee)}}\, \hat{\Delta}^{1/2}(\epsilon\boldsymbol{\xi})\hat{f}(\epsilon\boldsymbol{\xi}). \tag{5.1b}$$

Here $\det(\mathcal{Q}^\vee) := \det(\alpha_1^\vee, \ldots, \alpha_N^\vee) = (2\pi)^{-N}|W|\mathrm{Vol}(\mathbf{A})$.

Let $\mathrm{H}_\epsilon := J_\epsilon(\mathcal{H}) \subset \mathrm{H}_0$ and let $\hat{\mathrm{H}}_\epsilon := \hat{J}_\epsilon(\hat{\mathcal{H}}) \subset \hat{\mathrm{H}}_0$. The eigenfunction transform $\mathcal{F} : \mathcal{H} \to \hat{\mathcal{H}}$ (4.17a), (4.17b) and its inverse $\hat{\mathcal{F}} : \mathcal{H} \to \hat{\mathcal{H}}$ (4.18a), (4.18b) lift under the embeddings J_ϵ and $\hat{J}_\epsilon$, respectively, to a corresponding transform $F_\epsilon : \mathrm{H}_\epsilon \to \hat{\mathrm{H}}_\epsilon$ and its inverse $\hat{F}_\epsilon : \hat{\mathrm{H}}_\epsilon \to \mathrm{H}_\epsilon$ of the form

$$\hat{f}_\epsilon(\boldsymbol{\xi}) = (F_\epsilon f)(\boldsymbol{\xi}) := \int_{\mathbf{C}} f(\mathbf{x})\overline{\Phi_{[\epsilon^{-1}\mathbf{x}]}(\epsilon\boldsymbol{\xi})}\mathrm{d}\mathbf{x} \tag{5.2a}$$

and

$$f_\epsilon(\mathbf{x}) = (\hat{F}_\epsilon \hat{f})(\mathbf{x}) := \frac{1}{(2\pi)^N}\int_{\mathbf{C}} \hat{f}(\boldsymbol{\xi})\Phi_{[\epsilon^{-1}\mathbf{x}]}(\epsilon\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{5.2b}$$

with a kernel given by

$$\begin{aligned} \Phi_{[\epsilon^{-1}\mathbf{x}]}(\epsilon\boldsymbol{\xi}) &= \Delta^{1/2}_{[\epsilon^{-1}\mathbf{x}]}\hat{\Delta}^{1/2}(\epsilon\boldsymbol{\xi})\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})\Psi_{[\epsilon^{-1}\mathbf{x}]}(\epsilon\boldsymbol{\xi}) \\ &= \Delta^{1/2}_{[\epsilon^{-1}\mathbf{x}]}\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})\sum_{w\in W} S_\epsilon^{1/2}(\boldsymbol{\xi}_w)e^{i\epsilon\langle[\epsilon^{-1}\mathbf{x}],\boldsymbol{\xi}_w\rangle}, \end{aligned} \tag{5.3a}$$

where

$$S_\epsilon(\boldsymbol{\xi}) = \prod_{\alpha\in\boldsymbol{R}^+}\frac{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle - ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle + ig_\alpha)}, \tag{5.3b}$$

and with $\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})$ denoting the characteristic function of the rescaled alcove $\epsilon^{-1}\mathbf{A} \subset \mathbf{C}$. It follows from Theorem 4.3 that the transform F_ϵ and its inverse $\hat{F}_\epsilon$ define a unitary Hilbert space isomorphism between the closed subspaces $\mathrm{H}_\epsilon \subset \mathrm{H}_0$ and

$\hat{\mathrm{H}}_\epsilon \subset \hat{\mathrm{H}}_0$. In other words, we have the following commutative diagram of unitary Hilbert space isomorphisms

$$\begin{array}{ccc} \mathrm{H}_\epsilon & \overset{F_\epsilon,\hat{F}_\epsilon}{\longleftrightarrow} & \hat{\mathrm{H}}_\epsilon \\ \updownarrow_{J_\epsilon} & & \updownarrow_{\hat{J}_\epsilon} \\ \mathcal{H} & \overset{\mathcal{F},\hat{\mathcal{F}}}{\longleftrightarrow} & \hat{\mathcal{H}} \end{array} \qquad \begin{array}{cc} \hat{F}_\epsilon F_\epsilon = \mathbf{I}_{\mathrm{H}_\epsilon} & F_\epsilon \hat{F}_\epsilon = \mathbf{I}_{\hat{\mathrm{H}}_\epsilon} \\ & \\ \hat{\mathcal{F}}\mathcal{F} = \mathbf{I}_{\mathcal{H}} & \mathcal{F}\hat{\mathcal{F}} = \mathbf{I}_{\hat{\mathcal{H}}} \end{array} . \tag{5.4}$$

The orthogonal projections $\Pi_\epsilon : \mathrm{H}_0 \to \mathrm{H}_\epsilon$ and $\hat{\Pi}_\epsilon : \hat{\mathrm{H}}_0 \to \hat{\mathrm{H}}_\epsilon$ on the closed subspaces $\mathrm{H}_\epsilon \subset \mathrm{H}_0$ and $\hat{\mathrm{H}}_\epsilon \subset \hat{\mathrm{H}}_0$, respectively, are given explicitly by

$$(\Pi_\epsilon f)(\mathbf{x}) = \frac{\epsilon^{-N}}{\det(\mathcal{P})} \int_{\mathbf{T}^{(\epsilon)}([\epsilon^{-1}\mathbf{x}])} f(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{5.5a}$$

with $\mathbf{T}^{(\epsilon)}(\lambda) := \{\mathbf{x} \in \mathbf{E} \mid \epsilon\langle\lambda, \alpha_j^\vee\rangle \leq \langle \mathbf{x}, \alpha_j^\vee\rangle < \epsilon(\langle\lambda,\alpha_j^\vee\rangle + 1),\ j = 1,\ldots,N\}$, and by

$$(\hat{\Pi}_\epsilon \hat{f})(\boldsymbol{\xi}) = \chi_{\mathbf{A}}^{(\epsilon)}(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}). \tag{5.5b}$$

If we extend the definitions of F_ϵ and $\hat{F}_\epsilon$ in Eqs. (5.2a) and (5.2b) to arbitrary $f \in \mathrm{H}_0$ and $\hat{f} \in \hat{\mathrm{H}}_0$, respectively, then clearly

$$F_\epsilon \Pi_\epsilon = F_\epsilon \quad \text{and} \quad \hat{F}_\epsilon \hat{\Pi}_\epsilon = \hat{F}_\epsilon. \tag{5.6}$$

This gives rise to the following commutative diagrams of bounded transformations

$$\begin{array}{ccc} \mathrm{H}_0 & \overset{F_\epsilon}{\longrightarrow} & \hat{\mathrm{H}}_0 \\ \uparrow_{J_\epsilon} & & \uparrow_{\hat{J}_\epsilon} \\ \mathcal{H} & \overset{\mathcal{F}}{\longrightarrow} & \hat{\mathcal{H}} \end{array} \qquad \begin{array}{ccc} \mathrm{H}_0 & \overset{\hat{F}_\epsilon}{\longleftarrow} & \hat{\mathrm{H}}_0 \\ \uparrow_{J_\epsilon} & & \uparrow_{\hat{J}_\epsilon} \\ \mathcal{H} & \overset{\hat{\mathcal{F}}}{\longleftarrow} & \hat{\mathcal{H}} \end{array} , \tag{5.7}$$

with $F_\epsilon : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ and $\hat{F}_\epsilon : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$ being contractive operators in the sense that $\forall f \in \mathrm{H}_0$ and $\forall \hat{f} \in \hat{\mathrm{H}}_0$

$$\|F_\epsilon f\|_{\hat{\mathrm{H}}_0} \leq \|f\|_{\mathrm{H}_0} \quad \text{and} \quad \|\hat{F}_\epsilon \hat{f}\|_{\mathrm{H}_0} \leq \|\hat{f}\|_{\hat{\mathrm{H}}_0} \tag{5.8}$$

(where $\|\cdot\|_{\mathrm{H}_0} := \langle\cdot,\cdot\rangle_{\mathrm{H}_0}^{1/2}$ and $\|\cdot\|_{\hat{\mathrm{H}}_0} := \langle\cdot,\cdot\rangle_{\hat{\mathrm{H}}_0}^{1/2}$).

5.2. **The continuum limit $\epsilon \to 0$: eigenfunction transform.** For $\mathbf{x}$ and $\boldsymbol{\xi}$ in the interior of the Weyl chamber $\mathbf{C}$, it is straightforward to check that in the limit $\epsilon \to 0$ the kernel function $\Phi_{[\epsilon^{-1}\mathbf{x}]}(\epsilon\boldsymbol{\xi})$ (5.3a) degenerates pointwise to

$$\begin{aligned} \Phi_0(\mathbf{x};\boldsymbol{\xi}) &= \hat{\Delta}_0^{1/2}(\boldsymbol{\xi})\,\Psi_0(\mathbf{x};\boldsymbol{\xi}) \\ &= \sum_{w\in W} S_0^{1/2}(\boldsymbol{\xi}_w) e^{i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}, \end{aligned} \tag{5.9a}$$

with

$$S_0(\boldsymbol{\xi}) = \prod_{\alpha\in\boldsymbol{R}^+} \frac{\langle\alpha,\boldsymbol{\xi}\rangle - ig_\alpha}{\langle\alpha,\boldsymbol{\xi}\rangle + ig_\alpha}.$$

So, formally the eigenfunction transform F_ϵ (5.2a) and its adjoint $\hat{F}_\epsilon$ (5.2b) degenerate in this limit to

$$\hat{f}_0(\boldsymbol{\xi}) = (F_0 f)(\boldsymbol{\xi}) := \int_{\mathbf{C}} f(\mathbf{x})\overline{\Phi_0(\mathbf{x};\boldsymbol{\xi})}\mathrm{d}\mathbf{x} \tag{5.10a}$$

and its adjoint

$$f_0(\mathbf{x}) = (\hat{F}_0\hat{f})(\mathbf{x}) := \frac{1}{(2\pi)^N}\int_{\mathbf{C}} \hat{f}(\boldsymbol{\xi})\Phi_0(\mathbf{x};\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{5.10b}$$

respectively. From the fact that $|S_0(\boldsymbol{\xi})| = 1$ combined with the Plancherel property of the Fourier transform on $L^2(\mathbf{E})$, it follows that the integral transforms in Eqs. (5.10a) and (5.10b) define bounded operators $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ and $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$. The following two lemmas provide a precise meaning to the intuitive idea that for $\epsilon \to 0$

$$\mathrm{H}_\epsilon \to \mathrm{H}_0, \quad \hat{\mathrm{H}}_\epsilon \to \hat{\mathrm{H}}_0 \qquad \text{and} \qquad F_\epsilon \to F_0, \quad \hat{F}_\epsilon \to \hat{F}_0. \tag{5.11}$$

Lemma 5.1 (Continuum Limit: the Hilbert Space)**.** *One has that*

$$s - \lim_{\epsilon \to 0} \Pi_\epsilon = \mathbf{I}_{\mathrm{H}_0} \qquad \textit{and} \qquad s - \lim_{\epsilon \to 0} \hat{\Pi}_\epsilon = \mathbf{I}_{\hat{\mathrm{H}}_0} \tag{5.12}$$

(strongly).

Proof. Since Π_ϵ is a projection operator, it is obvious that $\|\Pi_\epsilon\|_{\mathrm{H}_0} \leq 1$ uniformly $\forall \epsilon > 0$. Hence, for validating the first limit in Eq. (5.12), it suffices to show that $\lim_{\epsilon \to 0} \Pi_\epsilon \phi = \phi$ for any ϕ in the dense subspace $C_0^\infty(\mathbf{C}) \subset \mathrm{H}_0$. It is obvious from the definition in Eq. (5.5a) that, for any test function $\phi \in C_0^\infty(\mathbf{C})$, the staircase approximation $(\Pi_\epsilon \phi)(\mathbf{x})$ converges pointwise to $\phi(\mathbf{x})$ when ϵ tends to 0. Moreover, since ϕ has compact support it is clear that the difference $|\Pi_\epsilon \phi - \phi|$ admits an L^2 upper bound that is uniform in ϵ (for $\epsilon \leq 1$ say). The desired convergence $\lim_{\epsilon \to 0} \|\Pi_\epsilon \phi - \phi\|_{\mathrm{H}_0} = 0$ thus follows by the dominated convergence theorem of Lebesgue. To demonstrate the second limit in Eq. (5.12), we simply observe that for any $\hat{f} \in \hat{\mathrm{H}}_0$

$$\lim_{\epsilon \to 0} \|\hat{\Pi}_\epsilon \hat{f} - \hat{f}\|^2_{\hat{\mathrm{H}}_0} = \lim_{\epsilon \to 0} (2\pi)^{-N} \int_{\mathbf{C}} \big(1 - \chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})\big) |\hat{f}(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi},$$

which converges to zero (again by Lebesgue's dominated convergence theorem). □

Lemma 5.2 (Continuum Limit: the Eigenfunction Transform)**.** *One has that* i)

$$\forall f \in \mathrm{H}_0 : \qquad \lim_{\epsilon \to 0} (F_\epsilon f)(\boldsymbol{\xi}) = (F_0 f)(\boldsymbol{\xi}), \qquad \xi \in \mathbf{C} \tag{5.13a}$$

(pointwise) and that ii)

$$s - \lim_{\epsilon \to 0} \hat{F}_\epsilon = \hat{F}_0 \tag{5.13b}$$

(strongly).

Proof. i). The action of F_ϵ on $f \in \mathrm{H}_0$ reads

$$(F_\epsilon f)(\boldsymbol{\xi}) = \sum_{w \in W} \chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi}) S_\epsilon^{1/2}(-\boldsymbol{\xi}_w) \int_{\mathbf{C}} f(\mathbf{x}) \Delta^{1/2}_{[\epsilon^{-1}\mathbf{x}]} e^{-i\epsilon \langle [\epsilon^{-1}\mathbf{x}], \boldsymbol{\xi}_w \rangle} \mathrm{d}\mathbf{x}.$$

For any $\mathbf{x}, \boldsymbol{\xi} \in \mathbf{C}$, we have that for $\epsilon \to 0$

$$\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi}) \to 1, \quad S_\epsilon(\boldsymbol{\xi}_w) \to S_0(\boldsymbol{\xi}_w), \quad \Delta_{[\epsilon^{-1}\mathbf{x}]} \to 1, \quad e^{i\epsilon \langle [\epsilon^{-1}\mathbf{x}], \boldsymbol{\xi}_w \rangle} \to e^{i \langle \mathbf{x}, \boldsymbol{\xi}_w \rangle} \tag{5.14}$$

pointwise. Since $|e^{-i\epsilon \langle [\epsilon^{-1}\mathbf{x}], \boldsymbol{\xi}_w \rangle}| = 1$ and $\Delta_{[\epsilon^{-1}\mathbf{x}]} \leq 1$, the pointwise limit in Eq. (5.13a) follows by Lebesgue's dominated convergence theorem.

ii). The action of $\hat{F}_\epsilon$ on any $\hat{f} \in \hat{\mathrm{H}}_0$ is given by

$$(\hat{F}_\epsilon \hat{f})(\mathbf{x}) = \frac{1}{(2\pi)^N} \sum_{w \in W} \Delta_{[\epsilon^{-1}\mathbf{x}]} \int_{\mathbf{C}} \hat{f}(\boldsymbol{\xi}) S_\epsilon^{1/2}(\boldsymbol{\xi}_w) e^{i\epsilon \langle [\epsilon^{-1}\mathbf{x}], \boldsymbol{\xi}_w \rangle} \chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}.$$

The pointwise limit $\lim_{\epsilon \to 0} (\hat{F}_\epsilon \hat{f})(\mathbf{x}) = (\hat{F}_0 \hat{f})(\mathbf{x})$ thus follows by dominated convergence from the pointwise convergence in Eq. (5.14) combined with the bounds $|S_\epsilon(\boldsymbol{\xi}_w)| = 1$, $|e^{-i\epsilon \langle [\epsilon^{-1}\mathbf{x}], \boldsymbol{\xi}_w \rangle}| = 1$ and $|\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})| \leq 1$. It remains to show that the transition $\hat{F}_\epsilon \to \hat{F}_0$ converges in fact strongly. Since $\hat{F}_\epsilon$ is uniformly bounded in ϵ in view of Eq. (5.8), it suffices to show that $\lim_{\epsilon\to 0}\hat{F}_\epsilon\hat{\phi} = \hat{F}_0\hat{\phi}$ for any $\hat{\phi}$ in the dense subspace $C_0^\infty(\mathbf{C}) \subset \hat{\mathrm{H}}_0$. The latter limit follows from the estimate

$$|(\hat{F}_\epsilon\hat{\phi})(\mathbf{x})| \leq C(1+\|\mathbf{x}\|^{2N})^{-1} \tag{5.15}$$

uniformly in ϵ for ϵ sufficiently small. Indeed, the already established pointwise convergence $\lim_{\epsilon\to 0}(\hat{F}_\epsilon\hat{\phi})(\mathbf{x}) = (\hat{F}_0\hat{\phi})(\mathbf{x})$ combined with the L^2-bound in Eq. (5.15) guarantees the convergence of the limit in the Hilbert space H_0 by the bounded convergence theorem. In order to verify the estimate in Eq. (5.15), we note that from the explicit formula for the action of $\hat{F}_\epsilon$ it is clear that

$$\epsilon^{2N}\|[\epsilon^{-1}\mathbf{x}]\|^{2N}|(\hat{F}_\epsilon\hat{\phi})(\mathbf{x})| \leq \frac{1}{(2\pi)^N}\sum_{w\in W}\int_{\mathbf{C}} |\nabla^{2N}_{\boldsymbol{\xi}}\big(S_\epsilon^{1/2}(\boldsymbol{\xi}_w)\hat{\phi}(\boldsymbol{\xi})\big)|\mathrm{d}\boldsymbol{\xi}, \tag{5.16}$$

provided ϵ is sufficiently small so as to ensure that the support of $\hat{\phi}$ is contained in $\epsilon^{-1}\mathbf{A}$. Now let $\partial_{\boldsymbol{\xi}_1},\ldots,\partial_{\boldsymbol{\xi}_N}$ be the partial derivatives associated to an orthonormal basis $\mathbf{e}_1,\ldots,\mathbf{e}_N$ of $\mathbf{E}$. Then $\nabla^2_{\boldsymbol{\xi}} = \partial^2_{\boldsymbol{\xi}_1} + \cdots, \partial^2_{\boldsymbol{\xi}_N}$. Hence, to show that the bound (5.15) follows from (5.16) it suffices to check that the partial derivatives $\partial^{m_j}_{\boldsymbol{\xi}_j}\partial^{m_k}_{\boldsymbol{\xi}_k}S_\epsilon^{1/2}(\boldsymbol{\xi})$ are bounded in ϵ on the support of $\hat{\phi}$. The partial derivatives in question are sums of products of expressions of the form

$$\partial^{n_j}_{\boldsymbol{\xi}_j}\partial^{n_k}_{\boldsymbol{\xi}_k}\Big(\frac{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle - ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle + ig_\alpha)}\Big)^{1/2}. \tag{5.17}$$

The derivatives in Eq. (5.17) are in turn sums of products built of expressions of the form $\Big(\frac{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle-ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle+ig_\alpha)}\Big)^{1/2}$, $\frac{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle+ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle-ig_\alpha)}$, $\frac{\epsilon\langle\alpha,\mathbf{e}_j\rangle\cos\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle-ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle+ig_\alpha)}$, $\frac{\epsilon\langle\alpha,\mathbf{e}_j\rangle\cos\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle+ig_\alpha)}{\sin\frac{\epsilon}{2}(\langle\alpha,\boldsymbol{\xi}\rangle+ig_\alpha)}$, and $\epsilon\langle\alpha,\mathbf{e}_j\rangle$, which remain bounded as $\epsilon\to 0$. □

With the aid of Lemmas 5.1 and 5.2, we are now in the position to push through the continuum limit $\epsilon \to 0$ at the level of the Plancherel formula.

Proposition 5.3 (Isometry)**.** *The transformation* $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$ *constitutes an isometry with left-inverse* $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$.

Proof. The transform $\hat{F}_0$ inherits from $\hat{F}_\epsilon$ the property that it is an isometry in view of Lemmas 5.1 and 5.2. Indeed, for any $\hat{f} \in \hat{\mathrm{H}}_0$

$$\begin{aligned}\|\hat{F}_0\hat{f}\|_{\mathrm{H}_0} &\overset{\text{Eq. (5.13b)}}{=} \lim_{\epsilon\to 0}\|\hat{F}_\epsilon\hat{f}\|_{\mathrm{H}_0} \overset{\text{Eq. (5.12)}}{=} \lim_{\epsilon\to 0}\|\hat{F}_\epsilon\hat{\Pi}_\epsilon\hat{f}\|_{\mathrm{H}_0}\\ &\overset{\text{Eq. (5.4)}}{=} \lim_{\epsilon\to 0}\|\hat{\Pi}_\epsilon\hat{f}\|_{\hat{\mathrm{H}}_0} \overset{\text{Eq. (5.12)}}{=} \|\hat{f}\|_{\hat{\mathrm{H}}_0},\end{aligned}$$

whence $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$ is an isometry. To see that F_0 is a left-inverse of $\hat{F}_0$, we consider the identity (cf. the commutative diagram in Eq. (5.4))

$$F_\epsilon\hat{F}_\epsilon\hat{\phi} = \hat{\Pi}_\epsilon\hat{\phi} \tag{5.18}$$

for $\hat{\phi} \in C_0^\infty(\mathbf{C})$. The l.h.s. of this identity is given by

$$\sum_{w\in W}\chi^{(\epsilon)}_{\mathbf{A}}(\boldsymbol{\xi})S_\epsilon^{1/2}(-\boldsymbol{\xi}_w)\int_{\mathbf{C}}(\hat{F}_\epsilon\hat{\phi})(\mathbf{x})\Delta^{1/2}_{[\epsilon^{-1}\mathbf{x}]}e^{-i\epsilon\langle[\epsilon^{-1}\mathbf{x}],\boldsymbol{\xi}_w\rangle}\mathrm{d}\mathbf{x}. \tag{5.19}$$

We know from the second part of the proof of Lemma 5.2 that $(\hat{F}_\epsilon\hat{\phi})(\mathbf{x})$ admits an L^2-bound that is uniform in ϵ for ϵ sufficiently small (cf. Eq. (5.15)) and that for

$\epsilon \to 0$ it converges pointwise to $(\hat{F}_0\hat{\phi})(\mathbf{x})$. By following the steps in the first part of the proof of Lemma 5.2, we readily infer from the expression in Eq. (5.19) that $\lim_{\epsilon\to 0}(F_\epsilon \hat{F}_\epsilon \hat{\phi})(\boldsymbol{\xi}) = (F_0\hat{F}_0\hat{\phi})(\boldsymbol{\xi})$ (pointwise). On the other hand, it follows from (the proof of) Lemma 5.1 that $\lim_{\epsilon\to 0}(\hat{\Pi}_\epsilon\hat{\phi})(\boldsymbol{\xi}) = \hat{\phi}(\boldsymbol{\xi})$. We thus conclude that for $\epsilon \to 0$ the identity in Eq. (5.18) degenerates to

$$F_0\hat{F}_0\hat{\phi} = \hat{\phi},$$

whence $F_0\hat{F}_0 = \mathbf{I}_{\hat{\mathrm{H}}_0}$ (since the subspace $C_0^\infty(\mathbf{C})$ is dense in $\hat{\mathrm{H}}_0$ and the operators involved are bounded). □

In other words, Proposition 5.3 states that $\hat{F}_0$ is a unitary Hilbert space isomorphism between $\hat{\mathrm{H}}_0$ and the closed subspace $\hat{F}_0(\hat{\mathrm{H}}_0) \subset \mathrm{H}_0$. The following proposition ensures that in fact $\hat{F}_0(\hat{\mathrm{H}}_0) = \mathrm{H}_0$.

Proposition 5.4 (Completeness)**.** *The transformation $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$ is surjective, i.e. $\hat{F}_0(\hat{\mathrm{H}}_0) = \mathrm{H}_0$.*

Proof. For proving the surjectivity of $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$ it is enough to show that $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ is injective (in view of Proposition 5.3). This injectivity is verified in Appendix A below. □

Combination of Propositions 5.3 and 5.4 entails that the transformation $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ constitutes a unitary Hilbert space isomorphism with inverse $\hat{F}_0 : \hat{\mathrm{H}}_0 \to \mathrm{H}_0$:

$$\mathrm{H}_0 \overset{F_0,\hat{F}_0}{\longleftrightarrow} \hat{\mathrm{H}}_0, \qquad \hat{F}_0F_0 = \mathbf{I}_{\mathrm{H}_0}, \quad F_0\hat{F}_0 = \mathbf{I}_{\hat{\mathrm{H}}_0}. \tag{5.20}$$

The Plancherel formula in Theorem 3.2 is now immediate upon performing the gauge transformation $\hat{f} \mapsto \hat{\Delta}_0^{1/2}\hat{f}$ at the spectral side, so as to trade the Lebesgue measure $\mathrm{d}\boldsymbol{\xi}$ for the Plancherel measure $\hat{\Delta}_0(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}$.

5.3. **The continuum limit $\epsilon \to 0$: Laplacian.** Let $\hat{E}_{\sigma,\epsilon}$ and $\hat{E}_0$ be multiplication operators in $\hat{\mathrm{H}}_0$ of the form

$$(\hat{E}_{\sigma,\epsilon}\hat{f})(\boldsymbol{\xi}) = \hat{E}_{\sigma,\epsilon}(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}) \quad \text{and} \quad (\hat{E}_0\hat{f})(\boldsymbol{\xi}) = \hat{E}_0(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi}), \tag{5.21a}$$

with

$$\hat{E}_{\sigma,\epsilon}(\boldsymbol{\xi}) = \epsilon^{-2}\sum_{\nu\in W(\sigma)}\bigl(1-\cos(\epsilon\langle\nu,\boldsymbol{\xi}\rangle)\bigr) \quad \text{and} \quad \hat{E}_0(\boldsymbol{\xi}) = \|\boldsymbol{\xi}\|^2. \tag{5.21b}$$

We introduce the operators $L_{\sigma,\epsilon}$ and L_0 in H_0 as the pullbacks of $\hat{E}_{\sigma,\epsilon}$ and $\hat{E}_0$ with respect to the eigenfunction transforms $F_\epsilon : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ and $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$:

$$L_{\sigma,\epsilon} := \hat{F}_\epsilon\hat{E}_{\sigma,\epsilon}F_\epsilon, \tag{5.22a}$$

$$L_0 := \hat{F}_0\hat{E}_0F_0. \tag{5.22b}$$

The operator L_0 (5.22b) amounts to the Laplacian $-\nabla_{\mathbf{x}}^2$ in the Weyl chamber $\mathbf{C}$ with boundary conditions at the walls of the form in Eq. (3.4b), and the operator $L_{\sigma,\epsilon}$ (5.22a) corresponds to the lift of $\epsilon^{-2}\bigl(|W(\sigma)| - L_\sigma/2 - L_{w_0(\sigma)}/2\bigr)$ from $\mathcal{H}$ to H_ϵ:

$$L_{\sigma,\epsilon}J_\epsilon = \frac{1}{2\epsilon^2}\left(2|W(\sigma)| - L_\sigma - L_{w_0(\sigma)}\right), \tag{5.23}$$

where L_σ denotes the discrete Laplacian defined in Eqs. (4.1a)–(4.1c). The following proposition states that, in the continuum limit $\epsilon \to 0$, the discrete difference operator $L_{\sigma,\epsilon}$ (5.22a) tends (up to a positive factor) to the differential operator L_0 (5.22b) in the strong resolvent sense.

Proposition 5.5 (Continuum Limit: the Laplacian)**.** *Let $z \in \mathbb{C} \setminus [0,\infty)$. Then*

$$s - \lim_{\epsilon\to 0}(L_{\sigma,\epsilon} - z\mathbf{I}_{\mathrm{H}_0})^{-1} = (c_\sigma L_0 - z\mathbf{I}_{\mathrm{H}_0})^{-1}$$

for some positive constant c_σ.

Proof. From the limit

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\sum_{\nu\in W(\sigma)}\bigl(1-\cos(\epsilon\langle\nu,\boldsymbol{\xi}\rangle)\bigr) = \frac{1}{2}\sum_{\nu\in W(\sigma)}|\langle\nu,\boldsymbol{\xi}\rangle|^2 = c_\sigma\|\boldsymbol{\xi}\|^2$$

for some positive constant c_σ, one concludes that $\lim_{\epsilon\to 0}\hat{E}_{\sigma,\epsilon}(\boldsymbol{\xi}) = c_\sigma\hat{E}_0(\boldsymbol{\xi})$ pointwise. Hence, for any $\hat{f}\in\hat{\mathrm{H}}_0$ and $z\in\mathbb{C}\setminus[0,\infty)$

$$\lim_{\epsilon\to 0}(\hat{E}_{\sigma,\epsilon} - z\mathbf{I}_{\hat{\mathrm{H}}_0})^{-1}\hat{f} = (c_\sigma\hat{E}_0 - z\mathbf{I}_{\hat{\mathrm{H}}_0})^{-1}\hat{f}$$

strongly, by the dominated convergence theorem. The proposition now follows from the telescope

$$\begin{aligned}
&\|(L_{\sigma,\epsilon} - z\mathbf{I}_{\mathrm{H}_0})^{-1}f - (c_\sigma L_0 - z\mathbf{I}_{\mathrm{H}_0})^{-1}f\|_{\mathrm{H}_0}\\
&\quad\le \|(L_{\sigma,\epsilon} - z\mathbf{I}_{\mathrm{H}_0})^{-1}(\hat{F}_0 - \hat{F}_\epsilon)F_0 f\|_{\mathrm{H}_0}\\
&\quad+ \|\hat{F}_\epsilon[(\hat{E}_{\sigma,\epsilon} - z\mathbf{I}_{\hat{\mathrm{H}}_0})^{-1} - (c_\sigma\hat{E}_0 - z\mathbf{I}_{\hat{\mathrm{H}}_0})^{-1}]F_0 f\|_{\mathrm{H}_0}\\
&\quad+ \|(\hat{F}_\epsilon - \hat{F}_0)(c_\sigma\hat{E}_0 - z\mathbf{I}_{\hat{\mathrm{H}}_0})^{-1}F_0 f\|_{\mathrm{H}_0}
\end{aligned}$$

upon sending ϵ to zero (and invoking of Lemma 5.2). □

Appendix A. The Inversion Formula: Continuous Case

In the proof of Proposition 5.4 we needed the fact that the transformation $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$ in Eq. (5.10a)—or equivalently the transformation $\mathcal{F}_0 : \mathcal{H}_0 \to \hat{\mathcal{H}}_0$ in Eq. (3.9a)—is injective. In principle this injectivity follows from the analysis by Heckman and Opdam. Indeed, it was shown in Ref. [HO] that

$$\hat{\mathcal{F}}_0\mathcal{F}_0 = \mathbf{I}_{\mathcal{H}_0} \tag{A.1}$$

upon restriction to the dense subspace $C_0^\infty(\mathbf{C}) \subset \mathcal{H}_0$ (cf. also the comments just after Theorem 3.2). Since all operators involved are bounded, the inversion formula in Eq. (A.1) is readily extended from $C_0^\infty(\mathbf{C})$ to the whole of $\mathcal{H}_0$ (by taking the closure), whence the transformation $\mathcal{F}_0 : \mathcal{H}_0 \to \hat{\mathcal{H}}_0$ (and thus the transformation $F_0 : \mathrm{H}_0 \to \hat{\mathrm{H}}_0$) is injective.

The proof of Eq. (A.1) indicated in [HO] is quite sophisticated and hinges on a deep result due to Peetre regarding the characterization of differential operators as support preserving operators on smooth test functions [P1, P2]. In this appendix we present an elementary proof for this inversion formula.

Let $\phi \in C_0^\infty(\mathbf{C})$. Then

$$\begin{aligned}
(\mathcal{F}_0\phi)(\boldsymbol{\xi}) &= \int_{\mathbf{C}} \phi(\mathbf{x})\overline{\Psi_0(\mathbf{x};\boldsymbol{\xi})}\mathrm{d}\mathbf{x} \\
&= \sum_{w\in W} \mathcal{C}_0(-\boldsymbol{\xi}_w) \int_{\mathbf{C}} \phi(\mathbf{x}) e^{-i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}\mathrm{d}\mathbf{x} \\
&= \sum_{w\in W} \mathcal{C}_0(-\boldsymbol{\xi}_w)\, \breve{\phi}(\boldsymbol{\xi}_w),
\end{aligned} \tag{A.2a}$$

where

$$\mathcal{C}_0(\boldsymbol{\xi}) = \prod_{\alpha\in \boldsymbol{R}^+} \frac{\langle \alpha,\boldsymbol{\xi}\rangle - i g_\alpha}{\langle \alpha,\boldsymbol{\xi}\rangle} \tag{A.2b}$$

and

$$\breve{\phi}(\boldsymbol{\xi}) = \int_{\mathbf{E}} \phi(\mathbf{x}) e^{-i\langle \mathbf{x},\boldsymbol{\xi}\rangle}\mathrm{d}\mathbf{x}. \tag{A.2c}$$

Substitution of $\hat{\phi} = \mathcal{F}_0\phi$ into

$$\begin{aligned}
(\hat{\mathcal{F}}_0\hat{\phi})(\mathbf{x}) &= \frac{1}{(2\pi)^N} \int_{\mathbf{C}} \hat{\phi}(\boldsymbol{\xi})\Psi_0(\mathbf{x};\boldsymbol{\xi})\Delta_0(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^N} \sum_{w\in W} \int_{\mathbf{C}} \hat{\phi}(\boldsymbol{\xi})\mathcal{C}_0(\boldsymbol{\xi}_w) e^{i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}\Delta_0(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^N} \sum_{w\in W} \int_{\mathbf{C}} \hat{\phi}(\boldsymbol{\xi}) \frac{1}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} e^{i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}\mathrm{d}\boldsymbol{\xi}
\end{aligned} \tag{A.3}$$

yields

$$\begin{aligned}
(\hat{\mathcal{F}}_0\mathcal{F}_0\phi)(\mathbf{x}) &= \frac{1}{(2\pi)^N} \sum_{w_1,w_2\in W} \int_{\mathbf{C}} \frac{\mathcal{C}_0(-\boldsymbol{\xi}_{w_1})}{\mathcal{C}_0(-\boldsymbol{\xi}_{w_2})} \breve{\phi}(\boldsymbol{\xi}_{w_1}) e^{i\langle \mathbf{x},\boldsymbol{\xi}_{w_2}\rangle}\mathrm{d}\boldsymbol{\xi} \\
&= \frac{1}{(2\pi)^N} \sum_{w\in W} \int_{\mathbf{E}} \frac{\mathcal{C}_0(-\boldsymbol{\xi})}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} \breve{\phi}(\boldsymbol{\xi}) e^{i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}\mathrm{d}\boldsymbol{\xi},
\end{aligned} \tag{A.4a}$$

where

$$\begin{aligned}
\frac{\mathcal{C}_0(-\boldsymbol{\xi})}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} &= \prod_{\alpha\in\boldsymbol{R}^+} \frac{\langle\alpha,\boldsymbol{\xi}\rangle + i g_\alpha}{\langle\alpha,\boldsymbol{\xi}\rangle} \prod_{\alpha\in\boldsymbol{R}^+} \frac{\langle\alpha,\boldsymbol{\xi}_w\rangle}{\langle\alpha,\boldsymbol{\xi}_w\rangle + i g_\alpha} \\
&= \prod_{\alpha\in\boldsymbol{R}^+\cap w^{-1}(\boldsymbol{R}^-)} \frac{\langle\alpha,\boldsymbol{\xi}\rangle + i g_\alpha}{\langle\alpha,\boldsymbol{\xi}\rangle - i g_\alpha}.
\end{aligned} \tag{A.4b}$$

The inversion formula $\hat{\mathcal{F}}_0\mathcal{F}_0\phi = \phi$ for $\phi \in C_0^\infty(\mathbf{C})$ is now immediate from Eqs. (A.4a), (A.4b) combined with the fact that for $\mathbf{x}\in\mathbf{C}$ and $w\in W$

$$\frac{1}{(2\pi)^N} \int_{\mathbf{E}} \frac{\mathcal{C}_0(-\boldsymbol{\xi})}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} \breve{\phi}(\boldsymbol{\xi}) e^{i\langle \mathbf{x},\boldsymbol{\xi}_w\rangle}\mathrm{d}\boldsymbol{\xi} = \begin{cases} \phi(\mathbf{x}) & \text{if} \quad w = \boldsymbol{Id}, \\ 0 & \text{if} \quad w \neq \boldsymbol{Id}. \end{cases} \tag{A.5}$$

To infer the equality in Eq. (A.5), let us first note that the case $w = \boldsymbol{Id}$ is clear as it amounts to the standard Fourier inversion formula on $\mathbf{E}$. The case $w \neq \boldsymbol{Id}$ is verified with the aid of the following straightforward observations.

(i) For $\phi \in C_0^\infty(\mathbf{C})$ the Fourier transform $\breve{\phi}(\boldsymbol{\xi})$ (A.2c) is entire in $\boldsymbol{\xi}$ and rapidly decreasing on the tubular domain $\mathbf{E} - i\mathbf{C}^\vee$, where $\mathbf{C}^\vee$ denotes the open convex cone dual to $\mathbf{C}$, generated by the positive roots (i.e. $\mathbf{C}^\vee := \mathrm{Span}_{\mathbb{R}_+}(\boldsymbol{R}^+)$).

(ii) The parameter restriction $g_\alpha > 0$ ensures that the quotient $\mathcal{C}_0(-\boldsymbol{\xi})/\mathcal{C}_0(-\boldsymbol{\xi}_w)$ (A.4b) is holomorphic and bounded on the tubular domain $\mathbf{E} - i\mathbf{C}_w$, where $\mathbf{C}_w := \{\boldsymbol{\xi} \in \mathbf{E} \mid \langle \boldsymbol{\xi}, \alpha\rangle > 0,\ \forall \alpha \in \boldsymbol{R}^+ \cap w^{-1}(\boldsymbol{R}^-)\}$.

(iii) For all $\mathbf{x} \in \mathbf{C}$ and $\vartheta \in \mathbf{C}_w^\vee := \mathbf{C}^\vee \cap w^{-1}(-\mathbf{C}^\vee) = \mathrm{Span}_{\mathbb{R}_+}(\boldsymbol{R}^+ \cap w^{-1}(\boldsymbol{R}^-))$, one has that $\langle \mathbf{x}, \vartheta_w\rangle < 0$.

Indeed, we conclude from (i) and (ii) and the Cauchy integral theorem that for an arbitrary but fixed $\vartheta \in \mathbf{C}_w \cap \mathbf{C}_w^\vee$

$$\int_{\mathbf{E}} \frac{\mathcal{C}_0(-\boldsymbol{\xi})}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} \breve{\phi}(\boldsymbol{\xi}) e^{i\langle \mathbf{x}, \boldsymbol{\xi}_w\rangle} \mathrm{d}\boldsymbol{\xi} = \int_{\mathbf{E} - is\vartheta} \frac{\mathcal{C}_0(-\boldsymbol{\xi})}{\mathcal{C}_0(-\boldsymbol{\xi}_w)} \breve{\phi}(\boldsymbol{\xi}) e^{i\langle \mathbf{x}, \boldsymbol{\xi}_w\rangle} \mathrm{d}\boldsymbol{\xi} \tag{A.6}$$

for all $s \geq 0$. Furthermore, it follows from (i), (ii) and (iii) that for $s \to \infty$ the r.h.s. of Eq. (A.6) tends to zero, whence the case $w \neq \boldsymbol{Id}$ of the equality in Eq. (A.5) follows.

Note. To convince oneself that the cone $\mathbf{C}_w \cap \mathbf{C}_w^\vee$ is nonempty for any $w \in W \setminus \{\boldsymbol{Id}\}$, we observe that it contains the nonzero vector $\rho - \rho_{w^{-1}}$ (where $\rho = \sum_{\alpha \in \boldsymbol{R}^+} \alpha/2$). Indeed, we have on the one hand that $\rho \in \mathbf{C}$ and $\rho_{w^{-1}} \in w^{-1}(\mathbf{C})$, so $\rho - \rho_{w^{-1}} \in \mathbf{C}_w$, while on the other hand

$$\begin{aligned} \rho - \rho_{w^{-1}} &= \frac{1}{2} \sum_{\alpha \in \boldsymbol{R}^+} \alpha - \frac{1}{2} \sum_{\alpha \in \boldsymbol{R}^+ \cap w^{-1}(\boldsymbol{R}^+)} \alpha + \frac{1}{2} \sum_{\alpha \in \boldsymbol{R}^+ \cap w^{-1}(\boldsymbol{R}^-)} \alpha \\ &= \sum_{\alpha \in \boldsymbol{R}^+ \cap w^{-1}(\boldsymbol{R}^-)} \alpha, \end{aligned}$$

so $\rho - \rho_{w^{-1}} \in \mathbf{C}_w^\vee$.

Appendix B. Macdonald's Orthogonality Relations

In this appendix we outline the proof of Macdonald's orthogonality relations [M1, M3]

$$(\Psi_\lambda, \Psi_\mu)_{\hat{\mathcal{H}}} = \begin{cases} \Delta_\lambda^{-1} & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu, \end{cases} \tag{B.1}$$

for the Bethe wave functions $\Psi_\lambda(\boldsymbol{\xi})$ of Theorem 4.2. The proof, which follows Macdonald's treatment in Ref. [M3], hinges on the following two lemmas.

Lemma B.1 (Triangularity)**.** *The Bethe wave functions $\Psi_\lambda(\boldsymbol{\xi})$, $\lambda \in \mathcal{P}^+$ (cf. (4.11)) expand triangularly on the basis of monomial symmetric functions $m_\mu(\boldsymbol{\xi})$, $\mu \in \mathcal{P}^+$ (cf. (4.21)):*

$$\Psi_\lambda(\boldsymbol{\xi}) = \Delta_\lambda^{-1} m_\lambda(\boldsymbol{\xi}) + \sum_{\mu \in \mathcal{P}^+,\, \mu \prec \lambda} a_{\lambda\mu}\, m_\mu(\boldsymbol{\xi}),$$

with Δ_λ given by Eq. (4.12) and $a_{\lambda\mu} \in \mathbb{C}$.

Proof. Starting from $\Psi_\lambda(\boldsymbol{\xi})$ (4.7a)–(4.7c) with $\mathcal{C}(\boldsymbol{\xi})$ of the form in Eq. (4.9), one readily derives that

$$\begin{aligned}\Psi_\lambda(\boldsymbol{\xi}) &= \frac{1}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w e^{i\langle\rho+\lambda,\boldsymbol{\xi}_w\rangle}\prod_{\alpha\in\boldsymbol{R}^+}(1-t_\alpha e^{-i\langle\alpha,\boldsymbol{\xi}_w\rangle})\\ &= \frac{1}{\delta(\boldsymbol{\xi})}\sum_{w\in W}(-1)^w\Bigl(\sum_{X\subset\boldsymbol{R}^+}(-1)^{|X|}e^{i\langle\rho(X^c)-\rho(X)+\lambda,\boldsymbol{\xi}_w\rangle}\prod_{\alpha\in X}t_\alpha\Bigr)\\ &= \sum_{X\subset\boldsymbol{R}^+}(-1)^{w_X}(-1)^{|X|}\Bigl(\prod_{\alpha\in X}t_\alpha\Bigr)\chi_{\lambda(X)}(\boldsymbol{\xi}),\end{aligned}\tag{B.2}$$

where we have introduced the notation

$$\rho(X):=\frac{1}{2}\sum_{\alpha\in X}\alpha,\qquad \rho(X^c):=\frac{1}{2}\sum_{\alpha\in\boldsymbol{R}^+\setminus X}\alpha,$$
$$w_X:=w_{\rho(X^c)-\rho(X)+\lambda},\qquad \lambda(X):=w_X(\rho(X^c)-\rho(X)+\lambda)-\rho,$$

and where $\chi_\mu(\boldsymbol{\xi})$ denotes the Weyl character defined in Eq. (4.23). Since

$$\begin{aligned}\lambda(X) &= w_X(\lambda)+w_X(\rho(X^c)-\rho(X))-\rho\\ &= w_X(\lambda)-\sum_{\alpha\in\boldsymbol{R}^+\cap w_X(X)}\alpha+\sum_{\alpha\in\boldsymbol{R}^-\cap w_X(X^c)}\alpha\\ &\preceq w_X(\lambda)\preceq\lambda,\end{aligned}$$

it follows from Eq. (B.2) that the Bethe function $\Psi_\lambda(\boldsymbol{\xi})$ expands triangularly on the basis of Weyl characters:

$$\Psi_\lambda(\boldsymbol{\xi})=\sum_{\mu\in\mathcal{P}^+,\,\mu\preceq\lambda}b_{\lambda\mu}\,\chi_\mu(\boldsymbol{\xi}),$$

for certain complex coefficients $b_{\lambda\mu}$. To compute the leading coefficient $b_{\lambda\lambda}$ it is needed to collect all terms in Eq. (B.2) for which $\lambda(X)=\lambda$. These terms correspond to those subsets $X\subset\boldsymbol{R}^+$ for which $w_X(\lambda)=\lambda$ and $w_X^{-1}(\rho)=\rho(X^c)-\rho(X)$, or equivalently, to those subsets X for which $X=\{\alpha\in\boldsymbol{R}^+\mid w(\alpha)\in\boldsymbol{R}^-\}$ with $w\in W_\lambda$. We thus find that $b_{\lambda\lambda}$ is given by the Poincaré series of the stabilizer W_λ:

$$b_{\lambda\lambda}=\sum_{w\in W_\lambda}t_s^{\ell_s(w)}t_l^{\ell_l(w)}.$$

The lemma now follows from the well-known fact that the Weyl characters expand unitriangularly on the monomials $\chi_\lambda=m_\lambda+\sum_{\mu\in\mathcal{P}^+,\,\mu\prec\lambda}c_{\lambda\mu}\,\chi_\mu$, combined with Macdonald's celebrated product formula for the Poincaré series in question [M2, M3]

$$\sum_{w\in W_\lambda}t_s^{\ell_s(w)}t_l^{\ell_l(w)}=\prod_{\substack{\alpha\in\boldsymbol{R}^+\\ \langle\lambda,\alpha^\vee\rangle=0}}\frac{1-t_\alpha t_s^{\mathrm{ht}_s(\alpha)}t_l^{\mathrm{ht}_l(\alpha)}}{1-t_s^{\mathrm{ht}_s(\alpha)}t_l^{\mathrm{ht}_l(\alpha)}}.$$

□

Lemma B.2 (Bi-orthogonality Relations)**.** *The Bethe wave functions $\Psi_\lambda(\boldsymbol{\xi})$, $\lambda\in\mathcal{P}^+$ and the monomial symmetric functions $m_\mu(\boldsymbol{\xi})$, $\mu\in\mathcal{P}^+$ satisfy the bi-orthogonality relations*

$$(\Psi_\lambda,m_\mu)_{\hat{\mathcal{H}}}=\begin{cases}1 & \text{if } \mu=\lambda,\\ 0 & \text{if } \mu\not\preceq\lambda.\end{cases}$$

Proof. Starting from $\Psi_\lambda(\boldsymbol{\xi})$ (4.11) we obtain

$$\begin{aligned}(\Psi_\lambda, m_\mu)_{\hat{\mathcal{H}}} &= \frac{1}{|W|\,\mathrm{Vol}(\mathbf{A})\,|W_\mu|} \\ &\times \int_{\mathbf{A}} \hat{\Delta}(\boldsymbol{\xi}) \sum_{w_1\in W}\Bigl(e^{i\langle\lambda,\boldsymbol{\xi}_{w_1}\rangle}\prod_{\alpha\in\boldsymbol{R}^+}\frac{1-t_\alpha e^{-i\langle\alpha,\boldsymbol{\xi}_{w_1}\rangle}}{1-e^{-i\langle\alpha,\boldsymbol{\xi}_{w_1}\rangle}}\Bigr)\sum_{w_2\in W} e^{-i\langle\mu,\boldsymbol{\xi}_{w_2}\rangle}\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{\mathrm{Vol}(\mathbf{A})\,|W_\mu|}\sum_{w\in W}\int_{\mathbf{A}} e^{i\langle\lambda,\boldsymbol{\xi}\rangle - i\langle\mu,\boldsymbol{\xi}_w\rangle}\prod_{\alpha\in\boldsymbol{R}^+}\Bigl(\frac{1-e^{i\langle\alpha,\boldsymbol{\xi}\rangle}}{1-t_\alpha e^{i\langle\alpha,\boldsymbol{\xi}\rangle}}\Bigr)\mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{\mathrm{Vol}(\mathbf{A})\,|W_\mu|}\sum_{w\in W}\int_{\mathbf{A}} e^{i\langle\lambda-\mu_w,\boldsymbol{\xi}\rangle}\prod_{\alpha\in\boldsymbol{R}^+}\Bigl(1+\sum_{n_\alpha=1}^{\infty}(t_\alpha^{n_\alpha}-t_\alpha^{n_\alpha-1})e^{in_\alpha\langle\alpha,\boldsymbol{\xi}\rangle}\Bigr)\mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

The integral on the last line picks up the constant term of the integrand multiplied by the volume of the Weyl alcove. It is clear that a nonzero constant term can occur only if $\lambda - \mu_w \in -\mathcal{Q}^+$ for some $w \in W$. If $\mu \not\succeq \lambda$, then for all $w \in W$ also $\mu_w \not\succeq \lambda$ (since $\mu_w \preceq \mu$). Hence, in this situation the constant term vanishes. If $\mu = \lambda$, then the constant part of the term labelled by w is nonzero (namely equal to 1) if and only if $w \in W_\lambda$. By summing all these contributions originating from the stabilizer the lemma follows. □

It is immediate from Lemmas B.1 and B.2 that $(\Psi_\lambda, \Psi_\lambda)_{\hat{\mathcal{H}}} = (\Psi_\lambda, \Delta_\lambda^{-1} m_\lambda)_{\hat{\mathcal{H}}} = \Delta_\lambda^{-1}$ and that $(\Psi_\lambda, \Psi_\mu)_{\hat{\mathcal{H}}} = 0$ if $\mu \not\succeq \lambda$. But then $(\Psi_\lambda, \Psi_\mu)_{\hat{\mathcal{H}}}$ must in fact vanish for all $\mu \neq \lambda$ in view of the symmetry $(\Psi_\lambda, \Psi_\mu)_{\hat{\mathcal{H}}} = \overline{(\Psi_\mu, \Psi_\lambda)_{\hat{\mathcal{H}}}}$, which completes the proof of the orthogonality relations.

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