Homological representations of the Iwahori-Hecke algebra associated with a Selberg type integral

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Abstract

We realize a series of irreducible representations of the Iwahori-Hecke algebra on the space of homology group with coefficients in the local system associated with a Selberg type integral. This gives an affirmative answer to the conjecture by R. Lawrence [11].

A *Selberg type integral* is

$$\int_{\gamma(z_1,\ldots,z_n)} \prod_{1\le i<j\le m} (t_i - t_j)^{\nu} \prod_{1\le i\le m} \prod_{1\le j\le n} (t_i - z_j)^{\lambda_j} dt_1 \cdots dt_m, \tag{1}$$

where ν and λ_j are complex parameters, and γ is a suitable cycle. This is a natural generalization of the Selberg integral [14] and the Euler integral of Gauss' hypergeometric function. It is known that the integral (1) gives a conformal block of the minimal model [5] and that of $su(2)$ Wess-Zumino-Witten model [4] [6] [16] in conformal field theory.

Associated with the integral (1), a homology group with coefficients in local system can be defined. The purpose of this paper is to give a family of representations of the Iwahori-Hecke algebra in terms of the cycle of such a homology group. More precisely, we first describe an action of the braid group B_n on specific cycles (Theorem 1) and give linear relations among such cycles (Theorem 2); next, after imposing the condition on the exponents ν and λ_j's, which corresponds to a degenerate case of its homology group and is called a *resonant case* (see [12]), we extract a representation of the Iwahori-Hecke algebra $H(\mathfrak{S}_n)$ as a submodule of the previous representation of the braid group. This representation turns out to be the irreducible one parametrized by the two-row Young diagram $(n-m, m)$ for $n \ge 2m$ (Theorem 3). This is an affirmative answer to the conjecture by R. Lawrence (Conjecture 2.4 in [11], where she proved $m = 2$ case). We note that our irreducible representation corresponds to that of Tsuchiya-Kanie [18] in the conformal field theory, where they derive it without integral. We also note that the representation of B_n in Theorem 1 corresponds to the R-matrix for the m-th symmeric tensor representation of $U_q(sl_2)$, as is proved in [17].

1 Preliminary and results

Let $\mathrm{Conf}_n(\mathbb{C})$ be the configuration space of n distinct points of $\mathbb{C}$:

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \, ; \, z_i \neq z_j \text{ if } i \neq j \, \}.$$

For each point $z = (z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C})$, let T_z be a complex manifold $\mathbb{C}^m \backslash D_z$, where

$$D_z = \cup_{1 \leq i < j \leq m} \{ \, t_i - t_j = 0 \, \} \cup \cup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \{ \, t_i - z_j = 0 \, \}.$$

Let $u(t)$ be a multivalued holomorphic function

$$u(t) = \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\nu} \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} (t_i - z_j)^{\lambda_j}$$

defined on T_z. Let $\mathcal{L}_z$ be a sheaf consisting of the local solutions L of $dL = L\omega$ for $\omega = du(t)/u(t)$. Let $H^{lf}_m(T_z, \mathcal{L}_z)$ be the m-th locally finite homology group with coefficients in $\mathcal{L}_z$. Elements of this homology group are represented by ∂-closed locally finite chains

$$C = \sum_{\rho} a_\rho \rho \otimes v_\rho \quad (a_\rho \in \mathbb{C}),$$

where each ρ is an m-simplex and v_ρ a section of $\mathcal{L}_z$ on ρ. The boundary operator ∂ is defined to be a $\mathbb{C}$-linear mapping satisfying $\partial(\rho \otimes v) = \sum_{i=0}^m (-1)^i \rho^i \otimes v|_{\rho^i}$, where ρ is an m-simplex, ρ^i denotes the i-th face of ρ, and $v|_{\rho^i}$ is the restriction of v on ρ^i. This is a realization of the homology group with *local coefficients* defined by N. Steenrod [15]. We refer the reader [2] for more references on the application of such homology (and cohomology) theory to the hypergeometric type functions.

The action of the symmetric group $\mathfrak{S}_m$ of degree m on the space T_z is defined by $\sigma t = (t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(m)})$ for $\sigma \in \mathfrak{S}_m$ and $t = (t_1, \ldots, t_m) \in T_z$. This induces the action on $H^{lf}_m(T_z, \mathcal{L}_z)$. Take a chain $\gamma(t) = \rho(t) \otimes v_\rho(t)$. Then, for each $\sigma \in \mathfrak{S}_m$, $(\sigma v_\rho)(t) = v_\rho(\sigma^{-1} t)$ turns out to be a section of $\mathcal{L}_z$ on $(\sigma\rho)(t) = \rho(\sigma^{-1} t)$. Hence $(\sigma\gamma)(t) = \gamma(\sigma^{-1} t)$ gives a representation of $\mathfrak{S}_m$ on $H^{lf}_m(T_z, \mathcal{L}_z)$. In this paper we are interested in the $\mathfrak{S}_m$-invariant part $H^{lf}_m(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$.

The braid group B_n of n-strands is algebraically presented by generators $g_1, \ldots, g_{n-1}$ and the defining relations

$$g_i g_j = g_j g_i \quad \text{if} \quad |i - j| \geq 2, \quad 1 \leq i, j \leq n-1, \quad \text{and}$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \leq i \leq n-2.$$

Let $\nu : B_n \to \mathfrak{S}_n$ be a homomorphism defined by $\nu(g_i) = \sigma_i \, , 1 \leq i \leq n-1$, where $\sigma_i = (i, i+1)$ is the transposition of i and $i+1$. The kernel of this homomorphism is called the pure braid group. The fundamental group of $\mathrm{Conf}_n(\mathbb{C})$ is known to be the pure braid group, and the fundamental group of the quotient space $\mathrm{Conf}_n(\mathbb{C})/\mathfrak{S}_n$ is known to be the braid group, where $\mathfrak{S}_n$ acts naturally on $\mathrm{Conf}_n(\mathbb{C})$, namely, $\sigma z = (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)})$ for $\sigma \in \mathfrak{S}_n$ and $z = (z_1, \ldots, z_n) \in \mathrm{Conf}_n(\mathbb{C})$. In what follows, we identify the elements of the braid group B_n with the closed paths of $\mathrm{Conf}_n(\mathbb{C})/\mathfrak{S}_n$.

Let V be the vector bundle $\cup_{z\in \mathrm{Conf_n}(\mathbb{C})} H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$ with fiber $H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$. For each $z \in \mathrm{Conf_n}(\mathbb{C})$, a path $\tau \in B_n$ with starting point z induces the map

$$\tau \;:\; H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m} \longrightarrow H_m^{lf}(T_{\nu(\tau)z}, \mathcal{L}_{\nu(\tau)z})^{\mathfrak{S}_m}.$$

Hence, we have a representation ρ of $\tau \in B_n$ on $H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$ defined by

$$(\rho(\tau).\gamma)(z) = \nu(\tau)(\tau.\gamma)(z) \quad \text{for} \quad \gamma(z) \in H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}.$$

To study such a representation of the braid group B_n concretely and without loss of generality, we first fix a point $z \in \mathrm{Conf_n}(\mathbb{C})$ as

$$-\infty < z_1 < z_2 < \cdots < z_n < +\infty,$$

and a path $\tau_i = (\zeta_1(t), \ldots, \zeta_n(t))\,(0 \le t \le 1) \in \mathrm{Conf_n}(\mathbb{C})$ as

$$\zeta_i(t) = \frac{z_{i+1} + z_i}{2} - e^{\pi t\sqrt{-1}}\frac{z_{i+1} - z_i}{2},\; \zeta_{i+1}(t) = \frac{z_{i+1} + z_i}{2} + e^{\pi t\sqrt{-1}}\frac{z_{i+1} - z_i}{2},$$

and $\zeta_k(t) = z_k\;(k \ne i, i+1)$. Each path τ_i corresponds to a generator of B_n and called the *half Dehn twist* (see Figure 1).

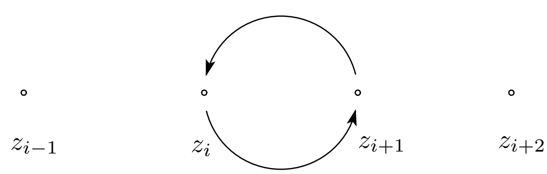


Figure 1

Let

$$\{\widetilde{\gamma}_{j_1\, j_2\, \ldots j_m}(z); 1 \le j_1 \le j_2 \le \cdots \le j_m \le n\}$$

be a set of elements of $H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$ (precise definition is given in Section 4). Then we have an expression of

$$\rho(\tau_i) \;:\; H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m} \longrightarrow H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m},$$

and relations among $\widetilde{\gamma}_{j_1\, j_2\, \ldots j_m}(z)$.

Theorem 1. *For $0 \leq i \leq n-1$, we have*

$$\rho(\tau_i)\widetilde{\gamma}_{1^{a_1}\cdots n^{a_n}}(z)$$

$$= \sum_{s=0}^{a_i} \Big(e(\lambda_{i+1} + a_{i+1}\frac{\nu}{2}); e(\frac{\nu}{2}) \Big)_{a_i - s} e(s(\lambda_{i+1} + a_{i+1}\frac{\nu}{2})) \begin{bmatrix} a_i \\ s \end{bmatrix}_{e(\frac{\nu}{2})}$$

$$\times \widetilde{\gamma}_{\cdots i^{a_{i+1}+a_i-s}\,(i+1)^s \cdots}(z),$$

where $\widetilde{\gamma}_{1^{a_1}\cdots n^{a_n}}$ *means*

$$\widetilde{\gamma}_{\underbrace{1\cdots 1}_{a_1}\underbrace{2\cdots 2}_{a_2}\cdots\underbrace{n\cdots n}_{a_n}}(z)$$

with $a_1 + \cdots + a_n = m$, *and*

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}} \quad \textit{with} \quad (a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i).$$

Theorem 2. *Suppose that $\lambda_1 + \cdots + \lambda_n + (m-1)\nu \notin \mathbb{Z}$. Then, for $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{m-1} \leq n$, we have*

$$\sum_{t=0}^{m-1} \sum_{s=j_t+1}^{j_{t+1}-1} (e(\lambda_1 + \cdots + \lambda_{s-1} + t\frac{\nu}{2}) - e(\lambda_1 + \cdots + \lambda_s + t\frac{\nu}{2}))$$
$$\times \widetilde{\gamma}_{j_1 \ldots j_t\, s\, j_{t+1} \cdots j_{m-1}}(z)$$
$$+ \sum_{t=1}^{m-1} (e(\lambda_1 + \cdots + \lambda_{j_t - 1} + (t-1)\frac{\nu}{2}) - e(\lambda_1 + \cdots + \lambda_{j_t} + t\frac{\nu}{2}))$$
$$\times \widetilde{\gamma}_{j_1 \ldots j_{t-1}\, j_t\, j_t\, j_{t+1} \cdots j_{m-1}}(z)$$

with $j_0 = 0$ and $j_m = n+1$.

Remark 1. The representation of Theorem 1 in the case of $m = 2$ (called the Lawrence-Krammer representation) is used in [3] [9] [10] to prove that the Braid groups are linear.
Remark 2. Contents of Theorem 1 corresponds to Proposition 5.1 in [7], where the action is considered as the one on the paths, not on the cycle of the (locally finite) homology with coefficients in the local system. See also [17].

Next, if we impose the condition that $\lambda_j + \dfrac{\nu}{2} \in \mathbb{Z}$ for $1 \leq j \leq n$, then $V = \oplus_{1 \leq j_1 < \cdots < j_m \leq n} \mathbb{C}\widetilde{\gamma}_{j_1 \cdots j_m}$ factors through the space of $H_m^{lf}(T_z, \mathcal{L}_z)^{\mathfrak{S}_m}$ as a submodule over the braid group B_n, and moreover V turns out to be a module over the Iwahori-Hecke algebra $H(\mathfrak{S}_n)$.

The Iwahori-Hecke algebra $H(\mathfrak{S}_n)$ is the associative $\mathbb{C}$-algebra with generators $g_1, \ldots, g_{n-1}$, a parameter $q \in \mathbb{C}$, and the defining relations

$$\begin{cases} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}\,, & 1 \le i \le n-2, \\ g_i g_j = g_j g_i\,, & |i-j| \ge 2, \\ (g_i - 1)(g_i + q) = 0, & q \in \mathbb{C}. \end{cases}$$

Under the condition

$$q(1+q)\cdots(1+q+\cdots+q^{n-1}) \neq 0, \tag{2}$$

called *n-regular*, $H(\mathfrak{S}_n)$ is isomorphic to the group algebra $\mathbb{C}\mathfrak{S}_n$; hence, the representation theory of $H(\mathfrak{S}_n)$ is equivalent to that of $\mathfrak{S}_n$, and, in particular, irreducible representations of $H(\mathfrak{S}_n)$ are parametrized by the Young diagram with n boxes (See [8] [19]).

At this stage, we can sate our main theorem.

Theorem 3. *Suppose that* $\lambda_j + \frac{\nu}{2} \in \mathbb{Z}$ $(1 \le j \le n)$, $\frac{\nu}{2}(2(m-1)-n) \notin \mathbb{Z}$ *and* $n \ge 2m$. *Then*

$$\sum_{1 \le j_1 < \cdots < j_m \le n} \mathbb{C}\widetilde{\gamma}_{j_1 \cdots j_m}$$

is an irreducible $H(\mathfrak{S}_n)$*-module with* $q = e(-\frac{\nu}{2})$ *parametrized by the Young diagram of type* $(n-m, m)$*, and its dimension is* $\binom{n}{m} - \binom{n}{m-1}$.

Remark 1. This is an affirmative answer for the Conjecture 2.4 of [11], where only the cases of $m = 1$ and $m = 2$ are proved.

Remark 2. In [6], the solution of the Knizhnik-Zamolodchikov equation for the $SU(2)$ Wess-Zumino-Witten model is given in terms of the Selberg type integral with

$$\lambda_j = -\frac{1}{\kappa}, \quad \frac{\nu}{2} = \frac{1}{\kappa} \quad \text{and} \quad \kappa = l + 2,$$

where l is the level of the model. We note that n-regular is equivalent to $l \neq 0, 1, \ldots, n-2$ and that the cases $l = n-1, n, n+1, \ldots$ are available. Compare our theorem with the result of [18].

Remark 3. Dimension $\binom{n}{m} - \binom{n}{m-1}$ is equal to the rank of the twisted cohomology $H^m(T_z, \mathcal{L}_z^\vee)^{\mathfrak{S}_m}$, where $\mathcal{L}_z^\vee$ is dual to $\mathcal{L}_z$ (See [1]).

A proof of these Theorems 1 and 2 in case of $m = 1$ will be given in Section 2. By using the technique in Section 2, a proof of the $m = 2$ case will be given in Section 3. And a proof of the general m case will be completed by induction in Section 4. Concerning with Theorem 3, we will give a proof in Section 5.

2 Case $m = 1$.

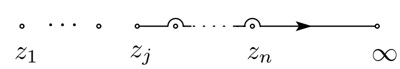
Set $u(t)$ to be

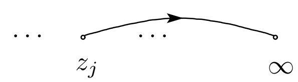
$$u(t) = \prod_{1\le j\le n} (t - z_j)^{\lambda_j},$$

where $\lambda_j \in \mathbb{C}$ $(j = 1, \ldots, n)$ and first $z = (z_1, \ldots, z_n)$ is fixed to be

$$-\infty < z_1 < \cdots < z_n < +\infty.$$

Let $\mathcal{L}_z$ be the local system on $T_z = \mathbb{C}\backslash\{z_1, \ldots, z_n\}$ determined by $u(t)$, that is, the sheaf of the local solutions of $dL/dt = \sum_{1\le j\le n} \dfrac{\lambda_j}{t - z_j} L$. For each $j = 1, \ldots, n$, define the element $\gamma_j(z) \in H_1^{lf}(T_z, \mathcal{L}_z)$ to be

$\gamma_j(z) = \Big\{$  $\Big\} \otimes u(t)$

$= \Big\{$  $\Big\} \otimes u(t),$

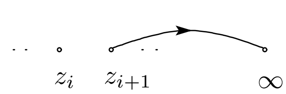
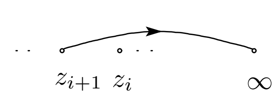
where the argument of each factor of $u(t)$ on the path is fixed to be $\arg(t - z_j) = 0$ for $t > z_n$. Then, the linear mapping τ_i is described as follows, which leads to Theorem 1 in the case of $m = 1$.

Proposition 1. *For $i = 1, \ldots, n - 1$, we have*

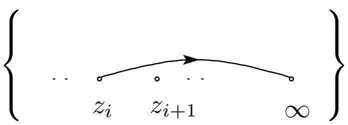
$$\begin{cases} \gamma_{i+1}(z) & \overset{\tau_i}{\longmapsto} \gamma_i(\ldots, z_{i+1}, z_i, \ldots), \\ \gamma_i(z) & \overset{\tau_i}{\longmapsto} (1 - e(\lambda_{i+1}))\gamma_i(\ldots, z_{i+1}, z_i, \ldots) + e(\lambda_{i+1})\gamma_{i+1}(\ldots, z_{i+1}, z_i, \ldots), \\ \gamma_k(z) & \overset{\tau_i}{\longmapsto} \gamma_k(\ldots, z_{i+1}, z_i, \ldots)\,, \qquad k \neq i, i+1, \end{cases}$$

where $e(\lambda) = \exp(2\pi\sqrt{-1}\lambda)$.

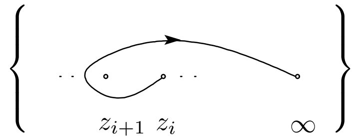
Proof. These relations are obtained by deforming the cycles appropriately. The third relation is trivially derived. For the first relation, it is enough to look at the following.

$\Big\{$  $\Big\} \otimes u(t) \overset{\tau_i}{\longmapsto} \Big\{$  $\Big\} \otimes u(t).$

For the second relation, we first describe the transformation of τ_i as

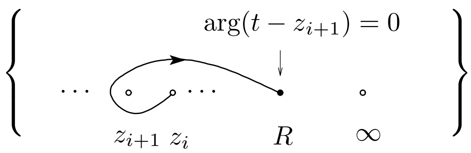


$\otimes u(t) \overset{\tau_i}{\longmapsto}$



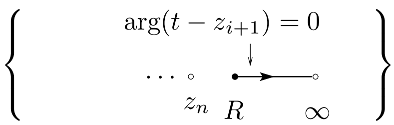
$\otimes u(t).$

The right hand side is regarded as the sum of the following two cycles.



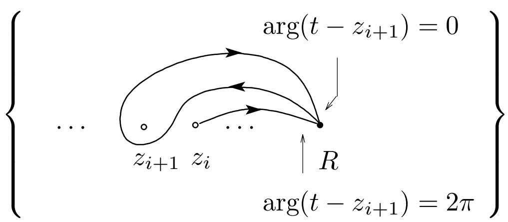
$\otimes u(t)$

$+$



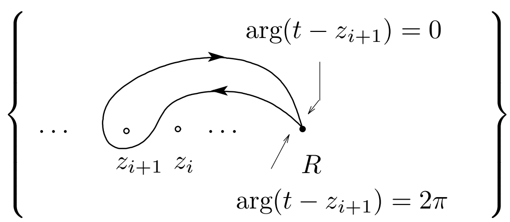
$\otimes u(t).$

Here $z_n < R$, and the first term is described as



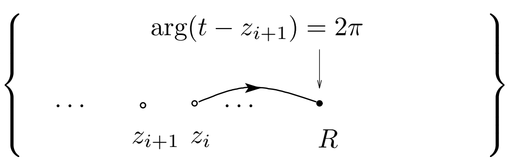
$\otimes u(t)$

$=$



$\otimes u(t)$

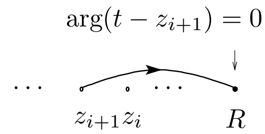
$+$



$\otimes u(t)\,.$ (3)

By using Lemma 1 below, the first term of the right hand side of (2) turns out to be

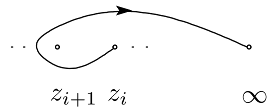
$$(1-e(\lambda_{i+1}))\Big\{$$



$$\Big\}\otimes u(t).$$

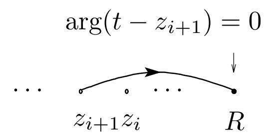
Hence we have

$$\Big\{$$



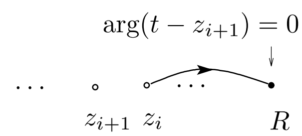
$$\Big\}\otimes u(t)$$

$$=(1-e(\lambda_{i+1}))\Big\{$$



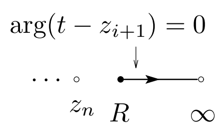
$$\Big\}\otimes u(t)$$

$$+\ e(\lambda_{i+1})\Big\{$$



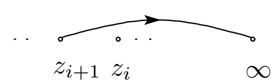
$$\Big\}\otimes u(t)$$

$$+\Big\{$$



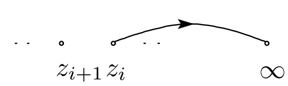
$$\Big\}\otimes u(t)$$

$$=(1-e(\lambda_{i+1}))\Big\{$$



$$\Big\}\otimes u(t)$$

$$+\ e(\lambda_{i+1})\Big\{$$

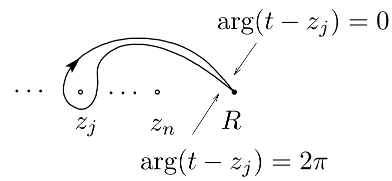


$$\Big\}\otimes u(t)$$

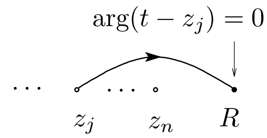
$$=(1-e(\lambda_{i+1}))\gamma_i(\ldots,z_{i+1},z_i,\ldots)+e(\lambda_{i+1})\gamma_{i+1}(\ldots,z_{i+1},z_i,\ldots).$$

This completes the proof.

Lemma 1. *Fix R to be $z_n < R$. Then, as a loaded cycle,*

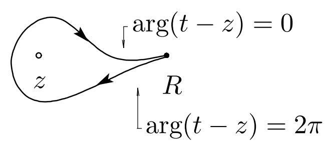
$\left\{ \right.$  $\left. \right\} \otimes u(t)$

is homologous to

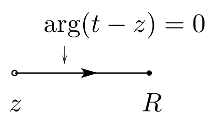
$(1 - e(\lambda_j)) \left\{ \right.$  $\left. \right\} \otimes u(t),$

where $e(\lambda) = \exp(2\pi\sqrt{-1}\lambda)$.

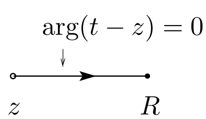
Proof. It is enough to prove that a loaded cycle

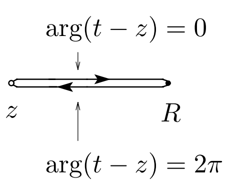
$\left\{ \right.$  $\left. \right\} \otimes (t - z)^\lambda$

for $z < R$ is homologous to

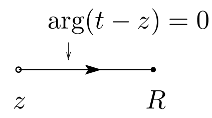
$(1 - e(\lambda)) \left\{ \right.$  $\left. \right\} \otimes (t - z)^\lambda.$

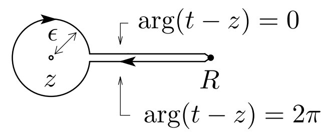
To prove it, we first note that

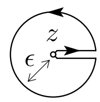
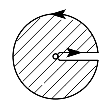
$(1 - e(\lambda)) \left\{ \right.$  $\left. \right\} \otimes (t - z)^\lambda$

$= \left\{ \right.$  $\left. \right\} \otimes (t - z)^\lambda.$

By using this, we have

$(1-e(\lambda))\ \{$  $\} \otimes (t-z)^{\lambda}$

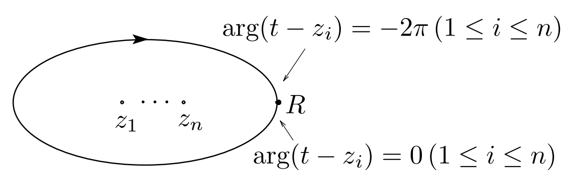
$-\ \{$  $\} \otimes (t-z)^{\lambda}$

$=\ \{$  $\} \otimes (t-z)^{\lambda} = \{\ \partial$  $\} \otimes (t-z)^{\lambda}.$

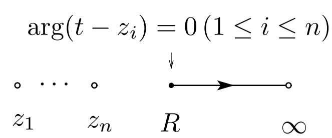
The right-most is the boundary of a locally finite loaded 2-chain supported by a slitted disc. This completes the proof of Lemma 1.

We prepare two more Lemmas for the proof of Theorem 2 in the case of $m=1$.

Lemma 2. *For a fixed real number R such that $z_n < R$, as a loaded cycle,*

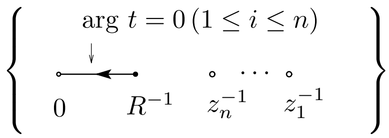
$\{$  $\} \otimes u(t)$

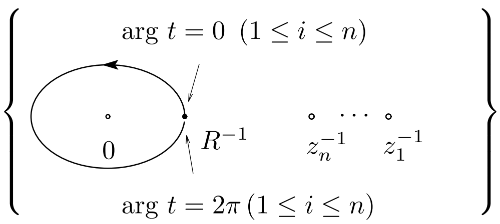
is homologous to

$(1-e(-\lambda_1-\cdots-\lambda_n))\ \{$  $\} \otimes u(t),$

where $e(\lambda)=\exp(2\pi\sqrt{-1}\lambda)$.

Proof. Change of the coordinate such that $t \to 1/t$ shows that the equality is equivalent to

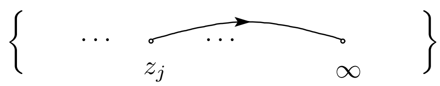
$(1 - e(-\lambda_1 - \cdots - \lambda_n))$  $\otimes\, u(t^{-1})$

$=$  $\otimes\, u(t^{-1})$.

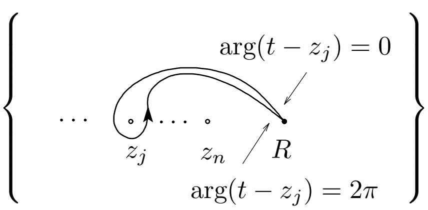
Here $u(t^{-1}) = t^{-\lambda_1 - \cdots - \lambda_n}(1 - z_1 t)^{\lambda_1} \cdots (1 - z_n t)^{\lambda_n}$. This equality follows from Lemma 1. This completes the proof of Lemma 2.

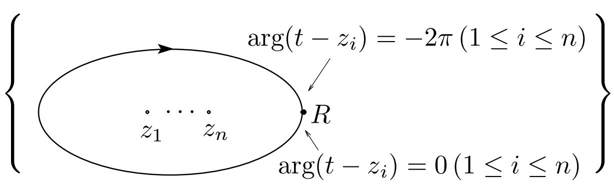
Combination of Lemma 1 and Lemma 2 implies the following.

Lemma 3. *Suppose that $\lambda_j \notin \mathbb{Z}\,(1 \le j \le n)$ and $\lambda_1 + \cdots + \lambda_n \notin \mathbb{Z}$. Then, for a fixed real number R such that $z_n < R$,*

$\gamma_j(z) =$  $\otimes\, u(t)$

is homologous to

$\frac{1}{e(\lambda_j) - 1}$  $\otimes\, u(t)$

$- \frac{1}{e(-\lambda_1 - \cdots - \lambda_n) - 1}$  $\otimes\, u(t)$,

as a loaded cycle.

At this stage, we prove the following, which is Theorem 2 in the case of $m = 1$.

Proposition 2. *Suppose that $\lambda_1 + \cdots + \lambda_n \notin \mathbb{Z}$. Then we have a liear relation among $\gamma_i(z)$ $(1 \leq i \leq n)$ such as*

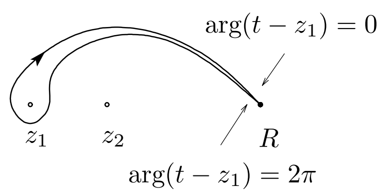
$$\sum_{j=1}^{n} (e(\lambda_1 + \cdots + \lambda_{j-1}) - e(\lambda_1 + \cdots + \lambda_j))\gamma_j(z) = 0.$$

Proof. For simplicity, we demonstrate the $n = 2$ case: $u(t) = (t - z_1)^{\lambda}(t - z_2)^{\lambda}$; the general n case is obtained in the same way.

Lemma 3 implies

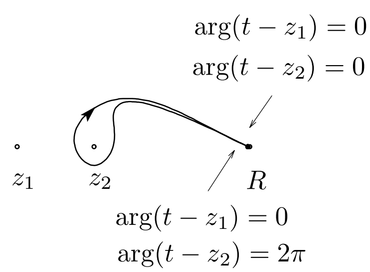
$$(1 - e(\lambda_1))\gamma_1(z) + (e(\lambda_1) - e(\lambda_1 + \lambda_2))\gamma_2(z)$$

$= \Big\{$



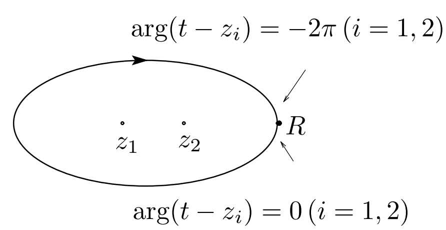
$\Big\} \otimes u(t)$

$+\, e(\lambda_1) \Big\{$



$\Big\} \otimes u(t)$

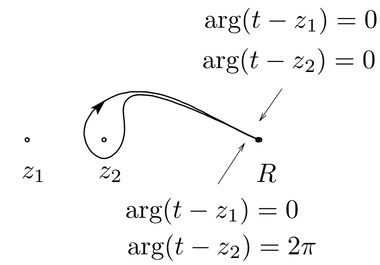
$-\, \dfrac{1 - e(\lambda_1 + \lambda_2)}{e(-\lambda_1 - \lambda_2) - 1} \Big\{$



$\Big\} \otimes u(t).$ (4)

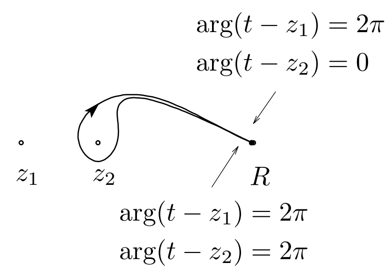
Since

$e(\lambda_1) \Bigg\{$



$\Bigg\} \otimes u(t)$

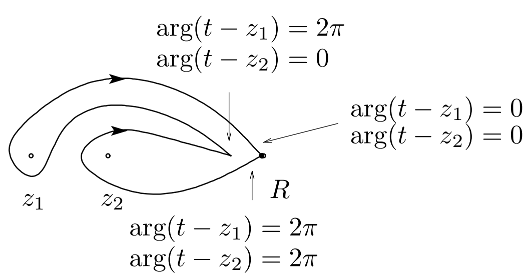
$= \Bigg\{$



$\Bigg\} \otimes u(t),$

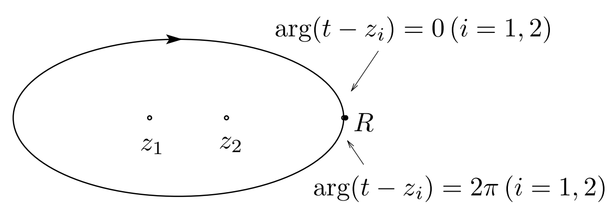
the sum of the first and the second term of the right handside of (3) is equal to

$\Bigg\{$



$\Bigg\} \otimes u(t)$

$= \Bigg\{$



$\Bigg\} \otimes u(t).$

Hence we have the result.

3 Case $m = 2$.

Set $u(t)$ to be

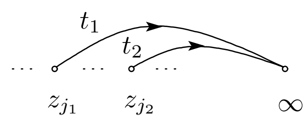
$$u(t) = (t_1 - t_2)^\nu \prod_{1\le i\le 2} \prod_{1\le j\le n} (t_i - z_j)^{\lambda_j},$$

where λ_j $(j = 1, \dots, n)$ and ν are complex numbers, and $z = (z_1, \dots, z_n)$ is first fixed to be

$$-\infty < z_1 < \cdots < z_n < +\infty.$$

For $1 \le j_1 \le j_2 \le n$, let $\gamma_{j_1\, j_2}(t; z)$ be the element

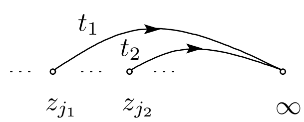
$\gamma_{j_1\, j_2}(t;z) = \Big\{$



$\Big\} \otimes u(t) \in H_2^{lf}(T_z, \mathcal{L}_z)\,,$

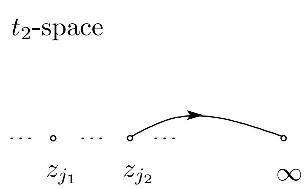
where the argument of $t_i - z_j$ on the path is fixed to be zero for $t_i > z_n$ and the argument of $t_1 - t_2$ on the path is fixed to be zero for $t_1 > t_2$. This picture describes two paths; one is the path in t_1-space , and the other one is the path in t_2-space:

$\Big\{$

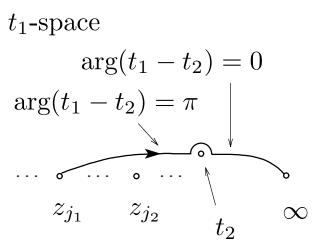


$\Big\} \otimes u(t)$

$= \Big\{$



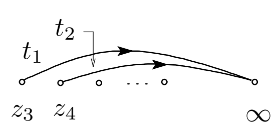
$\Big\}\Big\{$



$\Big\} \otimes u(t).$

The suffix $j_1 j_2$ does not indicate the points z_{j_1} and z_{j_2} but the j_1-th point and the j_2-th point. Hence, for instance,

$\gamma_{1\,2}(t; z_3, z_4, \dots) = \Big\{$



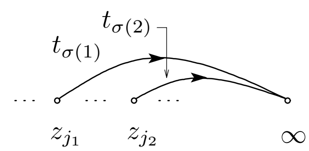
$\Big\} \otimes u(t)\,.$

A symmetrization of $\gamma_{j_1\,j_2}(t;z)$ with respect to the variables $t=(t_1,\,t_2)$ is defined to be

$$\widetilde{\gamma}_{j_1\,j_2}(z)=\sum_{\sigma\in\mathfrak{S}_2}\gamma_{j_1\,j_2}(t_{\sigma(1)},t_{\sigma(2)};z)$$

with

$$\gamma_{j_1\,j_2}(t_{\sigma(1)},t_{\sigma(2)};z)=\left\{\quad\right\}\otimes u(t_{\sigma(1)},t_{\sigma(2)})\,,$$

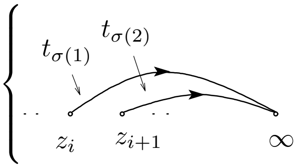


where the argument of $t_{\sigma(i)}-z_j$ on the path is fixed to be zero for $t_{\sigma(i)}>z_n$ and the argument of $t_{\sigma(1)}-t_{\sigma(2)}$ on the path is fixed to be zero for $t_{\sigma(1)}>t_{\sigma(2)}$. It is seen that $\widetilde{\gamma}_{j_1\,j_2}(z)$ is an element of the symmetric part of $H_2^{lf}(T_z,\mathcal{L}_z)$. The linear mapping $\tau_i:H_2^{lf}(T_z,\mathcal{L}_z)^{\mathfrak{S}_2}\mapsto H_2^{lf}(T_{\sigma z},\mathcal{L}_{\sigma z})^{\mathfrak{S}_2}$ is described as follows, which induces Theorem 1 in the case of $m=2$.

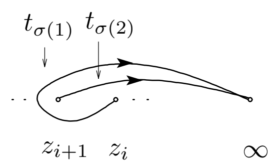
Proposition 3. *For $i=1,\ldots,n-1$, we have*

$$\begin{cases}
\widetilde{\gamma}_{i+1,i+1}(z) \stackrel{\tau_i}{\longmapsto} \widetilde{\gamma}_{i,i}(\ldots,z_{i+1},z_i,\ldots),\\
\widetilde{\gamma}_{i,i+1}(z) \stackrel{\tau_i}{\longmapsto} (1-e(\lambda_{i+1}+\frac{\nu}{2}))\,\widetilde{\gamma}_{i,i}(\ldots,z_{i+1},z_i,\ldots)\\
\qquad + e(\lambda_{i+1}+\frac{\nu}{2})\,\widetilde{\gamma}_{i,i+1}(\ldots,z_{i+1},z_i,\ldots),\\
\widetilde{\gamma}_{i,i}(z) \stackrel{\tau_i}{\longmapsto} (1-e(\lambda_{i+1}))(1-e(\lambda_{i+1}+\frac{\nu}{2}))\,\widetilde{\gamma}_{i,i}(\ldots,z_{i+1},z_i,\ldots)\\
\qquad + e(\lambda_{i+1})(1-e(\lambda_{i+1}))(1+e(\frac{\nu}{2}))\,\widetilde{\gamma}_{i,i+1}(\ldots,z_{i+1},z_i,\ldots)\\
\qquad + e(2\lambda_{i+1})\,\widetilde{\gamma}_{i+1,i+1}(\ldots,z_{i+1},z_i,\ldots),\\
\widetilde{\gamma}_{i+1,k_2}(z) \stackrel{\tau_i}{\longmapsto} \widetilde{\gamma}_{i,k_2}(\ldots,z_{i+1},z_i,\ldots),\quad i+1<k_2,\\
\widetilde{\gamma}_{k_1,i+1}(z) \stackrel{\tau_i}{\longmapsto} \widetilde{\gamma}_{k_1,i}(\ldots,z_{i+1},z_i,\ldots),\quad k_1<i,\\
\widetilde{\gamma}_{i,k_2}(z) \stackrel{\tau_i}{\longmapsto} (1-e(\lambda_{i+1}))\widetilde{\gamma}_{i,k_2}(\ldots,z_{i+1},z_i,\ldots)\\
\qquad + e(\lambda_{i+1})\widetilde{\gamma}_{i+1,k_2}(\ldots,z_{i+1},z_i,\ldots),\quad i+1<k_2,\\
\widetilde{\gamma}_{k_1,i}(z) \stackrel{\tau_i}{\longmapsto} (1-e(\lambda_{i+1}))\widetilde{\gamma}_{k_1,i}(\ldots,z_{i+1},z_i,\ldots)\\
\qquad + e(\lambda_{i+1})\widetilde{\gamma}_{k_1,i+1}(\ldots,z_{i+1},z_i,\ldots),\quad k_1<i,\\
\widetilde{\gamma}_{k_1,k_2}(z) \stackrel{\tau_i}{\longmapsto} \widetilde{\gamma}_{k_1,k_2}(\ldots,z_{i+1},z_i,\ldots),\quad k_1,k_2\neq i,i+1.
\end{cases}$$

Proof. The first relation is trivially derived. For the second relation, we start with the following.

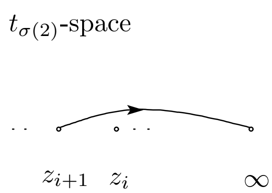
$$\sum_{\sigma \in \mathfrak{S}_2} \left\{ \right.$$


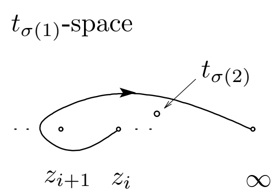
$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

$$\overset{\tau_i}{\longmapsto} \quad \sum_{\sigma \in \mathfrak{S}_2} \left\{ \right.$$


$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}). \qquad (5)$$

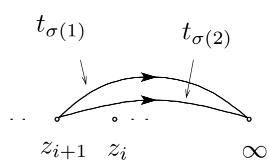
Here the summand of the right hand side is considered as

$$\left\{ \right.$$


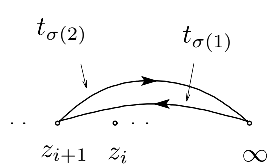
$$\left. \right\} \left\{ \right.$$


$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}).$$

Therefore, deformation of the cycle in $t_{\sigma(1)}$-space as in the proof of Proposition 1 shows that the summand of the right hand side of (4) is equal to

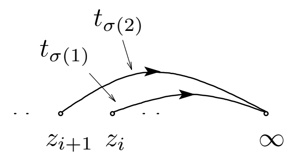
$$\left\{ \right.$$


$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

$$+ \, e(\lambda_{i+1} + \nu) \left\{ \right.$$


$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

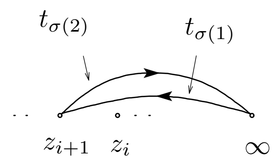
$$+\, e(\lambda_{i+1}+\nu) \Big\{$$



$$\Big\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}),$$

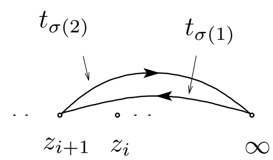
where $\arg(t_{\sigma(1)} - t_{\sigma(2)}) = 0$ for $t_{\sigma(1)} > t_{\sigma(2)}$ on the path and $\arg(t_i - z_n) = 0$ for $t_i > z_n$. On the other hand, Lemma 4 below gives the equality

$$\Big\{$$



$$\Big\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

$$= e(-\frac{\nu}{2}) \Big\{$$



$$\Big\} \otimes u(t_{\sigma(2)}, t_{\sigma(1)}).$$

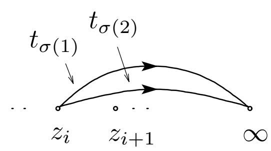
Consequently, the right hand side of (4) turns out to be

$$(1 - e(\lambda_{i+1} + \frac{\nu}{2}))\widetilde{\gamma}_{i,i}(\ldots, z_{i+1}, z_i, \ldots) + e(\lambda_{i+1} + \frac{\nu}{2})\widetilde{\gamma}_{i,i+1}(\ldots, z_{i+1}, z_i, \ldots).$$

This is a second relation of Proposition 3.

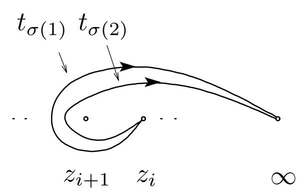
For the third relation, we start with the following:

$$\sum_{\sigma \in \mathfrak{S}_2} \Big\{$$



$$\Big\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

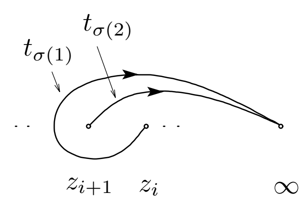
$$\stackrel{\tau_i}{\longmapsto} \quad \sum_{\sigma \in \mathfrak{S}_2} \Big\{$$



$$\Big\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}).$$

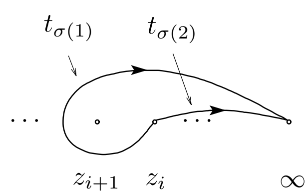
By applying the second relation of Proposition 1 to the path in the $t_{\sigma(2)}$-space, the right hand side turns out to be

$$(1-e(\lambda_{i+1}))\sum_{\sigma\in\mathfrak{S}_2}\left\{\right.$$



$$\left.\right\}\otimes u(t_{\sigma(1)},t_{\sigma(2)})$$

$$+\,e(\lambda_{i+1})\sum_{\sigma\in\mathfrak{S}_2}\left\{\right.$$

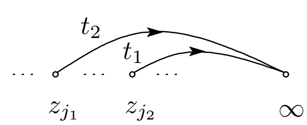


$$\left.\right\}\otimes u(t_{\sigma(1)},t_{\sigma(2)}).$$

Apply the second relation of the present Proposition to the path in $t_{\sigma(1)}$-space of the first term and the second relation of Proposition 1 to the path in $t_{\sigma(1)}$-space of the second term. Then we have the third desired relation. Other relations which remains to be proved are induced by Proposition 1. This completes the proof of Proposition 3.

Lemma 4. *The loaded cycle*

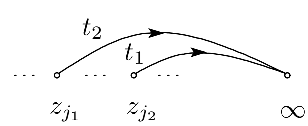
$$\left\{\right.$$



$$\left.\right\}\otimes u(t_1,t_2),$$

where $\arg(t_1-t_2)=0$ *on the path for* $t_1>t_2$ *and* $\arg(t_i-z_n)=0$ *for* $t_i>z_n$, *is homologous to*

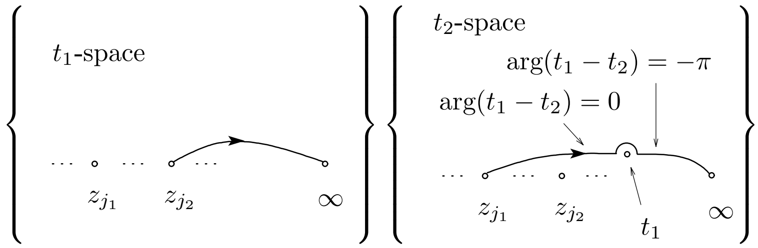
$$e(-\frac{\nu}{2})\left\{\right.$$



$$\left.\right\}\otimes u(t_2,t_1),$$

where $\arg(t_2-t_1)=0$ *on the path for* $t_2>t_1$ *and* $\arg(t_i-z_n)=0$ *for* $t_i>z_n$, *that is,* $\gamma_{j_1j_2}(t_2,t_1;z)$.

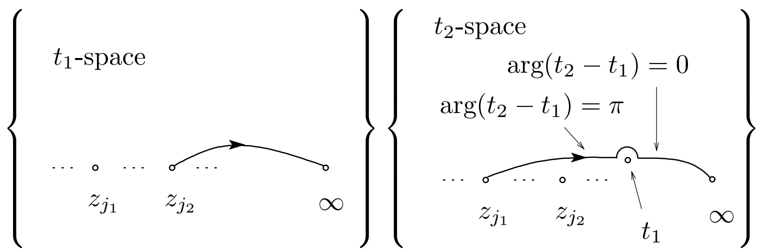
Proof. The left hand side is considered as



$\otimes\, u(t_1, t_2)$.

Hence, by considering $u(t)$ only as a function of t_2, this turns out to be

$e(-\frac{\nu}{2})$



$\otimes u(t_2, t_1)$,

which is the right hand side of the desired equality. This completes the proof of Lemma 4.

Next, linear relations among $\widetilde{\gamma}_{j_1\, j_2}(z)$ are given as follows, which is Theorem 2 in the case of $m = 2$.

Proposition 4. *Suppose that $\lambda_1 + \cdots + \lambda_n + \nu \notin \mathbb{Z}$. Then, for $1 \leq j \leq n$, we have*

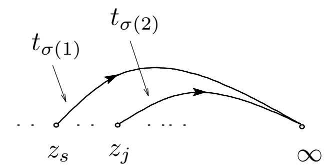
$$\sum_{s=1}^{j-1}(e(\lambda_1 + \cdots + \lambda_{s-1}) - e(\lambda_1 + \cdots + \lambda_s))\,\widetilde{\gamma}_{s,\,j}(z)$$

$$+\,(e(\lambda_1 + \cdots + \lambda_{j-1}) - e(\lambda_1 + \cdots + \lambda_j + \frac{\nu}{2}))\,\widetilde{\gamma}_{j,\,j}(z)$$

$$+\sum_{s=j+1}^{n}(e(\lambda_1 + \cdots + \lambda_{s-1} + \frac{\nu}{2}) - e(\lambda_1 + \cdots + \lambda_s + \frac{\nu}{2}))\,\widetilde{\gamma}_{j,\,s}(z) = 0.$$

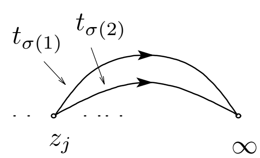
Proof. We have the equality

$$\left\{\sum_{s=1}^{j-1}(e(\lambda_1+\cdots+\lambda_{s-1})-e(\lambda_1+\cdots+\lambda_s))\right\{$$



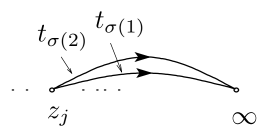
$$\Big\}$$

$$+\,e(\lambda_1+\cdots+\lambda_{j-1})\Big\{$$



$$\Big\}$$

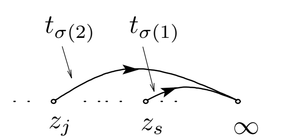
$$-\,e(\lambda_1+\cdots+\lambda_j+\nu)\Big\{$$



$$\Big\}$$

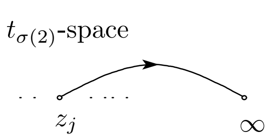
$$+\sum_{s=j+1}^{n}(e(\lambda_1+\cdots+\lambda_{s-1}+\nu)-e(\lambda_1+\cdots+\lambda_s+\nu))$$

$$\Big\{$$



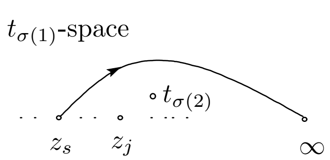
$$\Big\}\Big\}\otimes u(t_{\sigma(1)},t_{\sigma(2)})$$

$$=\Big\{$$

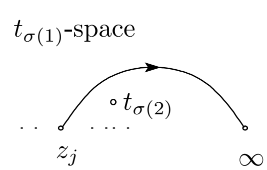


$$\Big\}\left\{\sum_{s=1}^{j-1}(e(\lambda_1+\cdots+\lambda_{s-1})-e(\lambda_1+\cdots+\lambda_s))\right.$$

$$\Big\{$$

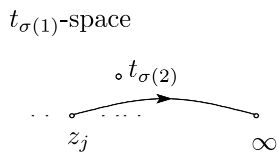


$$\Big\}+e(\lambda_1+\cdots+\lambda_{j-1})\Big\{$$



$$\Big\}$$

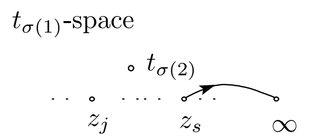
$$- e(\lambda_1 + \cdots + \lambda_j + \nu) \left\{ \right.$$



$$\left. \right\}$$

$$+ \sum_{s=j+1}^{n} (e(\lambda_1 + \cdots + \lambda_{s-1} + \nu) - e(\lambda_1 + \cdots + \lambda_s + \nu))$$

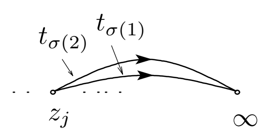
$$\left\{ \right.$$



$$\left. \right\} \Big\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}) = 0,$$

where $\arg(t_{\sigma(1)} - t_{\sigma(2)}) = 0$ for $t_{\sigma(1)} > t_{\sigma(2)}$ on the path. Under the assumption that $\lambda_1 + \cdots + \lambda_n + \nu \notin \mathbb{Z}$, the last equality follows from the same argument as in the proof of Proposition 2 (apply it to the path in $t_{\sigma(1)}$-space). On the other hand, Lemma 4 implies that the loaded cycles

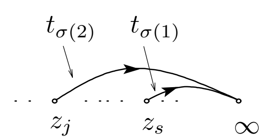
$$\left\{ \right.$$



$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)})$$

and

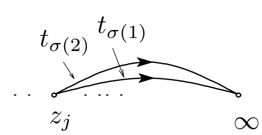
$$\left\{ \right.$$



$$\left. \right\} \otimes u(t_{\sigma(1)}, t_{\sigma(2)}),$$

where $\arg(t_{\sigma(1)} - t_{\sigma(2)}) = 0$ for $t_{\sigma(1)} > t_{\sigma(2)}$ on the path, are homologous to

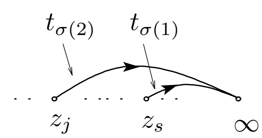
$$e(-\frac{\nu}{2}) \left\{ \right.$$



$$\left. \right\} \otimes u(t_{\sigma(2)}, t_{\sigma(1)})$$

and

$$e(-\frac{\nu}{2}) \left\{ \right.$$



$$\left. \right\} \otimes u(t_{\sigma(2)}, t_{\sigma(1)}),$$

where $\arg(t_{\sigma(2)} - t_{\sigma(1)}) = 0$ for $t_{\sigma(2)} > t_{\sigma(1)}$, respectively. Therefore, by taking a symmetric sum for $\sigma \in \mathfrak{S}_2$, we obtain the desired result.

4 General m case.

Let T_z be a complex manifold $\mathbb{C}^m \backslash D_z$, where

$$D_z = \cup_{1 \le i < j \le m} \{ t_i - t_j = 0 \} \cup \cup_{\substack{1 \le i \le m \\ 1 \le j \le n}} \{ t_i - z_j = 0 \}.$$

Let $u(t)$ be a multivalued holomorphic function

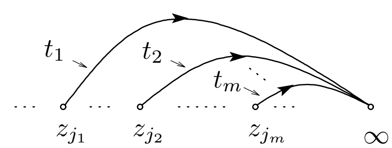
$$u(t) = \prod_{1 \le i < j \le m} (t_i - t_j)^{\nu} \prod_{1 \le i \le m} \prod_{1 \le j \le n} (t_i - z_j)^{\lambda_j}$$

defined on T_z. Let $\mathcal{L}_z$ be a sheaf consisting of the local solutions L of $dL = L\omega$ for $\omega = du(t)/u(t)$. Associated with the function $u(t)$, we define the elements, for $1 \le j_1 \le j_2 \le \cdots \le j_m \le n$,

$$\widetilde{\gamma}_{j_1 \, j_2 \, \dots j_m}(z) = \sum_{\sigma \in \mathfrak{S}_m} (\gamma_{j_1 \, j_2 \, \dots j_m})(t_{\sigma(1)}, \dots, t_{\sigma(m)}; z)$$

with

$$\gamma_{j_1 \, j_2 \, \dots j_m}(t;z) = \left\{ \vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right\} \otimes u(t) \in H^{lf}_m(T_z, \mathcal{L}_z)\,,$$

where $\arg(t_{\sigma(i)} - t_{\sigma(j)}) = 0$ for $t_{\sigma(i)} > t_{\sigma(j)}$ and $\arg(t_{\sigma(i)} - z_{\sigma(j)}) = 0$ for $t_{\sigma(i)} > z_n$. In what follows, we also use the symbol

$$\gamma_{1^{a_1} \, 2^{a_2} \, \dots \, n^{a_n}}(z) \quad \text{or} \quad \widetilde{\gamma}_{1^{a_1} \, 2^{a_2} \, \dots \, n^{a_n}}(z)$$

in stead of

$$\gamma_{\underbrace{1, \cdots, 1}_{a_1}, \underbrace{2, \cdots, 2}_{a_2}, \cdots, \underbrace{n, \cdots, n}_{a_n}}(z) \quad \text{or} \quad \widetilde{\gamma}_{\underbrace{1, \cdots, 1}_{a_1}, \underbrace{2, \cdots, 2}_{a_2}, \cdots, \underbrace{n, \cdots, n}_{a_n}}(z)$$

with $a_1 + \cdots + a_n = m$. Then we have the following, which implies Theorem 1 in the general m case.

Proposition 5. *For $1 \leq i \leq n-1$, we have*

$$\tau_i \widetilde{\gamma}_{1^{a_1} \dots n^{a_n}}(z)$$

$$= \sum_{s=0}^{a_i} \Big(e(\lambda_{i+1} + a_{i+1}\frac{\nu}{2}); e(\frac{\nu}{2}) \Big)_{a_i - s} e(s(\lambda_{i+1} + a_{i+1}\frac{\nu}{2})) \left[\begin{array}{c} a_i \\ s \end{array} \right]_{e(\frac{\nu}{2})}$$

$$\times \widetilde{\gamma}_{\dots i^{a_{i+1}+a_i-s},(i+1)^s \dots}(\dots, z_{i+1}, z_i, \dots),$$

where

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \quad \text{and} \quad \left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}.$$

Proof. Without loss of generality, we prove

$$\tau_i \widetilde{\gamma}_{i^k,(i+1)^{m-k}}(z)$$

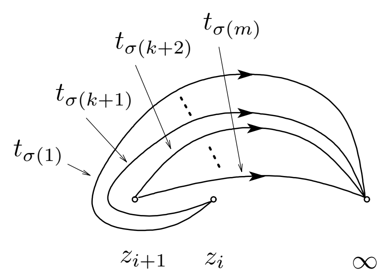
$$= \sum_{s=0}^{k} \Big(e(\lambda_{i+1} + (m-k)\frac{\nu}{2}); e(\frac{\nu}{2}) \Big)_{k-s} e(s(\lambda_{i+1} + (m-k)\frac{\nu}{2})) \left[\begin{array}{c} k \\ s \end{array} \right]_{e(\frac{\nu}{2})}$$

$$\times \widetilde{\gamma}_{i^{m-s},(i+1)^s}(\dots, z_{i+1}, z_i, \dots),$$

for $0 \leq k \leq m$. We prove this by induction on k. The case $k=0$: $\tau_i \widetilde{\gamma}_{(i+1)^m}(z) = \widetilde{\gamma}_{i^m}(\dots, z_{i+1}, z_i, \dots)$ is trivially derived. For the case of $k+1$, we start with the following:

$$\tau_i \widetilde{\gamma}_{i^{k+1},(i+1)^{m-k-1}}(z)$$

$$= \sum_{\sigma \in \mathfrak{S}_m} \left\{ \right.$$



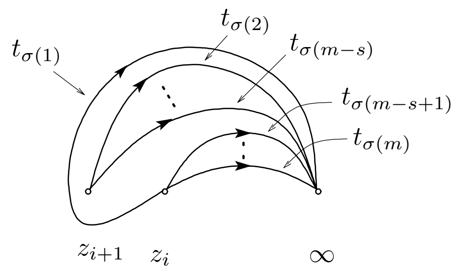
$$\left. \right\} \otimes u(t_{\sigma(1)}, \dots, t_{\sigma(m)}). \quad (6)$$

Apply the relation in the case of k to the paths in $t_{\sigma(i)}$-space for $2 \leq i \leq m$.

Then the right hand side of (5) turns out to be

$$\sum_{s=0}^{k} \Big(e(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}); e(\frac{\nu}{2})\Big)_{k-s} e(s(\lambda_{i+1} + (m-1-k)\frac{\nu}{2})) \begin{bmatrix} k \\ s \end{bmatrix}_{e(\frac{\nu}{2})}$$

$$\times \sum_{\sigma \in \mathfrak{S}_m} \left\{ \vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right\} \otimes u(t_{\sigma(1)}, \ldots, t_{\sigma(m)}).$$

By the same way as the proof of the second relation of Proposition 3 (along with Lemma 5 below), the sum over the symmetric group $\mathfrak{S}_m$ of the right hand side is shown to be

$$(1 - e(\lambda_{i+1} + (m-s-1)\frac{\nu}{2}))\widetilde{\gamma}_{i^{m-s},(i+1)^s}(\ldots, z_{i+1}, z_i, \ldots)$$

$$+ e(\lambda_{i+1} + (m-s-1)\frac{\nu}{2})\widetilde{\gamma}_{i^{m-s-1},(i+1)^{s+1}}(\ldots, z_{i+1}, z_i, \ldots).$$

Hence

$$\tau_i \widetilde{\gamma}_{i^{k+1},(i+1)^{m-k-1}}(z)$$

$$= \sum_{s=0}^{k} \Big(e(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}); e(\frac{\nu}{2})\Big)_{k-s+1} e(s(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}))$$

$$\times \begin{bmatrix} k \\ s \end{bmatrix}_{e(\frac{\nu}{2})} \widetilde{\gamma}_{i^{m-s},(i+1)^s}(\ldots, z_{i+1}, z_i, \ldots)$$

$$+ \sum_{s=0}^{k} \Big(e(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}); e(\frac{\nu}{2})\Big)_{k-s+1} e((s+1)(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}))$$

$$\times e((k-s)\frac{\nu}{2}) \begin{bmatrix} k \\ s \end{bmatrix}_{e(\frac{\nu}{2})} \widetilde{\gamma}_{i^{m-s-1},(i+1)^{s+1}}(\ldots, z_{i+1}, z_i, \ldots).$$

Change of the index s into $s-1$ of the sum of the second term of the right hand side leads to

$$\tau_i \widetilde{\gamma}_{i^{k+1},(i+1)^{m-k-1}}(z)$$

$$= \sum_{s=0}^{k+1} \Big(e(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}); e(\frac{\nu}{2}) \Big)_{k-s+1} e(s(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}))$$

$$\times \left\{ \left[\begin{array}{c} k \\ s \end{array} \right]_{e(\frac{\nu}{2})} + e(\frac{\nu}{2})^{k-s+1} \left[\begin{array}{c} k \\ s-1 \end{array} \right]_{e(\frac{\nu}{2})} \right\} \widetilde{\gamma}_{i^{m-s},(i+1)^s}(\ldots, z_{i+1}, z_i, \ldots)$$

$$= \sum_{s=0}^{k+1} \Big(e(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}); e(\frac{\nu}{2}) \Big)_{k-s+1} e(s(\lambda_{i+1} + (m-1-k)\frac{\nu}{2}))$$

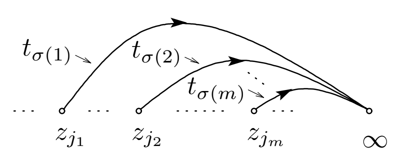
$$\times \left[\begin{array}{c} k+1 \\ s \end{array} \right]_{e(\frac{\nu}{2})} \widetilde{\gamma}_{i^{m-s},(i+1)^s}(\ldots, z_{i+1}, z_i, \ldots).$$

Here we have used the relation

$$\left[\begin{array}{c} k \\ s \end{array} \right]_q + q^{k-s+1} \left[\begin{array}{c} k \\ s-1 \end{array} \right]_q = \left[\begin{array}{c} k+1 \\ s \end{array} \right]_q.$$

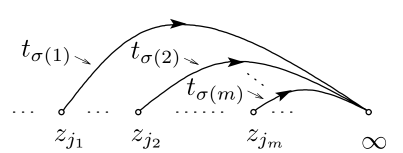
This completes the proof of Proposition 5.

Lemma 5. *For* $\sigma \in \mathfrak{S}_m$*, a loaded cycle*

$$\left\{ \right.$$


$$\left. \right\} \otimes u(t_1, \ldots, t_m),$$

where $\arg(t_i - t_j) = 0$ *for* $t_i > t_j$ *with* $1 \le i < j \le m$*, is homologous to*

$$e(-l(\sigma)\frac{\nu}{2}) \left\{ \right.$$


$$\left. \right\} \otimes u(t_{\sigma(1)}, \ldots, t_{\sigma(m)}),$$

where $\arg(t_{\sigma(i)} - t_{\sigma(j)}) = 0$ *for* $t_{\sigma(i)} > t_{\sigma(j)}$ *with* $1 \le i < j \le m$*, and* $l(\sigma)$ *stands for the inversion number of* $\sigma \in \mathfrak{S}_m$: $l(\sigma) = \mathrm{Card}\{ (i,j)\, ; \sigma(i) > \sigma(j) \,\mathrm{and}\, i < j \}$.

Proof. Apply Lemma 4 repeatedly. Then we obtain the result.

Next, the follwing linear relations among $\widetilde{\gamma}_{j_1 j_2 \dots j_m}(z)\,(1 \le j_1 \le j_2 \le \cdots \le j_m \le n)$ are obtained in the same way as in the proof of Proposition 4.

Proposition 6 (Theorem 2). *Suppose that $\lambda_1+\cdots+\lambda_n+(m-1)\nu \notin \mathbb{Z}$. Then, we have, for $1 \le j_1 \le j_2 \le \cdots \le j_{m-1} \le n$,*

$$\begin{aligned}
&\sum_{s=1}^{j_1-1}(e(\lambda_1+\cdots+\lambda_{s-1})-e(\lambda_1+\cdots+\lambda_s))\,\widetilde{\gamma}_{s\,j_1\,\dots\,j_{m-1}}(z)\\
&+(e(\lambda_1+\cdots+\lambda_{j_1-1})-e(\lambda_1+\cdots+\lambda_{j_1}+\frac{\nu}{2}))\,\widetilde{\gamma}_{j_1\,j_1\,\dots\,j_{m-1}}(z)\\
&+\sum_{s=j_1+1}^{j_2-1}(e(\lambda_1+\cdots+\lambda_{s-1}+\frac{\nu}{2})\\
&\qquad\qquad -e(\lambda_1+\cdots+\lambda_s+\frac{\nu}{2}))\,\widetilde{\gamma}_{j_1\,s\,j_2\,\dots\,j_{m-1}}(z)\\
&+\cdots\\
&+(e(\lambda_1+\cdots+\lambda_{j_{m-1}-1}+(m-2)\frac{\nu}{2})\\
&\qquad\qquad -e(\lambda_1+\cdots+\lambda_{j_{m-1}}+(m-1)\frac{\nu}{2}))\,\widetilde{\gamma}_{j_1\,\dots\,j_{m-1}\,j_{m-1}}(z)\\
&+\sum_{s=j_{m-1}+1}^{n}(e(\lambda_1+\cdots+\lambda_{s-1}+(m-1)\frac{\nu}{2})\\
&\qquad\qquad\qquad -e(\lambda_1+\cdots+\lambda_s+(m-1)\frac{\nu}{2}))\,\widetilde{\gamma}_{j_1\,\dots\,j_{m-1}\,s}(z)=0.
\end{aligned}$$

5 Proof of Theorem 3

In what follows, we suppose the condition, called n-regular,

$$q(1+q)\cdots(1+q+\cdots+q^{n-1}) \neq 0. \tag{7}$$

Let Ψ be a $H(\mathfrak{S}_n)$-module spanned by the elements $\psi_{j_1\cdots j_m}$ for $1 \le j_1 < \cdots < j_m \le n$ endowed with the action determined by

$$\psi_{j_1\cdots j_m} \overset{\tau_i}{\longmapsto} \begin{cases} \psi_{\cdots\, j_s-1\,\cdots}, \quad \text{if} \quad i \neq j_{s-1},\, i+1=j_s,\\ (1-q)\psi_{j_1\cdots j_m}+q\psi_{j_1\,\cdots\,j_s+1\,\cdots\,j_m}, \quad \text{if} \quad i=j_s,\, i+1\neq j_{s+1},\\ \psi_{j_1\cdots j_m}, \quad \text{if} \quad (i,i+1)=(j_s,j_{s+1}) \quad \text{or} \quad i,i+1\neq j_1,\dots,j_m. \end{cases}$$

It is seen that

$$\Psi = \sum_{1\le j_1<\cdots<j_m\le n} \mathbb{C}\psi_{j_1\cdots j_m}$$

is generated by $\psi_{12\cdots m}$ as an $H(\mathfrak{S}_n)$-module, and that the element $\psi_{12\cdots m}$ is invariant with respect to the action of $\tau_1, \ldots, \tau_{m-1}, \tau_{m+1}, \ldots, \tau_{m-1} \in H(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$. Hence, as $H(\mathfrak{S}_n)$-modules,

$$\Psi \simeq \mathrm{Ind}_{H(\mathfrak{S}_m \times \mathfrak{S}_{n-m})}^{H(\mathfrak{S}_n)} 1,$$

where 1 means the trivial representation of $H(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$.

Let $R_{j_1 \cdots j_{m-1}}$ be the elements defined by

$$R_{j_1 \cdots j_{m-1}} = \sum_{s \in \{1,\ldots,n\} \setminus \{j_1, \cdots, j_{m-1}\}} q^{s-1-\ell(s;\{j_1, \cdots, j_{m-1}\})} \psi_{:\{j_1, \cdots, j_{m-1}\} \cup \{s\}:}$$

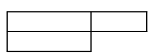
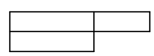
for $1 \le j_1 < \cdots < j_{m-1} \le n$, where $\ell(s;\{j_1, \cdots, j_{m-1}\}) = k$ and $:\{j_1, \cdots, j_{m-1}\} \cup \{s\} := j_1 \cdots j_k\, s\, j_{k+1} \cdots j_{m-1}$ if $j_k + 1 \le s \le j_{k+1} - 1$ (generally, for a set $A = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$, a normal ordered set $: A :$ means the ordered sequence $a_1\, a_2\, \ldots\, a_m$). Then it is shown that

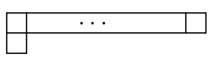
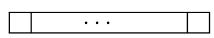
$$R_{j_1 \cdots j_{m-1}} \overset{\tau_i}{\longmapsto} \begin{cases} R_{\cdots\, j_s - 1\, \cdots}, & \text{if} \quad i \neq j_{s-1},\ i+1 = j_s, \\ (1-q) R_{j_1 \cdots j_{m-1}} + q R_{j_1 \cdots j_s + 1 \cdots j_{m-1}}, & \text{if} \quad i = j_s,\ i+1 \neq j_{s+1}, \\ R_{j_1 \cdots j_{m-1}}, & \text{if} \quad (i, i+1) = (j_s, j_{s+1}) \quad \text{or} \quad i, i+1 \neq j_1, \ldots, j_{m-1}. \end{cases}$$

Hence

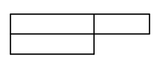
$$\sum_{1 \le j_1 < \cdots < j_{m-1} \le n} \mathbb{C} R_{j_1 \cdots j_{m-1}} \simeq \mathrm{Ind}_{H(\mathfrak{S}_{m-1} \times \mathfrak{S}_{n-m+1})}^{H(\mathfrak{S}_n)} 1,$$

as $H(\mathfrak{S}_n)$-modules. Furthermore, it is known that $\mathrm{Ind}_{H(\mathfrak{S}_m \times \mathfrak{S}_{n-m})}^{H(\mathfrak{S}_n)} 1$ has an irreducible decomposition such as

$\mathrm{Ind}_{H(\mathfrak{S}_m \times \mathfrak{S}_{n-m})}^{H(\mathfrak{S}_n)} 1 =$  $n-m$ m $\oplus$  $n-m+1$ $m-1$ $\oplus \cdots$

$\cdots \oplus$  $n-1$ 1 $\oplus$  n .

As a consequence, we have

$\Psi / (\sum_{1 \le j_1 < \cdots < j_{m-1} \le n} \mathbb{C} R_{j_1 \cdots j_{m-1}}) \simeq$  $n-m$ m .

On the other hand, if the condition $\lambda_i + \dfrac{\nu}{2} \in \mathbb{Z}$ $(1 \le i \le n)$ is imposed, then Theorem 1 and Theorem 2 are reduced to the following.

Proposition 7. *Suppose that* $\lambda_j + \dfrac{\nu}{2} \in \mathbb{Z}$ $(1 \le j \le n)$. *Then, for* $1 \le i \le n-1$ *and* $1 \le j_1 < \cdots < j_m \le n$, *we have*

$$\widetilde{\gamma}_{j_1 \ldots j_m} \overset{\rho(\tau_i)}{\longmapsto} \begin{cases} \widetilde{\gamma}_{\cdots\, j_s - 1\, \cdots}, & \text{if} \quad i \neq j_{s-1},\ i+1 = j_s, \\ (1 - e(-\frac{\nu}{2})) \widetilde{\gamma}_{j_1 \cdots j_m} + e(-\frac{\nu}{2}) \widetilde{\gamma}_{j_1 \cdots j_s + 1 \cdots j_m}, & \text{if} \quad i = j_s,\ i+1 \neq j_{s+1}, \\ \widetilde{\gamma}_{j_1 \cdots j_m}, & \text{if} \quad (i, i+1) = (j_s, j_{s+1}) \quad \text{or} \quad i, i+1 \neq j_1, \ldots, j_m. \end{cases}$$

Proposition 8. *Suppose that* $\lambda_j + \frac{\nu}{2} \in \mathbb{Z}$ $(1 \leq j \leq n)$, $\frac{\nu}{2} \notin \mathbb{Z}$ *and* $\frac{\nu}{2}(2(m-1)-n) \notin \mathbb{Z}$. *Then, for* $1 \leq j_1 < j_2 < \cdots < j_{m-1} \leq n$, *we have*

$$\begin{aligned}
&\sum_{s=1}^{j_1-1} e(-(s-1)\frac{\nu}{2})\widetilde{\gamma}_{s,\,j_1,\,j_2,\,\ldots,\,j_{m-1}}(z) \\
&+\sum_{s=j_1+1}^{j_2-1} e(-(s-2)\frac{\nu}{2})\widetilde{\gamma}_{j_1,\,s,\,j_2\,\ldots,\,j_{m-1}}(z) \\
&+\cdots \\
&+\sum_{s=j_{m-1}+1}^{n} e(-(s-m)\frac{\nu}{2})\widetilde{\gamma}_{j_1,\,\ldots,\,j_{m-2},\,j_{m-1},\,s}(z) = 0.
\end{aligned}$$

At this stage, it is clear that there is an $H(\mathfrak{S}_n)$-isomorphism

$$\Psi/(\sum_{1\leq j_1<\cdots<j_{m-1}\leq n} \mathbb{C}R_{j_1\cdots j_{m-1}}) \longrightarrow V$$

determined by $\psi_{j_1\cdots j_m} \mapsto \widetilde{\gamma}_{j_1\cdots j_m}$ and $q \mapsto e(-\frac{\nu}{2})$. This completes the proof of Theorem 3.

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