Existence Theory for Hyperbolic Systems of Conservation Laws with General Flux-Functions

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Abstract

For the Cauchy problem associated with a nonlinear, strictly hyperbolic system of conservation laws in one-space dimension we establish a general existence theory in the class of functions with sufficiently small total variation (say less than some constant c). To begin with, we assume that the flux-function $f(u)$ is **piecewise genuinely nonlinear,** in the sense that it exhibits finitely many (at most p, say) points of lack of genuine nonlinearity along each wave curve. Importantly, our analysis applies *arbitrary large* p, in the sense that the constant c restricting the total variation is *independent* of p. Second, by an approximation argument, we prove that the existence theory above extends to *more general flux-functions* $f(u)$ that can be approached by a sequence of piecewise genuinely nonlinear flux-functions $f^\epsilon(u)$.

The main contribution in this paper is the derivation of *uniform estimates* for the wave curves and wave interactions (which are entirely independent of the properties of the flux-function) together with a new **wave interaction potential** which is decreasing in time and is a fully local functional depending upon the angle made by any two propagating discontinuities. Our existence theory applies, for instance, to the p-system of gas dynamics for general pressure-laws $p = p(v)$ satisfying solely the hyperbolicity condition $p'(v) < 0$ but no convexity assumption.

1. Introduction

We are concerned with the initial value problem for general **nonlinear hyperbolic systems of conservation laws** in one space variable, that is,

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in I\!R^N,\ x \in I\!R,\ t > 0. \tag{1.1}$$

As is usual, we restrict attention to solutions taking their values in a neighborhood of some constant state in $I\!R^N$. We denote by $\mathbf{B}_\delta(v)$ the open ball in $I\!R^N$ with center $v \in I\!R^N$ and radius $\delta > 0$, and we set $\mathbf{B}_\delta := \mathbf{B}_\delta(0)$. Without loss of generality we assume that $u(x,t) \in \mathbf{B}_{\delta_1}$ for some $\delta_1 > 0$. In (1.1), the flux $f : \mathbf{B}_{\delta_1} \to I\!R^N$ is a given smooth, strictly hyperbolic mapping, that is, the Jacobian matrix $A(u) := Df(u)$ admits N real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$ for each $u \in \mathbf{B}_{\delta_1}$. It is well-known that solutions to nonlinear hyperbolic equations generally exhibit propagating discontinuities in finite time, and that weak solutions in the sense of distributions only can be sought. Moreover, for the sake of uniqueness, solutions must be constrained by a suitable admissibility condition. For background material on the subject, we refer the reader to [19], [29], [11], [22].

In the particular case of **genuinely nonlinear** (GNL) flux-functions, that is, when

$$\nabla\lambda_j(u) \cdot r_j(u) \neq 0, \quad u \in \mathbf{B}_{\delta_1}, \tag{1.2}$$

has been extensively studied. A typical example is given by the inviscid Burgers equation for which $N = 1$ and $f(u) = u^2/2$. Under the assumption (1.2), it is well-known that **entropy solutions** should satisfy Lax shock admissibility inequalities (Lax [19]). Another special case is given by **linearly degenerate** (LD) fields, which by definition satisfy

$$\nabla\lambda_j(u) \cdot r_j(u) \equiv 0, \quad u \in \mathbf{B}_{\delta_1}. \tag{1.3}$$

The first existence result for nonlinear hyperbolic systems was obtained in Glimm's pioneering work [14], establishing the existence of an entropy solution $u = u(x,t)$ to the initial value problem associated with (1.1) for every initial data with sufficiently small total variation, provided each characteristic field of f is GNL or LD.

In addition, solutions to (1.1) have bounded variation in x for each time $t \geq 0$, and satisfy the Lipschitz continuity estimate for some uniform constant $C > 0$ and for all times $t, t' \geq 0$

$$\|u(t) - u(t')\|_{L^1(I\!R)} \leq C\,|t - t'|.$$

One of the key ideas in Glimm's existence theory [14] is to rely on the *decrease* of the **wave interaction potential**

$$Q(u(t)) := \sum_{x<y} \big|u_+(x,t) - u_-(x,t)\big|\,\big|u_+(y,t) - u_-(y,t)\big| \tag{1.4}$$

in order to control the possible increase of the total variation

$$TV(u(t)) := \sum_{x \in I\!R} \big|u_+(x,t) - u_-(x,t)\big|.$$

Here, for simplicity in the discussion, we express these functionals for piecewise constant functions and summations are over all jump points. The notation $u_\pm(x,t)$ stands for the left- and right-hand traces of the function $x \mapsto u(x,t)$.

In a subsequent work, Liu [24], [25], [26] introduced a generalized notion of entropy condition (see also [27], [18], [30] for earlier developments), and established an existence result, as well as discussed the structure and asymptotic behavior of solutions, when the GNL condition is violated at finitely many points only, and by tacitly assuming that the elementary wave curves of (1.1) are *sufficiently smooth.* By examining the arguments in [26] it appears that it is sufficient for the wave curves to be continuously differentiable and to admit bounded second-order derivatives.

As a matter of fact, wave curves associated with GNL fields (Lax [19]) are twice continuously differentiable, and wave curves associated with **convex-concave** or **concave-convex** (CC) fields are solely continuously differentiable with bounded second-order derivatives [16], [22]. Recall from [22] that a j-characteristic field is said to be concave-convex (the model situation being $N = 1$ and $f(u) = u^3$) if the GNL condition (1.2) fails along a smooth $(N-1)$-dimensional manifold $\mathcal{M}_j \subset I\!R^N$

$$\mathcal{M}_j := \big\{ u \in \mathbf{B}_{\delta_1} \,/\, \nabla\lambda_j(u) \cdot r_j(u) = 0 \big\}$$

with

$$\big(\nabla(\nabla\lambda_j \cdot r_j) \cdot r_j\big)(u) > 0. \tag{1.5}$$

At this juncture, it must be pointed out that Liu's existence theory may not cover systems beyond those having GNL, LD, or CC characteristic fields. This is so because the arguments in [26] require the total variation of the solutions to be sufficiently small (as is, of course, the case in all works concerning systems (1.1)) and so, in general, we cannot exclude that actual solutions will take their values in a subset of the u-space where the quantities $(\nabla\lambda_j \cdot r_j)(u) = 0$ vanish *at most once* along each wave curve, that is, the system may admit only GNL, LD, or CC fields in the region covered by Liu's theorem. Convex-concave characteristic fields have also been studied by Hayes and LeFloch [16], Chern [8], Ancona and Marson [1], [2], and LeFloch [22].

An important alternative approach to Glimm's scheme is provided by the wave front tracking scheme which was introduced by Dafermos [10] for scalar conservation laws and extended to systems by DiPerna [13], Bressan [7], Risebro [28], and Baiti and Jenssen [3].

In addition, it has been recently pointed out by Bianchini [5] that, for hyperbolic systems for which the GNL condition fails at "two points" at least, wave curves fail to be continuously differentiable and are solely Lipschitz continuous. Interestingly enough, this is also the regularity of the wave curves constructed earlier in [20], [21], [12] (for hyperbolic systems in nonconservative form) and [16] (for nonclassical solutions). This collection of results provide further ground to the second author's claim in [22] that, in its final form, the existence (and uniqueness) theory for nonlinear hyperbolic systems should encompass a large class of solutions (classical, nonclassical) and systems (conservative, nonconservative). Having in mind this degree of generality provides a useful guideline in developing the theory.

Recall also that Glimm's theory was extended to **nonconservative systems** by LeFloch and Liu [23], using the generalized Rankine-Hugoniot relations discovered by Dal Maso, LeFloch, and Murat [12]. Some generalization of Glimm's theory to (undercompressive) **nonclassical solutions** is found in [4], [22].

Our purpose in the present paper is, building strongly on Liu's insightful ideas [26] but using several new key observations (in particular a new definition of wave interaction potential) to establish a general existence theory of entropy solutions for nonlinear hyperbolic systems (1.1) for initial data with sufficiently small total variation. Our main existence result applies to initial data u_0 at time $t = 0$ having sufficiently small total variation, say

$$TV(u_0) < c, \tag{1.6}$$

and to a **sequence of flux-functions** $f = f^\epsilon(u)$ that is "non-degenerate", in the sense that the condition

$$\text{If } (\nabla\lambda_j^\epsilon \cdot r_j^\epsilon)(u) = 0 \text{ at } u \in \mathbf{B}_{\delta_1}, \text{ then } (\nabla(\nabla\lambda_j^\epsilon \cdot r_j^\epsilon) \cdot r_j^\epsilon)(u) \neq 0 \tag{Hyp.1}$$

holds. It will convenient to refer to such hyperbolic systems as being **piecewise genuinely nonlinear** (PGNL), as we will prove below (Section 5) that the corresponding Riemann solution are made of finitely many waves only. Call p^ϵ the largest number of waves in a Riemann solution associated with the flux f^ϵ (that is, the solution of a Cauchy problem with piecewise constant data). Importantly, our main theorem applies uniformly to *arbitrary large* p^ϵ, that is, the constant c restricting the total variation is *independent of* p^ϵ. In consequence, by a straightforward approximation argument, the main theorem allows us to encompass **more general flux-functions** f by approximation in the C^2 norm by a sequence of non-degenerate flux-functions f^ϵ. The precise assumptions for the theory are stated shortly at the end of this introductory section.

The following features of our proof are to be noted. Note that all estimates derived in this paper are *uniform* with respect to the number p^ϵ introduced above.

- We construct each j-wave curve $\mathcal{W}_j(u_0)$ issuing from any $u_0 \in \mathbf{B}_{\delta_1}$, by an induction procedure. We determine the wave curve locally near the base point u_0 and, then, extend it as we move away from u_0. (See Section 5 for details.)
- This explicit approach provides us with precise control of the wave curves, in particular it allows us to carefully investigate their regularity in a way which is essentially independent from the properties of the flux. We establish that the wave curves are **globally Lipschitz continuous** but are actually **more regular near the base point** u_0; see the statement in Theorem 6.2. We can also solve the Riemann problem by relying on the implicit function theorem for Lipschitz continuous mappings [9].
- We then consider the sequence of approximate solutions $u = u^{\epsilon,h}(x,t)$ constructed by Glimm scheme.

- To obtain a uniform bound for the total variation of the approximation solutions, we introduce here new notions of **generalized angle** and **generalized wave interaction potential.** The latter is an extension of Glimm's definition [14]. Our potential is inspired by Liu's proposal in [26] who first suggested that the angle between two waves was relevant. Our functional is different from the one used in [26] since our definition is *fully local* and depends upon the angle between two arbitrary waves, only (rather than between all of the intermediate waves). See the formula (7.3).
- The central argument this paper is contained in Section 7 which provides us with the uniform wave interaction estimates. The interaction of waves of different families (Lemma 7.2) is delicate since the wave curves are solely Lipschitz continuous. The interaction of waves of the same family (and of the "same sign") requires the use of the generalized angle (Lemma 7.3).
- Our main existence theorem is concerned with a sequence of systems (1.1) with PGNL fields satisfying some natural uniform bounds. Henceforth; we can encompass general flux-functions f by approximation by PGNL functions f^ϵ. Since all of our estimates are *independent of* ϵ, we can justify the passage to the limit $\epsilon \to 0$ and arrive at a solution for the general system (1.1).

An outline of the paper follows. In Section 2, we discuss our approach in the (comparatively much simpler) situation of a scalar conservation law. In Section 3 we derive well-known properties of wave curves. In Section 4 we discuss the entropy criterion for general characteristic fields. In Section 5 we give a detailed construction of the solution of the Riemann problem. In Section 6 we prove the existence of the solution to the Riemann problem for (1.1) and we derive the additional regularity estimates on the wave curves. In Section 7 we discuss the interaction of elementary waves and establish the necessary estimates on wave strengths. Finally in Sections 8 and 9 we state and prove our main existence results for the Cauchy problem associated with (1.1).

To close this introduction, note that in a recent preprint by Bianchini and Bressan [6], the existence of entropy solutions for hyperbolic systems with general flux is also established. The authors prove that vanishing viscosity approximations to (1.1) converge to entropy solutions, by a technique which is completely different from Glimm's approach followed in the present paper. It is interesting to note that, in the spirit of the earlier work [12], [20], [21], [23], Bianchini and Bressan [6] cover the (larger) class of nonconservative systems which need not take the form of conservation laws. By contrast, in the present paper we restrict attention to systems of conservation laws and strongly rely on the conservative form of the Rankine-Hugoniot relations, as is clear when deriving the interaction estimates in Section 7. It would be very interesting to extend our arguments to the Dal Maso-LeFloch-Murat's generalized jump relations.

We collect here some notations which will be useful throughout this paper and we state our main assumptions. We denote by $r_j(u)$ and $l_j(u)$ the left- and right-eigenvectors associated with $\lambda_j(u)$, i.e.,

$$A(u)\,r_j(u) = \lambda_j(u)\,r_j(u), \quad l_j(u)\,A(u) = \lambda_j(u)\,l_j(u),$$

and normalized so that

$$\begin{aligned} &|r_j(u)| = 1, \quad l_j(u)\,r_j(u) = 1, \\ &l_i(u)\,r_j(u) = 0 \quad \text{for } i \neq j. \end{aligned} \tag{1.7}$$

Since system (1.1) is strictly hyperbolic, λ_j, r_j, and l_j $(j = 1, \dots, N)$ are smooth functions on $\mathbf{B}_{\delta_1}$.

We also introduce the averaging matrix $\overline{A}(u,v)$ for $u, v \in \mathbf{B}_{\delta_1}$ by

$$\overline{A}(u,v) := \int_0^1 Df(s\,u + (1-s)\,v)\,ds = \int_0^1 A(s\,u + (1-s)\,v)\,ds. \tag{1.8}$$

It is easy to see that $\overline{A}$ is a smooth function of u and v. We will assume that for each $u, v \in \mathbf{B}_{\delta_1}$ the matrix $\overline{A}(u,v)$ admits N distinct eigenvalues

$$\overline{\lambda}_1(u,v) < \cdots < \overline{\lambda}_N(u,v). \tag{1.9}$$

For $u, v \in \mathbf{B}_{\delta_1}$ we denote by $\overline{r}_j(u,v)$ and $\overline{l}_j(u,v)$ left- and right-eigenvectors associated with $\overline{\lambda}_j(u,v)$ and normalized so that

$$\begin{aligned} &|\overline{r}_j(u,v)| = 1, \quad \overline{l}_j(u,v)\,\overline{r}_j(u,v) = 1, \\ &\overline{l}_i(u,v)\,\overline{r}_j(u,v) = 0 \quad \text{for } i \neq j. \end{aligned} \tag{1.10}$$

It follows from (1.9) that $\overline{\lambda}_j$, $\overline{r}_j$, and $\overline{l}_j$ $(j = 1, \dots, N)$ are smooth functions on $\mathbf{B}_{\delta_1} \times \mathbf{B}_{\delta_1}$. (Note that, by a continuity argument, the conditions on $\overline{A}$ could be deduced from the ones on $A(u)$ by replacing $\mathbf{B}_{\delta_1}$ with a smaller ball, if necessary.)

The estimates derived in the paper will be uniform for a whole class of flux-functions f^ϵ. So, we now use the notation: $\overline{A}^\epsilon$, etc. We will work with a family of **uniformly strictly hyperbolic flux-functions** f^ϵ in the sense that

$$\begin{aligned} &\text{(i)}\ \ \lambda_j^{\min} \leq \overline{\lambda}_j^\epsilon(u,v) \leq \lambda_j^{\max}, \quad u, v \in \mathbf{B}_{\delta_1} \\ &\text{(ii) All families of } N \text{ vectors } \overline{r}_j^\epsilon(u,v)\ (1 \leq j \leq N,\ u, v \in \mathbf{B}_{\delta_1}) \\ &\qquad\qquad \text{are linearly independent,} \end{aligned} \tag{Hyp.2}$$

where the constants $\lambda_j^{\min}, \lambda_j^{\max}$ are given independent of ϵ with

$$\lambda_1^{\min} \leq \lambda_1^{\max} < \lambda_2^{\min} \leq \ldots \leq \lambda_{N-1}^{\max} < \lambda_N^{\min} \leq \lambda_N^{\max}.$$

The amplitude of the solution will be directly determined by the constant $K > 0$ in

$$\sup_{\substack{u,v \in \mathbf{B}_{\delta_1} \\ 1 \leq j \leq N}} |D_u \overline{r}_j^\epsilon(u,v)| \leq K \tag{Hyp.3}$$

(which also controls $D_v \overline{r}_j^\epsilon$ since the averaging matrix is symmetric in u, v). Finally, it is assumed that the second-order derivatives of f^ϵ are uniformly bounded:

$$\sup_{u,v \in \mathbf{B}_{\delta_1}} |D^2 f^\epsilon(u,v)| \leq \mathcal{O}(1) \qquad \text{(Hyp.4)}$$

(which gives also a control of $|D\overline{A}^\epsilon(u,v)|$). For instance, (Hyp.4) implies that all of the first-order derivatives of $\overline{\lambda}_j^\epsilon, \overline{r}_j^\epsilon, \ldots$ are uniformly bounded.

We will show that the Cauchy problem for (1.1) admits a global solution when the amplitude of the initial data does not exceed some $\delta_5 \leq \delta_1$ and the total variation does not exceed some constant c. From our proofs, it will be clear that

$$\delta_1, \delta_5, c \text{ can be chosen arbitrarily large when } K \to 0.$$

Obviously, when $K = 0$, the eigenvectors $r_j(u)$ are constants independent of u and system (1.1) decouples into N scalar equations for which no restriction should be imposed on the data. Throughout this paper, we denote by C_K constants that can be made arbitrary small by taking K arbitrary small.

2. Scalar Conservation Laws

In this section, we restrict attention to scalar conservation laws and we motivate our general strategy with this comparatively simpler situation. We establish the existence of entropy solutions for general flux-functions, by first dealing with functions having finitely many inflection points and by then covering arbitrary functions by some straightforward approximation argument. The existence theory for scalar equations is, of course, already well-established by Kruzkov's work [18]. In addition, we introduce in this section a new *interaction potential,* which will be needed later on (in a suitable generalized form) to deal with systems of equations. Note that, in this section, the solutions may have arbitrary large amplitude and arbitrary total variation.

We begin by assuming that the flux $f : I\!R \to I\!R$ admits finitely many inflection points, only. For any constants u_l and u_r the solution of the **Riemann problem**

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in I\!R,\ x \in I\!R,\ t > 0, \qquad (2.1)$$

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \qquad (2.2)$$

can be explicitly constructed by using finitely many elementary waves. As usual, we are interested in piecewise smooth solutions satisfying the **Oleinik**

entropy inequalities: at each discontinuity connecting two states u_- and u_+ and propagating with speed λ we must have

$$\frac{f(v) - f(u_-)}{v - u_-} \geq \lambda \quad \text{for all } v \text{ between } u_- \text{ and } u_+. \tag{2.3}$$

Of course, the speed is given by the **Rankine-Hugoniot relation** (where $a := f'$)

$$\lambda = \overline{a}(u_-, u_+) := \frac{f(u_+) - f(u_-)}{u_+ - u_-} = \int_0^1 a(u_- + s(u_+ - u_-))\, ds. \tag{2.4}$$

The construction of the Riemann solution is based on the convex hull (when $u_l < u_r$) or the concave hull (when $u_l > u_r$) of the function f in the interval limited by the Riemann data u_l and u_r. Denoting this envelop by $\widehat{f}$ and assuming (for instance) that $u_l > u_r$ we can decompose the interval $[u_r, u_l]$ by introducing finitely many states $u_l = u^1 \geq u^2 > \ldots > u^{N-1} > u^N = u_r$ such that for all relevant values of p

$$\begin{aligned} \widehat{f}(u) &= f(u), \quad u \in \big(u^{2p}, u^{2p+1}\big), \\ \widehat{f}(u) &> f(u), \quad u \in \big(u^{2p+1}, u^{2p+2}\big). \end{aligned}$$

The intervals in which $\widehat{f}$ coincides with f correspond to rarefaction fans in the solution of the Riemann problem; the intervals where $\hat{f}$ is strictly below f correspond to shock waves. Setting $g := \big(\widehat{f}\,'\big)^{-1}$ it is not difficult to check that

$$u(x,t) = \begin{cases} u_l, & x < t\,\widehat{f}(u_l), \\ g(x/t), & t\,\widehat{f}(u_l) < x < t\,\widehat{f}(u_r), \\ u_r, & x > t\,\widehat{f}(u_r), \end{cases} \tag{2.5}$$

is the solution of the Riemann problem (2.1) and (2.2). Observe that $\widehat{f}$ is concave (but not necessarily strictly concave) and, therefore, $\widehat{f}\,'$ is non-decreasing and its inverse is well-defined but may be discontinuous. Note also that the formula (2.5) cover the particular case $\widehat{f}\,'(u_l) = \widehat{f}\,'(u_r) = \overline{a}(u_l, u_r)$ when the Riemann solution contains a single shock wave.

In passing, we observe that the explicit formula (2.5) also makes sense for general functions f having infinitely many inflection points; the number of waves in the Riemann solution may then be unbounded.

We now turn to the issue of the existence of entropy solutions for the **Cauchy problem**

$$\partial_t u + \partial_x f(u) = 0, \quad u = u(x,t) \in I\!R,\ x \in I\!R,\ t > 0, \tag{2.6}$$

$$u(x,0) = u_0(x), \quad x \in I\!R, \tag{2.7}$$

where the initial data $u_0 : I\!R \to I\!R$ are supposed to be of bounded total variation, denoted by $TV(u_0)$. Based on the Riemann solver just described and following Dafermos's pioneering work [10], we can construct **wave front tracking approximations**, that is, a sequence of piecewise constant, approximate solutions to the Cauchy problem (2.6) and (2.7), as explained now. Consider any approximate flux $f^\epsilon : I\!R \to I\!R$ having *finitely many* inflection points and satisfying the properties

$$\begin{aligned} &\text{(i)} \quad \sup |f^\epsilon - f| \to 0, \\ &\text{(ii)} \quad \|f^{\epsilon\prime}\|_{L^\infty} \to \|f'\|_{L^\infty}, \qquad \text{as } \epsilon \to 0. \end{aligned} \tag{2.8}$$

In (2.8), the supremum and the L^∞ norms are taken over the range of the initial data u_0 under consideration. For instance, since a polynomial has only finitely many inflection points, f^ϵ could be a polynomial approximation of f with degree less than $1/\epsilon$. Let us set $a^\epsilon := f^{\epsilon\prime}$ and

$$\overline{a}^\epsilon(u_-, u_+) := \int_0^1 a^\epsilon(u_- + s(u_+ - u_-))\, ds.$$

Fix a sequence $h \to 0+$ and some initial data u_0. For each $h > 0$ let $u_0^h : I\!R \to I\!R$ be any piecewise constant approximation with compact support, containing at most $1/h$ jump discontinuities and satisfying

$$\begin{aligned} &\inf u_0 \le u_0^h \le \sup u_0, \\ &TV(u_0^h) \le TV(u_0), \\ &u_0^h \to u_0 \ \text{ in the } L^1 \text{ norm, as } h \to 0. \end{aligned} \tag{2.9}$$

At each jump point x of u_0^h and locally in time (at least), solve the Riemann problem associated with the initial data $u_0^h(x\pm)$ and the flux f^ϵ. Of course, the Riemann solution is not always truly piecewise constant and may contain rarefaction fans. Replace any rarefaction fan, centered at some point $(x,t) = (x_0, 0)$ and connecting two states u_1 and u_2, with the *single* **rarefaction front**

$$\begin{cases} u_1, & x - x_0 < t\,\overline{a}^\epsilon(u_1, u_2), \\ u_2, & x - x_0 > t\,\overline{a}^\epsilon(u_1, u_2), \end{cases}$$

if its strength $|u_2 - u_1|$ is less than or equal to h. If $|u_2 - u_1| > h$, we replace the rarefaction fan with *several* rarefaction fronts with small strength:

$$\begin{cases} u_1, & x - x_0 < t\,\overline{a}^\epsilon(u_1, w_1), \\ w_k, & t\,\overline{a}^\epsilon(w_{p-1}, w_p) < x - x_0 < t\,\overline{a}^\epsilon(w_p, w_{p+1}) \quad (1 \le p \le P-1), \\ u_2, & x - x_0 > t\,\overline{a}^\epsilon(w_{P-1}, u_2), \end{cases}$$

where the integer P is such that $|u_2 - u_1|/P < h$ and

$$w_p := u_1 + \frac{p}{P}(u_2 - u_1) \qquad \text{for } p = 0, \dots, P.$$

By the construction, each jump travels with the speed determined by the Rankine-Hugoniot relation (2.4). Finally, patch together these local solutions and obtain an approximate solution $u^{\epsilon,h} = u^{\epsilon,h}(x,t)$ defined up to the first interaction time t_1 at which two waves from different Riemann solutions meet.

At this first interaction point we face again a Riemann problem which is solved with possibly several shock waves and rarefaction fans. Rarefaction fans are replaced with small rarefaction fronts, each having a strength less than or equal to h and traveling with the Rankine-Hugoniot speed. At the second interaction time t_2 we proceed similarly. The construction is continued inductively by handling each interaction one by one.

We point out that the number of outgoing waves at each interaction is finite, since f^ϵ has finitely many inflection points and, therefore, a Riemann solution always contains finitely many waves. However, it is not clear at this stage that our construction can be continued for all times since the number of waves may well increase at interactions and, possibly, become infinite in finite time. An additional difficulty comes from the possibility that the number of interaction points may be unbounded. Furthermore, it is convenient to assume in the forthcoming discussion that, at any given time, there are *at most one* interaction point and *exactly two* waves interacting. This is *not* essential in our analysis but will simplify the notation. Up to minor changes, all of the forthcoming arguments and calculations can be extended to the general situation.

In the following, we denote by $u^{\epsilon,h}$ the sequence of approximations associated with the approximate initial data u_0^h and the approximate flux f^ϵ.

Theorem 2.1. (Existence theory for scalar conservation laws.) *Consider the Cauchy problem* (2.6) *and* (2.7) *where the flux-function* $f : I\!R \mapsto I\!R$ *is Lipschitz continuous and the initial data* u_0 *have bounded variation. Suppose that* f^ϵ *be a sequence of functions of class* $\mathcal{C}^1$ *having finitely many inflection points and satisfying* (2.8). *Let* $u_0^h : I\!R \to I\!R$ *be a sequence of piecewise constant approximate initial data with compact support, containing at most* $1/h$ *jump discontinuities and satisfying* (2.9).

(i) *Then, the corresponding approximations* $u^{\epsilon,h} = u^{\epsilon,h}(x,t)$ *determined are well defined globally in time. The total number of waves in* $x \mapsto u^{\epsilon,h}(x,t)$ *is uniformly bounded with respect to* t *(but maybe unbounded as* h *or* ϵ *tend to* 0*). The approximate solutions satisfy the uniform estimates*

$$\begin{aligned}
&(a) \quad \inf u_0 \le u^{\epsilon,h}(x,t) \le \sup u_0, && x \in I\!R,\ t > 0,\\
&(b) \quad TV(u^{\epsilon,h}(t)) \le TV(u_0), && t \ge 0,\\
&(c) \quad \|u^{\epsilon,h}(t) - u^{\epsilon,h}(s)\|_{L^1(I\!R)} \le TV(u_0)\, \|f^{\epsilon\prime}\|_{L^\infty}\, |t-s|, && s,\ t \ge 0,
\end{aligned} \tag{2.10}$$

where the sup-norm of $f^{\epsilon\prime}$ *is taken over the range determined by* (2.10a).

(ii) *As* $h \to 0$ *the sequence* $u^{\epsilon,h}$ *converges strongly to a weak solution* $u^\epsilon = u^\epsilon(x,t)$ *of the Cauchy problem associated with the flux* f^ϵ*:*

$$u^{\epsilon,h}(t) \to u^\epsilon(t) \quad \text{as } h \to 0 \text{ (in } L^1_{\text{loc}} \text{ for each } t)$$

and the following entropy inequalities hold:

$$\partial_t U(u^\epsilon) + \partial_x F^\epsilon(u^\epsilon) \le 0, \qquad F^{\epsilon\prime} := U'\, f^{\epsilon\prime}, \quad U'' \ge 0.$$

(iii) *As* $\epsilon \to 0$ *the sequence* u^ϵ *converges to a weak solution of the Cauchy problem* (2.6) *and* (2.7)*:*

$$u^\epsilon(t) \to u(t) \quad \text{as } \epsilon \to 0 \text{ (in } L^1_{\text{loc}} \text{ for each } t),$$

with, moreover,

$$\begin{aligned}
&(a) \quad \inf u_0 \le u(x,t) \le \sup u_0, && x \in I\!R,\ t > 0,\\
&(b) \quad TV(u(t)) \le TV(u_0), && t \ge 0,\\
&(c) \quad \|u(t) - u(s)\|_{L^1(I\!R)} \le TV(u_0)\, \|f^{\epsilon\prime}\|_{L^\infty}\, |t-s|, && s,\, t \ge 0,
\end{aligned} \tag{2.11}$$

and

$$\|u(t) - u_0\|_{L^1(I\!R)} \le t\, TV(u_0)\, \|f'\|_{L^\infty}, \quad t \ge 0.$$

Furthermore, the solution u *satisfies all of the entropy inequalities:*

$$\partial_t U(u) + \partial_x F(u) \le 0, \qquad F' := U'\, f', \quad U'' \ge 0. \tag{2.12}$$

Proof of Theorem 2.1. First of all, we check that the total number of waves in $u^{\epsilon,h}$ remains finite (h and ϵ being kept fixed). Consider an arbitrary interaction, involving a left-hand wave connecting u_l to u_m and a right-hand wave connecting u_m to u_r. We distinguish between two cases:

- monotone incoming patterns: $(u_m - u_l)\,(u_r - u_m) > 0$,
- non-monotone incoming patterns: $(u_m - u_l)\,(u_r - u_m) < 0$.

For each time t (excluding however interaction times) we denote by $N_1(t)$ the total number of changes of monotonicity in $u^{\epsilon,h}(t)$. Observe that the function $N_1(t)$ diminishes at all interactions associated with a non-monotone pattern, precisely:

$$\begin{aligned}
\big[N_1(t)\big] &:= N_1(t+) - N_1(t-)\\
&= \begin{cases} 0 & \text{if there is a monotone incoming pattern} \\ & \text{at the interaction time } t,\\ -1 & \text{if there is a non-monotone incoming pattern}\\ & \text{at the interaction time } t. \end{cases}
\end{aligned}$$

Since $N_1(0+)$ is finite, this implies that the number of "non-monotone interactions" is finite. On the other hand, at each "monotone interaction" we can have only the following three possibilities:

- Both incoming waves are shocks and the outgoing pattern is a single shock.
- The incoming pattern contains a shock and a rarefaction and the outgoing pattern contains a single shock.
- The incoming and outgoing patterns both contain exactly one shock and one rarefaction.

Therefore, "monotone interactions" cannot increase the number of waves. In turn, we deduce that the *total number of waves* is finite.

Call $N_2(t)$ the total number of waves in $u^{\epsilon,h}(t)$. Suppose that there exists a point (x_0, t_0) at which infinitely many interactions take place. As noted above, the total number of "non-monotone interactions" is also finite, so only "monotone interactions" take place in a backward neighborhood of the point (x_0, t_0). Since a Riemann solution contains at most two waves, it is clear geometrically that one wave must be cancelled and the number of waves must decrease strictly at that point, that is,

$$N_2(t_0) < \lim_{\substack{t \to t_0 \\ t < t_0}} N_2(t).$$

Therefore, there can be *at most finitely many points* at which infinitely many interactions take place. Finally, we can "pass through" any of these interactions by observing that since the singularity is localized at isolated points, on any given line $t = t_0$ we have

$$\lim_{\substack{t \to t_0 \\ t < t_0}} u^{\epsilon,h}(x,t) = u^{\epsilon,h}(x,t_0)$$

for all $x \neq x_0$. This completes the proof that the approximations $u^{\epsilon,h}$ are well-defined globally in time.

Deriving the uniform bounds (2.10) on $u^{\epsilon,h}$ is a standard matter. It is well-known also that, when $h \to 0$, the functions $u^{\epsilon,h}$ converge to some solution u^ϵ of the Cauchy problem associated with the flux f^ϵ. Next, observing that the total variation of the limiting function u^ϵ is uniformly bounded, we can use a Helly's compactness theorem to show that (a sequence of) u^ϵ converges to some function u with bounded variation satisfying the estimates (2.11). Finally, since

$$\partial_t u^\epsilon + \partial_x f^\epsilon(u^\epsilon) = 0$$

and f^ϵ approaches f in the uniform norm (by (2.8)), it follows that u is a solution of the original Cauchy problem with flux f. This completes the proof of Theorem 2.1. □

In the rest of this section, we derive an additional property of entropy solutions based on the notion of *interaction potential* which will play a central role later in our study of systems. Motivated by similar in spirit – but different– definitions introduced by Glimm [14] and Liu [26] we propose here the following notion.

A wave pattern is made of a combination of shocks and rarefactions; we use the notation $(u_-, u_+) \in \mathcal{S}$ and $(u_-, u_+) \in \mathcal{R}$, respectively. We introduce the following new notion of **generalized angle** between two elementary waves:

$$\theta^\epsilon(u_-, u_+; u'_-, u'_+)$$
$$= \begin{cases} \Big(\overline{a}^\epsilon(u'_-, u'_+) - \overline{a}^\epsilon(u_-, u_+)\Big)^-, & (u_-, u_+), (u'_-, u'_+) \in \mathcal{S}, \\ \frac{1}{u_+ - u_-} \int_{u_-}^{u_+} \Big(\overline{a}^\epsilon(u'_-, u'_+) - a^\epsilon(u)\Big)^- du, & \\ & (u_-, u_+) \in \mathcal{R},\ (u'_-, u'_+) \in \mathcal{S}, \\ \frac{1}{u'_+ - u'_-} \int_{u'_-}^{u'_+} \Big(a^\epsilon(u') - \overline{a}^\epsilon(u_-, u_+)\Big)^- du', & \\ & (u_-, u_+) \in \mathcal{S},\ (\sqcap'_-, \sqcap'_+) \in \mathcal{R}, \\ \frac{1}{(u_+ - u_-)(u'_+ - u'_-)} \int_{u'_-}^{u'_+} \int_{u_-}^{u_+} \Big(a^\epsilon(u') - a^\epsilon(u)\Big)^- du du', & \\ & (u_-, u_+), (v_-, v_+) \in \mathcal{R}, \end{cases}$$

where by definition $s^- := \max(0, -s)$. From this, we also define the **weight** Θ^ϵ by

$$\Theta^\epsilon\big(u_-, u_+; u'_-, u'_+\big) := \begin{cases} 1, & \big(u_+ - u_-\big)\big(u'_+ - u'_-\big) < 0, \\ \theta^\epsilon(u_-, u_+; u'_-, u'_+), & \big(u_+ - u_-\big)\big(u'_+ - u'_-\big) > 0. \end{cases} \tag{2.13a}$$

Given a piecewise constant function $u = u(x)$ we define the **interaction potential**

$$Q^\epsilon(u) := \sum_{x<y} \Theta^\epsilon\big(u_-(x), u_+(x); u_-(y), u_+(y)\big) \times |u_+(x) - u_-(x)||u_+(y) - u_-(y)|. \tag{2.13b}$$

Observe that

$$Q^\epsilon(u) \le \sup\big(1, 2\,\|f^{\epsilon\prime}\|_{L^\infty}\big)\, TV(u)^2.$$

Observe that Glimm [14] was using the product of the two incoming waves only, while Liu [26] introduced the notion of angle but was taking all intermediate waves located between x and y into account. Our definition only involves the speeds at the points x and y, and the generalized angle θ^ϵ in (2.13) vanishes if the two waves are moving away from each other.

Theorem 2.2. (Interaction potential for scalar conservation laws.) *Consider the Cauchy problem* (2.6) *and* (2.7) *under the assumption of Theorem 2.1. Then the interaction potential associated with the approximations* $u^{\epsilon,h} = u^{\epsilon,h}(x,t)$ *is strictly decreasing at each interaction time* t*, involving two incoming fronts connecting some states* u_l*,* u_m*, and* u_r*:*

$$\big[Q^\epsilon(u^{\epsilon,h}(t))\big] \le -\Theta^\epsilon(u_l, u_m; u_m, u_r)|u_l - u_m||u_m - u_r|. \tag{2.14}$$

This theorem provides us with a strictly decreasing functional. (The potential remains constant outside interaction times.) For systems, a suitable generalization of this functional will be essential to control the possible increase in wave strengths at interactions.

Proof. The interaction takes place at some point (x,t). Denote by u'_k ($k = 0, 1, \dots, K, K+1$) the values achieved by the Riemann solution leaving from (x,t), with $u'_0 = u_l$ and $u'_{K+1} = u_r$. Since, by our definition of the potential the outgoing waves within a Riemann solution are non-interacting, in the sense that

$$\Theta^\epsilon\big(u'_k, u'_{k+1}; u'_p, u'_{p+1}\big) = 0, \quad 0 \le k \le p \le K,$$

we see that

$$\begin{aligned}\big[Q^\epsilon(u^{\epsilon,h}(t))\big] &= -\Theta^\epsilon(u_l, u_m; u_m, u_r)\, |u_l - u_m|\, |u_m - u_r| \\ &\quad + \sum_{y \ne x} B(y)\, |u_-^{\epsilon,h}(y) - u_+^{\epsilon,h}(y)|.\end{aligned}$$

Here, we have set for $y > x$ (for instance)

$$\begin{aligned}B(y) := \sum_{k=0}^{K} &\Theta^\epsilon\big(u'_k, u'_{k+1}; u_-^{\epsilon,h}(y), u_+^{\epsilon,h}(y)\big)\, |u'_{k+1} - u'_k| \\ &-\Theta^\epsilon\big(u_l, u_m; u_-^{\epsilon,h}(y), u_+^{\epsilon,h}(y)\big)\, |u_l - u_m| \\ &-\Theta^\epsilon\big(u_m, u_r; u_-^{\epsilon,h}(y), u_+^{\epsilon,h}(y)\big)\, |u_m - u_r|.\end{aligned}$$

It is sufficient to show that $B(y)$ is non-positive for all y. We will show this for $y > x$, for instance. We distinguish between monotone and non-monotone interactions. It is sufficient to deal with the case that the wave at y is a shock, since the speed

$$\Lambda := \overline{a}^\epsilon(u_-^{\epsilon,h}(y), u_+^{\epsilon,h}(y))$$

will play no special role in the forthcoming calculations, so that all of our discussion below remains valid if one replaces Λ by an average of the characteristic speed a, as would be needed to deal with a rarefaction at y.

First, when the sequence u_l, u_m, u_r is monotone, without loss of generality we can assume that $u_l < u_m < u_r$, and we distinguish between two subcases. If $u_-^{\epsilon,h}(y) > u_+^{\epsilon,h}(y)$, then we find

$$B(y) = (u_r - u_l) - (u_l - u_m) - (u_r - u_m) \equiv 0.$$

If $u_-^{\epsilon,h}(y) < u_+^{\epsilon,h}(y)$, then we must use the generalized angle. We check that $B(y)$ is non-positive as follows, by distinguishing between three types of interaction.

If the incoming waves are both shocks, then it is easily checked that the outgoing wave is also a shock wave. The sum in the expression of $B(y)$

reduces to a single term and, using that $\alpha^- = (\alpha + |\alpha|)/2$,

$$\begin{aligned} B(y) &= \Big(\Lambda - \overline{a}^\epsilon(u_l, u_r)\Big)^- (u_r - u_l) \\ &\quad -\Big(\Lambda - \overline{a}^\epsilon(u_l, u_m)\Big)^- (u_m - u_l) - \Big(\Lambda - \overline{a}^\epsilon(u_m, u_r)\Big)^- (u_r - u_m) \\ &= \frac{1}{2}\Big|\Lambda - \overline{a}^\epsilon(u_l, u_r)\Big| (u_r - u_l) \\ &\quad -\frac{1}{2}\Big|\Lambda - \overline{a}^\epsilon(u_l, u_m)\Big| (u_m - u_l) - \frac{1}{2}\Big|\Lambda - \overline{a}^\epsilon(u_m, u_r)\Big| (u_r - u_m) \\ &= \frac{1}{2}\Big|\Lambda\,(u_r - u_l) - f^\epsilon(u_r) + f^\epsilon(u_l)\Big| \\ &\quad -\frac{1}{2}\Big|\Lambda\,(u_m - u_l) - f^\epsilon(u_m) + f^\epsilon(u_l)\Big| \\ &\quad -\frac{1}{2}\Big|\Lambda\,(u_r - u_m) - f^\epsilon(u_r) + f^\epsilon(u_m)\Big|, \end{aligned}$$

which is non-positive by the triangle inequality.

Now, if the incoming pattern is made of a shock followed by a rarefaction, then it can be checked that the outgoing pattern is made either of a single shock (and we can argue exactly as was done above), or else of a shock plus a rarefaction. In the latter, denoting by $u_l^\natural$ the point within the interval (u_l, u_r) at which the shock speed $\overline{a}^\epsilon(u_l, u_l^\natural)$ coincides with the characteristic speed $f'(u_l^\natural)$, we find

$$\begin{aligned} B(y) = &\Big(\Lambda\,(u_l^\natural - u_l) - f^\epsilon(u_l^\natural) + f^\epsilon(u_l)\Big)^- + \int_{u_l^\natural}^{u_r} \Big((\Lambda - f^{\epsilon\prime}(u)\Big)^- du \\ &-\Big(\Lambda\,(u_m - u_l) - f^\epsilon(u_m) + f^\epsilon(u_l)\Big)^- - \int_{u_m}^{u_r} \Big(\Lambda - f^{\epsilon\prime}(u)\Big)^- du. \end{aligned}$$

Consider the following function of the variables (v, Λ):

$$\begin{aligned} G(v, \Lambda) &:= \Big(\Lambda\,(v - u_l) - f^\epsilon(v) + f^\epsilon(u_l)\Big)^- + \int_v^{u_r} \Big(\Lambda - f^{\epsilon\prime}(u)\Big)^- du \\ &= \Big(\Lambda - \overline{a}^\epsilon(u_l, v)\Big)^- (v - u_l) + \int_v^{u_r} \Big(\Lambda - f^{\epsilon\prime}(u)\Big)^- du, \end{aligned}$$

where we restrict attention to values $v \in [u_m, u_l^\natural]$. Note that $f'(v)$ is increasing for v in the interval $[u_m, u_l^\natural]$ and that

$$f^{\epsilon\prime}(v) \le \overline{a}^\epsilon(u_l, v).$$

We will prove that the function G is non-increasing in v. We distinguish between three cases. If $\Lambda < f^{\epsilon\prime}(v)$, we have

$$\begin{aligned} G(v, \Lambda) &= f^\epsilon(v) - f^\epsilon(u_l) - \Lambda\,(v - u_l) + f(u_r) - f(v) - \Lambda\,(u_r - v) \\ &= f^\epsilon(u_r) - f^\epsilon(u_l) - \Lambda\,(u_r - u_l), \end{aligned}$$

which is constant. If ${f^\epsilon}'(v) \le \Lambda < \min\big(\overline{a}^\epsilon(u_l, v), {f^\epsilon}'(u_r)\big)$, then denoting by w the point where $\Lambda = {f^\epsilon}'(w)$ we have

$$G(v, \Lambda) = f^\epsilon(v) - f^\epsilon(u_l) - \Lambda\,(v - u_l) + f(u_r) - f(w) - \Lambda\,(u_r - w)$$

and thus

$$\frac{\partial G}{\partial v}(v, \Lambda) = {f^\epsilon}'(v) - \Lambda \le 0.$$

If $\overline{a}^\epsilon(u_l, v) \le \Lambda < {f^\epsilon}'(u_r)$, then denoting by w the point where $\Lambda = {f^\epsilon}'(w)$ we have again

$$G(v, \Lambda) = f(u_r) - f(w) - \Lambda\,(u_r - w),$$

which is constant. If ${f^\epsilon}'(u_r) \le \Lambda < \overline{a}^\epsilon(u_l, v)$, then

$$G(v, \Lambda) = f^\epsilon(v) - f^\epsilon(u_l) - \Lambda\,(v - u_l),$$

which is non-increasing. Finally, if $\Lambda \ge \max\big(\overline{a}^\epsilon(u_l, v), {f^\epsilon}'(u_r)\big)$, we have

$$G(v, \Lambda) \equiv 0.$$

This completes the proof that $B(y) = G(u_l^\natural, \Lambda) - G(u_m, \Lambda) \le 0$ in the monotone case.

Suppose next that the sequence u_l, u_m, u_r is non-monotone, for instance that $u_m < u_l < u_r$ and the wave connecting u_l to u_m is a rarefaction while the wave connecting u_m to u_r is a shock. Other cases can be treated in exactly the same manner. We have two possibilities: either $u_-^{\epsilon,h}(y) > u_+^{\epsilon,h}(y)$ and we find easily

$$B(y) \le \sum_{k=0}^{K} (u'_{k+1} - u'_k) - \,(u_r - u_m) = -\,(u_l - u_m) \le 0,$$

or else $u_-^{\epsilon,h}(y) < u_+^{\epsilon,h}(y)$ which is the situation of interest in the rest of this discussion. Let us fix the notation: the outgoing Riemann solution contains a rarefaction (possibly trivial) connecting $u_0 = u_l$ to u_1, followed by a shock wave connecting u_1 to u_2, $\cdots$, and, finally, a shock connecting u_{2P-1} to $u_{2P} = u_r$ where $K := 2P - 1$. Then, we can write

$$\begin{aligned} B(y) = &\sum_{p=1}^{P} \int_{u'_{2p-2}}^{u'_{2p-1}} \Big(\Lambda - a^\epsilon(u)\Big)^-\,du \\ &+ \sum_{p=1}^{P} \Big(\Lambda - \overline{a}^\epsilon(u'_{2p-1}, u'_{2p})\Big)^-\,(u'_{2p} - u'_{2p-1}) \\ &- (u_l - u_m) - \Big(\Lambda - \overline{a}^\epsilon(u_m, u_r)\Big)^-\,(u_r - u_m). \end{aligned}$$

We observe that the speeds occurring in the integral and in the summation are in *increasing order.* In particular the sequence of shock speeds $\overline{a}^\epsilon(u'_{2p-1}, u'_{2p})$ is monotone increasing in p, with

$$\overline{a}^\epsilon(u_l, u_1) \le \overline{a}^\epsilon(u'_{2p-1}, u'_{2p}) \le \overline{a}^\epsilon(u_m, u_r),$$

and the characteristic speeds $a^\epsilon(u)$ is increasing in each interval of interest, with

$$\overline{a}^\epsilon(u'_{2p-3}, u'_{2p-2}) \le a^\epsilon(u) \le \overline{a}^\epsilon(u'_{2p-1}, u'_{2p}), \quad u \in \big[u'_{2p-2}, u'_{2p-1}\big].$$

This monotonicity property allows us to distinguish between several cases, by discussing upon the value of Λ. Disregarding the two (trivial) extreme cases where Λ is greater than or less than all speeds under consideration, we can suppose that for some p_0

$$\overline{a}^\epsilon(u'_{2p_0-3}, u'_{2p_0-2}) \le \Lambda \le \overline{a}^\epsilon(u'_{2p_0-1}, u'_{2p_0}),$$

so that $\lambda = a^\epsilon(w)$ for some $w \in \big[u'_{2p_0-2}, u'_{2p_0-1}\big]$. We find

$$\begin{aligned}
B(y) = & \int_w^{u'_{2p_0-1}} \Big(-\Lambda + a^\epsilon(u)\Big)\, du + \sum_{p=p_0+1}^{P} \int_{u'_{2p-2}}^{u'_{2p-1}} \Big(-\Lambda + a^\epsilon(u)\Big)\, du \\
& + \sum_{p=p_0}^{P} \Big(-\Lambda + \overline{a}^\epsilon(u'_{2p-1}, u'_{2p})\Big)\,(u'_{2p} - u'_{2p-1}) \\
& -(u_l - u_m) - \Big(-\Lambda + \overline{a}^\epsilon(u_m, u_r)\Big)\,(u_r - u_m) \\
= & \Big(-\Lambda + \overline{a}^\epsilon(w, u_r)\Big)\,(u_r - w) - (u_l - u_m) \\
& - \Big(-\Lambda + \overline{a}^\epsilon(u_m, u_r)\Big)\,(u_r - u_m) \\
\le & \Big(-\Lambda + \overline{a}^\epsilon(w, u_r)\Big)\,(u_r - w) - \Big(-\Lambda + \overline{a}^\epsilon(u_m, u_r)\Big)\,(u_r - w) \\
= & -\Big(\overline{a}^\epsilon(u_m, u_r) - \overline{a}^\epsilon(w, u_r)\Big)\,(u_r - w) \\
\le & \ 0.
\end{aligned}$$

This completes the proof of Theorem 2.2. $\square$

3. Properties of Rarefaction and Shock Curves

In this section we derive basic properties of *rarefaction* and *shock curves.* We investigate solutions u of system (1.1) whose range is included in a small ball $\mathbf{B}_{\delta_1}$ centered at $u = 0$. As we will show, it is usually necessary to replace $\mathbf{B}_{\delta_1}$ by a smaller ball $\mathbf{B}_{\delta'}$ with $\delta' \le \delta_1$. The limit case $\delta_1 = \delta' = +\infty$ is allowed when $K = 0$ and the assumptions made in Section 1 on the data $f(u), r_j(u), \overline{r}_j(u_-, u_+), \dots$ hold for arbitrary values u.

On one hand, the following properties of *integral curves* associated with system (1.1) are immediate.

Proposition 3.1. (Integral curves.) *For each $u_- \in \mathbf{B}_{\delta_1}$ and each $j = 1, \dots, N$ and for some $s_\star = s_\star(u_-) < 0$ and $s^\star = s^\star(u_-) > 0$, one can define the j*-**integral curve**

$$\mathcal{O}_j(u_-) = \big\{ w_j(s; u_-) \,/\, s_\star \le s \le s^\star \big\}$$

by

$$w_j' = r_j(w_j), \quad w_j(0; u_-) = u_-, \tag{3.1}$$

where w_j is a smooth function of s and u_- satisfying $w_j(s; u_-) \in \mathbf{B}_{\delta_1}$ $(s_\star < s < s^\star)$ and $w_j(s_\star; u_-), w_j(s^\star; u_-) \in \partial\mathbf{B}_{\delta_1}$. In particular, as $s \to 0$ we have

$$w_j(s; u_-) = u_- + s\, r_j(u_-) + \frac{s^2}{2}\,(Dr_j r_j)(u_-) + C_K\, \mathcal{O}(s^3). \tag{3.2}$$

The analysis of the Hugoniot curves associated with (1.1) is more technical and is the subject of Proposition 3.2 and Lemmas 3.3 to 3.5. Some left-hand state $u_- \in \mathbf{B}_{\delta_1}$ being fixed we consider the **Hugoniot set** $\mathcal{H}(u_-)$ consisting of all right-hand states u_+ satisfying the Rankine-Hugoniot relation

$$-\lambda\,(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{3.3}$$

for some shock speed λ.

Proposition 3.2. (Hugoniot curves.) *There exists a positive $\delta_2 \le \delta_1$ (which can approach $+\infty$ together with δ_1 as K approaches 0) such that the following hold. For each $u_- \in \mathbf{B}_{\delta_2}$ and each $j = 1, \dots, N$, and for some $s_\star = s_\star(u_-) < 0$ and $s^\star = s^\star(u_-) > 0$, one can define the j-**Hugoniot curve***

$$\mathcal{H}_j(u_-) := \big\{ v_j(s; u_-) \,/\, s_\star \le s \le s^\star \big\}$$

by

$$\begin{aligned} &-\overline{\lambda}_j(u_-, v_j(s; u_-))\,\big(v_j(s; u_-) - u_-\big) + f(v_j(s; u_-)) - f(u_-) = 0, \\ &v_j(s; u_-) = u_- + s\,\overline{r}_j(u_-, v_j(s; u_-)), \end{aligned} \tag{3.4}$$

where v_j is a smooth function of s and u_- and satisfies $v_j(s; u_-) \in \mathbf{B}_{\delta_2}$ $(s_\star < s < s^\star)$ and $v_j(s_\star; u_-), v_j(s^\star; u_-) \in \partial\mathbf{B}_{\delta_2}$. Moreover, as $s \to 0$ we have

$$v_j(s; u_-) = u_- + s\, r_j(u_-) + \frac{s^2}{2}\,(Dr_j r_j)(u_-) + C_K\, \mathcal{O}(s^3) \tag{3.5}$$

and the corresponding shock speed $\overline{\lambda}_j(s; u_-) := \overline{\lambda}(u_-, v_j(s; u_-))$ satisfies

$$\begin{aligned} \overline{\lambda}_j(s; u_-) = \lambda_j(u_-) &+ \frac{s}{2}\,(\nabla\lambda_j \cdot r_j)(u_-) \\ &+ \frac{s^2}{6}\Big(\nabla(\nabla\lambda_j \cdot r_j)\cdot r_j + \frac{\nabla\lambda_j \cdot r_j}{2}\, l_j D r_j r_j\Big)(u_-) \\ &+ C_K\, \mathcal{O}(s^3). \end{aligned} \tag{3.6}$$

A possible choice of δ_2 in terms of δ_1 and K is provided in (3.7) below.

Proof. Note first that the Rankine-Hugoniot relation is equivalent to

$$\big(\lambda\,\mathbf{I} - \overline{A}(u_-,u_+)\big)\,(u_+ - u_-) = 0,$$

which shows that $u_+ - u_-$ is an eigenvector of the matrix $\overline{A}(u_-,u_+)$. Therefore, u_+ belongs to the Hugoniot set $\mathcal{H}(u_-)$ if and only if for some index $j = 1,\dots,N$ and $s \in I\!R$ such that $u_+ - u_- = s\,\overline{r}_j(u_-,u_+)$. Taking this into account, for $j = 1,\dots,N$ we define the mappings $F_j = F_j(u;s,u_-)$ and $G_j = G_j(u,s,u_-)$ by

$$\begin{aligned}F_j(u;s,u_-) &:= u_- + s\,\overline{r}_j(u_-,u),\\ G_j(u,s,u_-) &:= u - F_j(u;s,u_-).\end{aligned}$$

Since $|\overline{r}_j(u_-,u)| = 1$ we have

$$\begin{aligned}&|F_j(u_1;s,u_-) - F_j(u_2;s,u_-)| \le |s|\,\|D\overline{R}\|_{C(\mathbf{B}_{\delta_1}\times\mathbf{B}_{\delta_1})}\,|u_1 - u_2|,\\ &|F_j(u_1;s,u_-) - u_-| = |s|\end{aligned}$$

for any $u_1, u_2, u_- \in \mathbf{B}_{\delta_1}$, $s \in I\!R$, and $j = 1,\dots,N$. In view of this property, consider the positive constants s_1 and δ_2 defined by

$$s_1 := \min\Big(\frac{2}{3}\,\delta_1, \frac{1}{2\,K}\Big), \qquad \delta_2 := \frac{s_1}{2}. \tag{3.7}$$

Then, we have $\mathbf{B}_{\delta_2} \subset \mathbf{B}_{s_1}(u_-) \subset \mathbf{B}_{\delta_1}$ for $u_- \in \mathbf{B}_{\delta_2}$, and the mapping $F_j(\cdot;s,u_-)$ is a contraction on $\mathbf{B}_{s_1}(u_-)$ if $u_- \in \mathbf{B}_{\delta_2}$ and $|s| \le s_1$. Therefore, for each $j = 1,\dots,N$, $u_- \in \mathbf{B}_{\delta_2}$, and $s \in [-s_1, s_1]$ there exists $v_j(s;u_-) \in \mathbf{B}_{s_1}(u_-)$ such that

$$v_j(s;u_-) = F_j(v_j(s;u_-);s,u_-).$$

Moreover, G_j is a smooth mapping on $\mathbf{B}_{\delta_1} \times I\!R \times \mathbf{B}_{\delta_1}$ and

$$\begin{aligned}&G_j(v_j(s;u_-),s,u_-) = 0,\\ &(D_u G_j)(v_j(s;u_-),s,u_-) = \mathbf{I} - s\,(D_{u_+}\overline{r}_j)(u_-, v_j(s;u_-)) \quad \text{is invertible.}\end{aligned}$$

This together with the implicit function theorem implies that v_j is smooth on $(-s_1,s_1)\times\mathbf{B}_{\delta_2}$ for $j = 1,2,\dots,N$. This completes the proof of Proposition 3.2. □

We now derive some important *monotonicity property* of the shock speed along the Hugoniot curve and of its critical values. The conclusions in Lemmas 3.3 to 3.5 below will play an essential role later in Section 5 for the construction of the wave curves (since the entropy criterion is closely related to the monotonicity of the shock speed).

Lemma 3.3. (Monotonicity of the shock speed along the Hugoniot curves.) *There exist a positive* $\delta_3 \le \delta_2$ *(which can approach* $+\infty$ *together with* δ_1 *and* δ_2 *as* K *approaches* 0*) and a function* $\kappa_1 = \kappa_1(s; u_-)$ *(which is smooth with respect to* s *and* u_- *and satisfies* $c_* \le \kappa_1(s; u_-) \le c^*$ *for some numerical constants* $0 < c_* < c^*$*) such that for all*

$$u_-, v_j(s; u_-) \in \mathbf{B}_{\delta_3},$$

we have the identity

$$s\,\overline{\lambda}_j'(s; u_-) = \kappa_1(s; u_-)\left(\lambda_j(v_j(s; u_-)) - \overline{\lambda}_j(s; u_-)\right). \tag{3.8}$$

A possible choice of δ_3 in terms of δ_2 and K is given in (3.10) below. The main consequence of (3.8) is that the critical points of the shock speed $\overline{\lambda}_j(s; u_-)$ are exactly those points where the shock speed coincides with the characteristic speed $\lambda_j(v_j(s; u_-))$. Observe in passing that $\kappa_1(s; u_-) = 1 + \mathcal{O}(s)$.

Proof. Throughout this proof and the following ones we simply write $v_j(s) := v_j(s; u_-)$ and $\overline{\lambda}_j(s) := \overline{\lambda}_j(s; u_-)$. Differentiating the first equation in (3.4) with respect to s, we obtain

$$\overline{\lambda}_j'(s)\,(v_j(s) - u_-) = \left(Df(v_j(s)) - \overline{\lambda}_j(s)\right) v_j'(s). \tag{3.9}$$

Multiplying this identity by the left-eigenvector $l_j(v_j(s))$ and using the second equation in (3.4), we deduce that

$$s\,\overline{\lambda}_j'(s) = \frac{l_j(v_j(s)) v_j'(s)}{l_j(v_j(s))\overline{r}_j(u_-, v_j(s))}\left(\lambda_j(v_j(s)) - \overline{\lambda}_j(s)\right).$$

On the other hand, differentiating the second equation in (3.4) with respect to s, we obtain

$$v_j'(s) = \left(\mathbf{I} - s\left(D_{u_+}\overline{r}_j\right)(u_-, v_j(s))\right)^{-1} \overline{r}_j(u_-, v_j(s)),$$

which implies that

$$\begin{aligned} &v_j'(s) - r_j(v_j(s)) \\ &= \left(\mathbf{I} - s\left(D_{u_+}\overline{r}_j\right)(u_-, v_j(s))\right)^{-1} \Big(\overline{r}_j(u_-, v_j(s)) - \overline{r}_j(v_j(s), v_j(s)) \\ &\qquad\qquad + s\left(D_{u_+}\overline{r}_j\right)(u_-, v_j(s))\,\overline{r}_j(v_j(s), v_j(s))\Big). \end{aligned}$$

In particular, by (Hyp.3) we see that

$$|v_j'(s) - r_j(v_j(s))| \le \left(1 - |s|\,K\right)^{-1} 2\,|s|\,K.$$

We can also get

$$|\overline{r}_j(u_-, v_j(s)) - r_j(v_j(s))| \le |s|\,K.$$

We now define the constant δ_3 by

$$\delta_3 := \min\Big(\delta_2, \frac{1}{16K}\left(1 + \sup_{u \in \mathbf{B}_{\delta_1}} |l_j(u)|\right)^{-1}\Big). \tag{3.10}$$

If the point $v_j(s)$ on the Hugoniot curve is in the ball $\mathbf{B}_{\delta_3}$, then we have

$$1 \le 2\, l_j(v_j(s))\, v_j'(s) \le 3, \quad 1 \le 2\, l_j(v_j(s))\, \overline{r}_j(u_-, v_j(s)) \le 3.$$

This completes the proof of Lemma 3.3. □

We now consider *critical points* of the shock speed.

Lemma 3.4. (Critical values of the shock speed I.) *There exist functions $\kappa_2 = \kappa_2(s; u_-)$ and $\kappa_3 = \kappa_3(s; u_-)$ (which are smooth with respect to s and u_- and satisfy $c_* \le \kappa_2(s; u_-) \le c^*$ and $c_* \le \kappa_3(s; u_-) \le c^*$) such that if*

$$\overline{\lambda}_j'(s_0; u_-) = 0, \quad u_-, v_j(s_0; u_-) \in \mathbf{B}_{\delta_3},$$

then we have the properties

$$\begin{aligned} &v_j'(s_0; u_-) = \kappa_2(s_0; u_-)\, r_j(v_j(s_0; u_-)), \\ &s_0\, \overline{\lambda}_j''(s_0; u_-) = \kappa_3(s_0; u_-)\, (\nabla\lambda_j \cdot r_j)(v_j(s_0; u_-)). \end{aligned} \tag{3.11}$$

Proof. It follows from Lemma 3.3 that $\overline{\lambda}_j(s_0) = \lambda_j(v_j(s_0))$. Multiplying (3.9) by the left-eigenvector $l_k(v_j(s_0))$ and setting $s = s_0$, we obtain

$$\big(\lambda_k(v_j(s_0)) - \lambda_j(v_j(s_0))\big)\, l_k(v_j(s_0))\, v_j'(s_0) = 0,$$

which implies that $l_k(v_j(s_0))\, v_j'(s_0) = 0$ for $k \ne j$. Therefore, we can simply write $v_j'(s_0) = \kappa_2(s_0; u_-)\, r_j(v_j(s_0))$ with of course $\kappa_2(s_0; u_-) = l_j(v_j(s_0))\, v_j'(s_0)$. Moreover, differentiating (3.8) with respect to s and setting $s = s_0$, we obtain

$$\begin{aligned} s_0\, \overline{\lambda}_j''(s_0) &= \kappa_1(s_0)\, \nabla\lambda_j(v_j(s_0)) \cdot v_j'(s_0) \\ &= \kappa_1(s_0)\, l_j(v_j(s_0))\, v_j'(s_0)\, (\nabla\lambda_j \cdot r_j)(v_j(s_0)), \end{aligned}$$

which completes the proof of the lemma. □

Next, consider the case when the *first* and *second derivatives* of the shock speed vanish at some point.

Lemma 3.5. (Critical values of the shock speed II.) *There exists a function $\kappa_4 = \kappa_4(s; u_-)$, which is smooth with respect to s and u_- and satisfies $c_* \le \kappa_4(s; u_-) \le c^*$, such that if*

$$\overline{\lambda}_j'(s_0; u_-) = \overline{\lambda}_j''(s_0; u_-) = 0, \quad u_-, v_j(s_0; u_-) \in \mathbf{B}_{\delta_3},$$

then we have

$$s_0\, \overline{\lambda}_j'''(s_0; u_-) = \kappa_4(s_0; u_-)\, \big(\nabla(\nabla\lambda_j \cdot r_j) \cdot r_j\big)(v_j(s_0; u_-)). \tag{3.12}$$

In the forthcoming sections, we will first focus attention to the (generic) case that $\nabla\lambda_j \cdot r_j$ and $\nabla(\nabla\lambda_j \cdot r_j)\cdot r_j$ cannot vanish at the same point. (See our assumption at the beginning of Section 5.) Therefore, we will never have

$$\overline{\lambda}_j'(s_0;u_-) = \overline{\lambda}_j''(s_0;u_-) = \overline{\lambda}_j'''(s_0;u_-) = 0$$

and it is unnecessary to discuss here third- and higher-order derivatives of the shock speed. In the course of the proof of Lemma 3.5 we will also obtain a formula for the second-order derivative

$$v_j''(s_0;u_-) = (\kappa_2(s_0;u_-))^2\,(Dr_j\,r_j)(v_j(s_0)) + \beta(s_0;u_-)\,r_j(v_j(s_0;u_-)),$$

where $\beta = \beta(s;u_-)$ is a smooth functions with respect to s and u_-.

Proof. It follows from Lemmas 3.3 and 3.4 that

$$\overline{\lambda}_j(s_0) = \lambda_j(v_j(s_0)), \quad (\nabla\lambda_j \cdot r_j)(v_j(s_0)) = 0$$

and that

$$v_j'(s_0) = \kappa_2(s_0) r_j(v_j(s_0)).$$

Differentiating (3.9) with respect to s once again, we get

$$\begin{aligned}&\overline{\lambda}_j''(s)\,(v_j(s) - u_-) + 2\,\overline{\lambda}_j'(s)\,v_j'(s)\\ &\quad = D^2 f(v_j(s))\,\big[v_j'(s), v_j(s)\big] + (Df(v_j(s)) - \overline{\lambda}_j(s))v_j''(s),\end{aligned}$$

where $D^2 f(v)\,\big[u,w\big] = \big(D^2 f(v)\,u\big)w$. Differentiating the identity

$$Df(v_j(s))\,r_j(v_j(s)) = \lambda_j(v_j(s))\,r_j(v_j(s))$$

with respect to s in the direction $v_j'(s)$ yields

$$\begin{aligned}&D^2 f(v_j(s))\,\big[r_j(v_j(s)), v_j'(s)\big]\\ &\quad = (\nabla\lambda_j(v_j(s))\cdot v_j'(s))\,r_j(v_j(s))\\ &\qquad + \big(\lambda_j(v_j(s)) - Df(v_j(s))\,Dr_j(v_j(s))\big)v_j'(s).\end{aligned}$$

Setting $s = s_0$ in these two equations, we see that

$$\begin{aligned}&(\kappa_2(s_0))^2 Df(v_j(s_0))[r_j(v_j(s_0)), r_j(v_j(s_0))]\\ &\qquad\qquad + (Df(v_j(s_0)) - \lambda_j(v_j(s_0)))v_j''(s_0) = 0,\\ &D^2 f(v_j(s_0))[r_j(v_j(s_0)), r_j(v_j(s_0))]\\ &\qquad\qquad = (\lambda_j(v_j(s_0)) - Df(v_j(s_0)))(Dr_j r_j)(v_j(s_0)),\end{aligned}$$

which implies that

$$(Df(v_j(s_0)) - \lambda_j(v_j(s_0)))\big(v_j''(s_0) - (\kappa_2(s_0))^2(Dr_j r_j)(v_j(s_0))\big) = 0.$$

Therefore, by using similar arguments as above we can write

$$\begin{aligned}v_j''(s_0) &= (\kappa_2(s_0))^2(Dr_j r_j)(v_j(s_0))\\ &\quad + l_j(v_j(s_0))\big(v_j''(s_0) - (\kappa_2(s_0))^2(Dr_j r_j)(v_j(s_0))\big)r_j(v_j(s_0)).\end{aligned}$$

Moreover, differentiating (3.8) twice with respect to s and setting $s = s_0$, we see that

$$\begin{aligned} s_0 \overline{\lambda}_j'''(s_0) &= \kappa_1(s_0)(\kappa_2(s_0))^2 (D^2\lambda_j[r_j, r_j] + \nabla\lambda_j \cdot (Dr_j r_j))(v_j(s_0)) \\ &= \kappa_1(s_0)(\kappa_2(s_0))^2 (\nabla(\nabla\lambda_j \cdot r_j) \cdot r_j)(v_j(s_0)). \end{aligned}$$

This completes the proof of Lemma 3.5. □

4. Global Parameter and Entropy Criterion

From now on we use a **globally defined parameter** $u \mapsto \mu_j(u)$ satisfying the normalization

$$\nabla\mu_j(u) \cdot r_j(u) > 0 \tag{4.1}$$

and we always parametrize the integral and Hugoniot curves using this parameter μ_j, as follows. Consider for the instance the integral curves $s \mapsto w_j(s; u_0)$ defined in Section 3. Thanks to (3.1) and (4.1) and the implicit function theorem, we can find a smooth function $\phi_j = \phi_j(m; u_0)$ that is monotone with respect to m and satisfies along the integral curves

$$\mu_j(w_j(\phi_j(m; u_0); u_0)) = m.$$

In consequence, the function ϕ_j satisfies

$$\begin{aligned} &\phi_j(\mu_j(u_0); u_0) = 0, \\ &\phi_j'(m; u_0) = \Big(\frac{1}{\nabla\mu_j \cdot r_j}\Big)\big(w_j(\phi_j(n; u_0); u_0))\big). \end{aligned}$$

In the rest of this paper, it is always assumed that the integral curve is parametrized with this new variable m and, for simplicity in the notation, *we write* $w_j(m; u_0)$ *in place of* $w_j(\phi_j(m; u_0); u_0)$:

$$w_j(\phi_j(m; u_0); u_0) \rightsquigarrow w_j(m; u_0).$$

With the new parametrization we have simply

$$\mu_j(w_j(m; u_0)) = m \tag{4.2}$$

and

$$\begin{aligned} &w_j\big(\mu_j(u_0); u_0\big) = u_0, \\ &w_j'(m; u_0) = \Big(\frac{r_j}{\nabla\mu_j \cdot r_j}\Big)(w_j(m; u_0)). \end{aligned}$$

Clearly, the notation simplifies if we **renormalize the eigenvectors** by setting

$$\widetilde{r}_j(u_0) := \Big(\frac{r_j}{\nabla\mu_j \cdot r_j}\Big)(u_0). \tag{4.3}$$

Then, the Taylor expansion (3.2) along the integral curve simply becomes (after a tedious but straightforward calculation)

$$w_j(m;u_0) = u_0 + \big(m - \mu_j(u_0)\big)\,\widetilde{r}_j(u_0) + \frac{1}{2}\,\big(m - \mu_j(u_0)\big)^2\,\big(D\widetilde{r}_j\,\widetilde{r}_j\big)(u_0) \\ +C_K\,\mathcal{O}\big(m - \mu_j(u_0)\big)^3. \tag{4.4}$$

Similarly, we re-parametrize the Hugoniot curve $s \mapsto v_j(s;u_0)$ defined in Section 3 so that we can ensure that $\mu_j(v_j(\phi_j(m;u_0);u_0)) = m$. Henceforth, modulo the identification

$$v_j(\phi_j(m;u_0);u_0) \rightsquigarrow v_j(m;u_0),$$

we have

$$\mu_j(w_j(m;u_0)) = m, \tag{4.5}$$

and the Taylor expansions (3.5) and (3.6) for the Hugoniot curve and shock speed become

$$v_j(m;u_0) = u_0 + \big(m - \mu_j(u_0)\big)\,\widetilde{r}_j(u_0) + \frac{1}{2}\,\big(m - \mu_j(u_0)\big)^2\,\big(D\widetilde{r}_j\,\widetilde{r}_j\big)(u_0) \\ +C_K\,\mathcal{O}(m - \mu_j(u_0))^3 \tag{4.6}$$

and

$$\overline{\lambda}_j(m;u_0) = \lambda_j(u_0) + \frac{1}{2}\,\big(m - \mu_j(u_0)\big)\,\big(\nabla\lambda_j\cdot\widetilde{r}_j\big)(u_0) \\ +\frac{1}{6}\,\big(m - \mu_j(u_0)\big)^2\,\Big(\nabla(\nabla\lambda_j\cdot\widetilde{r}_j)\cdot\widetilde{r}_j + \frac{\nabla\lambda_j\cdot\widetilde{r}_j}{2}\,l_j D\widetilde{r}_j\widetilde{r}_j\Big)(u_0) \\ +\mathcal{O}(m - \mu_j(u_0))^3, \tag{4.7}$$

respectively.

We restrict attention to solutions satisfying the following entropy criterion proposed by Liu [26]. This criterion reduces to Kruzkov and Oleinik's entropy conditions in the scalar case [18], [27] and to a condition used earlier by Wendroff [30] for systems of compressible gas dynamics.

Definition 4.1. (Entropy criterion.) We say that a pair of states (u_-, u_+), such that for some $j = 1, \dots, N$

$$u_+ \in \mathcal{H}_j(u_-),$$

satisfies the entropy criterion (or **entropy admissible discontinuity**) if we have

$$\begin{aligned}&\overline{\lambda}_j(u_-, u_+) \le \overline{\lambda}_j(u_-, u) \text{ for all } u \in \mathcal{H}_j(u_-) \text{ such that}\\ &\mu_j(u) \text{ lies in the interval limited by } \mu_j(u_-) \text{ and } \mu_j(u_+).\end{aligned} \tag{4.8}$$

In other words, it is required that the minimal value of the shock speed on the interval limited by $\mu_j(u_-)$ and $\mu_j(u_+)$ is achieved at the point $\mu_j(u_+)$. By setting $m_- := \mu_j(u_-)$ and $m_+ := \mu_j(u_+)$, when $u_+ \in \mathcal{H}_j(u_-)$ we can write

$$u_+ = v_j(m_+; u_-)$$

and the entropy criterion reads

$$\overline{\lambda}_j(u_-, u_+) \leq \overline{\lambda}_j(u_-, v_j(m; u_-)) \quad \text{for all } m \text{ between } m_- \text{ and } m_+. \tag{4.8'}$$

In connection with the above definition, we state and prove two technical results of constant use in the following.

At this juncture, we will make the assumption (relaxed in the general existence theory in Section 8) that the flux-function is **non-degenerate** in the sense of the hypothesis (Hyp.1) stated in the introduction which, obviously, is equivalent to saying

$$(\widetilde{\text{Hyp.1}}) : \text{If } (\nabla\lambda_j \cdot \widetilde{r}_j)(u) = 0 \text{ at } u \in \mathbf{B}_{\delta_1}, \text{ then } (\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(u) \neq 0.$$

Lemma 4.2. (Equivalent formulation of the entropy criterion.) *Suppose that the flux is non-degenerate in the sense* (Hyp.1). *If a pair of states* (u_-, u_+) *with* $u_+ \in \mathcal{H}_j(u_-)$ *is entropy admissible, then it satisfies also*

$$\begin{aligned}&\overline{\lambda}_j(u_-, u_+) \geq \overline{\lambda}_j(u, u_+) \text{ for all } u \in \mathcal{H}_j(u_+) \text{ such that}\\ &\mu_j(u) \text{ lies in the interval limited by } \mu_j(u_-) \text{ and } \mu_j(u_+).\end{aligned} \tag{4.9}$$

Conversely, if (4.9) *holds, then the discontinuity is entropy admissible.*

The condition (4.9) means that the maximal value of the shock speed along the Hugoniot curve $\mathcal{H}_j(u_+)$ on the interval limited by $\mu_j(u_-)$ and $\mu_j(u_+)$ is achieved at the point $\mu_j(u_+)$. Of course, checking Lemma 4.2 in the scalar case $N = 1$ is straightforward from the identity (with the notation of Section 2)

$$\overline{a}(u_-, u_+) - \overline{a}(u, u_+) = \frac{u - u_-}{u_+ - u} \big(\overline{a}(u, u_-) - \overline{a}(u_-, u_+)\big), \tag{4.10}$$

in which $(u - u_-)/(u_+ - u)$ remains positive when u varies in the interval limited by u_- and u_+.

Lemma 4.3. (Propagating discontinuities with coinciding speeds.) *Suppose that* $u_0, u_1, u_2 \in \mathbf{B}_{\delta_3}$ *and* $\Lambda \in I\!R$ *satisfy the conditions*

$$\begin{aligned}&u_1 = v_j(m_1; u_0), \quad u_2 = v_j(m_2; u_1),\\ &\overline{\lambda}_j(u_0, u_1) = \overline{\lambda}_j(u_1, u_2) = \Lambda,\end{aligned} \tag{4.11}$$

for some m_1 *and* m_2 *satisfying* $\mu_j(u_0) < m_1 < m_2$. *Then, we have*

$$u_2 = v_j(m_2; u_0), \quad \overline{\lambda}_j(u_0, u_2) = \Lambda. \tag{4.12}$$

In addition, if the flux is non-degenerate in the sense (Hyp.1) *and if both discontinuities* (u_0, u_1) *and* (u_1, u_2) *are entropy admissible, then the discontinuity* (u_0, u_2) *is also admissible.*

We now give the proof of Lemma 4.2. Since Lemma 4.3 can be proved along very similar lines, we omit its proof.

Proof of Lemma 4.2. Set $m_- := \mu_j(u_-)$ and $m_+ := \mu_j(u_+)$ and, for definiteness, assume that $m_- < m_+$. Our task is to show that

$$\Lambda := \overline{\lambda}_j(u_-, u_+) \geq \overline{\lambda}_j(m; u_+), \quad m_- \leq m \leq m_+. \tag{4.13}$$

If this would not hold, then there would exist $m_0 \in [m_-, m_+)$ such that

$$\begin{aligned} &\Lambda \geq \overline{\lambda}_j(m; u_+), \quad m_- \leq m \leq m_0, \\ &\Lambda = \overline{\lambda}_j(m; u_+), \quad m = m_0, \\ &\Lambda < \overline{\lambda}_j(m; u_+), \quad 0 < m - m_0 \ll 1, \end{aligned} \tag{4.14}$$

On the other hand, since the shock connecting u_- to u_+ is admissible, we have

$$\Lambda \leq \overline{\lambda}_j(m; u_-), \quad m_- \leq m \leq m_+. \tag{4.15}$$

We consider two cases according the value of m_0.

Case 1: Suppose that $m_0 = m_-$. By Lemma 3.3 we have

$$(m_- - m_+)\overline{\lambda}_j'(m_-; u_+) = \kappa_1 \big(\lambda_j(u_-) - \overline{\lambda}_j(m_-; u_+)\big), \tag{4.16}$$

where κ_1 is a positive constant. It follows from (4.14) that $\overline{\lambda}_j'(m_-; u_+) \geq 0$, which together with (4.16) implies that $\lambda_j(u_-) \leq \overline{\lambda}_j(m_-; u_+) = \Lambda$. By (4.15) we also have $\Lambda \leq \lambda_j(u_-)$. Therefore, it holds that $\lambda_j(u_-) = \Lambda$, which together with (4.16) again yields that $\overline{\lambda}_j(m_-; u_+) = 0$. Hence, we have the expansion

$$\begin{aligned} \overline{\lambda}_j(m; u_+) = \Lambda + \frac{(m - m_-)^2}{2}\,\overline{\lambda}_j''(m_-; u_+) + \frac{(m - m_-)^3}{6}\,\overline{\lambda}_j'''(m_-; u_+) \\ + \mathcal{O}(m - m_-)^4. \end{aligned} \tag{4.17}$$

By Lemma 3.4 we also have

$$(m_- - m_+)\overline{\lambda}_j''(m_-; u_+) = \kappa_3\, (\nabla\lambda_j \cdot \widetilde{r}_j)(u_-), \tag{4.18}$$

where κ_3 is a positive constant. (4.14) and (4.17) imply that $\overline{\lambda}_j''(m_-; u_+) \geq 0$ so that $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_-) \leq 0$. On the other hand, (4.7) and (4.15) imply that $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_-) = \overline{\lambda}_j'(m_-; u_-) \geq 0$. Therefore, we obtain $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_-) = 0$ and $\overline{\lambda}_j''(m_-; u_+) = 0$. Hence, (4.14) and (4.17) imply that $\lambda_j'''(m_-; u_+) \geq 0$, and Lemma 3.5 implies that

$$(m_- - m_+)\,\overline{\lambda}_j'''(m_-; u_+) = \kappa_4 \big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j\big)(u_-) \leq 0.$$

This together with our hypothesis yields that $\big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j\big)(u_-) < 0$. Here, by (4.7) we have

$$\begin{aligned}\overline{\lambda}_j(m; u_-) &= \Lambda + \frac{(m - m_-)^2}{6} \big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j\big)(u_-) + \mathcal{O}(m - m_-)^3 \\ &< \Lambda\end{aligned}$$

for all $0 < m - m_- \ll 1$, which contradicts (4.15).

Case 2: Suppose next that $m_- < m_0 < m_+$. It follows from (4.15) that $\overline{\lambda}_j'(m_0; u_-) = 0$, which together with Lemma 3.3 implies that $\lambda_j(u_0) = \overline{\lambda}_j(m_0; u_-) = \Lambda$ and that $\overline{\lambda}_j'(m_0; u_+) = 0$. Therefore, by Lemma 3.4 we have

$$(m_0 - m_+)\, \overline{\lambda}_j''(m_0; u_+) = \kappa_3' \,(\nabla\lambda_j \cdot \widetilde{r}_j)(u_0),$$

where κ_3' is a positive constant, and the expansion

$$\begin{aligned}\overline{\lambda}_j(m; u_+) = \Lambda &+ \frac{(m - m_0)^2}{2}\, \overline{\lambda}_j''(m_0; u_+) + \frac{(m - m_-)^3}{6}\, \overline{\lambda}_j'''(m_0; u_+) \\ &+ \mathcal{O}(m - m_0)^4.\end{aligned}$$

It follows from this and (4.14) that $\overline{\lambda}_j''(m_0; u_+) = 0$ and $\overline{\lambda}_j'''(m_0; u_+) \le 0$. Now, we can use Lemma 3.5 to get

$$(m_0 - m_+)\, \overline{\lambda}_j'''(m_0; u_+) = \kappa_4' \,\big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j\big)(u_0) \ge 0,$$

where κ_4' is a positive constant. Since $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_0) = 0$, by our hypothesis (Hyp.1) we obtain $\big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j\big)(u_0) > 0$. Moreover, by Lemmas 3.4 and 3.5 again we see that $\overline{\lambda}_j''(m_0; u_-) = 0$ and $\overline{\lambda}_j'''(m_0; u_-) > 0$ so that

$$\overline{\lambda}_j(m; u_-) < \overline{\lambda}_j(m_0; u_-) = \Lambda, \quad 0 < m_0 - m \ll 1.$$

This contradicts (4.5).

In both cases, we have reached a contradiction. Hence, (4.13) hold true and the proof of Lemma 4.2 is completed. □

5. Construction of the Wave Curves

We are now ready to construct the wave curves associated with system (1.1) under the assumption that the flux is non-degenerate. Later, in Section 8, we will actually be able to relax this assumption. We will construct the ***j*-wave curve** issuing from a given left-hand state $u_0 \in \mathbf{B}_{\delta_3}$:

$$\mathcal{W}_j(u_0) = \big\{\psi_j(m; u_0) \,/\, m_{j\star}(u_0) \le m \le m_j^\star(u_0)\big\}, \tag{5.1}$$

where the end points of the curve satisfy

$$m_{j\star}(u_0) < \mu_j(u_0) < m_j^\star(u_0). \tag{5.2}$$

Recall that δ_3 was introduced in the previous section (Lemma 3.3).

Theorem 5.1. (Existence of the wave curves.) *Suppose that system* (1.1) *is uniformly strictly hyperbolic with non-degenerate characteristic fields, in the sense of* (Hyp.1)–(Hyp.4) *in* $\mathbf{B}_{\delta_1}$. *Let* δ_3 *be the positive constant defined in* (3.10). *Consider a global parameter* μ_j *for* $j = 1, \dots, N$ *satisfying* (4.1). *Then, for all* $u_0 \in \mathbf{B}_{\delta_3}$ *there exist* $m_\star = m_{j\star}(u_0)$ *and* $m^\star = m_j^\star(u_0)$ *satisfying* (5.2) *and a mapping* $\psi_j = \psi_j(m; u_0)$ *defined for* $m_\star \le m \le m^\star$ *and Lipschitz continuous in both arguments such that the following properties hold:*

- *It satisfies*
$$\begin{aligned} &\psi_j(m; u_0) \in \mathbf{B}_{\delta_3}, \quad m_\star < m < m^\star, \\ &\psi_j(\mu_j(u_0); u_0) = u_0, \\ &\psi_j(m_\star; u_0),\ \psi_j(m_\star; u_0) \in \partial \mathbf{B}_{\delta_3}. \end{aligned}$$
- *For each* $m \in [m_\star, m^\star]$ *there exist an integer* P *and a sequence of states* $u_1, u_2, \dots, u_{2P+1} \in \mathbf{B}_{\delta_3}$ *such that*
$$u_{2P+1} = \psi_j(m; u_0) \tag{5.3}$$
and (*for all relevant values of* p) u_{2p} *is connected to* u_{2p+1} *by a* j*-rarefaction wave while* u_{2p+1} *is connected to* u_{2p+2} *by an admissible* j*-shock with*
$$\begin{aligned} \lambda_j(u_0) \le \lambda_j(u_1) \le \overline{\lambda}_j(u_1, u_2) \le \lambda_j(u_2) \le \cdots \\ \cdots \le \overline{\lambda}_j(u_{2P-1}, u_{2P}) \le \lambda_j(u_{2P}) \le \lambda_j(u_{2P+1}). \end{aligned} \tag{5.4}$$

In the course of the proof of Theorem 5.1, we will see that the wave curve is actually **piecewise smooth.** It is made up from pieces of Hugoniot curves, rarefaction curves, and mixed curves which all are of class C^2 (the latter being determined by combining a rarefaction curve with a Hugoniot curve). We will see that the wave curve is of class C^1 at every point where these curves connect *except* at points where *two Hugoniot curves* connect. At the latter, the structure of the Riemann solution is "unstable" –There is, typically, a transition from a two-shock pattern to a single shock pattern.– and the curves connect Lipschitz continuously, only. Some important *additional regularity* will be established later (in Theorem 6.2).

Of course, the first rarefaction may be trivial, that is, $u_0 = u_1$, or/and the last rarefaction may be trivial, that is, $u_{2P} = u_{2P+1}$. The waves connecting u_{2p+1} to u_{2p+2} for $2 \le 2p + 2 \le 2P$ satisfy

$$\lambda_j(u_{2p+1}) = \overline{\lambda}_j(u_{2p+1}, u_{2p+2}) = \lambda_j(u_{2p+2})$$

and are referred to as **contact discontinuities**. When $u_0 \ne u_1$ and $u_{2P} \ne u_{2P+1}$, the same is true for the shock connecting u_1 to u_2 and for the shock connecting u_{2P-1} to u_{2P}. If $u_0 = u_1$ but $u_{2P} \ne u_{2P+1}$, the wave connecting $u_1(= u_0)$ to u_2 satisfy solely

$$\overline{\lambda}_j(u_1, u_2) = \lambda_j(u_2)$$

and is referred to as **right-contact**. If now $u_0 \neq u_1$ but $u_{2P} = u_{2P+1}$, the wave connecting u_{2P-1} to $u_{2P}\big(= u_{2P+1}\big)$ satisfy solely

$$\lambda_j(u_{2P-2}) = \overline{\lambda}_j(u_{2P-1}, u_{2P})$$

and is referred to as a **left-contact**. In the particular case where the j-characteristic field is linearly degenerate, the integral curve $\mathcal{O}_j(u_0)$, the Hugoniot curve $\mathcal{H}_j(u_0)$, and the wave curve $\mathcal{W}_j(u_0)$ coincide together.

The rest of this section is devoted to a proof of Theorem 5.1. For definiteness we restrict attention to the case

$$(\nabla\lambda_j \cdot \widetilde{r}_j)(u_0) > 0, \tag{5.5}$$

and we construct the part $m > \mu_j(u_0)$ of the wave curve. Discussing the range $m < \mu_j(u_0)$ or/and the case $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_0) \leq 0$ is similar. We are going to now construct the wave curve step by step, using first the integral curve $\mathcal{O}_j(u_0)$ locally near u_0, then continuing with a **mixed curve** determined by combination of integral and Hugoniot curves. Next, depending on the behavior of the characteristic and shock speeds along the mixed curve we continue along a Hugoniot or integral curve. This will lead us, eventually, to a global picture for the wave curve.

Step 1. *The Mixed Curve.*

Clearly, the wave curve coincides locally with a rarefaction curve for $m > \mu_j(u_0)$. If we try to extend further the wave curve then, by Proposition 3.1, there exists $\mu^1 = \mu^1(u_0) > \mu_j(u_0)$ such that

- *either the rarefaction curve reaches the boundary* $\partial\mathbf{B}_{\delta_3}$ (**Case 1**), with

$$\begin{aligned}
&w_j(\mu^1(u_0); u_0) \in \partial\mathbf{B}_{\delta_3},\\
&w_j(m; u_0) \in \mathbf{B}_{\delta_3}, \quad \mu_j(u_0) \leq m < \mu^1(u_0),\\
&(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_0)) > 0, \quad \mu_j(u_0) \leq m < \mu^1(u_0).
\end{aligned} \tag{5.6}$$

- *or else* $\mu^1(u_0)$ *is the "first point" where* $\nabla\lambda_j \cdot r_j$ *vanishes* (**Case 2**)*, with*

$$\begin{aligned}
&(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(\mu^1(u_0); u_0)) = 0,\\
&w_j(m; u_0) \in \mathbf{B}_{\delta_3}, \quad \mu_j(u_0) \leq m \leq \mu^1(u_0),\\
&(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_0)) > 0, \quad \mu_j(u_0) \leq m < \mu^1(u_0).
\end{aligned} \tag{5.7}$$

In both cases, the wave curve coincides with the *integral curve* $\mathcal{O}_j(u_0)$ up to the value $\mu^1(u_0)$:

$$\psi_j(m; u_0) := w_j(m; u_0), \quad \mu_j(u_0) \leq m \leq \mu^1(u_0). \tag{5.8}$$

If we are in **Case 1**, the construction is now completed. So, the rest of the discussion focuses on **Case 2** for which we need introduce a mixed curve. In **Case 2**, (5.7) together with the hypothesis (Hyp.1) implies

$$(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(w_j(\mu^1(u_0); u_0)) < 0.$$

Let us check that μ^1 *is a smooth function of its argument.* Consider the function

$$F_j(m;u) := (\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m;u)),$$

which is smooth with respect to m and u. When the point u_0 is given so that $F_j(\mu^1(u_0);u_0) = 0$, under the assumption (Hyp.1) we have

$$(\partial_m F_j)(\mu^1(u_0);u_0) < 0.$$

Therefore, by the implicit function theorem, the function $u \mapsto \mu^1(u)$ is well-defined in a neighborhood of u_0, at least, and depends smoothly upon u.

To extend the wave curve in **Case 2**, we use shock waves (actually left-contacts) and we continue the rarefaction curve with a mixed curve. So, for all meaningful values $(n,m;u)$ let us consider the function

$$G_j(m,n;u) := \begin{cases} \dfrac{1}{m-n}\big(\overline{\lambda}_j(m;w_j(n;u)) - \lambda_j(w_j(n;u))\big), & m \neq n, \\ \dfrac{1}{2}\Big(\nabla\lambda_j \cdot \widetilde{r}_j\Big)(w_j(m;u)), & m = n. \end{cases} \tag{5.9}$$

In view of the expansion (3.6) (in Proposition 3.2) for the shock speed along the Hugoniot curve, it is easy to see that G is smooth with respect to m, n, and u. Moreover, at the point $(m,n;u) = (\mu^1(u_0),\mu^1(u_0);u_0)$ we have

$$G_j(\mu^1(u_0),\mu^1(u_0);u_0) = \frac{1}{2}\Big(\nabla\lambda_j \cdot \widetilde{r}_j\Big)(w_j(\mu^1(u_0);u_0)) = 0,$$
$$(\partial_n G_j)(\mu^1(u_0),\mu^1(u_0);u_0) = \frac{1}{3}\Big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j)\cdot \widetilde{r}_j\Big)(w_j(\mu^1(u_0);u_0)) < 0.$$

Therefore, by the implicit function theorem applied to the function $(m,n;u)$ $\mapsto G_j(m,n;u)$ near the point $(m,n;u) = (\mu^1(u_0),\mu^1(u_0);u_0)$, we see that there exists a smooth function $(m;u) \mapsto \nu^1 = \nu^1(m;u)$ defined locally near $(m;u) = (\mu^1(u_0);u_0)$ such that

$$G_j(m,\nu^1(m;u_0);u_0) = 0, \quad \nu^1(\mu^1(u_0);u_0) = \mu^1(u_0),$$
$$(\partial_n G_j)(m,\nu^1(m;u_0);u_0) < 0.$$

In the following we often write

$$\mu^1 := \mu^1(u_0), \quad \nu^1(m) := \nu^1(m;u_0).$$

We note that

$$(\partial_m G_j)(\mu^1,\mu^1,u_0) = \frac{1}{6}\Big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j)\cdot \widetilde{r}_j\Big)(w_j(\mu^1;u_0))$$

and

$$\partial_m \nu^1(\mu^1) = -\frac{\partial_m G_j(\mu^1,\mu^1;u_0)}{\partial_n G_j(\mu^1,\mu^1;u_0)} = -\frac{1}{2}. \tag{5.10}$$

Now, we proceed to consider the shock speed of an arbitrary discontinuity connecting $w_j(n;u_0)$ to $v_j(m;w_j(n;u_0))$, that is,

$$\Lambda_j(m,n) := \overline{\lambda}_j(m;w_j(n;u_0)).$$

By construction, at $n=\nu^1(m)$ the shock speed coincides with the characteristic speed:

$$\Lambda_j(m,\nu^1(m)) = \lambda_j(w_j(\nu^1(m);u_0)). \tag{5.11}$$

The shock speed and the right-hand limit at the left contact point $n=\nu^1(m)$ admit a critical value with respect to their second argument, as stated now.

Lemma 5.2. (Properties at the critical point.) *Set*

$$V_j(m,n) := v_j(m;w_j(n;u_0)).$$

Then, at the point $n=\nu^1(m)$ and for all relevant m we have

$$\big(\partial_n\Lambda_j\big)(m,\nu^1(m)) = 0, \quad \big(\partial_n V_j\big)(m,\nu^1(m)) = 0.$$

For scalar conservation laws these properties are clear geometrically since the line connecting the points of the graph of f with coordinates n and m is *tangent* to the graph of f precisely at the point $n=\nu^1(m)$.

Proof. In the following we simply write $w_j(n)=w_j(n;u_0)$. By definition we have

$$\Lambda_j(m,n)\big(V_j(m,n)-w_j(n)\big) = f(V_j(m,n)) - f(w_j(n)).$$

Differentiating this relation with respect to n and setting $n=\nu^1(m)$ yields

$$\begin{aligned}&\big(\partial_n\Lambda_j\big)(m,\nu^1(m))\,\big(V_j(m,\nu^1(m)) - w_j(\nu^1(m))\big)\\ &\quad= \Big(Df\big(V_j(m,\nu^1(m))\big) - \lambda_j\big(w_j(\nu^1(m))\big)\Big)\,\big(\partial_n V_j\big)(m,\nu^1(m)).\end{aligned}$$

On the other hand, differentiating with respect to m and setting $n=\nu^1(m)$ yields we find

$$\begin{aligned}&\big(\partial_m\Lambda_j\big)(m,\nu^1(m))\,\big(V_j(m,\nu^1(m)) - w_j(\nu^1(m))\big)\\ &\quad= \Big(Df\big(V_j(m,\nu^1(m))\big) - \lambda_j\big(w_j(\nu^1(m))\big)\Big)\,\big(\partial_m V_j\big)(m,\nu^1(m))\big).\end{aligned}$$

These two equations imply that

$$\begin{aligned}&\Big(Df\big(V_j(m,\nu^1(m))\big) - \lambda_j\big(w_j(\nu^1(m))\big)\Big)\\ &\quad\times\bigg(\big(\partial_m\Lambda_j\big)(m,\nu^1(m))\,\big(\partial_n V_j\big)(m,\nu^1(m))\\ &\qquad\qquad - \big(\partial_n\Lambda_j\big)(m,\nu^1(m))\,\big(\partial_m V_j\big)(m,\nu^1(m))\bigg) = 0.\end{aligned}$$

In the situation under consideration we have

$$\lambda_j\big(V_j(m,\nu^1(m))\big) < \lambda_j\big(w_j(\nu^1(m))\big),$$

which implies that

$$\begin{aligned}&\big(\partial_m \Lambda_j\big)(m,\nu^1(m))\,\big(\partial_n V_j\big)(m,\nu^1(m))\\ &\quad= \big(\partial_n \Lambda_j\big)(m,\nu^1(m))\,\big(\partial_m V_j\big)(m,\nu^1(m)).\end{aligned} \tag{5.12}$$

By the definition of our global parameter μ_j we have $\mu_j\big(V_j(m,n)\big) = m$. Differentiating this with respect to m and n yields

$$\nabla\mu_j\big(V_j(m,n)\big)\cdot\big(\partial_m V_j\big)(m,n) = 1 \text{ and } \nabla\mu_j\big(V_j(m,n)\big)\cdot\big(\partial_n V_j\big)(m,n) = 0,$$

respectively. Taking the inner product of (5.12) with $\nabla\mu_j\big(V_j(m,\nu^1(m))\big)$ and noting that (see (5.15) below) $\big(\partial_m \Lambda_j\big)(m,\nu^1(m)) < 0$ we obtain the desired relations. This completes the proof of Lemma 5.2. □

Now, differentiating (5.11) with respect to m leads to

$$\partial_m\nu^1(m) = \frac{1}{(\nabla\lambda_j\cdot\widetilde{r}_j)\big(w_j(\nu^1(m);u_0)\big)}\,\big(\partial_m \Lambda_j\big)(m,\nu^1(m)). \tag{5.13}$$

On the other hand, by the definition of the function G_j we have

$$\big(\partial_n G_j\big)(m,\nu^1(m);u_0) = -\frac{1}{m-\nu^1(m)}\Big(\nabla\lambda_j\cdot\widetilde{r}_j\Big)\big(w_j(\nu^1(m);u_0)\big), \tag{5.14}$$

which has been observed to be strictly negative in a neighborhood of $\mu^1(u_0)$ at least. Since ${\nu^1}'(\mu^1) = -1/2$, we can assume that ${\nu^1}'(m) < 0$ for m sufficiently close to μ^1, therefore,

$$\big(\partial_m \Lambda_j\big)(m,\nu^1(m)) < 0, \quad m > \mu^1(u_0) \tag{5.15}$$

for m sufficiently close to μ^1 at least. This means that the speed of the left-contact decreases as m increases.

Next, we want to extend the domain of definition of the function $\nu^1 = \nu^1(m;u_0)$ as long as the basic constraint (5.11) makes sense. We now show that, roughly speaking, ν^1 is defined until $\partial_m\Lambda_j$ eventually vanishes. To this end, assume that, for some value m^*, the function ν^1 has been constructed for all $\mu^1(u_0)\le m\le m^*$ and satisfies the relations (5.11) and (5.15) for all $\mu^1(u_0)\le m\le m^*$ and $\mu_j(u_0) < \nu^1(\mu^2(u_0);u_0)$. Then, it follows from (5.13) and (5.14) that ${\nu^1}'(m) < 0$ for $\mu^1(u_0)\le m\le m^*$ and, in particular,

$$G_j(m^*,\nu^1(m^*;u_0);u_0) = 0, \quad (\partial_n G_j)(m^*,\nu^1(m^*;u_0);u_0) < 0.$$

Therefore, it is clear that the implicit function theorem applies again and we can extend $\nu^1 = \nu^1(m^*;u_0)$ in a neighborhood of m^*.

Step 2. *The Mixed Curve is Entropy Admissible.*
Repeating the above procedure, we eventually reach a value $\mu^2 = \mu^2(u_0)$ at which one of the following three possibilities arises:

- either the state $v_j(\mu^2; w_j(\nu^1(\mu^2; u_0); u_0))$ belongs to the boundary $\partial \mathbf{B}_{\delta_3}$ (**Case 2.1**),
- or the derivative of the shock speed $(\partial_m \Lambda_j)(\mu^2, \nu^1(\mu^2; u_0))$ equals zero (**Case 2.2**),
- or else the parameter $\nu^1(\mu^2; u_0)$ coincides with the "initial value" $\mu_j(u_0)$ (**Case 2.3**).

Using the mapping $m \mapsto v_j(m; w_j(\nu^1(m; u_0); u_0))$, we extend the wave curve $\psi_j = \psi_j(m; u_0)$ for $m > \mu^1(u_0)$ and get

$$\psi_j(m; u_0) = \begin{cases} w_j(m; u_0), & \mu_j(u_0) \le m < \mu^1(u_0), \\ v_j(m; w_j(\nu^1(m; u_0); u_0)), & \mu^1(u_0) \le m \le \mu^2(u_0). \end{cases} \tag{5.16}$$

It follows from Lemma 5.2 and Propositions 3.1 and 3.2 that

$$\begin{aligned} \frac{d}{dm} v_j\big(m; w_j(\nu^1(m; u_0); u_0)\big)\Big|_{m=\mu^1(u_0)} &= v_j'\big(\mu^1(u_0); w_j(\mu^1(u_0); u_0)\big) \\ &= \widetilde{r}_j\big(w_j(\mu^1(u_0); u_0)\big) \\ &= w_j'(\mu^1(u_0); u_0). \end{aligned}$$

Moreover, both v_j and w_j are smooth mappings and it has been checked already that $\mu^1(u_0)$ and $\nu^1(m; u_0)$ depend smoothly upon their arguments, so that this mapping $\psi_j = \psi_j(m; u_0)$ is of class $\mathcal{C}^1$ in m and u_0.

We now prove that the corresponding shock is admissible.

Lemma 5.3. *For each $m \in (\mu^1, \mu^2)$ we can connect the state $w_j(\nu^1(m); u_0)$ to the right-hand state $v_j(m; w_j(\nu^1(m); u_0))$ by an entropy admissible discontinuity.*

Proof. We fix $\overline{m} \in (\mu^1, \mu^2)$ arbitrarily and set

$$\overline{u} := w_j(\nu^1(\overline{m}); u_0).$$

It is sufficient to establish that the shock speed remains above the characteristic speed at the point $\overline{u}$:

$$\lambda_j(\overline{u}) \le \overline{\lambda}_j(\overline{u}, v_j(m; \overline{u})) = \Lambda_j(m, \nu^1(\overline{m})), \quad \nu^1(\overline{m}) < m < \overline{m}. \tag{5.17}$$

It is clear that $\lambda_j(\overline{u}) = \overline{\lambda}_j(\overline{u}, v_j(m; \overline{u}))$ for both $m = \nu^1(\overline{m})$ and $m = \overline{m}$. Furthermore, by the expansion (3.6) in Proposition 3.2 we have

$$\overline{\lambda}_j(\overline{u}, v_j(m; \overline{u})) = \lambda_j(\overline{u}) + \frac{1}{2}\big(m - \nu^1(\overline{m})\big)\big(\nabla\lambda_j \cdot \widetilde{r}_j\big)(\overline{u}) + \mathcal{O}\big(m - \nu^1(\overline{m})\big)^2,$$

hence $\lambda_j(\overline{u}) < \overline{\lambda}_j(\overline{u}, v_j(m; \overline{u}))$ for all m with $0 < m - \nu^1(\overline{m}) \ll 1$.

Let us first establish the property (5.17) when $0 < \overline{m} - \nu^1(\overline{m}) \ll 1$. If (5.17) would not hold, then there would exist $\widetilde{m} \in (\nu^1(\overline{m}), \overline{m})$ such that

$$\lambda_j(\overline{u}) = \overline{\lambda}_j(\overline{u}, v_j(\widetilde{m}; \overline{u}))$$

and then

$$G_j(\widetilde{m}, \nu^1(\overline{m}); u_0) = \frac{1}{\widetilde{m} - \nu^1(\overline{m})} \big(\overline{\lambda}_j(\overline{m}, v_j(\widetilde{m}; \overline{u})) - \lambda_j(\overline{u})\big) = 0.$$

Therefore, by the implicit function theorem we would have $\nu^1(\widetilde{m}) = \nu^1(\overline{m})$, and hence $\widetilde{m} = \overline{m}$, which would lead to a contradiction. Next, we prove by contradiction that (5.17) holds for any $\overline{m} \in (\mu^1, \mu^2)$. Suppose that there exists $\overline{m} \in (\mu^1, \mu^2)$ such that (5.17) does not hold. Then, the value

$$\overline{m}_* := \inf\big\{\overline{m} \in (\mu^1, \mu^2) \,/\, (5.17) \text{ holds } \big\}$$

would be well-defined, with $\mu^1 < \overline{m}_* < \mu^2$. By a continuity argument we see that the set

$$\big\{\overline{m} \in (\mu^1, \mu^2) \,/\, (5.17) \text{ does not hold for } \overline{m}\big\}$$

is open. Therefore, we can find a decreasing sequence $\{\overline{m}_p\}_{p=1,2,\cdots}$ such that (5.17) does not hold for $\overline{m} = \overline{m}_p$ and that $\lim_{p\to\infty} \overline{m}_p = \overline{m}_*$. Furthermore, for each p there exists $m_p \in (\nu^1(\overline{m}), \overline{m}_p)$ such that

$$\lambda_j(w_j(\nu^1(\overline{m}_p); u_0)) > \Lambda_j(m_p, \nu^1(\overline{m}_p)).$$

Without loss of generality, we can assume that the limit

$$m_* := \lim_{p\to\infty} m_p$$

exists and satisfies $\nu^1(\overline{m}) < m_* \le \overline{m}_*$. Letting $p \to \infty$ in the above inequality we obtain

$$\lambda_j(w_j(\nu^1(\overline{m}_*); u_0) \ge \Lambda_j(m_*, \nu^1(\overline{m}_*)).$$

Since we know that (5.17) holds for $\overline{m} = \overline{m}_*$, we have

$$\lambda_j(w_j(\nu^1(\overline{m}_*); u_0)) \le \Lambda_j(m, \nu^1(\overline{m}_*)), \quad \nu^1(\overline{m}) \le m \le \overline{m}_*.$$

Therefore, we see that $\Lambda_j(m_*, \nu^1(\overline{m}_*)) = \lambda_j(w_j(\nu^1(\overline{m}_*); u_0))$ and hence

$$(\partial_m \Lambda_j)(m_*, \nu^1(\overline{m}_*)) = 0.$$

In particular, we get $\nu^1(\overline{m}) < m_* < \overline{m}_*$. Finally, consider the function $g = g(m)$ defined by

$$g(m) := \Lambda_j(m_*, \nu^1(m)) - \lambda_j(w_j(\nu^1(m); u_0)).$$

It is easy to see that $g(\overline{m}_*) = 0$ and

$$g'(\overline{m}_*) = -{\nu^1}'(\overline{m}_*)\big(\nabla\lambda_j \cdot \widetilde{r}_j\big)\big(w_j(\nu^1(\overline{m}_*); u_0)\big) > 0,$$

so that $g(\overline{m}_* - \varepsilon) < 0$ for $0 < \varepsilon \ll 1$, which implies that

$$\lambda_j(w_j(\nu^1(\overline{m}_* - \varepsilon); u_0)) > \Lambda_j(m_*, \nu^1(\overline{m}_* - \varepsilon)).$$

This means that (5.17) does not hold for $\overline{m} = \overline{m}_* - \varepsilon$, which contradicts the definition of $\overline{m}_*$. $\square$

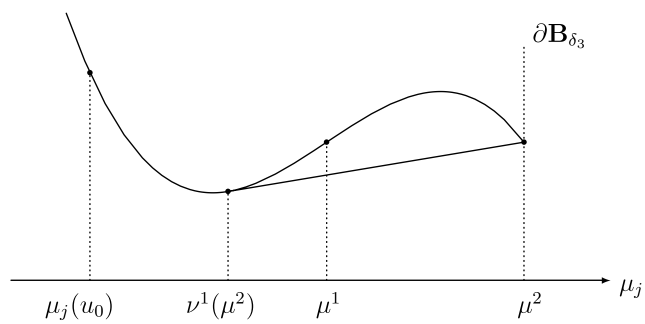

Fig. 1. Case 2.1

Step 3. *Extending the Wave Curve Beyond the Mixed Curve.*
We now distinguish between the three cases that were listed earlier.

Case 2.1: *The mixed curve reaches the boundary* $\partial\mathbf{B}_{\delta_3}$.
It might be that the function $\nu^1 = \nu^1(m)$ satisfying (5.11) is defined for $\mu^1(u_0) \le m \le \mu^2(u_0)$ with $\mu^2 = \mu^2(u_0)$ such that

$$\begin{aligned}
&\mu_j(u_0) < \nu^1(\mu^2),\\
&(\partial_m \Lambda_j)(m, \nu^1(m)) < 0, \quad \mu^1 \le m < \mu^2,\\
&v_j(m; w_j(\nu^1(m); u_0)) \in \mathbf{B}_{\delta_3}, \quad \mu^1 \le m < \mu^2,\\
&v_j(\mu^2; w_j(\nu^1(\mu^2); u_0)) \in \partial\mathbf{B}_{\delta_3}.
\end{aligned} \tag{5.18}$$

The construction of the wave curves is now completed in this case.

Case 2.2: *The shock speed may reach a critical value.*
We consider here the case when the function $\nu^1 = \nu^1(m; u_0)$ satisfying (5.11) is defined for $\mu^1(u_0) \le m \le \mu^2(u_0)$ with $\mu^2(u_0)$ such that

$$\begin{aligned}
&\nu^1(\mu^2(u_0), u_0) > \mu_j(u_0),\\
&(\partial_m \Lambda_j)(m, \nu^1(m; u_0)) < 0, \quad \mu^1(u_0) \le m < \mu^2(u_0),\\
&(\partial_m \Lambda_j)(\mu^2(u_0), \nu^1(\mu^2(u_0); u_0)) = 0,\\
&v_j(m; w_j(\nu^1(m; u_0); u_0)) \in \mathbf{B}_{\delta_3}, \quad \mu^1(u_0) \le m \le \mu^2(u_0).
\end{aligned} \tag{5.19}$$

By the definition of the function ν^1 and Lemma 3.3, the shock wave is actually a contact discontinuity. The shock speed coincides with the characteristic speed of its left- and right-hand states:

$$\lambda_j\big(w_j(\nu^1(\mu^2); u_0)\big) = \overline{\lambda}_j\big(\mu^2; w_j(\nu^1(\mu^2); u_0)\big) = \lambda_j\big(v_j(\mu^2; w_j(\nu^1(\mu^2); u_0))\big). \tag{5.20}$$

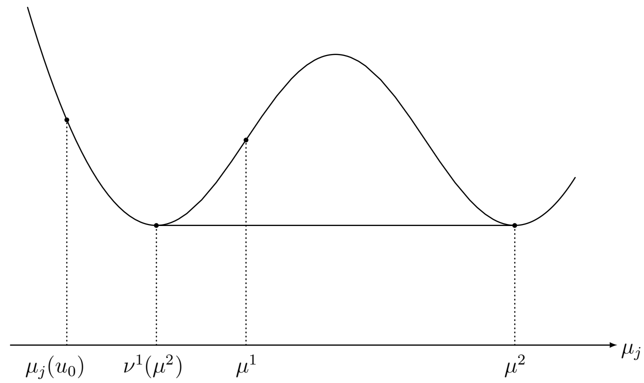

Fig. 2. Case 2.2a

Therefore, by setting $u_1 := w_j(\nu^1(\mu^2); u_0)$ it follows from Lemma 3.4 that

$$\begin{aligned}
&\overline{\lambda}_j(u_1, v_j(m; u_1)) = \Lambda_j(m, \nu^1(\mu^2)) \\
&= \Lambda_j(\mu^2, \nu^1(\mu^2)) + (m - \mu^2)(\partial_m \Lambda_j)(\mu^2, \nu^1(\mu^2)) \\
&\quad + \frac{(m-\mu^2)^2}{2} (\partial_m^2 \Lambda_j)(\mu^2, \nu^1(\mu^2)) + \mathcal{O}(m - \mu^2)^3 \\
&= \overline{\lambda}_j(u_1, v_j(\mu^2; u_1)) + (m - \mu^2)^2 \, \widetilde{\kappa}_3(\mu^2) \big(\nabla \lambda_j \cdot \widetilde{r}_j\big)(v_j(\mu^2; u_1)) \\
&\quad + \mathcal{O}(m - \mu^2)^3,
\end{aligned}$$

where $\widetilde{\kappa}_3(\mu^2)$ is a positive constant.

Since the states u_1 and $v_j(\mu^2; u_1)$ can be connected by an admissible j-discontinuity, we have that $(\nabla \lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) \geq 0$. This allows us to distinguish between two subcases depending whether $(\nabla \lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_1))$ is positive (**Case 2.2a**) or equal to zero (**Case 2.2b**), respectively.

Case 2.2a: *Consider the non-degenerate case when* $(\nabla \lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) > 0$.

By Lemma 3.4 and (5.13) we obtain

$$v_j'(\mu^2; u_1) = \widetilde{r}_j\big(v_j(\mu^2; u_1)\big), \qquad {\nu^1}'(\mu^2) = 0.$$

Taking these into account, we define the function $g = g(m; u)$ by

$$g(m; u) := \lambda_j\big(v_j(m; w_j(\nu^1(m; u); u))\big) - \lambda_j\big(w_j(\nu^1(m; u); u)\big).$$

It is easy to see that $g = g(m; u)$ depends smoothly upon its argument and that at the point $(\mu^2(u_0); u_0)$ we have

$$\begin{aligned}
&g(\mu^2(u_0); u_0) = 0, \\
&(\partial_m g)(\mu^2(u_0); u_0) = \big(\nabla \lambda_j \cdot \widetilde{r}_j\big) \, \big(v_j(\mu^2(u_0); u_1)\big) > 0.
\end{aligned}$$

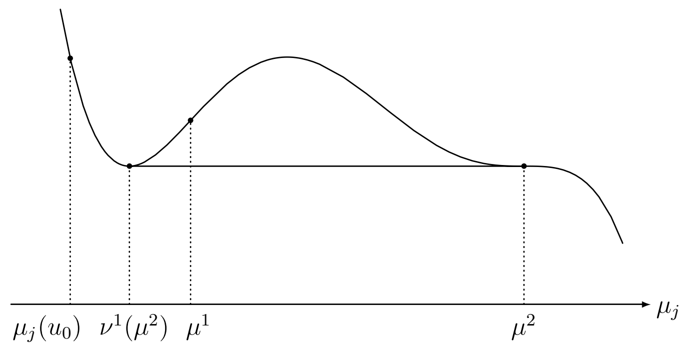

Fig. 3. Case 2.2b

Therefore, by the implicit function theorem we see that the function $u \mapsto \mu^2 = \mu^2(u)$ is smooth. In this case, the wave curve could be further extended by using an integral curve, similarly to what was done earlier to determine the wave curve locally near u_0. We will return to this part of the construction in the end of this section.

Case 2.2b: *Consider the degenerate case when* $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) = 0$.

By Lemma 3.4, we have

$$(\partial_m \Lambda_j)(\mu^2, \nu^1(\mu^2)) = (\partial_m^2 \Lambda_j)(\mu^2, \nu^1(\mu^2)) = 0.$$

It follows from Lemma 3.5 that

$$\begin{aligned} &\overline{\lambda}_j(u_1, v_j(m; u_1)) \\ &= \Lambda_j(\mu^2, \nu^1(\mu^2)) + \frac{(m-\mu^2)^3}{6}\,(\partial_m^3 \Lambda_j)(\mu^2, \nu^1(\mu^2)) + \mathcal{O}(m-\mu^2)^4 \\ &= \overline{\lambda}_j(u_1, v_j(\mu^2; u_1)) + \frac{(m-\mu^2)^3}{6}\,\widetilde{\kappa}_4(\mu^2)\,(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) \\ &\quad + \mathcal{O}(m-\mu^2)^4, \end{aligned} \tag{5.21}$$

where $\widetilde{\kappa}_4(\mu^2)$ is a positive constant.

Since the states u_1 and $v_j(\mu^2; u_1)$ can be connected by an admissible j-discontinuity, we have

$$(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) \leq 0,$$

hence, by the hypothesis (Hyp.1),

$$(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) < 0.$$

On the other hand, by (5.14) we have $(\partial_n G_j)(\mu^2, \nu^1(\mu^2); u_0) < 0$, which guarantees that we can extend the function $\nu^1 = \nu^1(m)$ for $m > \mu^2$ by applying the implicit function theorem once again. Moreover, in this case we also have

$$\nu^{1\prime}(\mu^2) = \nu^{1\prime\prime}(\mu^2) = 0 \tag{5.22}$$

because of (5.13).

Then, for the function $g(m) := (\partial_m \Lambda_j)(m, \nu^1(m))$ we see that

$$\begin{aligned} & g(\mu^2) = g'(\mu^2) = 0, \\ & g''(\mu^2) = (\partial_m^3 \Lambda_j)(\mu^2, \nu^1(\mu^2)) \\ & \qquad\qquad = \widetilde{\kappa}_4(\mu^2)\,(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^2; u_1)) < 0, \\ & (\partial_m \Lambda_j)(m, \nu^1(m)) = \tfrac{1}{2}\, g''(\mu^2)\,(m-\mu^2)^2 + \mathcal{O}(m-\mu^2)^3 \\ & \qquad\qquad\qquad\qquad < 0, \quad 0 < m - \mu^2 \ll 1. \end{aligned} \tag{5.23}$$

Therefore, the situation is the same as one at the beginning of **Case 2**. We could again distinguish here between two cases similar to **Cases 2.1, 2.2a, and 2.2b**. Repeating this procedure, we can eventually extend the function $\nu^1 = \nu^1(m)$ up to a point m at which

$$(\partial_m \Lambda_j)(m, \nu^1(m)) = 0.$$

This is done in finitely many steps. In fact, if this procedure would continue infinitely many times, then we could find a monotone increasing sequence $\{m_p\}_{p=1,2,\cdots}$ converging to some value m_∞ $(< \infty)$ satisfying

$$\begin{aligned} & (\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(m_p; w_j(\nu^1(m_p); u_0))) = 0, \\ & (\partial_m \Lambda_j)(m_p, \nu^1(m_p)) = \nu^{1\prime}(m_p) = 0, \quad p = 1, 2, \ldots . \end{aligned} \tag{5.24}$$

By the expansion in (3.5) we see that

$$\begin{aligned} & v_j(m_{p+1}; w_j(\nu^1(m_{p+1}); u_0)) \\ & = v_j(m_p; w_j(\nu^1(m_p); u_0)) \\ & \quad + (m_{p+1} - m_p)\, \widetilde{r}_j\big(v_j(m_p; w_j(\nu^1(m_p)))\big) + \mathcal{O}(m_{p+1} - m_p)^2, \end{aligned}$$

hence

$$\begin{aligned} 0 &= \frac{1}{m_{p+1} - m_p} \big((\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(m_{p+1}; w_j(\nu^1(m_{p+1}); u_0))) \\ & \qquad\qquad - (\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(m_p; w_j(\nu^1(m_p); u_0))) \big) \\ &= \Big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j \Big) \big(v_j(m_p; w_j(\nu^1(m_p); u_0)) \big) + \mathcal{O}(m_{p+1} - m_p). \end{aligned}$$

Letting $p \to \infty$ yields

$$0 = \big(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j \big) (v_j(m_\infty; w_j(\nu^1(m_\infty); u_0))).$$

On the other hand, by (5.24) we also obtain

$$(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(m_\infty; w_j(\nu^1(m_\infty); u_0))) = 0,$$

which contradicts our hypothesis (Hyp.1).

Case 2.3: ν^1 *may also reach the initial value* $\mu_j(u_0)$.

We treat here the case when the function $\nu^1 = \nu^1(m; u_0)$ satisfying (5.11) is defined for $\mu^1(u_0) \le m \le \mu^2(u_0)$ where $\mu^2 = \mu^2(u_0)$ is such that

$$\begin{aligned} &\nu^1(\mu^2(u_0); u_0) = \mu_j(u_0), \\ &(\partial_m \Lambda_j(m, \nu^1(m; u_0)) \le 0, \quad \mu^1(u_0) \le m \le \mu^2(u_0), \\ &v_j(m; w_j(\nu^1(m; u_0); u_0)) \in \mathbf{B}_{\delta_3}, \quad \mu^1(u_0) \le m \le \mu^2(u_0). \end{aligned} \tag{5.25}$$

Note that

$$\overline{\lambda}_j'(\mu^2; u_0) = (\partial_m \overline{\lambda}_j)(\mu^2, \nu^1(\mu^2)) \le 0.$$

Next, we will distinguish between two main situations depending whether $\overline{\lambda}_j'(\mu^2, \nu^1(\mu^2))$ is negative (**Case 2.3a**) or zero (**Case 2.3b**).

Let us first restrict attention to **Case 2.3a**. We wish to extend the wave curve by using the Hugoniot curve $\mathcal{H}_j(u_0)$. By Proposition 3.2 there exists now $\mu^3(u_0) > \mu^2(u_0)$ at which one of the following two possibilities arises:

- the state $v_j(\mu^3; u_0)$ lies on the boundary $\partial \mathbf{B}_{\delta_3}$ while the derivative $\overline{\lambda}_j'(m; u_0)$ keeps a constant sign (**Case 2.3a-i**),
- the derivative of the shock speed $\overline{\lambda}_j'(\mu^3; u_0)$ is zero (**Case 2.3a-ii**).

Using the Hugoniot curve $m \mapsto v_j(m; u_0)$, we extend the wave curve $\psi_j = \psi_j(m; u_0)$ for $m > \mu^2(u_0)$ as follows:

$$\psi_j(m; u_0) = \begin{cases} w_j(m; u_0), & \mu_j(u_0) \le m < \mu^1(u_0), \\ v_j(m; w_j(\nu^1(m; u_0); u_0)), & \mu^1(u_0) \le m < \mu^2(u_0), \\ v_j(m; u_0), & \mu^2(u_0) \le m \le \mu^3(u_0). \end{cases} \tag{5.26}$$

In this discussion (**Case 2.3a**) we have $\overline{\lambda}_j'(\mu^2, \nu^1(\mu^2)) < 0$. Let us check that the mapping $\psi_j(m; u_0)$ is of class $\mathcal{C}^1$ in m and u_0. To this end it is sufficient to show that the value $\mu^2(u_0)$ depends smoothly upon u_0, because both v_j and w_j are smooth mappings and $\mu^1(u_0)$, $\nu^1(m; u_0)$, and $\mu^2(u_0)$ depend smoothly upon their arguments and we can apply Lemma 5.2. It follows from (5.13) that ${\nu^1}'(\mu^2) < 0$. Taking this into account, we consider the function

$$F_j(m; u) := \nu^1(m; u) - \mu_j(u).$$

It is easy to see that $F_j = F_j(m; u)$ is smooth with respect to m and u, and that

$$\begin{aligned} &F_j(\mu^2(u_0); u_0) = \nu^1(\mu^2(u_0); u_0) - \mu_j(u) = 0, \\ &(\partial_m F_j)(\mu^2(u_0); u_0) = {\nu^1}'(\mu^2(u_0); u_0) > 0. \end{aligned}$$

Therefore, by the implicit function theorem, the function $\mu^2 = \mu^2(u)$ is well-defined and smooth in a neighborhood of u_0 at least, with

$$F_j(\mu^2(u), u) = \nu^1(\mu^2(u); u) - \mu_j(u) = 0.$$

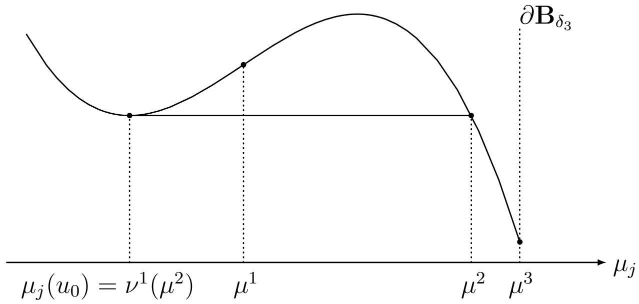

Fig. 4. Case 2.3a-i

We further consider the following subcases.

Case 2.3a-i: *Suppose that* $\overline{\lambda}_j'(m; u_0) < 0$ *for all* $\mu^2 \le m < \mu^3$.

By an earlier discussion we know that for each $m \in [\mu^1, \mu^2]$ the states $w_j(\nu^1(m); u_0)$ and $v_j(m; w_j(\nu^1(m); u_0))$ are connected by an admissible j-discontinuity. In particular, we have

$$\overline{\lambda}_j(\mu^2; u_0) \le \overline{\lambda}_j(m; u_0), \quad \mu^1 \le m \le \mu^2.$$

On the other hand, in this case the function $\overline{\lambda}_j(m; u_0)$ is assumed to be monotone decreasing for $\mu^2 \le m \le \mu^3$, so that

$$\overline{\lambda}_j(\overline{m}; u_0) \le \overline{\lambda}_j(m; u_0), \quad \mu^2 \le m \le \overline{m} \le \mu^3.$$

Therefore, for each $\overline{m} \in [\mu^2, \mu^3]$ we see that

$$\overline{\lambda}_j(\overline{m}; u_0) \le \overline{\lambda}_j(\mu^2; u_0) \le \overline{\lambda}_j(m; u_0), \quad \mu_j(u_0) \le m \le \mu^2,$$

and hence, $\overline{\lambda}_j(\overline{m}; u_0) \le \overline{\lambda}_j(m; u_0)$ for all $\mu_j(u_0) \le m \le \overline{m}$, which means that the couple of two states u_0 and $v_j(m; u_0)$ is an admissible j-discontinuity.

The construction of the waves curves is now completed in this case.

Case 2.3a-ii: *Suppose next that* μ^3 *is such that* $\overline{\lambda}_j'(m; u_0) < 0$ *for* $\mu^2 \le m < \mu^3$ *and* $\overline{\lambda}_j'(\mu^3; u_0) = 0$.

By Lemma 3.4 we see that

$$\begin{aligned}\overline{\lambda}_j'(m; u_0) &= \overline{\lambda}_j(\mu^3; u_0) + (m - \mu^3)\,\overline{\lambda}_j''(\mu^3; u_0) + \mathcal{O}(m - \mu^3)^2 \\ &= (m - \mu^3)\,\widetilde{\kappa}_3(\mu^3)\,\Big(\nabla\lambda_j \cdot \widetilde{r}_j\Big)\big(v_j(\mu^3; u_0)\big) + \mathcal{O}(m - \mu^3)^2.\end{aligned}$$

Therefore, we have $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^3; u_0)) \ge 0$.

If $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^3; u_0)) > 0$, then we can define the wave curve as in (5.28). In fact, we consider the function

$$F_j(m; u) := \lambda_j(v_j(m; u)) - \overline{\lambda}_j(m; u).$$

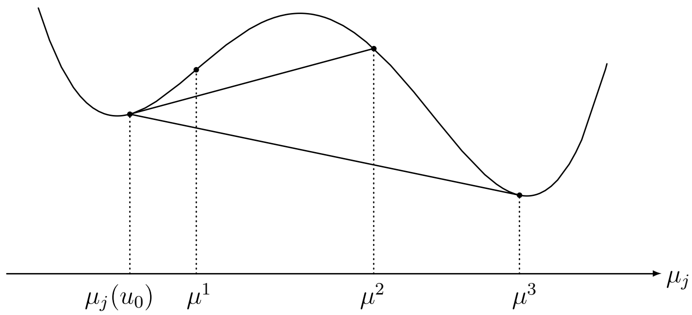

Fig. 5. Case 2.3a-ii

It is easy to see that F_j is smooth with respect to m and u, and that by Lemma 3.4

$$F_j(\mu^3; u_0) = 0, \quad (\partial_m F_j)(\mu^3; u_0) = \Big(\nabla\lambda_j \cdot \widetilde{r}_j\Big)\big(v_j(\mu^3; u_0)\big) > 0.$$

Therefore, by the implicit function theorem, the function $\mu_2 = \mu_2(u)$ is well-defined and smooth in a neighborhood of u_0, with

$$F_j(\mu_2(u), u) = \lambda_j(v_j(\mu_2(u); u)) - \overline{\lambda}_j(\mu_2(u); u) = 0.$$

We will further extend the wave curve below.

If $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^3; u_0)) = 0$, then we have

$$\overline{\lambda}'_j(\mu^3; u_0) = \overline{\lambda}''_j(\mu^3; u_0) = 0.$$

Therefore, by Lemma 3.5 we see that

$$\begin{aligned}\overline{\lambda}'_j(m; u_0) &= \overline{\lambda}'_j(\mu^3; u_0) + (m-\mu^3)\,\overline{\lambda}''_j(\mu^3; u_0) \\ &\quad + \frac{(m-\mu^3)^2}{2}\,\overline{\lambda}'''_j(\mu^3; u_0) + \mathcal{O}(m-\mu^3)^3 \\ &= \frac{(m-\mu^3)^2}{2}\,\widetilde{\kappa}_4(\mu^3)\,(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^3; u_0)) \\ &\quad + \mathcal{O}(m-\mu^3)^3.\end{aligned}$$

Hence, by taking the hypothesis (Hyp.1) into account we obtain

$$(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^3; u_0)) < 0,$$

which implies that $\overline{\lambda}_j(m; u_0) < 0$ for $0 < m - \mu^3 \ll 1$. Now, the situation is almost the same as one at the beginning of **Case 2.3a** so that we consider each **Cases 2.3a-i** and **2.3a-ii** again with μ^2 replaced by $\mu^3 + \varepsilon$, where ε is a sufficiently small positive constant. As in the same way in **Case 2.3a-i**, we can show that the corresponding shock satisfies the entropy criterion. Moreover, we can check as in the same way in **Case 2.2** that this procedure

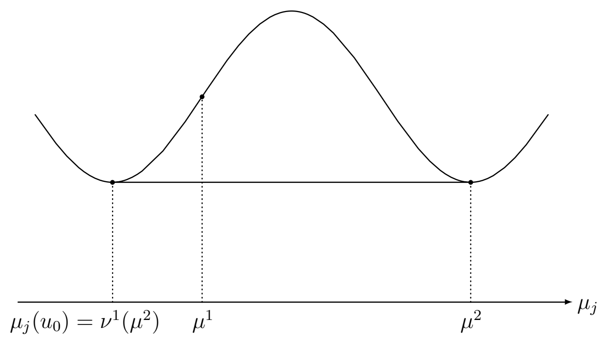

Fig. 6. Case 2.3b-i

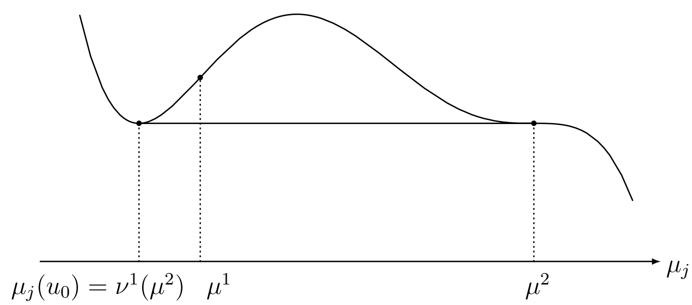

Fig. 7. Case 2.3b-ii

can be done in finitely many times. Further construction of the wave curves will be explained below.

Case 2.3b: *Suppose next that* $\overline{\lambda}_j'(\mu^2; u_0) = 0.$

By the same argument as the ones in **Case 2.3a-ii**, we see that $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_0)) \geq 0$.

Case 2.3b-i: *Suppose that* $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_0)) > 0.$

We can define the wave curve issuing from u_0 up to $v_j(\mu^2; u_0)$ by using the functions in (5.16). However, in this case it is not so immediate to analyze the regularity of the corresponding function $\psi_j = \psi_j(m; u)$ of the wave curve, because the structure of the wave curve is *not stable by perturbation of* u near u_0. The construction of the wave curve is continued below.

Case 2.3b-ii: *Suppose now that* $(\nabla\lambda_j \cdot \widetilde{r}_j)(v_j(\mu^2; u_0)) = 0.$

By the same arguments as in **Case 2.2b** we find

$$(\nabla(\nabla\lambda_j \cdot \widetilde{r}_j) \cdot \widetilde{r}_j)(v_j(\mu^2; u_0)) < 0,$$

hence, $\overline{\lambda}_j(m; u_0) < 0$ for $0 < m - \mu^2 \ll 1$. Therefore, the situation is essentially the same as the one in the beginning of **Case 2.3a** and we may follow the same arguments.

Step 4. *Conclusion.*

We have to continue the arguments of **Cases 2.2a, 2.3a-ii**, and **2.3b-i** to extend the wave curve up to the boundary $\partial \mathbf{B}_{\delta_3}$. In the following, we concentrate to continue the argument of **Case 2.2a**. The other cases can be treated in a similar fashion. Set

$$u_1 := w_j(\nu^1(\mu^2(u); u); u), \quad u_2 := v_j(\mu^2(u); u_1). \tag{5.27}$$

We see that $u_1 = u_1(u)$ and $u_2 = u_2(u)$ are smooth mappings in u defined in a neighborhood of u_0 and that $(\nabla\lambda_j \cdot \widetilde{r}_j)(u_2) > 0$. Roughly speaking, we may go back to the beginning of the argument and re-use the same arguments to extend the wave curve.

The essential difference is the treatment of **Case 2.3**. To explain this more precisely, we will repeat some of the arguments.

We can distinguish between two behaviors: Let $\mu^3 = \mu^3(u_0) > 0$ be such that

- either (**Case A**) the rarefaction curve reaches the boundary $\partial \mathbf{B}_{\delta_3}$ with

$$\begin{aligned} & w_j(\mu^3(u_0); u_2) \in \partial \mathbf{B}_{\delta_3}, \\ & w_j(m; u_2) \in \mathbf{B}_{\delta_3}, \quad \mu^2(u_0) \le m < \mu^3(u_0), \\ & (\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_2)) > 0, \quad \mu^2(u_0) \le m < \mu^3(u_0). \end{aligned} \tag{5.28}$$

- or else (**Case B**) $\mu^3(u_0)$ is the "first point" where $\nabla\lambda_j \cdot \widetilde{r}_j$ vanishes with

$$\begin{aligned} & (\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(\mu^3(u_0); u_2)) = 0, \\ & w_j(m; u_2) \in \mathbf{B}_{\delta_3}, \quad \mu^2(u_0) \le m \le \mu^3(u_0), \\ & (\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_2)) > 0, \quad \mu^2(u_0) \le m < \mu^3(u_0). \end{aligned} \tag{5.29}$$

In both cases, the wave curve coincides with the *integral curve* $\mathcal{O}_j(u_2)$ up to the value $\mu^3(u_0)$. We extend the wave curve $\psi_j = \psi_j(m; u_0)$ for $m > \mu^2(u_0)$ as follows:

$$\psi_j(m; u_0) = \begin{cases} w_j(m; u_0), & \mu_j(u_0) \le m < \mu^1(u_0), \\ v_j(m; w_j(\nu^1(m; u_0); u_0)), & \mu^1(u_0) \le m < \mu^2(u_0), \\ w_j(m; u_2), & \mu^2(u_0) \le m \le \mu^3(u_0). \end{cases} \tag{5.30}$$

Case A: *Consider first the case when* $(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_2)) > 0$ *for* $\mu^2 \le m < \mu^3$.

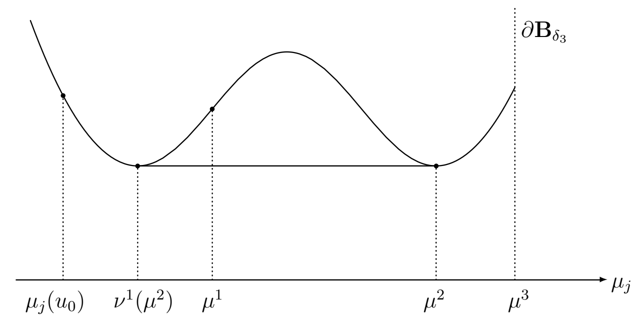

Fig. 8. Case A

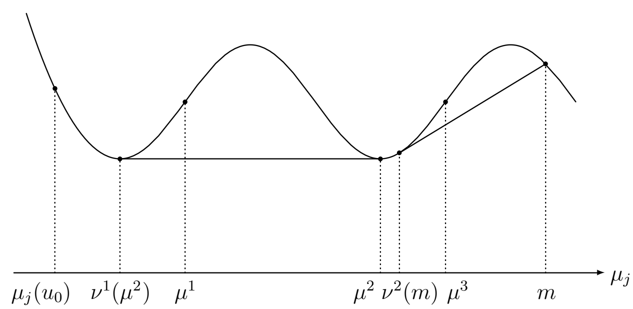

Fig. 9. Case B

We can prove in this case that the wave curve $\psi_j = \psi_j(m; u_0)$ defined by (5.30) is actually of class $\mathcal{C}^1$ in m and u_0, especially at $m = \mu^2(u_0)$. However, by Lemma 5.2 and then by Lemmas 3.3 and 3.4 we see that

$$\begin{aligned}\frac{d}{dm} v_j(m; w_j(\nu^1(m); u_0))\Big|_{m=\mu^2} &= v_j'(\mu_2; w_j(\nu^1(\mu^2); u_0)) \\ &= \widetilde{r}_j\big(v_j(\mu^2; w_j(\nu^1(\mu^2); u_0))\big) \\ &= w_j'(\mu_2; u_2).\end{aligned}$$

This implies the desired regularity.

Case B: *Suppose next that* $(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(m; u_2(u_0))) > 0$ *for* $\mu^2 \le m < \mu^3$ *and that*

$$(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(\mu^3; u_2(u_0))) = 0.$$

By applying the implicit function theorem to the function

$$F_j(s, u) := (\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(s; u))$$

at $(m;u) = (\mu^3, u_2(u_0))$, we see that the function $\mu^3 = \mu^3(u)$ is well-defined and smooth in a neighborhood of u_0, with

$$(\nabla\lambda_j \cdot \widetilde{r}_j)(w_j(\mu^3(u); u_2(u))) = 0.$$

Hence, we can extend the wave curve up to $w_j(\mu_3(u_0); u_2(u_0))$ by the formula (5.30). By a similar argument used in **Case 2**, there exists a smooth function $\nu^2 = \nu^2(m;u)$ such that $\nu_2(\mu^3(u);u) = \mu^3(u)$ and for $u_3(m,u) := w_j(\nu^2(m;u);u)$ we have

$$\overline{\lambda}_j(m; u_3(m,u)) = \lambda_j(u_3(m,u)). \tag{5.31}$$

The treatments of **Cases 1** and **2** are almost the same as before, so that we omit the details and concentrate on **Case 3**. That is, we assume that there exist $\mu^4 = \mu^4(u_0) > \mu^3(u_0)$ and the function $\nu^2(m) = \nu^2(m;u_0)$ satisfying (5.31) with $u = u_0$ such that $\nu^2(\mu^4) = \mu^2$ and that $\overline{\lambda}_j'(m; u_3(m,u_0)) \leq 0$ and $v_j(m; w_j(\nu^2(m); u_2(u_0))) \in \mathbf{B}_{\delta_3}$ for $\mu^3 \leq m \leq \mu^4$. Now, we know that the couples of two states $(u_1(u_0), u_2(u_0))$ and $(u_2(u_0), v_j(\mu^4; u_2(u_0))$ are admissible j-discontinuities and that

$$\overline{\lambda}_j(u_1(u_0), u_2(u_0)) = \overline{\lambda}_j(u_2(u_0), v_j(\mu^4, u_2(u_0))) = \lambda_j(u_2(u_0)).$$

Thanks to Lemma 3.5 we see that $(u_1(u_0), v_j(\mu^4; u_2(u_0)))$ is an admissible j-discontinuity. Therefore, the situation is almost the same as in **Case 2.2b** and we can argue in the same way as before and extend the wave curve. In this case, the wave curve is (Lipschitz) continuous but fails to have continuous first-order derivatives. This is due to the fact that the tangents do not coincide at point where two Hugoniot curves are connected together.

By similar arguments as in **Case 2.2b**, we can show that finitely waves only are needed to complete the construction of the wave curve in that case. This concludes the construction of the wave curves $\mathcal{W}_j(u_0)$ in all cases and completes the proof of Theorem 5.1.

6. The Riemann Problem and Refined Regularity Estimates

Based on the construction of the elementary wave curves given in Section 5 we can now establish the existence of the entropy solution to the Riemann problem associated with (1.1), that is, to the Cauchy problem with initial data of the form

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{6.1}$$

where u_l, u_r are constant states.

Theorem 6.1. (The Riemann problem.) *Suppose that system* (1.1) *is uniformly strictly hyperbolic with non-degenerate characteristic fields in the sense* (Hyp.1)–(Hyp.4)*. Then there exists a constant* $\delta_4 < \delta_3$ (*which can approach* $+\infty$ *together with* δ_1*,* δ_2*, and* δ_3 *as* K *approaches* 0) *such that for any Riemann data* $u_l, u_r \in \mathbf{B}_{\delta_4}$ *the Riemann problem* (1.1) *and* (6.1) *admits a self-similar solution made of elementary* (*rarefaction, admissible shock*) *waves. More precisely, there exists a Lipschitz continuous mapping*

$$\sigma = \sigma(u_l, u_r), \quad u_l, u_r \in \mathbf{B}_{\delta_4}$$

such that

$$F(\sigma(u_l, u_r); u_l, u_r) = 0. \tag{6.2}$$

We can actually prove some additional regularity property which supplements the Lipschitz continuity of the wave curves obtained in Theorem 5.1.

Theorem 6.2. (Additional regularity of the wave curves.) *Under the assumptions* (Hyp.1)–(Hyp.4)*, for each* $j = 1, \dots, N$ *the* **first-order derivatives** *of the mapping* ψ_j *with respect to both arguments* m *and* u *are* **Lipschitz continuous at the base point** u*, in the sense that*

$$\psi_j'(m; u) = \widetilde{r}_j(u) + C_K\, \mathcal{O}(m - \mu_j(u)) \tag{6.3}$$

and

$$(D_u \psi_j)(m; u) = \mathbf{I} - \big(\widetilde{r}_j \otimes \nabla \mu_j\big)(u) + C_K\, \mathcal{O}(m - \mu_j(u)), \tag{6.4}$$

where $\otimes$ *denotes the tensor product of two vectors, that is,* $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T$.

Recall here that

$$\widetilde{r}_j(u) := \Big(\frac{r_j}{\nabla \mu_j \cdot r_j}\Big)(u),$$

and note that the error term $C_K\, \mathcal{O}(m - \mu_j(u))$ vanishes when $K \to 0$.

Proof of Theorem 6.1. Let

$$F(m; u_l, u_r) := \psi_N(m_N; \cdots \psi_2(m_2; \psi_1(m_1; u_l)) \cdots) - u_r \tag{6.5}$$

where $m := (m_1, m_2, \dots, m_N) \in I\!R^N$ and $u_l, u_r \in \mathbf{B}_{\delta_3}$. In view of (6.3) and (6.4), the function F is Lipschitz continuous at least and that the matrix

$$D_m F(m; u_l, u_r) = \Big(r_j(0)\Big)_{1 \le j \le N} + o(1)$$

is invertible. Therefore, by applying the implicit function theorem for Lipschitz continuous mappings [9] to the function F at the point $(m; u_l, u_r) = (0; 0, 0)$, we immediately arrive at the statement in Theorem 6.1. □

The proof of Theorem 6.2 is based on the following two technical observations.

Lemma 6.3. *Let Y_p be a sequence of $N \times N$ matrices satisfying the property*

$$Y_p^2 = Y_p, \quad p = 1, 2, \ldots$$

Then, for all $n = 2, 3, \ldots$ we have

$$|Y_1 \, Y_2 \ldots Y_n| \le |Y_1| \, \exp\left(\sum_{p=1}^{n-1} |Y_{p+1} - Y_p| \right), \tag{6.6}$$

and

$$\begin{aligned} &|Y_1 \, Y_2 \ldots Y_n - Y_n| \\ &\le \left(\max_{1 \le p \le n} |Y_p| \right) \left(\sum_{p=1}^{n-1} |Y_{p+1} - Y_p| \right) \exp\left(\sum_{p=1}^{n-1} |Y_{p+1} - Y_p| \right). \end{aligned} \tag{6.7}$$

Lemma 6.4. *For each $p = 1, 2, \ldots, n$, consider a matrix X_p decomposed as a projection part Y_p and an error part E_p satisfying the following properties*:

$$\begin{aligned} &X_p = Y_p + E_p, \quad Y_p^2 = Y_p, \quad |Y_p| \le C_1, \\ &\sum_{p=1}^{n-1} |Y_{p+1} - Y_p| \le \delta, \quad \sum_{p=1}^{n} |E_p| \le \epsilon \end{aligned}$$

for some positive constants C_1, δ and ϵ. Set $C_2 := \max\{1, C_1 \, e^{\delta}\}$. Then, if $2 \, C_2 \, \epsilon \le 1$, we have

$$|X_1 \, X_2 \ldots X_n - Y_1 \, Y_2 \ldots Y_n| \le 2 \, C_2^2 \, \epsilon, \tag{6.8}$$

and

$$|X_1 \, X_2 \ldots X_n - Y_n| \le 2 \, C_2^2 \, \epsilon + C_2 \, \delta. \tag{6.9}$$

Proof of Theorem 6.2. By construction, for each m we can find intermediate states u_p, $p = 0, 1, 2, \ldots, P$, such that u_p is connected to u_{p+1} by an admissible j-shock wave or by a j-rarefaction wave and that $u_0 = u$ and $u_P = \psi_j(m; u)$. In the following, for definiteness we will consider the case where $m > \mu_j(u)$ and $P = 2M + 2$ and that u_{2p-1} is connected to u_{2p} by an admissible j-shock wave and u_{2p} is connected to u_{2p+1} by a rarefaction wave. Moreover, without loss of generality we can assume that the wave pattern is stable in the sense that it does not change under small perturbations of m and u. This is possible since the set of u for which the structure is unstable has measure zero thanks to our non-degeneracy assumption (Hyp.1). Note that u_p, $p = 1, 2, \ldots, 2M$ depends only on u while u_{2M+1} depends also on m. We can write

$$\begin{aligned} \psi_j(m; u) &= v_j(m; u_{2M+1}) \\ &= v_j(m; w_j(\nu^{M+1}(m; u); u_{2M})). \end{aligned} \tag{6.10}$$

Differentiating this identity with respect to m and applying Lemma 5.2 yield

$$\psi_j'(m; u) = v_j'(m; u_{2M+1}),$$

which together with (4.6) implies directly (6.3).

We now give the proof of (6.4) which is much more involved. To this end we have to drive an expression of the derivative $D_u\psi_j$ which should lead an uniform estimate with respect to the number of inflection points. Note that

$$u_{2p} = v_j(\mu^{2p}; u_{2p-1}) = v_j(\mu^{2p}; w_j(\nu^p(\mu^{2p}; u); u_{2p-2})),$$

where for instance $\mu^{2p} = \mu_j(u_{2p})$, etc. Differentiating this identity with respect to u and applying Lemma 5.2 again yield

$$\begin{aligned} D_u u_{2p} = &\, v_j'(\mu^{2p}; u_{2p-1}) \otimes \nabla\mu^{2p}(u) \\ &+ (D_u v_j)(\mu^{2p}; u_{2p-1})\,(D_u w_j)(\mu^{2p-1}; u_{2p-2})\,(D_u u_{2p-2}). \end{aligned}$$

Here, $(\nabla\mu^{2p})^T = (\nabla\mu_j)^T\,(D_u u_{2p})$ and, by Lemma 3.4, we also have

$$v_j'(\mu^{2p}; u_{2p-1}) = \widetilde{r}_j(u_{2p}).$$

Therefore, we obtain

$$\begin{aligned} &\big(\mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u_{2p})\big)(D_u u_{2p}) \\ &\quad = (D_u v_j)(\mu^{2p}; u_{2p-1})(D_u w_j)(\mu^{2p-1}; u_{2p-2})(D_u u_{2p-2}). \end{aligned} \tag{6.11}$$

Similarly, it follows from (6.10) that

$$(D_u\psi_j)(m; u) = (D_u v_j)(m; u_{2M+1})(D_u w_j)(\mu^{2M+1}; u_{2M})(D_u u_{2M}). \tag{6.12}$$

On the other hand, by the choice of our global parametrization, along the rarefaction curve we have the identity

$$w_j(m; u) = w_j(m; w_j(n; u)).$$

Differentiating this with respect to n and letting $n = \mu_j(u)$, we obtain the identity

$$(D_u w_j)(m; u)\,\widetilde{r}_j(u) = 0,$$

which holds for all m and u. In particular, we have

$$(D_u w_j)(m; u) = (D_u w_j)(m; u)\big(\mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u)\big). \tag{6.13}$$

In view of (6.11) and (6.12), inductively, together with (6.13), we arrive at

$$\begin{aligned} (D_u\psi_j)(m; u) = &\,(D_u v_j)(m; u_{2M+1})(D_u w_j)(\mu^{2M+1}; u_{2M}) \\ &(D_u v_j)(\mu^{2M}; u_{2M-1})(D_u w_j)(\mu^{2M-1}; u_{2M-2}) \\ &\ldots(D_u v_j)(\mu^2; u_1)(D_u w_j)(\mu^1; u). \end{aligned} \tag{6.14}$$

Since both $D_u w_j$ and $D_u v_j$ approach the matrix $\mathbf{I} - \widetilde{r}_j \otimes \nabla\mu_j$ thanks to the expansions (4.4) and (4.6), it can be easily conjectured that (6.14) should lead to the desired estimate (6.4). A rigorous derivation requires precisely our Lemmas 6.3 and 6.4: the error term may depend on M which is, roughly speaking, the number of inflection points. Hence, we have to evaluate carefully the error term in order to obtain an uniform estimate. We will make use of the property

$$\big(\mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u)\big)^2 = \mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u).$$

Set

$$\begin{aligned} &X_1 := (D_u v_j)(m; u_{2M+1}), \quad X_2 := (D_u w_j)(\mu^{2M+1}; u_{2M}), \dots, \\ &X_{2M+1} := (D_u v_j)(\mu^2; u_1), \quad X_{2M+2} := (D_u w_j)(\mu^1; u) \end{aligned}$$

and

$$\begin{aligned} &Y_1 := \mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u_{2M+1}), \quad Y_2 := \mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u_{2M}), \dots, \\ &Y_{2M+1} := \mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u_1), \quad Y_{2M+2} := \mathbf{I} - (\widetilde{r}_j \otimes \nabla\mu_j)(u). \end{aligned}$$

Then, by (4.4) and (4.6) it is easy to check that

$$\sum_{p=1}^{2M+1} |Y_{p+1} - Y_p| + \sum_{p=1}^{2M+2} |X_p - Y_p| \le C\,|m - \mu_j(u)| \ll 1,$$

where the constant C *does not depend* on M. Therefore, we can apply Lemma 6.4 and obtain the desired estimate (6.4). This completes the proof of Theorem 6.2. □

Proof of Lemma 6.3. Setting $a_n := |Y_1\, Y_2 \cdots Y_n|$, we see that

$$\begin{aligned} a_{n+1} &= |Y_1\, Y_2 \cdots Y_n + Y_1\, Y_2 \cdots Y_n \big(Y_{n+1} - Y_n\big)| \\ &\le a_n \big(1 + |Y_{n+1} - Y_n|\big). \end{aligned}$$

Using this inductively, we obtain

$$a_n \le a_1 \prod_{p=1}^{n-1} \big(1 + |Y_{p+1} - Y_p|\big) \le |Y_1| \exp\left(\sum_{p=1}^{n-1} |Y_{p+1} - Y_p|\right),$$

which is the first estimate in the lemma. To show the second one it is sufficient to write

$$\begin{aligned} Y_1\, Y_2 \cdots Y_n - Y_n = {} & (Y_1 - Y_2)\, Y_2\, Y_3 \cdots Y_n + (Y_2 - Y_3)\, Y_3\, Y_4 \cdots Y_n \\ & + \cdots + (Y_{n-2} - Y_{n-1})\, Y_{n-1} + (Y_{n-1} - Y_n) \end{aligned}$$

and to apply the first estimate. □

Proof of Lemma 6.4. We note that

$$\begin{aligned}
&X_1\,X_2\cdots X_n - Y_1\,Y_2\cdots Y_n\\
&= \sum_{1\le k_1\le n} Y_1\cdots Y_{k_1-1}\,E_{k_1}\,Y_{k_1+1}\cdots Y_n\\
&\quad + \sum_{1\le k_1<k_2\le n} Y_1\cdots Y_{k_1-1}\,E_{k_1}\,Y_{k_1+1}\cdots Y_{k_2-1}\,E_{k_2}\,Y_{k_2+1}\cdots Y_n\\
&\quad + \;\cdots\; + E_1\,E_2\cdots E_n.
\end{aligned}$$

Here, by lemma 6.3 we have

$$|Y_p\,Y_{p+1}\cdots Y_{p+k}| \le C_1\,e^{\delta} \le C_2.$$

Therefore, we see that

$$\begin{aligned}
&|X_1\,X_2\cdots X_n - Y_1\,Y_2\cdots Y_n|\\
&\le\; C_2^2 \sum_{1\le k_1\le n} |E_{k_1}| + C_2^3 \sum_{1\le k_1<k_2\le n} |E_{k_1}|\,|E_{k_2}|\\
&\quad + C_2^4 \sum_{1\le k_1<k_2<k_3\le n} |E_{k_1}|\,|E_{k_2}|\,|E_{k_3}| + \cdots + C_2^{n+1}\,|E_1|\,|E_2|\cdots|E_n|\\
&\le\; C_2^2\left(1 + C_2\,\epsilon + (C_2\epsilon)^2 + \cdots + (C_2\epsilon)^{n-1}\right)\epsilon\\
&\le\; 2\,C_2^2\,\epsilon.
\end{aligned}$$

This is the first estimate of the lemma. The second one is a direct consequence of the first estimate and Lemma 6.3. □

7. Wave Interaction Estimates

In this section we derive key estimates which will be needed in Sections 8 and 9 to derive a uniform bound on the total variation of a sequence of approximate solutions to system (1.1). Several estimates will be stated and proved.

First of all, consider an interaction between two elementary waves and let us compare the total sum of the wave strengths before and after the interaction. To control the possible increase, we introduce an *interaction potential* which provides a control of the increase of the wave strengths at the interaction. Our potential depends upon the angle made by the two incoming waves. A wave pattern is made of a combination of shocks and rarefactions. We introduce the following new notion of **generalized angle** between two elementary i-waves:

$$\theta_i(u_-, u_+; v_-, v_+) = \begin{cases} \Big(\overline{\lambda}_i(v_-, v_+) - \overline{\lambda}_i(u_-, u_+)\Big)^-, & u_+ \in \mathcal{H}_i(u_-),\ v_+ \in \mathcal{H}_i(v_-), \\ \frac{1}{\mu_i(u_+)-\mu_i(u_-)} \int_{\mu_i(u_-)}^{\mu_i(u_+)} \Big(\overline{\lambda}_i(v_-, v_+) - \lambda_i(w_i(\nu; u_-))\Big)^- d\nu, & u_+ \in \mathcal{O}_i(u_-),\ v_+ \in \mathcal{H}_i(v_-), \\ \frac{1}{\mu_i(v_+)-\mu_i(v_-)} \int_{\mu_i(v_-)}^{\mu_i(v_+)} \Big(\lambda_i(w_i(\xi; v_-)) - \overline{\lambda}_i(u_-, u_+)\Big)^- d\xi, & u_+ \in \mathcal{H}_i(u_-),\ v_+ \in \mathcal{O}_i(v_-), \\ \frac{1}{(\mu_i(v_+)-\mu_i(v_-))(\mu_i(u_+)-\mu_i(u_-))} \int_{\mu_i(u_-)}^{\mu_i(u_+)} \int_{\mu_i(v_-)}^{\mu_i(v_+)} \Big(\lambda_i(w_i(\xi; v_-)) - \lambda_i(w_i(\nu; u_-))\Big)^- d\nu d\xi, & u_+ \in \mathcal{O}_i(u_-),\ v_+ \in \mathcal{O}_i(v_-). \end{cases} \tag{7.1}$$

Recall that δ_4 was introduced in Section 3.

Theorem 7.1. (Interaction between two elementary waves.) *For all u_l, u_m, and $u_r \in \mathbf{B}_{\delta_4}$ and $1 \le i, j \le N$ we have the following property. Suppose that u_l is connected to u_m by an i-wave and that u_m is connected to u_r by a j-wave. Then, the wave strengths $\sigma_k(u_l, u_r)$ of the outgoing Riemann solution connecting the left-hand state u_l to the right-hand state u_r satisfy*

$$\begin{aligned} &\sigma_k(u_l, u_r) \\ &= \sigma_k(u_l, u_m) + \sigma_k(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \quad 1 \le k \le N, \\ &= \begin{cases} \sigma_i(u_l, u_m) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), & k = i \ne j, \\ \sigma_j(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), & k = j \ne i, \\ \sigma_i(u_l, u_m) + \sigma_j(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), & k = j = i, \\ C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), & \textit{otherwise}, \end{cases} \end{aligned} \tag{7.2}$$

where the **generalized interaction potential** *associated with the two incoming waves is*

$$Q(u_l, u_m, u_r) := \Theta_{ij}(u_l, u_m, u_r)\, |\sigma_i(u_l, u_m)\, \sigma_j(u_m, u_r)| \tag{7.3}$$

the weight being defined by

$$\Theta_{ij}(u_l, u_m, u_r) := \begin{cases} 0, & i < j, \\ 1, & i > j, \\ 1, & i = j \ \textit{and}\ \sigma_i(u_l, u_m)\, \sigma_i(u_m, u_r) < 0, \\ \theta_i(u_l, u_m; u_m, u_r), & i = j \ \textit{and}\ \sigma_i(u_l, u_m)\, \sigma_i(u_m, u_r) > 0. \end{cases} \tag{7.4}$$

To establish Theorem 7.1, we will treat first the cases when $i \neq j$ or when $i = j$ but the incoming waves have opposite signs, which are easier to handle since, in these cases, the potential $Q(u_l, u_m, u_r)$ reduces to the quadratic term $|\sigma_i(u_l, u_m)\, \sigma_j(u_m, u_r)|$. To begin with, we will recall Glimm's standard argument which applies if the mappings $\psi_j = \psi_j(m; u_0)$ are of class $W^{2,\infty}$. Under this assumption, the wave strengths are also of class $W^{2,\infty}$ and (when $i \neq j$ or when $i = j$ but the incoming waves have opposite signs) the estimates (7.2) follow from a standard division argument. Of course, relaxing the $W^{2,\infty}$ regularity is actually one new feature in Theorem 7.1.

Precisely, let us describe the set of solutions under consideration by fixing the left-hand state u_l and using the wave strengths

$$s_i := \sigma_i(u_l, u_m), \quad s_j := \sigma_j(u_m, u_r)$$

as parameters. The state u_r is regarded as a function of s_i and s_j, that is,

$$u_r := \psi_j(\mu_j(u_m) + s_j; u_m), \quad u_m := \psi_i(\mu_i(u_l) + s_i; u_l).$$

We set

$$H_k(s_i, s_j) := \sigma_k(u_l, u_r) - \begin{cases} s_i, & k = i \neq j, \\ s_j, & k = j \neq i, \\ s_i + s_j, & k = j = i, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, if either $s_i = 0$ or $s_j = 0$ one of the incoming wave is trivial and we have $H_k(s_j, s_i) = 0$, so we expect that

$$\big|H_k(s_j, s_i)\big| \le C\, |s_i\, s_j| \tag{7.5}$$

for all relevant values s_j, s_i and for some uniform constant $C > 0$. Indeed, we have

$$\begin{aligned} H_k(s_j, s_i) &= H_k(s_j, 0) + \int_0^{s_i} \frac{\partial H_k}{\partial s_i}(s_j, \sigma'')\, d\sigma'' \\ &= H_k(s_j, 0) + \int_0^{s_i} \left(\frac{\partial H_k}{\partial s_i}(0, \sigma'') + \int_0^{s_j} \frac{\partial^2 H_k}{\partial s_j \partial s_i}(\sigma', \sigma'')\, d\sigma' \right) d\sigma'', \end{aligned}$$

which gives (7.5) with

$$C := \sup \Big| \frac{\partial^2 H_k}{\partial s_j \partial s_i} \Big|,$$

since $H_k(s_j, 0) = H_k(0, s_i) = 0$ and *under the regularity assumption* (not satisfied in general) the functions σ_k and ψ_k have bounded second-order derivatives.

The novelty of our method is that we will establish a similar result when the wave curves are *Lipschitz continuous only* and their first-order derivatives are Lipschitz continuous at the base point (Theorem 6.2). The proof of Theorem 7.1 will be decomposed into three lemmas, as follows.

Lemma 7.2. *Under the assumptions in Theorem 7.1, the estimate* (7.2) *holds when $i \neq j$, that is, for all $1 \le k \le N$*

$$\sigma_k(u_l, u_r) = \sigma_k(u_l, u_m) + \sigma_k(u_m, u_r) + C_K\,\mathcal{O}(1)\ |\sigma_i(u_l, u_m)\,\sigma_j(u_m, u_r)|.$$

Furthermore, the estimate (7.2) *holds when $i = j$ and, $\sigma_i(u_l, u_m)$ and $\sigma_i(u_m, u_r)$ are of opposite signs:*

$$\sigma_i(u_l, u_r) = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r) + C_K\,\mathcal{O}(1)\,|\sigma_i(u_l, u_m)\,\sigma_j(u_m, u_r)|$$

and for all $k \neq i$

$$\sigma_k(u_l, u_r) = C_K\,\mathcal{O}(1)\,|\sigma_i(u_l, u_m)\,\sigma_j(u_m, u_r)|.$$

To cover the case $i = j$ when the incoming waves are *both negative or both positive,* we distinguish between the following three possibilities:

(a) either both incoming waves are shock waves,
(b) or the wave connecting u_l to u_m is a shock wave while the wave connecting u_m to u_r is a rarefaction,
(c) or else the wave connecting u_l to u_m is a rarefaction wave while the wave connecting u_m to u_r is a shock wave.

Case (c) turns out to be very similar to Case (b) and, in the following, we will omit the statements and proofs for this case. We reformulate the interaction estimates first in an approximate sense.

Lemma 7.3. (Approximate Riemann solution.) *In Case* (a) (*shock/shock interaction*) *there exists a state $u_r' \in \mathcal{H}_i(u_l)$ such that the i-shock wave connecting u_l to u_r' is admissible and*

$$\begin{aligned}
&\sigma_i(u_l, u_r') = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r),\\
&|u_r' - u_r| = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r),\\
&\sigma_i(u_l, u_r')\,\overline{\lambda}_i(u_l, u_r') = \sigma_i(u_l, u_m)\,\overline{\lambda}_i(u_l, u_m) + \sigma_i(u_m, u_r)\,\overline{\lambda}_i(u_m, u_r)\\
&\qquad\qquad\qquad\qquad\qquad + C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r).
\end{aligned} \tag{7.6}$$

In Case (b) (*shock/rarefaction interaction*) *there exist two states $u_m' \in \mathcal{O}_i(u_m)$ (with possibly $u_m' = u_r$) and $u_l' \in \mathcal{H}_i(u_m')$ such that the i-shock wave connecting u_l' to u_m' is admissible, $u_r = \psi_i(\mu_i(u_r); u_l')$ and that*

$$\begin{aligned}
&\sigma_i(u_l', u_m') + \sigma_i(u_m', u_r) = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r),\\
&|\sigma_i(u_m', u_r)| < |\sigma_i(u_m, u_r)|,\\
&|u_l' - u_l| = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r),\\
&\sigma_i(u_l', u_m')\,\overline{\lambda}_i(u_l', u_m') = \sigma_i(u_l, u_m)\,\overline{\lambda}_i(u_l, u_m) + \sigma_i(u_m, u_m')\,\widehat{\lambda}_i(u_m, u_m')\\
&\qquad\qquad\qquad\qquad\qquad + C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r),
\end{aligned} \tag{7.7}$$

where $\widehat{\lambda}_i(u_m, u'_m)$ *is the* **averaged speed of the rarefaction wave** *defined by*

$$\widehat{\lambda}_i(u_m, u'_m) = \frac{1}{\mu_i(u'_m) - \mu_i(u_m)} \int_{\mu_i(u_m)}^{\mu_i(u'_m)} \lambda_i(w_i(\nu; u_m))\, d\nu.$$

It will be convenient to introduce the following notation: when u_- is connected by $u_+ \in \mathcal{W}_i(u_-)$ by a sequence of rarefaction waves (connecting u^{2p-1} to u^{2p}, say) and shock waves (connecting u^{2p} to u^{2p+1}, say), we set

$$\begin{aligned} \Gamma_i(u_-, u_+) := & \int_{\mu_i(u^{2p-1})}^{\mu_i(u^{2p})} \lambda_i(w_i(\nu; u^{2p-1}))\, d\nu \\ & + \sum_p \sigma_i(u^{2p}, u^{2p+1})\, \overline{\lambda}_i(u^{2p}, u^{2p+1}). \end{aligned}$$

More generally, we denote by $\Gamma_i(u_l, u_r)$ the quantity associated with the i-wave fan within the Riemann solution associated with the data u_l, u_r.

The lemma above shows that the quantity Γ_i changes continuously at interactions, up to an error of order $Q(u_l, u_m, u_r)$. That is, in both Cases (a) and (b) we just proved:

$$\Gamma_i(u_l, u_r) = \Gamma_i(u_l, u_m) + \Gamma_i(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r). \tag{7.8}$$

The following lemma provides a generalization of this formula to the exact Riemann solution.

Lemma 7.4. (Exact Riemann solution.) *In Case* (a)*, the outgoing pattern determined by the Riemann solver contains a* (*large*) *admissible i-shock wave connecting some states u'_l and $u'_r \in \mathcal{H}_i(u_l)$, plus other waves* (*in other j-families*) *with total strength $C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r)$. Precisely, we have*

$$\begin{aligned} & \sigma_i(u'_l, u'_r) = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \\ & |u'_l - u_l| + |u'_r - u_r| = C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \\ & \Gamma_i(u_l, u_r) = \sigma_i(u'_l, u'_r)\, \overline{\lambda}_i(u'_l, u'_r) \\ & \qquad = \Gamma_i(u_l, u_m) + \Gamma_i(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r) \\ & \qquad = \sigma_i(u_l, u_m)\, \overline{\lambda}_i(u_l, u_m) + \sigma_i(u_m, u_r)\, \overline{\lambda}_i(u_m, u_r) \\ & \qquad\qquad + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r). \end{aligned} \tag{7.9}$$

In Case (b)*, the outgoing pattern determined by the Riemann solver contains small j-waves* ($j < i$) *connecting u_l to some state u'_l, a* (*large*) *admissible i-shock wave connecting u'_l to some $u'_m \in \mathcal{H}_i(u'_l)$ plus other i-rarefactions and shocks connecting u'_m to u'_r, followed by other* (*small*) *waves in other j-families* ($j > i$)*. The small waves have total strength $C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r)$.*

Precisely, we have

$$
\begin{aligned}
&\sigma_i(u_l', u_m') + \sigma_i(u_m', u_r') \\
&\quad = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \\
&|\sigma_i(u_m', u_r')| \le |\sigma_i(u_m, u_r)| + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \\
&|u_l' - u_l| + |u_m' - u_m| + |u_r' - u_r| = C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r), \\
&\Gamma_i(u_l, u_r) = \Gamma_i(u_l', u_m') + \Gamma_i(u_m', u_r') + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r) \\
&\qquad\qquad = \Gamma_i(u_l, u_m) + \Gamma_i(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r).
\end{aligned}
\tag{7.10}
$$

Finally, we note that the general case of two Riemann patterns can be obtained by induction from Theorem 7.1.

Theorem 7.5. (Interaction between two Riemann solutions.) *For all u_l, u_m, and $u_r \in \mathbf{B}_{\delta_4}$ and $1 \le i, j \le N$, the wave strengths $\sigma_k(u_l, u_r)$ of the outgoing Riemann solution connecting the left-hand state u_l to the right-hand state u_r satisfy*

$$
\begin{aligned}
&\sigma_k(u_l, u_r) = \sigma_k(u_l, u_m) + \sigma_k(u_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r),\ 1 \le k \le N, \\
&\Gamma_k(u_l, u_r) = \Gamma_k(u_l, u_m) + \Gamma_m, u_r) + C_K\, \mathcal{O}(1)\, Q(u_l, u_m, u_r),\ 1 \le k \le N,
\end{aligned}
$$

where the **interaction potential** *associated with the two Riemann solutions is*

$$
\begin{aligned}
&Q(u_l, u_m, u_r) \\
&:= \sum_{(u^k, u^{k+1}), (u^p, u^{p+1})} \Theta(u^k, u^{k+1}; u^p, u^{p+1})\, |\sigma_i(u^k, u^{k+1})\, \sigma_j(u^p, u^{p+1})|,
\end{aligned}
$$

where the summation is over all elementary i-waves (u^k, u^{k+1}) and j-waves (u^p, u^{p+1}) in the incoming Riemann solutions (the i-wave being located on the left-hand side of the j-wave) and the weight is defined by

$$
\begin{aligned}
&\Theta_{ij}(u^k, u^{k+1}; u^p, u^{p+1}) \\
&:= \begin{cases}
0, & i < j, \\
1, & i > j, \\
1, & i = j \ \text{and } \sigma_i(u^k, u^{k+1})\, \sigma_i(u^p, u^{p+1}) < 0, \\
\theta_i(u^k, u^{k+1}; u^p, u^{p+1}), & i = j \ \text{and } \sigma_i(u^k, u^{k+1})\, \sigma_i(u^p, u^{p+1}) > 0.
\end{cases}
\end{aligned}
$$

We will now give the proof of Lemmas 7.2 to 7.4.

Proof of Lemma 7.2. We are using the notation

$$
\begin{aligned}
u_m &= u_m(s_j) := \psi_j\big(\mu_j(u_l) + s_j; u_l\big), \\
u_r &= u_r(s_i, s_j) := \psi_i\big(\mu_i(u_m) + s_i; u_m\big),
\end{aligned}
$$

and, for the intermediate states u_k^* which determine the outgoing pattern,

$$\begin{aligned}
&u_1^* := u_l,\\
&u_{k+1}^* = u_{k+1}^*(s_i, s_j) := \psi_k\big(\mu_k(u_k^*) + t_k; u_k^*\big),\\
&u_{N+1}^* = u_{N+1}^*(s_i, s_j) := u_r,
\end{aligned}$$

where $t_k = t_k(s_i, s_j)$ denote the strengths of the outgoing k-waves. We will prove that

$$|t_i(s_i, s_j) - s_i| + |t_j(s_i, s_j) - s_j| + \sum_{k \neq i,j} |t_k(s_i, s_j)| = C_K\, \mathcal{O}(s_i\, s_j), \tag{7.11}$$

relying solely on the following uniform properties of the wave curves (Theorem 6.2):

$$\begin{aligned}
&\psi_k'(\mu_k(u) + s; u) = \widetilde{r}_k(u) + C_K\, \mathcal{O}(s),\\
&(D_u\psi_k)(\mu_k(u) + s; u) = \mathbf{I} - \widetilde{r}_k(u) \otimes \nabla\mu_k(u) + C_K\, \mathcal{O}(s),
\end{aligned} \tag{7.12a}$$

which of course imply that the curves are uniformly Lipschitz continuous:

$$\psi_k'(m; u) = \mathcal{O}(1), \quad (D_u\psi_k)(m; u) = \mathcal{O}(1). \tag{7.12b}$$

Recall also that, since attention is restricted to a neighborhood of the origin in $I\!R^N$,

$$\begin{aligned}
&|u_k^*(s_i, s_j) - u_l| = o(1),\\
&|t_i(s_i, s_j) - s_i| + |t_j(s_i, s_j) - s_j| + |t_k(s_i, s_j)| = o(1), \quad k \neq i, j.
\end{aligned} \tag{7.12c}$$

To begin with, from the definitions we easily determine the partial derivatives of the states u_m and u_r:

$$\begin{aligned}
&\frac{\partial u_m}{\partial s_i} = 0,\\
&\frac{\partial u_m}{\partial s_j} = \psi_j'\big(\mu_j(u_l) + s_j; u_l\big),\\
&\frac{\partial u_r}{\partial s_i} = \psi_i'\big(\mu_i(u_m) + s_i; u_m\big),
\end{aligned} \tag{7.13a}$$

and

$$\begin{aligned}
\frac{\partial u_r}{\partial s_j} &= \psi_i'\big(\mu_i(u_m) + s_i; u_m\big)\,\Big(\nabla\mu_i(u_m) \cdot \frac{\partial u_m}{\partial s_j}\Big)\\
&\qquad + (D_u\psi_i)\big(\mu_i(u_m) + s_i; u_m\big)\frac{\partial u_m}{\partial s_j}\\
&= \psi_i'\big(\mu_i(u_m) + s_i; u_m\big)\,\Big(\nabla\mu_i(u_m) \cdot \psi_j'\big(\mu_j(u_l) + s_j; u_l\big)\Big)\\
&\qquad + (D_u\psi_i)\big(\mu_i(u_m) + s_i; u_m\big)\psi_j'\big(\mu_j(u_l) + s_j; u_l\big).
\end{aligned} \tag{7.13b}$$

On the other hand, for the outgoing wave pattern we find

$$\begin{aligned}
&\frac{\partial u_1^*}{\partial s_i} = 0, \\
&\frac{\partial u_{k+1}^*}{\partial s_i} = \psi_k'\big(\mu_k(u_k^*) + t_k; u_k^*\big) \Big(\nabla \mu_i(u_k^*) \cdot \frac{\partial u_k^*}{\partial s_i} + \frac{\partial t_k}{\partial s_i}\Big) \\
&\qquad\qquad + (D_u \psi_k)\big(\mu_k(u_k^*) + t_k; u_k^*\big) \, \frac{\partial u_k^*}{\partial s_i},
\end{aligned} \tag{7.14}$$

together with a entirely similar expression for the s_j-derivatives.

It will be notationally convenient to introduce the following mapping (which is based on replacing $\partial u_k^* / \partial s_i$ and $\partial t_k / \partial s_i$ by a constant vector $V \in \mathbb{R}^N$ and by zero, respectively, in the right-hand side above):

$$\begin{aligned}
&\Phi_k = \Phi_k(s_i, s_j) : \mathbb{R}^N \mapsto \mathbb{R}^N, \\
&\Phi_k(s_i, s_j)[V] := \psi_k'\big(\mu_k(u_k^*) + t_k; u_k^*\big) \Big(\nabla \mu_i(u_k^*) \cdot V\Big) \\
&\qquad\qquad + (D_u \psi_k)\big(\mu_k(u_k^*) + t_k; u_k^*\big) \, V.
\end{aligned}$$

Hence, (7.14) takes the form

$$\frac{\partial u_{k+1}^*}{\partial s_i} = \psi_k'\big(\mu_k(u_k^*) + t_k; u_k^*\big) \, \frac{\partial t_k}{\partial s_i} + \Phi_k \Big[\frac{\partial u_k^*}{\partial s_i}\Big]. \tag{7.14$'$}$$

Using the formula inductively, we are able to express $\partial u_r / \partial s_i$ in the form

$$\begin{aligned}
\frac{\partial u_r}{\partial s_i} &= \frac{\partial u_{N+1}^*}{\partial s_i} \\
&= \psi_N'\big(\mu_N(u_N^*) + t_N; u_N^*\big) \, \frac{\partial t_N}{\partial s_i} \\
&\quad + \Phi_N \Big[\psi_{N-1}'\big(\mu_{N-1}(u_{N-1}^*) + t_{N-1}; u_{N-1}^*\big)\Big] \, \frac{\partial t_{N-1}}{\partial s_i} \\
&\quad + \big(\Phi_N \circ \Phi_{N-1}\big) \Big[\psi_{N-2}'\big(\mu_{N-2}(u_{N-2}^*) + t_{N-2}; u_{N-2}^*\big)\Big] \, \frac{\partial t_{N-2}}{\partial s_i} \\
&\quad + \cdots \\
&\quad + \big(\Phi_N \circ \Phi_{N-1} \cdots \circ \Phi_2\big) \Big[\psi_1'\big(\mu_1(u_1^*) + t_1; u_1^*\big)\Big] \, \frac{\partial t_1}{\partial s_i},
\end{aligned} \tag{7.15}$$

Equivalently, (7.15) shows that the "unknown" coefficients $\partial t_k / \partial s_i$, $1 \le k \le N$, satisfy the following vector-valued equation:

$$a_1 \, \frac{\partial t_1}{\partial s_i} + a_2 \, \frac{\partial t_2}{\partial s_i} + \ldots + a_N \, \frac{\partial t_N}{\partial s_i} = \frac{\partial u_r}{\partial s_i} \tag{7.16a}$$

and, similarly, for the s_j-derivatives:

$$a_1 \, \frac{\partial t_1}{\partial s_j} + a_2 \, \frac{\partial t_2}{\partial s_j} + \ldots + a_N \, \frac{\partial t_N}{\partial s_j} = \frac{\partial u_r}{\partial s_j}, \tag{7.16b}$$

whose coefficients $a_k = a_k(s_i, s_j)$, $k = 1, 2, \dots, N$ are given by

$$\begin{aligned}
&a_N(s_i,s_j) := \psi'_N\big(\mu_N(u_N^*) + t_N; u_N^*\big),\\
&a_{N-1}(s_i,s_j) := \Phi_N\Big[\psi'_{N-1}\big(\mu_{N-1}(u_{N-1}^*) + t_{N-1}; u_{N-1}^*\big)\Big],\\
&a_{N-2}(s_i,s_j) := \big(\Phi_N \circ \Phi_{N-1}\big)\Big[\psi'_{N-2}\big(\mu_{N-2}(u_{N-2}^*) + t_{N-2}; u_{N-2}^*\big)\Big],\\
&\qquad\qquad\vdots\\
&a_1(s_i,s_j) := \big(\Phi_N \circ \Phi_{N-1} \cdots \circ \Phi_2\big)\Big[\psi'_1\big(\mu_1(u_1^*) + t_1; u_1^*\big)\Big].
\end{aligned}$$

Observe that these coefficients depend smoothly on the data, in the sense that there exist smooth (polynomial) functions $F_1, F_2, \cdots, F_N$ such that

$$\begin{aligned}
a_N &= F_N\Big(\psi'_N\big(\mu_N(u_N^*) + t_N; u_N^*\big)\Big),\\
a_{N-1} &= F_{N-1}\Big(\psi'_N\big(\mu_N(u_N^*) + t_N; u_N^*\big), D_u\psi_N\big(\mu_N(u_N^*) + t_N; u_N^*\big),\\
&\qquad \nabla\mu_N(u_N^*), \psi'_{N-1}\big(\mu_{N-1}(u_{N-1}^*) + t_{N-1}; u_{N-1}^*\big)\Big),\\
&\qquad\vdots\\
a_1 &= F_1\Big(\psi'_N\big(\mu_N(u_N^*) + t_N; u_N^*\big), D_u\psi_N\big(\mu_N(u_N^*) + t_N; u_N^*\big), \nabla\mu_N(u_N^*),\\
&\qquad \cdots \quad \cdots \quad \cdots\\
&\qquad \psi'_2\big(\mu_2(u_2^*) + t_2; u_2^*\big), D_u\psi_2\big(\mu_2(u_2^*) + t_2; u_2^*\big), \nabla\mu_2(u_2^*),\\
&\qquad \psi'_1\big(\mu_1(u_1^*) + t_1; u_1^*\big)\Big).
\end{aligned}$$

Next, we investigate the properties of the coefficients $a_k(s_i, s_j)$. From the definition of Φ_k and the uniform bounds (7.12b), we have immediately

$$\Phi_k(s_i,s_j)[V] = \mathcal{O}(1)\,|V|,$$

which implies that the coefficients a_k are uniformly bounded:

$$a_k(s_i,s_j) = \mathcal{O}(1). \tag{7.17}$$

Using the properties (7.12a) we obtain the more precise estimate:

$$\begin{aligned}
\Phi_k(s_i,s_j)[V] = &\,\widetilde{r}_k(u_k^*)\,\big(\nabla\mu_k(u_k^*)\cdot V\big) + \big(\mathbf{I} - \big(\widetilde{r}_j \otimes \nabla\mu_j\big)(u_k^*)\big)\,V\\
&+ C_K\,\mathcal{O}(1)\,|t_k(s_i,s_j)|\,|V|.
\end{aligned}$$

But, noticing that, for any three vectors $a, b, c \in I\!R^N$, $\big(a \otimes b\big)\,c = a\,\big(b\cdot c\big)$, and thus

$$\big(\widetilde{r}_j \otimes \nabla\mu_j\big)(u_k^*)\,V = \widetilde{r}_k(u_k^*)\,\big(\nabla\mu_k(u_k^*)\cdot V\big),$$

the above estimate is rewritten as

$$\Phi_k(s_i,s_j)[V] = V + C_K\,\mathcal{O}(1)\,|t_k(s_i,s_j)|\,|V|$$

or, regarding $\Phi_k(s_i, s_j)$ as an $N \times N$ matrix,

$$\Phi_k(s_i, s_j) = \mathbf{I} + C_K\, \mathcal{O}(1)\, |t_k(s_i, s_j)|. \tag{7.18}$$

It then follows from (7.17), the definition of a_k, and (7.12a) that

$$a_k(s_i, s_j) = \widetilde{r}_k(u_k^*(s_i, s_j)) + C_K\, \mathcal{O}(1) \sum_{p=k}^{N} |t_p(s_i, s_j)|,$$

thus with (7.12c)

$$\begin{aligned} a_k(s_i, s_j) - \widetilde{r}_k(u_l) &= C_K\, \mathcal{O}(1)\, |u_k^*(s_i, s_j) - u_l| + C_K\, \mathcal{O}(1) \sum_{p=k}^{N} |t_p(s_i, s_j)| \\ &= o(1). \end{aligned}$$

In particular, the determinant of both linear systems (7.16) is uniformly bounded away from zero:

$$\det\Big(a_1(s_i, s_j), \dots, a_k(s_i, s_j)\Big) = \det\Big(\widetilde{r}_1(u_l), \dots, \widetilde{r}_k(u_l)\Big) + o(1). \tag{7.19}$$

Observe now that using (7.12b) within the expressions (7.13) yields a uniform bound for the source-terms $\partial u_r/\partial s_i$ and $\partial u_r/\partial s_j$ of the systems (7.16):

$$\Big|\frac{\partial u_r}{\partial s_i}\Big| + \Big|\frac{\partial u_r}{\partial s_j}\Big| = \mathcal{O}(1).$$

Therefore, relying also on (7.17) and (7.19), we deduce that the (unique) solutions of (7.16) are also uniformly bounded:

$$\Big|\frac{\partial t_k}{\partial s_j}\Big| + \Big|\frac{\partial t_k}{\partial s_j}\Big| = \mathcal{O}(1), \quad 1 \le k \le N. \tag{7.20}$$

In turn, since by definition

$$\begin{aligned} &t_i(s_i, 0) = s_i, \quad t_k(s_i, 0) = 0, \quad k \neq i, \\ &t_j(0, s_j) = s_j, \quad t_k(0, s_j) = 0, \quad k \neq j, \end{aligned}$$

we deduce that the outgoing wave strengths satisfy (at least)

$$\begin{aligned} t_k(s_i, s_j) &= \mathcal{O}(s_j), \quad k \neq i, \\ t_k(s_i, s_j) &= \mathcal{O}(s_i), \quad k \neq j. \end{aligned} \tag{7.21}$$

Then, returning to (7.14') we deduce from (7.12b) and (7.20) (by induction on k):

$$\Big|\frac{\partial u_k^*}{\partial s_i}\Big| + \Big|\frac{\partial u_k^*}{\partial s_j}\Big| = \mathcal{O}(1),$$

and, by integration,

$$\begin{aligned} \Big|u_k^*(s_i,s_j)-u_k^*(0,s_j)\Big| &= \mathcal{O}(s_i),\\ \Big|u_k^*(s_i,s_j)-u_k^*(s_i,0)\Big| &= \mathcal{O}(s_j). \end{aligned} \tag{7.22}$$

We now distinguish between two cases.

Case 1 : Suppose that $|s_i| \le |s_j|$. We claim that

$$\begin{aligned} \Big|a_k(s_i,s_j)-a_k(s_i,0)\Big| &= C_K\mathcal{O}(s_j),\\ \Big|\tfrac{\partial u_r}{\partial s_i}(s_i,s_j)-\tfrac{\partial u_r}{\partial s_i}(s_i,0)\Big| &= C_K\mathcal{O}(s_j). \end{aligned} \tag{7.23}$$

Of course, if the claim is established, relying on the fact that the linear system (7.16a) has smooth (polynomial) coefficients and that its determinant is uniformly bounded away from zero, we deduce that

$$\Big|\frac{\partial t_k}{\partial s_i}(s_i,s_j)-\frac{\partial t_k}{\partial s_i}(s_i,0)\Big| = C_K\,\mathcal{O}(s_j).$$

Using Kronecker's symbols, this result reads

$$\Big|\frac{\partial}{\partial s_i}\Big(t_k(s_i,s_j)-\delta_{ki}\,s_i-\delta_{kj}\,s_j\Big)\Big| = C_K\,\mathcal{O}(s_j),$$

and, after integration in s_i,

$$\Big|t_k(s_i,s_j)-\delta_{ki}\,s_i-\delta_{kj}\,s_j\Big| = C_K\,\mathcal{O}(s_i\,s_j), \quad 1\le k\le N,$$

which is the desired estimate (7.11).

Finally, to establish (7.23) we proceed as follows. Let us, for instance, deal with a_1 (which is the coefficient having the most intricate expression). We have

$$\begin{aligned} &\Big|a_1(s_i,s_j)-F_1\Big(\widetilde{r}_N(u_N^*),\mathbf{I}-\big(\widetilde{r}_j\otimes\nabla\mu_j\big)(u_N^*),\nabla\mu_N(u_N^*),\cdots,\\ &\qquad\qquad\qquad \widetilde{r}_2(u_2^*),\mathbf{I}-\big(\widetilde{r}_j\otimes\nabla\mu_j\big)(u_2^*),\nabla\mu_2(u_2^*),\widetilde{r}_1(u_1^*)\Big)\Big|\\ &\quad = \sum_k C_K\,\mathcal{O}(t_k) = C_K\,\mathcal{O}(s_i)+C_K\,\mathcal{O}(s_j) = C_K\,\mathcal{O}(s_j). \end{aligned}$$

The function F_1 is defined in terms of the functions $\Phi_k[V]$ which in view of the property (7.12a) can be replaced by

$$\widetilde{r}_k(u_k^*)\,\nabla\mu_k(u_k^*)\cdot V+\mathbf{I}-\big(\widetilde{r}_j\otimes\nabla\mu_j\big)(u_k^*),$$

which, of course, is identically equal to V. Therefore we have

$$\big|a_1(s_i,s_j)-\widetilde{r}_1(u_1^*)\big| = C_K\,\mathcal{O}(s_j),$$

which yields

$$\begin{aligned}\big|a_1(s_i,s_j)-a_1(s_i,0)\big| &\le \big|a_1(s_i,s_j)-\widetilde{r}_1(u_1^*(s_i,s_j))\big|\\ &\quad+\big|a_1(s_i,0)-\widetilde{r}_1(u_1^*(s_i,0))\big|\\ &\quad+\big|\widetilde{r}_1(u_1^*(s_i,s_j))-\widetilde{r}_1(u_1^*(s_i,0))\big|\\ &= C_K\,\mathcal{O}(s_j)+O(1)\,\big|u_1^*(s_i,s_j)-u_1^*(s_i,0)\big|\\ &= C_K\,\mathcal{O}(s_j).\end{aligned}$$

This is the first statement in (7.23). The estimates for the other coefficients a_k and for $\partial u_r/\partial s_i$ are obtained in exactly the same way.

Case 2 : Suppose next that $|s_i| \ge |s_j|$. The corresponding claim

$$\begin{aligned}\Big|a_k(s_i,s_j)-a_k(0,s_j)\Big| &= C_K\,\mathcal{O}(s_i),\\ \Big|\tfrac{\partial u_r}{\partial s_j}(s_i,s_j)-\tfrac{\partial u_r}{\partial s_j}(0,s_j)\Big| &= C_K\,\mathcal{O}(s_i).\end{aligned} \tag{7.24}$$

can be checked exactly as in Case 1. Moreover, from these properties we get

$$\Big|\frac{\partial t_k}{\partial s_j}(s_i,s_j)-\frac{\partial t_k}{\partial s_j}(0,s_j)\Big| = C_K\,\mathcal{O}(s_i),$$

or

$$\Big|\frac{\partial}{\partial s_j}\Big(t_k(s_i,s_j)-\delta_{ki}\,s_i-\delta_{kj}\,s_j\Big)\Big| = C_K\,\mathcal{O}(s_i),$$

which, by integration in s_j yields the desired estimate (7.11).

Finally we observe that proving the interaction estimate for two waves of the same characteristic family but of opposite signs: can be done by following exactly the same steps as in the case of different families. We now have $i=j$ and s_i and s_j denote wave strengths of the same family. But, all the arguments given above remain valid. This completes the proof of Lemma 7.2. □

Proof of Lemma 7.3. Set $\mu_l := \mu_j(u_l)$, $\mu_m = \mu_i(u_m)$ and $\mu_r = \mu_i(u_r)$ and, for definiteness, consider the case that $\alpha := \sigma_i(u_l,u_m) \equiv \mu_m - \mu_l > 0$ and $\beta := \sigma_i(u_m,u_r) \equiv \mu_r - \mu_m > 0$. Other cases can be treated similarly.

Case (a): From the Rankine-Hugoniot relations

$$\overline{\lambda}_i(u_l,u_m)\,(u_l-u_m) = f(u_l)-f(u_m)$$

and

$$\overline{\lambda}_i(u_m,u_r)\,(u_m-u_r) = f(u_m)-f(u_r),$$

we deduce that

$$\overline{\lambda}_i(u_l,u_m)\,(u_l-u_m)+\overline{\lambda}_i(u_m,u_r)\,(u_m-u_r) = f(u_l)-f(u_r). \tag{7.25}$$

Defining the **averaged shock speed**

$$\Lambda := \frac{\alpha}{\alpha+\beta}\,\overline{\lambda}_i(u_l, u_m) + \frac{\beta}{\alpha+\beta}\,\overline{\lambda}_i(u_m, u_r), \tag{7.26}$$

we find that the states u_l and u_r satisfy the Rankine-Hugoniot relation in an approximate sense:

$$-\Lambda\,(u_l - u_r) + f(u_l) - f(u_r) = R, \tag{7.27}$$

where the remainder R is

$$\begin{aligned} R :=\ & \overline{\lambda}_i(u_l, u_m)\,(u_l - u_m) + \overline{\lambda}_i(u_m, u_r)\,(u_m - u_r) \\ & -(u_l - u_r)\,\left(\frac{\alpha}{\alpha+\beta}\,\overline{\lambda}_i(u_l, u_m) + \frac{\beta}{\alpha+\beta}\,\overline{\lambda}_i(u_m, u_r)\right) \\ =\ & \big(\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\big)\left(\frac{\alpha}{\alpha+\beta}(u_r - u_m) - \frac{\beta}{\alpha+\beta}(u_m - u_l)\right). \end{aligned}$$

Using a standard "division argument" with the function

$$\begin{aligned} \phi(\alpha,\beta) :=\ & \frac{\alpha}{\alpha+\beta}\big(v_i(\mu_l + \alpha + \beta; v_i(\mu_l + \alpha; u_l)) - v_i(\mu_l + \alpha; u_l)\big) \\ & - \frac{\beta}{\alpha+\beta}\big(v_i(\mu_l + \alpha; u_l) - u_l\big), \end{aligned}$$

which satisfies $\phi(\alpha, 0) = \phi(0, \beta) = 0$ and is of class $W^{2,\infty}$ at least (since it is determined from Hugoniot curves which are known to be smooth), we find

$$\begin{aligned} |R| & \le C_K\,\big|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\big|\,|u_l - u_m|\,|u_m - u_r| \\ & = C_K\,Q(u_l, u_m, u_r). \end{aligned} \tag{7.28}$$

Next, take $u_r' \in \mathcal{H}_i(u_l)$ such that $\mu_i(u_r') = \mu_i(u_r)$. Obviously, we have

$$\sigma_i(u_l, u_r') = \sigma_i(u_l, u_m) + \sigma_i(u_m, u_r).$$

By our choice of u_r' we have

$$-\lambda_i(u_l, u_r')\,(u_r' - u_l) + f(u_r') - f(u_l) = 0,$$

thus using (7.28) we arrive at the identity

$$-\Lambda\,(u_r' - u_r) + f(u_r') - f(u_r) = \big(\lambda_i(u_l, u_r') - \Lambda\big)\,(u_r' - u_l) + R, \tag{7.29}$$

which we now use to estimate the difference $|u_r' - u_r|$.

To this end, introduce the decomposition

$$u_r' - u_r =: \sum_j \alpha_j\,\overline{r}_j(u_r, u_r'),$$

so that

$$\sum_j \big(\overline{\lambda}_j(u_r, u_r') - \Lambda\big)\,\alpha_j\,\overline{r}_j(u_r, u_r') = \big(\lambda_i(u_l, u_r') - \Lambda\big)\,(u_r' - u_l) + R. \tag{7.30}$$

Multiplying (7.30) by $\bar{l}_j(u_r, u_r')$ we get

$$\big(\overline{\lambda}_j(u_r, u_r') - \Lambda\big)\, \alpha_j = \big(\lambda_i(u_l, u_r') - \Lambda\big)\, \bar{l}_j(u_r, u_r')(u_r' - u_l) + \bar{l}_j(u_r, u_r')R. \tag{7.31}$$

Here, by the definition of u_r' we see that

$$u_r' - u_l = \int_0^{\alpha+\beta} v_i'(\mu_l + s; u_l)ds = (\alpha + \beta)\,\big(\widetilde{r}_i(u_l) + C_K\, \mathcal{O}(\alpha + \beta)\big)$$

and that

$$\begin{aligned}\bar{l}_j(u_r, u_r') &= l_j(u_l) + C_K\, \mathcal{O}(|u_r - u_l| + |u_r' - u_l|)\\ &= l_j(u_l) + C_K\, \mathcal{O}(\alpha + \beta).\end{aligned}$$

Therefore, we find

$$\begin{aligned}&\big|\bar{l}_j(u_r, u_r')(u_r' - u_l)\big| \le C\,(\alpha + \beta)^2, \quad j \ne i,\\ &\frac{1}{C}\,(\alpha + \beta) \le \big|\bar{l}_i(u_r, u_r')(u_r' - u_l)\big| \le C\,(\alpha + \beta).\end{aligned} \tag{7.32}$$

Then, it follows from (7.31) that

$$|\alpha_j| \le C\,\big|\lambda_i(u_l, u_r') - \Lambda\big|(\alpha + \beta)^2 + C\,|R|, \quad j \ne i,$$

which (since $\mu_i(u_r') = \mu_i(u_r)$) is equivalent to saying

$$|u_r' - u_r| \le C\,\big|\lambda_i(u_l, u_r') - \Lambda\big|(\alpha + \beta)^2 + C\,|R|. \tag{7.33}$$

Moreover, in view of (7.31), (7.32) and the fact $|\alpha_i| \le C\,|u_r' - u_r|$ we also obtain

$$(\alpha + \beta)\,\big|\overline{\lambda}_i(u_l, u_r') - \Lambda\big| \le C\,\big(|u_r - u_r'| + |R|\big). \tag{7.34}$$

Finally, combining (7.28), (7.33), and (7.34) we arrive at

$$|u_r - u_r'| + (\alpha + \beta)\,\big|\overline{\lambda}_i(u_l, u_r') - \Lambda\big| \le C_K\, Q(u_l, u_m, u_r), \tag{7.35}$$

which establishes the second and third relations in (7.6).

It remain to prove that the shock wave connecting u_l to u_r' is admissible. By contradiction, suppose there would exist a state $u_* \in \mathcal{H}_i(u_l)$ such that

$$\mu_m < \mu_* := \mu_i(u_*) < \mu_r, \quad \overline{\lambda}_i(u_l, u_*) = \overline{\lambda}_i(u_l, u_r').$$

Of course for the interaction between the two incoming shock to actually take place one needs that

$$\overline{\lambda}_i(u_l, u_m) > \overline{\lambda}_i(u_m, u_r).$$

From

$$\overline{\lambda}_i(u_l, u_r')\,(u_r' - u_l) = f(u_r') - f(u_l)$$

and

$$\overline{\lambda}_i(u_*, u_l)\,(u_* - u_l) = f(u_*) - f(u_l),$$

we then deduce that

$$\overline{\lambda}_i(u_l, u_r')\,(u_r' - u_*) = f(u_r') - f(u_r).$$

Therefore, $u_* \in \mathcal{H}_i(u_r')$ and $u_r' \in \mathcal{H}_i(u_*)$ and $\overline{\lambda}_i(u_*, u_r') = \overline{\lambda}_i(u_l, u_r')$, and with (7.35)

$$(\alpha + \beta)\left|\overline{\lambda}_i(u_*, u_r') - \Lambda\right| \le C_K \left|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\right| \alpha\beta \tag{7.36}$$

On the other hand, since the shock connecting u_m to u_r is admissible, we have

$$\overline{\lambda}_i(u, u_r) \le \overline{\lambda}_i(u_m, u_r) \tag{7.37}$$

for every $u \in \mathcal{H}_i(u_r)$ satisfying $\mu_i(u_m) < \mu_i(u) < \mu_i(u_r)$. Now, we choose a state $u \in \mathcal{H}_i(u_r)$ as $u := v_i(\mu_*; u_r)$, which is close to u_* in the sense that

$$\begin{aligned}|u - u_*| &= \left|v_i(\mu_*; u_r) - v_i(\mu_*; u_r')\right| \\ &\le C|u_r - u_r'| \\ &\le C_K \left|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\right| \alpha\beta,\end{aligned}$$

therefore

$$\begin{aligned}\left|\overline{\lambda}_i(u_*, u_r') - \overline{\lambda}_i(u, u_r)\right| &\le C\left(|u - u_*| + |u_r - u_r'|\right) \\ &\le C_K \left|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_l)\right|\alpha\beta \end{aligned} \tag{7.38}$$

From the definition of Λ and by (7.36), (7.37) and (7.38), we then deduce that

$$\begin{aligned}&\alpha\left(\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\right) \\ &= (\alpha + \beta)\Big(\big(\Lambda - \overline{\lambda}_i(u_*, u_r')\big) + \big(\overline{\lambda}_i(u_*, u_r') - \overline{\lambda}_i(u, u_r)\big) \\ &\qquad - \big(\overline{\lambda}_i(u_m, u_r) - \overline{\lambda}_i(u, u_r)\big)\Big) \\ &\le (\alpha + \beta)\left(\left|\Lambda - \overline{\lambda}_i(u_*, u_r')\right| + \left|\overline{\lambda}_i(u_*, u_r') - \overline{\lambda}_i(u, u_r)\right|\right. \\ &\le C_K\left(\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\right)\alpha\beta.\end{aligned}$$

This leads us to $1 \le C_K\,\beta$, which is a contradiction when $|u_m - u_r|$ is sufficiently small. This completes the discussion of Case (a).

Case (b): The proof will be divided into three steps.

Step 1. Set $u_l^* := v_i(\mu_l; u_r)$ and define an averaged wave speed $\tilde{\lambda}$ by

$$\tilde{\lambda} := \frac{\alpha}{\alpha + \beta}\overline{\lambda}_i(u_l, u_m) + \frac{\beta}{\alpha + \beta}\overline{\lambda}_i(u_m, u_r).$$

We are going to show first that the state u_l and u_r satisfy the Rankine-Hugoniot relation in some approximate sense:

$$-\tilde{\lambda}(u_l - u_r) + f(u_l) - f(u_r) = R, \tag{7.39}$$

where the remainder R will be estimated. First, from the exact Rankine-Hugoniot relation

$$\overline{\lambda}_i(u_l, u_m)(u_l - u_m) = f(u_l) - f(u_m),$$

we can write the term R in the form

$$\begin{aligned} R \;&=\; \big(\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)\big)\Big(\frac{\beta}{\alpha+\beta}(u_l - u_m) - \frac{\alpha}{\alpha+\beta}(u_m - u_r)\Big) \\ &\qquad + \big(f(u_m) - f(u_r) - \overline{\lambda}_i(u_m, u_r)(u_m - u_r)\big) \\ &=: R_1 + R_2. \end{aligned}$$

The first part R_1 can be estimated in exactly the same way as in Case (a) and we find

$$|R_1| \le C\,|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)|\,\alpha\beta.$$

On the other hand, to estimate R_2 we express it in the form

$$R_2 = \big(\overline{A}(u_m, u_r) - \overline{\lambda}_i(u_m, u_r)\,\mathbf{I}\big)\,(u_m - u_r) \tag{7.40}$$

or equivalently

$$R_2 = -\int_{\mu_m}^{\mu_r} \big(\lambda_i(w_i(m; u_m)) - \overline{\lambda}_i(u_m, u_r)\big)\, w_i'(m; u_m)\, dm. \tag{7.41}$$

By (7.40) we have $l_i(u_m, u_r)R_2 = 0$. Since

$$|\lambda_i(w_i(m; u_m)) - \overline{\lambda}_i(u_m, u_r)| + |l_j(u_m, u_r)\, w_i'(m, u_m)| \le C\,\beta$$

for $\mu_i(u_m) \le m \le \mu_i(u_r)$ and for $j \ne i$, (7.41) gives $|l_j(u_m, u_r)R_2| \le C\,\beta^3$. Thus, we get $|R_2| \le C\,\beta^3$ so that

$$|R| \le C\big(|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)|\alpha\beta + \beta^3\big). \tag{7.42}$$

On the other hand, by the definition of u_l^* we have the Rankine-Hugoniot relation

$$\overline{\lambda}_i(u_l^*, u_r)\,(u_l^* - u_r) = f(u_l^*) - f(u_r). \tag{7.43}$$

Repeating the argument in the proof of Case (a) and using together (7.39), (7.42) and (7.43) we obtain

$$\begin{aligned} &\big|(\alpha+\beta)\,\overline{\lambda}_i(u_l^*, u_r) - \big(\alpha\,\overline{\lambda}_i(u_l, u_m) + \beta\,\overline{\lambda}_i(u_m, u_r)\big)\big| + |u_l^* - u_l| \\ &\qquad \le C\,\big(|\overline{\lambda}_i(u_l, u_m) - \overline{\lambda}_i(u_m, u_r)|\,\alpha\beta + \beta^3\big). \end{aligned}$$

In particular, we get

$$\begin{aligned}&\big|(\alpha+\beta)\,\overline{\lambda}_i(u_l^*,u_r)-\big(\alpha\,\overline{\lambda}_i(u_l,u_m)+\beta\,\lambda_i(u_m)\big)\big|+|u_l^*-u_l|\\&\qquad\le C\,\big(|\overline{\lambda}_i(u_l,u_m)-\lambda_i(u_m)|\,\alpha\beta+\beta^2\big).\end{aligned}\tag{7.44}$$

Step 2. For simplicity we introduce the notation

$$\begin{aligned}&w_i(\nu):=w_i(\nu;u_m),\quad V_i(\nu):=v_i(\mu_l;w_i(\nu)),\\&\Lambda_i(\nu):=\overline{\lambda}_i(V_i(\nu),w_i(\nu)),\quad \theta_i(\nu):=\Lambda_i(\nu)-\lambda_i(w_i(\nu)).\end{aligned}$$

Take ν and h such that $\mu_m\le\nu<\nu+h\le\mu_r$. Applying the result (7.44) in Step 1 to $u_m=w_i(\nu)$, $u_r=w_i(\nu+h)$ and $u_l=V_i(\nu)$ (in this case we have $u_l^*=V_i(\nu+h)$) we obtain

$$\begin{aligned}&\big|(\nu-\mu_l+h)\,\Lambda_i(\nu+h)-\big((\nu-\mu_l)\,\Lambda_i(\nu)+h\,\lambda_i(w_i(\nu))\big)\big|\\&\qquad+|V_i(\nu+h)-V_i(\nu)|\le C\,\big((\nu-\mu_l)\,h\,|\theta_i(\nu)|+h^2\big).\end{aligned}\tag{7.45}$$

Dividing both sides of this inequality by h and letting $h\to+0$ yield that

$$\big|\big((\nu-\mu_l)\,\Lambda_i(\nu)\big)'-\lambda_i(w_i(\nu))\big|+|V_i'(\nu)|\le C\,(\nu-\mu_l)\,|\theta_i(\nu)|.\tag{7.46}$$

Now, we define u_l' as follows. By our assumption it is obvious that $\theta_i(\mu_m)>0$. If $\theta_i(\nu)>0$ for all $\nu\in(\mu_m,\mu_r)$, we define $u_m':=u_r$, $u_l':=V_i(\mu_r)$ and $\nu':=\mu_r$. Otherwise, if there exists a zero of $\theta_i=\theta_i(\nu)$, then we set $u_m':=w_i(\nu')$ and $u_l':=V_i(\nu')$ where ν' is the first zero of θ_i in the interval (μ_m,μ_r). In any case, we have $\theta_i(\nu)>0$ for all $\mu_m<\nu<\nu'$.

In addition to $\theta_i(\nu)=\Lambda_i(\nu)-\lambda_i(w_i(\nu))$ we introduce the angle

$$\tilde{\theta}_i(\nu):=\Lambda_i(\mu_m)-\lambda_i(w_i(\nu)).$$

Note that $\Lambda(\mu_m)=\overline{\lambda}_i(u_m,u_l)$ and

$$\theta_i(\nu)-\tilde{\theta}_i(\nu)=\Lambda_i(\nu)-\Lambda_i(\mu_m).$$

Then, by (7.46) we obtain

$$\Big((\nu-\mu_l)\,\Lambda_i(\nu)\Big)'\le\lambda_i(w(\nu))+C\,(\nu-\mu_l)\,\theta_i(\nu)$$

and, equivalently,

$$\Big((\nu-\mu_l)\,\theta_i(\nu)\Big)'\le-(\nu-\mu_l)\,\lambda_i(w(\nu))'+C\,(\nu-\mu_l)\,\theta_i(\nu).$$

By integration over the interval $[\mu_m,\nu]$ we deduce

$$\begin{aligned}(\nu-\mu_l)\,\theta_i(\nu)\le(\mu_m-\mu_l)\,\theta_i(\mu_m)&-\int_{\mu_m}^{\nu}(s-\mu_l)\,\lambda_i(w(s))'\,ds\\&+C\int_{\mu_m}^{\nu}(s-\mu_l)\,\theta_i(s)\,ds.\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}(\mu_m - \mu_l)\, \theta_i(\mu_m) - \int_{\mu_m}^{\nu} (s - \mu_l)\, \lambda_i(w(s))'\, ds \\ = (\mu_m - \mu_l)\, \tilde{\theta}_i(\nu) - \int_{\mu_m}^{\nu} (s - \mu_m)\, \lambda_i(w(s))'\, ds\end{aligned}$$

in which the latter term is non-positive. Therefore we arrive at:

$$(\nu - \mu_l)\, \theta_i(\nu) \le (\mu_m - \mu_l)\, \tilde{\theta}_i(\nu) + C \int_{\mu_m}^{\nu} (s - \mu_l)\, \theta_i(s)\, ds.$$

Applying Gronwall lemma we have

$$\int_{\mu_m}^{\nu} (s - \mu_l)\, \theta_i(s)\, ds \le \mathcal{O}(1)\, (\mu_m - \mu_l) \int_{\mu_m}^{\nu} \tilde{\theta}_i(s)\, ds$$

for all $\mu_m \le \nu \le \nu'$. This, together with (7.46) after integration over the interval (μ_m, ν'), yields the desired estimates in (7.7).

Step 3. It remains to show that the i-shock wave connecting u_l' to u_m' is admissible and that $u_r = \psi_i(\mu_r; u_l')$. Define the set S of all ν for which $V_i(\nu)$ is connected to $w_i(\nu)$ by an admissible shock wave, that is,

$$S := \Big\{\nu \in [\mu_m, \nu'] \,/\, \text{shock wave connecting } V_i(\nu) \text{ to } w_i(\nu) \text{ is admissible}\Big\}.$$

By our assumption it is obvious that $\mu_m \in S$. Now, fix any arbitrary $\nu \in S \cap [\mu_m, \nu')$, and note that $\theta_i(\nu) > 0$. For clarity we set $\alpha' := \nu - \mu_l > 0$ and we consider $h > 0$ satisfying $\nu + h \le m'$. Then, by (7.45) we have

$$\begin{aligned}&|V_i(\nu + h) - V_i(\nu)| \le C\, (\theta_i(\nu)\, \alpha' h + h^2), \\ &|\varLambda_i(\nu + h) - \widetilde{\varLambda}| \le \frac{C}{\alpha' + h}\, (\theta_i(\nu)\, \alpha' h + h^2),\end{aligned} \tag{7.47}$$

where $\widetilde{\varLambda}$ is the **averaged speed** defined by

$$\widetilde{\varLambda} := \frac{\alpha'}{\alpha' + h}\, \varLambda_i(m) + \frac{h}{\alpha' + h}\, \lambda_i(w_i(m)).$$

Now, we will show that if h is sufficiently small, then the inequality

$$\varLambda_i(\nu + h) \le \overline{\lambda}_i(s; V_i(\nu + h)) \tag{7.48}$$

holds for any $s \in [\mu_l, \nu + h]$, which guarantees the admissibility of the corresponding shock wave. We set $u_* := v_i(s; V_i(\nu + h))$ and we consider two cases according to the value of s.

When $\mu_l \le s \le \nu$, we define a state u by $u := v_i(s; V_i(\nu))$. Then, by (7.47) we have

$$|u - u_*| \le C\, |V_i(\nu + h) - V_i(\nu)| \le C\, (\theta_i(\nu)\, \alpha' h + h^2). \tag{7.49}$$

Since the shock wave connecting $V_i(\nu)$ to $w_i(\nu)$ is admissible, we also have

$$\Lambda_i(\nu) \le \overline{\lambda}_i(V_i(\nu), u). \tag{7.50}$$

We write

$$\overline{\lambda}_i(V_i(\nu+h), u_*) - \Lambda_i(\nu+h) = \overline{\lambda}_i(V_i(\nu), u) - \Lambda_i(\nu) + E,$$

where

$$\begin{aligned} E &= (\Lambda_i(\nu) - \widetilde{\Lambda}) + (\widetilde{\Lambda} - \Lambda_i(\nu+h)) + \big(\overline{\lambda}_i(V_i(\nu+h), u_*) - \overline{\lambda}_i(V_i(\nu), u)\big) \\ &\ge \frac{h}{\alpha' + h}\,\theta_i(\nu) - |\widetilde{\Lambda} - \Lambda_i(\nu+h)| - \big|\overline{\lambda}_i(V_i(\nu+h), u_*) - \overline{\lambda}_i(V_i(\nu), u)\big| \\ &\ge \frac{h}{\alpha' + h}\,\theta_i(\nu) - \frac{C}{\alpha' + h}\big(\theta_i(\nu)\,\alpha' h + h^2\big) \\ &= \frac{h\theta_i(\nu)}{\alpha' + h}\bigg(1 - \Big(C\,\alpha' + \frac{C\,h}{\theta_i(\nu)}\Big)\bigg). \end{aligned}$$

In the above calculations we used (7.47) and (7.49). Here, without loss of generality we can assume that $C\alpha' \le 1/2$. Therefore, if h is sufficiently small compared with $\theta_i(m)$, then we have $E > 0$. This and (7.50) imply that (7.48) holds for $\mu_l \le s \le m$.

When $\nu < s \le \nu + h$, we evaluate the quantity under consideration as follows.

$$\begin{aligned} &\lambda_i(v_i(s; V_i(\nu+h))) - \lambda_i(s; V_i(\nu+h)) \\ &= -\theta_i(\nu) + \big(\lambda_i(v_i(s; V_i(\nu+h)) - \lambda_i(v_i(\nu; V_i(\nu)))\big) \\ &\qquad + \big(\overline{\lambda}_i(V_i(\nu), v_i(\nu; V_i(\nu))) - \overline{\lambda}_i(V_i(\nu+h), v_i(s; V_i(\nu+h))\big) \\ &\le -\theta_i(\nu) + C\,(|V_i(\nu+h) - V_i(\nu)| + |s-\nu|) \\ &\le -\theta_i(\nu) + C\,h < 0, \end{aligned}$$

if h is sufficiently small compared with $\theta_i(\nu)$. This, together with Lemma 3.3, implies that $\overline{\lambda}_i'(s; V_i(\nu+h)) < 0$ and that

$$\overline{\lambda}_i(s; V_i(\nu+h)) \ge \overline{\lambda}_i(\nu+h; V_i(\nu+h)) = \Lambda_i(\nu+h)$$

for $\nu \le s \le \nu + h$. This shows that (7.48) is also valid in this case, too.

Summarizing the above arguments, we see that for any $\nu \in S \cap [\mu_m, \nu')$ there exists a small constant $h > 0$ such that $[\nu, \nu+h] \subseteq S$. Moreover, it is easy to check that the set S is closed: if $\{m_l\}_{l=1}^{\infty} \subseteq S$ is a monotone increasing sequence converging to some value m_∞, then $m_\infty \in S$. Therefore, we can sweep out the interval so that the set S should be equal to the whole interval $[\mu_m, \nu']$. Particularly, we obtain $\nu' \in S$ which means that the shock wave connecting u_l' to u_m' is admissible. Finally, since the fact $\theta_i(\nu') = 0$ implies $\overline{\lambda}_i(u_l', u_m') = \lambda_i(u_m')$, it is obvious that $u_r = \psi_i(\mu_l; u_l')$. This conclude the discussion of Case (b). □

Proof of Lemma 7.4. We only consider Case (a). We rely on the proof of Lemma 7.3, in which we established the existence of a state u_r' such that the discontinuity connecting u_l to u_r' is an admissible shock and

$$u_r' - u_l = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r).$$

Actually, the arguments given there yield the following *stronger statement*: To each left-hand state $\overline{u}_l$ satisfying

$$\overline{u}_l - u_l = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r)$$

we can associate a right-hand state $\overline{u}_r$ such that the discontinuity connecting $\overline{u}_l$ to $\overline{u}_r$ is an admissible shock and

$$\overline{u}_r - u_r = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r).$$

Indeed, this result is immediate by replacing in the proof of Lemma 7.4 u_l by $\overline{u}_l$ in all of the expressions; all the statements are still valid up to an error of order $C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r)$ which, of course, does not modify our conclusion.

Finally, to conclude we consider the entropy admissible shock connecting u_l to some right-hand state u_r', which is given by Lemma 7.4 and solve the Riemann problem having u_l as a left-hand state, we connect first the given state u_l to some state $\overline{u}_l$ by using only waves of the families $1, \ldots, i-1$ with strengths $s_1, \ldots, s_{i-1}$, respectively. Then we connect $\overline{u}_l$ to the corresponding state $\overline{u}_r$ by a large admissible shock with total strength $s_i + \mu_i(u_r') - \mu_i(u_l)$ such that

$$s_i = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r).$$

We finally use waves of the families $i+1, \ldots, N$ with strengths $s_{i+1}, \ldots, s_N$, respectively, and reach some right-hand state denoted by $\varphi(s_1, \ldots, s_N)$. Since the wave curves are transversal and $\varphi(0, \ldots, 0) = u_r'$, by the implicit function theorem [9] the Lipschitz continuous mapping

$$(s_1, \ldots, s_N) \mapsto \varphi(s_1, \ldots, s_N)$$

maps a neighborhood of $0 \in I\!R^N$ to a neighborhood of u_r'. When the strengths are sufficiently small,

$$s_j = C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r), \quad 1 \le j \le N,$$

the corresponding Riemann solution by construction has the desired structure: it contains a large admissible i-shock plus small waves. In particular, since the prescribed right-hand state u_r is in a small neighborhood of size $C_K\,\mathcal{O}(1)\,Q(u_l, u_m, u_r)$ from the reference state $\overline{u}_r$, we conclude that we can connect u_l to u_r with the desired Riemann structure. □

8. Existence Theory

We are now in a position to establish the general existence theory for the **Cauchy problem**

$$\partial_t u + \partial_x f^\epsilon(u) = 0, \quad u = u^\epsilon(x,t) \in \mathbf{B}_{\delta_1},\ x \in I\!R,\ t \geq 0, \tag{8.1}$$

$$u(x,0) = u_0(x), \quad x \in I\!R. \tag{8.2}$$

Here, f^ϵ is a family of flux-functions depending on some parameter ϵ, satisfying the uniform bounds (Hyp.1)–(Hyp.4) stated in Section 1. We will use Glimm's scheme [14] to construct a family of approximate solutions denoted by $u^{\epsilon,h}(x,t)$ and then justify the passage to the limit.

We assume that the initial function $u_0 : I\!R \to \mathbf{B}_{\delta_5}$ take its values in a sufficiently small neighborhood of the origin and has bounded total variation denoted by $TV(u_0)$. Here, we will take $\delta_5 < \delta_4$ where $\delta_4 < \delta_1$ was introduced in Theorem 6.1.

We are given a mesh with space and time lengths h_x and h_t, respectively, such that the ratio h_t/h_x is fixed and satisfies the stability condition

$$\frac{h_t}{h_x} \max\big(|\lambda_1^{\min}|, |\lambda_N^{\max}|\big).$$

For simplicity we set $h := h_x$. Let also (ξ_n) be a (random) sequence taking its values in the interval $(-1,1)$.

Consider any piecewise constant approximation $u_0^h : I\!R \to \mathbf{B}_{\delta_5}$ having compact support and being constant on each interval $(m\,h_x, (m+2)\,h_x)$ for all even integers m, with

$$\begin{aligned} &u_0^h \to u_0 \quad \text{almost everywhere,} \\ &TV(u_0^h) \leq TV(u_0). \end{aligned} \tag{8.3}$$

The Cauchy problem associated with the initial data u_0^h can be solved explicitly using the Riemann solution described in Section 5, as long as waves issuing from two nearby discontinuities do not interact, that is, for all times $\leq h_t$ at least. At this time $t_1 := h_t$ we determine the new piecewise constant function $x \mapsto u^{\epsilon,h}(x, h_t+)$ by

$$\begin{aligned} u^{\epsilon,h}(x,t_1+) &:= u^{\epsilon,h}((m+1+\xi_1)\,h_x, h_t-), \\ &\qquad x \in (m\,h_x, (m+2)\,h_x),\ m+1 \text{ even.} \end{aligned}$$

Provided the range of the approximate solution remains in the region $\mathbf{B}_{\delta_4}$ (within which we can solve the Riemann problem by Theorem 6.1), we can continue this construction inductively by solving a sequence of Riemann problems at each time level $t_n := n\,h_t$. If the solution $x \mapsto u^{\epsilon,h}(x,t_n-)$ is known at some time level then we determine the new piecewise constant approximation by

$$\begin{aligned} u^{\epsilon,h}(x,t_n+) &:= u^{\epsilon,h}((m+1+\xi_n)\,h_x, t_n-), \\ &\qquad x \in (m\,h_x, (m+2)\,h_x),\ m+n \text{ even.} \end{aligned}$$

We can derive a uniform bound on the total variation of the approximate solutions which does not depend on h and ϵ, and we prove that the limit $u^{\epsilon,h}$ has a limit when $\epsilon, h \to 0$.

Theorem 8.1. (General existence theory.) *Given* δ_1 *and* $K > 0$, *there exists a positive* $\delta_5 \leq \delta_1$ *(which approaches* $+\infty$ *as* K *approaches* 0 *and* δ_1 *approaches* $+\infty$*), and constants* $c_1 \in (0, 1]$ *and* $C_1 \leq 1$ *such that for every sequence of systems of conservation laws* (1.1) *whose flux-functions* $f^\epsilon : \mathbf{B}_{\delta_1} \to I\!R^N$ *are uniformly strictly hyperbolic and non-degenerate in the sense of* (Hyp.1)–(Hyp.4)*, the associated Cauchy problem admits a solution* u^ϵ *in the weak sense for each initial condition* $u_0 : I\!R \to \mathbf{B}_{\delta_5}$ *satisfying*

$$TV(u_0) < c_1. \tag{8.4}$$

(1) *More precisely, consider any sequence of approximate initial data* $u_0^h : I\!R \to \mathbf{B}_{\delta_5}$ *satisfying* (8.3)*. Then, the Glimm scheme generates approximate solutions* $u^{\epsilon,h}$ *which are globally defined in time and satisfy the uniform estimates*

$$\begin{aligned} &u^{\epsilon,h}(x,t) \in \mathbf{B}_{\delta_1}, \quad x \in I\!R,\, t \geq 0, \\ &TV(u^{\epsilon,h}(t)) \leq C_1\, TV(u_0), \quad t \geq 0, \\ &\|u^{\epsilon,h}(t) - u^{\epsilon,h}(s)\|_{L^1(I\!R)} \leq \big(o(1) + C_1\, \widehat{\lambda}\, TV(u_0)\big)\, |t-s|, \quad t, s \geq 0, \end{aligned} \tag{8.5}$$

where $o(1) \to 0$ *as* $h \to 0$, *and*

$$\widehat{\lambda} := \max\big(|\lambda_1^{\min}|, |\lambda_N^{\max}|\big).$$

(2) *When* $h \to 0$, *after extracting a subsequence if necessary and for almost every sequence* (ξ_n), *the sequence* $u^{\epsilon,h}$ *converges* (*almost everywhere in* (x,t) *and in* L^1_{loc} *in* x *for all* t) *to a weak solution* u^ϵ *of the Cauchy problem* (8.1) *and* (8.2),

$$u^\epsilon := \lim_{h \to 0} u^{\epsilon,h}.$$

(3) *If now the sequence of flux-functions converges to some function* f *in the sense that*

$$\sup_{u \in \mathbf{B}_{\delta_1}} |f^\epsilon - f| + \sup_{u \in \mathbf{B}_{\delta_1}} |Df^\epsilon - Df| \to 0, \quad \epsilon \to 0, \tag{8.6}$$

then the limit

$$u := \lim_{\epsilon \to 0} u^\epsilon$$

exists (*almost everywhere in* (x,t) *and in* L^1_{loc} *for all* t) *and is a weak solution of the Cauchy problem associated with the initial data* u_0 *and the flux-function* f, *and*

$$\begin{aligned} &u(x,t) \in \mathbf{B}_{\delta_1}, \quad x \in I\!R,\, t \geq 0, \\ &TV(u(t)) \leq C_1\, TV(u_0), \quad t \geq 0, \\ &\|u(t) - u(s)\|_{L^1(I\!R)} \leq C_1\, \widehat{\lambda}\, TV(u_0)\, |t-s|, \quad t, s \geq 0. \end{aligned} \tag{8.7}$$

Section 9 is devoted to deriving the uniform bounds (8.5). Here we complete the proof of Theorem 8.1 by assuming (8.5). Note also that, by using Liu's technique one can prove that Glimm's scheme actually converges for every equidistributed sequence [25]. In addition, the weak solutions obtained in Theorem 8.1 can be proven to be entropy solutions; see [26], [17].

Proof. In view of the uniform estimates (8.5) we can apply Helly's compactness theorem. For each time $t > 0$ there exists a converging subsequence $u^{\epsilon,h}(t)$ and a limiting function $u(t)$. By a standard diagonal argument we can find a subsequence so that for all rational t

$$u^{\epsilon,h}(x,t) \to u(x,t) \text{ for almost every } x. \tag{8.8}$$

The uniform Lipschitz bound in (8.5) then implies that $u(t)$ is well defined for all t and that, in fact, $u^{\epsilon,h}(t)$ converges to $u(t)$ in L^1_{loc} for all t. The inequalities (8.7) follow immediately from (8.5). It is a standard matter to check that the limit u is a weak solution [14]. This completes the proof of Theorem 8.1 when the uniform estimates (8.5) holds. □

A direct application of Theorem 8.1 yields the general existence theory for the p-system.

Corollary 8.2. *Consider the Cauchy problem associated with the p-system of gas dynamics*

$$\begin{aligned} &\partial_t v - \partial_x u = 0, \\ &\partial_t u + \partial_x p(v) = 0, \end{aligned} \tag{8.9}$$

where $v = v(x,t) > 0$ and $u = u(x,t)$ represent the specific volume and the velocity of the fluid in the Lagrangian coordinates, respectively, and the pressure function $p = p(v)$ is a smooth function which solely satisfies the hyperbolicity condition

$$p'(v) < 0. \tag{8.10}$$

Then, given any constant states u_*, v_*, there exist $c > 0$ and $\delta > 0$ such that for any initial data u_0, v_0 satisfying

$$\begin{aligned} &u_0(x) \in \mathbf{B}_\delta(u_*),\, v_0(x) \in \mathbf{B}_\delta(v_*), \quad x \in I\!R, \\ &TV(u_0) + TV(v_0) < c, \end{aligned}$$

the corresponding Cauchy problem admits a solution in the weak sense.

To establish the corollary, it suffices to approximate the one-variable function p by a function p^ϵ having a finite number of non-degenerate inflection points and to apply Theorem 8.1. Precisely, we have:

Lemma 8.3. *Given a function $p : [a,b] \to I\!R$ of class $C^3 \cap W^{3,\infty}$, there exist a positive constant G and a sequence of functions $p^\epsilon : [a,b] \to I\!R$ of class C^3 satisfying*

(i) $\sup_{v\in[a,b]} |p^\epsilon(v) - p(v)| \leq G\,\epsilon^2, \quad \sup_{u\in[a,b]} |p^{\epsilon\prime}(v) - p'(v)| \leq G\,\epsilon,$

(ii) $\sup_{v\in[a,b]} |p^{\epsilon\prime\prime}(v)| \leq \sup_{v\in[a,b]} |p''(v)| + G,$

(iii) *for every $v \in [a,b]$ such that $p^{\epsilon\prime\prime}(v) = 0$ we have $p^{\epsilon\prime\prime\prime}(v) \neq 0$.*

Proof of Lemma 8.3. We define the approximate function p^ϵ by

$$p^\epsilon(v) := p(v) + \epsilon^2\, G\, \sin(v/\epsilon),$$

where $G > 0$ will be determined shortly. Then, it is easy to see that these approximate functions satisfy the properties (i) and (ii) in the lemma. Suppose that

$$p^{\epsilon\prime\prime}(v_0) = p^{\epsilon\prime\prime\prime}(v_0) = 0$$

at some point v_0. Then one deduce that

$$p''(v_0) - G\,\sin(v_0/\epsilon) = p'''(v_0) - \frac{G}{\epsilon}\,\cos(v_0/\epsilon) = 0.$$

Therefore, we have

$$|\cos(v_0/\epsilon)| \leq \frac{\epsilon}{G}\,\|p'''\|_{L^\infty} \quad \text{and} \quad |\sin(v_0/\epsilon)| \leq \frac{1}{G}\,\|p''\|_{L^\infty},$$

which yields that

$$\begin{aligned} 1 &= \cos^2(v_0/\epsilon) + \sin^2(v_0/\epsilon) \\ &\leq \frac{1}{G^2}\big(\epsilon^2\|p'''\|^2_{L^\infty} + \|p''\|^2_{L^\infty}\big). \end{aligned}$$

Now, it is sufficient to take G to be sufficiently large, precisely

$$G^2 > \epsilon^2\,\|p'''\|^2_{L^\infty} + \|p''\|^2_{L^\infty},$$

in order to arrive at a contradiction. This completes the proof of Lemma 8.3.
□

9. Uniform Total Variation Bound

This section provides a proof of the technical estimates in Theorem 8.1. We use the notation given after the statement of Theorem 8.1. To simplify the notation we often suppress the explicit dependence in h, ϵ, and/or t.

The total strength of waves in a piecewise constant approximation $u = u(x,t)$ is controlled by the **total variation functional**

$$V^\epsilon(t) = \sum_{x\in\mathcal{J}(t)} |\sigma(x,t)|,$$

where $\sigma(x,t)$ is the strength of the wave located at x and $\mathcal{J}(t)$ denotes the set of jump points in $u(t)$. The possible increase due to wave interactions will be measured by the **generalized wave interaction potential**

$$Q^\epsilon(t) = \sum_{(x,y)\in\mathcal{J}(t)\times\mathcal{J}(t)} \Theta^\epsilon(x,y,t)\,|\sigma(x,t)\,\sigma(y,t)|,$$

which should be regarded as an extension of Glimm's definition (1.4). Here, the sum is over every pair of waves and $\Theta^\epsilon(x,y,t)$ is a non-negative weight specified now which is motivated by the interaction estimates derived in Section 7. We count in $Q^\epsilon(t)$ all products of between **approaching waves**, that is,

- waves of different families, provided the wave on the left-hand side is faster than the wave on the right-hand side,
- and waves of the same family.

In other words, we take into account pairs (x,y) of having either $x<y$ and $0\le i(y,t)<i(x,t)\le N+1$, or else $x<y$ and $1\le i(x,t)=i(y,t)\le N$. As in Section 7, we introduce the **generalized angle**

$$\Theta^\epsilon(x,y,t) := \begin{cases} 0, & i(x,t)<i(y,t),\\ 1, & i(x,t)>i(y,t),\\ 1, & i(x,t)=i(y,t),\quad \sigma(x,t)\,\sigma(y,t)<0,\\ \theta_i^\epsilon\big(v_-(y),v_+(y);v_-(x),v_+(x)\big), & i(x,t)=i(y,t),\quad \sigma(x,t)\,\sigma(y,t)>0. \end{cases}$$

In view of the following property of the wave strengths

$$\frac{1}{C_2}\big|u_+(x,t)-u_-(x,t)\big| \le |\sigma(x,t)| \le C_2\big|u_+(x,t)-u_-(x,t)\big|, \tag{9.1}$$

V^ϵ is equivalent to the usual total variation :

$$\frac{1}{C_2}\,TV(u(t)) \le V^\epsilon(t) \le C_2\,TV(u(t)), \quad t\ge 0. \tag{9.2}$$

Estimating V^ϵ and Q^ϵ is based on the wave interaction estimates derived in Section 7. On one hand, the wave strengths are increased by a small amount at interactions. On the other hand, the function Q^ϵ decreases at interactions by the same amount at least. To take advantage of this property, we consider the functional

$$V^\epsilon(t) + C_3\,Q^\epsilon(t).$$

By choosing a sufficiently large constant C_3, the increase of $V^\epsilon(t)$ can be compensated by the decrease of $Q^\epsilon(t)$. This is Glimm's key idea [14].

Theorem 9.1. (Decreasing functional.) *For $C_3 > 0$ sufficiently large the (piecewise constant)* **generalized Glimm functional**

$$t \mapsto V^\epsilon(t) + C_3\, Q^\epsilon(t)$$

is decreasing at each time level $t = t_n$.

Proof. For simplicity in the notation we suppress the explicit dependence in ϵ. We follow Glimm's strategy [14] and we use Glimm's fundamental decomposition of the plane into diamonds Δ. It is sufficient to show the desired estimate along two space-like curves separated by a single diamond Δ. We use simply the notation $\left[V^\epsilon + C_3\, Q^\epsilon\right]$ for the variation of the functional across the diamond. In addition, it is sufficient to treat the situation where the waves entering the diamond are elementary waves; the general case follows by induction in light of Theorem 7.5.

Assume that the wave entering the diamond is an i^α-wave with strength s^α (located on the left-hand side) meet with an i^β-wave with strength s^β. Let s^γ be the strengths of the waves leaving the diamond, where γ describes a finite set of indices.

First of all, since V is the sum of all wave strengths which possibly increase at the time t but by (at most) the interaction potential between the two incoming waves (Theorem 7.1), we have

$$\sum_\gamma |s^\gamma| \le |s^\alpha| + |s^\beta| + C_4\, \Theta^{\alpha,\beta}\, |s^\alpha\, s^\beta|,$$

where $\Theta^{\alpha,\beta}$ is the weight associated with the two incoming waves. Thus, the total increase for the total variation is

$$[V] \le C_4\, \Theta^{\alpha,\beta}\, |s^\alpha\, s^\beta|. \tag{9.3}$$

Suppose first that the interaction involves two waves of *different* families, for which we simply have $\Theta^{\alpha,\beta} = 1$. By definition of the potential, the quadratic term $|s^\alpha\, s^\beta|$ is counted in Q before the interaction, but is no longer taken into account after it, since the two waves are no longer approaching after the interaction. Moreover, by Theorem 7.1, waves in the other families are of quadratic order at most. Therefore, we find for two waves of the same family (for which $\Theta^{\alpha,\beta} = 1$)

$$[Q] \le -\Theta^{\alpha,\beta}\, |s^\alpha\, s^\beta| + C_5\, \Theta^{\alpha,\beta}\, V(t-)\, |s^\alpha\, s^\beta|. \tag{9.4}$$

Consider next the case of two waves of the *same family* $i = i^\alpha = i^\beta$ for which the generalized angle plays a key role and let us derive the same estimate (9.4). We will use the following notation. Call u_l, u_m, u_r the states associated with the two waves entering the diamonds with

$$s^\alpha = \sigma_i(u_l, u_m) = \mu_i(u_m) - \mu_i(u_l)$$
$$s^\beta = \sigma_i(u_m, u_r) = \mu_i(u_r) - \mu_i(u_m)$$

and set also

$$Q^- := \Theta_i^{\alpha,\beta}\,|s^\alpha\, s^\beta| = \Theta_i(u_l,u_m;u_m,u_r)\,|\sigma_i(u_l,u_m)|\,|\sigma_i(u_m,u_r)|.$$

Denote by u_k', $k=0,1,\dots,K+1$ the values achieved by the i-waves of the Riemann solution exiting the diamond. In particular, by Theorem 7.1 we have

$$u_0' = u_l + C_K\,\mathcal{O}(1)\,Q^-, \quad u_{K+1}' = u_r + C_K\,\mathcal{O}(1)\,Q^-.$$

Since there is no interaction between two (ordered) waves issued from the same Riemann solution, we have

$$\Theta_i\big(u_k',u_{k+1}';u_l',u_{l+1}'\big) = 0, \quad 0 \le k \le l \le K.$$

Denote by J^L and J^R the sets of waves located on the left- and right-hand sides of the diamond, respectively, which are not involved in the interaction. By Theorem 7.1 we find

$$\begin{aligned}[Q] = &-\Theta_i(u_l,u_m;u_m,u_r)\,|\sigma_i(u_,u_m)|\,|\sigma_i(u_m,u_r)|\\ &+ \sum_{(u_-,u_+)\in J^L\cup J^R} B(u_-,u_+)\,\sigma(u_-,u_+),\end{aligned}$$

where, for example for J^R,

$$\begin{aligned}B(u_-,u_+) = &\sum_{k=1}^{K} q\big(u_k',u_{k+1}';u_-,u_+\big)\,|\sigma_i(u_k,u_{k+1})|\\ &-\Theta\big(u_l,u_m;u_-,u_+\big)\,|\sigma_i(u_l,u_m)|\\ &-\Theta\big(u_m,u_r;u_-,u_+\big)\,|\sigma_i(u_m,u_r)|\end{aligned} \tag{9.5}$$

and (for instance) $\Theta\big(u_l,u_m;u_-,u_+\big)$ denotes the generalized angle associated with the pair of waves (u_l,u_m) and (u_-,u_+).

It is sufficient to derive the estimate

$$B(u_-,u_+) = C_K\,\mathcal{O}(1)\,Q^-,$$

since then the principal term in $[Q]$ above becomes dominant, provided the total variation

$$\sum_{(u_-,u_+)\in J^L\cup J^R} |\sigma(u_-,u_+)| = C_K\,\mathcal{O}(1)\,TV(u^h(t))$$

is sufficiently small, and this leads us back to (9.4).

In the following we restrict attention to the waves in J^R; dealing with the other set of waves is completely similar.

To begin with, we observe that if (u_-,u_+) is a j-wave with $j>i$, then

$$\Theta\big(u_k',u_{k+1}';u_-,u_+\big) = \Theta\big(u_l,u_m;u_-,u_+\big) = \Theta\big(u_m,u_r;u_-,u_+\big) = 0.$$

If (u_-, u_+) is a j-wave with $j < i$, then

$$\Theta\big(u'_k, u'_{k+1}; u_-, u_+\big) = \Theta\big(u_l, u_m; u_-, u_+\big) = \Theta\big(u_m, u_r; u_-, u_+\big) = 1$$

and the desired estimate follows immediately from Theorem 7.1. Therefore, in the rest of the discussion we assume that $i = j$.

First, consider the *non-monotone* interactions, for which we simply have

$$\Theta(u_l, u_m; u_m, u_r) = 1.$$

Suppose for instance that

$$\mu_i(u_m) < \mu_i(u_l) < \mu_i(u_r)$$

and the wave connecting u_l to u_m is a i-rarefaction while the wave connecting u_m to u_r is a i-shock. Other cases can be treated in the same way. If $\mu_i(u_-) > \mu_i(u_+)$, we find easily

$$\begin{aligned} B(u_-, u_+) = &\ (\mu_i(u_l) - \mu_i(u_r)) - (\mu_i(u_r) - \mu_i(u_m)) \\ & -q\big(u_m, u_r; u_-, u_+\big)\,(\mu_i(u_l) - \mu_i(u_m)) \\ & +C_K\,\mathcal{O}(1)\,Q^- \\ & \le -|\mu_i(u_l) - \mu_i(u_m)|\,\Big(1 - C_K\,\mathcal{O}(1)\,|\mu_i(u_r) - \mu_i(u_m)|\Big) \\ & = -|\mu_i(u_l) - \mu_i(u_m)|\,\Big(1 - C_K\,\mathcal{O}(1)\,TV(u^h)\Big) \le 0. \end{aligned} \tag{9.6}$$

If now $\mu_i(u_-) < \mu_i(u_+)$, we will set

$$\Lambda := \overline{\lambda}_i(u_-, u_+)$$

and restrict attention for definiteness to the case that (u_-, u_+) is a shock. No property of Λ will be used in the following discussion. Therefore, the discussion below generalizes immediately to rarefactions for which an average of the characteristic speed should be used.

Let us fix the notation: the outgoing i-wave pattern contains a rarefaction (possibly trivial) connecting u'_0 to u'_1, followed by a shock wave connecting u'_1 to u'_2, $\cdots$, followed, finally, by a shock connecting u'_{2P-1} to u'_{2P} (with $K := 2P - 1$). We can write

$$\begin{aligned} B(u_-, u_+) = & \sum_{p=1}^{P} \int_{\mu_i(u'_{2p-2})}^{\mu_i(u'_{2p-1})} \Big(\Lambda - \lambda_i(w_i(\nu; u'_{2p-2}))\Big)^- \,d\nu \\ & + \sum_{p=1}^{P} \Big(\Lambda - \overline{\lambda}_i(u'_{2p-1}, u'_{2p})\Big)^- \,(\mu_i(u'_{2p}) - \mu_i(u'_{2p-1})) \\ & - (\mu_i(u_l) - \mu_i(u_m)) - \Big(\Lambda - \overline{\lambda}_i(u_m, u_r)\Big)^- \,(\mu_i(u_r) - \mu_i(u_m)). \end{aligned}$$

The argument is based on the fact that the speeds arising in the expression of $B(u_-, u_+)$ (that is, $\lambda_i(w_i(\nu; u'_{2p-2}))$ and $\overline{\lambda}_i(u'_{2p-1}, u'_{2p})$) belong to the same Riemann solution and, therefore, can be ordered monotonically in some obvious manner. Therefore, we can suppose that for some p_0

$$\overline{\lambda}_i(u'_{2p_0-3}, u'_{2p_0-2}) \le \varLambda \le \overline{\lambda}_i(u'_{2p_0-1}, u'_{2p_0}) \tag{9.7}$$

with, for some $\nu_0 \in \big(\mu_i(u'_{2p-2}), \mu_i(u'_{2p-1})\big)$,

$$\varLambda = \lambda_i(w_i(\nu_0; u'_{2p_0-2})).$$

(Other situations are easier to handle.) We then get

$$\begin{aligned} B(u_-,u_+) = &\int_{\nu_0}^{\mu_i(u'_{2p_0-1})} \Big(- \varLambda + \lambda_i(w_i(\nu; u'_{2p_0-2}))\Big)\, d\nu \\ &+ \sum_{p=p_0+1}^{P} \int_{\mu_i(u'_{2p-2})}^{\mu_i(u'_{2p-1})} \Big(- \varLambda + \lambda_i(w_i(\nu; u'_{2p-2}))\Big)\, d\nu \\ &+ \sum_{p=p_0}^{P} \Big(- \varLambda + \overline{\lambda}_i(u'_{2p-1}, u'_{2p})\Big) \, (\mu_i(u'_{2p}) - \mu_i(u'_{2p-1})) \\ &- (\mu_i(u_l) - \mu_i(u_m)) - \Big(-\varLambda + \overline{\lambda}_i(u_m, u_r)\Big)\, (\mu_i(u_r) - \mu_i(u_l)) \end{aligned}$$

thus

$$\begin{aligned} B(u_-,u_+) = &- \int_{\nu_0}^{\mu_i(u'_{2p_0-1})} \Big|\overline{\lambda}_i(u_m, u_r) - \lambda_i(w_i(\nu; u'_{2p_0-2}))\Big|\, d\nu \\ &- \sum_{p=p_0+1}^{P} \int_{\mu_i(u'_{2p-2})}^{\mu_i(u'_{2p-1})} \Big|\overline{\lambda}_i(u_m, u_r) - \lambda_i(w_i(\nu; u'_{2p-2}))\Big|\, d\nu \\ &- \sum_{p=p_0}^{P} \Big|\overline{\lambda}_i(u_m, u_r) - \overline{\lambda}_i(u'_{2p-1}, u'_{2p})\Big| \\ &- (\mu_i(u_l) - \mu_i(u_m)) - \Big|\overline{\lambda}_i(u_m, u_r) - \varLambda\Big| (\nu_0 - \mu_i(u_l)) \\ &\le 0, \end{aligned}$$

which completes the discussion of non-monotone interactions.

When the sequence $\mu_i(u_l), \mu_i(u_m), \mu_i(u_r)$ is *monotone,* without loss of generality, we can assume that

$$\mu_i(u_l) < \mu_i(u_m) < \mu_i(u_r).$$

There are two main subcases, as before. If $\mu_i(u_-) > \mu_i(u_+)$, we have immediately

$$\begin{aligned} B(u_-,u_+) &= (\mu_i(u_r) - \mu_i(u_l)) - (\mu_i(u_l) - \mu_i(u_m)) \\ &\quad - (\mu_i(u_r) - \mu_i(u_m)) + C_K\, \mathcal{O}(1)\, Q^- \\ &\equiv C_K\, \mathcal{O}(1)\, Q^-. \end{aligned}$$

If $\mu_i(u_-) < \mu_i(u_+)$, we distinguish between several types of interactions.

If the incoming waves are both shocks, then the outgoing i-wave is also a shock wave. Using Lemma 7.4, we can write

$$\begin{aligned}
B(u_-, u_+) &= \Big(\Lambda - \overline{\lambda}_i(u_l, u_r)\Big)^- \, (\mu_i(u_r) - \mu_i(u_l)) \\
&\quad - \Big(\Lambda - \overline{\lambda}_i(u_l, u_m)\Big)^- \, (\mu_i(u_m) - \mu_i(u_l)) \\
&\quad - \Big(\Lambda - \overline{\lambda}_i(u_m, u_r)\Big)^- \, \big(\mu_i(u_r) - \mu_i(u_m)\big) + C_K \, \mathcal{O}(1) \, Q^- \\
&= \Big(\Lambda \, (\mu_i(u_r) - \mu_i(u_l)) - \overline{\lambda}_i(u_l, u_m) \, (\mu_i(u_m) - \mu_i(u_l)) \\
&\qquad\qquad - \overline{\lambda}_i(u_m, u_r) \, (\mu_i(u_r) - \mu_i(u_m))\Big)^- \\
&\quad - \Big(\Lambda - \overline{\lambda}_i(u_l, u_m)\Big)^- \, (\mu_i(u_m) - \mu_i(u_l)) \\
&\quad - \Big(\Lambda - \overline{\lambda}_i(u_m, u_r)\Big)^- \, \big(\mu_i(u_r) - \mu_i(u_m)\big) + C_K \, \mathcal{O}(1) \, Q^- \\
&= \frac{1}{2} \Big| \Lambda \, (\mu_i(u_r) - \mu_i(u_l)) - \overline{\lambda}_i(u_l, u_m) \, (\mu_i(u_m) - \mu_i(u_l)) \\
&\qquad\qquad - \overline{\lambda}_i(u_m, u_r) \, (\mu_i(u_r) - \mu_i(u_m)) \Big| \\
&\quad - \frac{1}{2} \Big| \Lambda - \overline{\lambda}_i(u_l, u_m) \Big| \, (\mu_i(u_m) - \mu_i(u_l)) \\
&\quad - \frac{1}{2} \Big| \Lambda - \overline{\lambda}_i(u_m, u_r) \Big| \, \big(\mu_i(u_r) - \mu_i(u_m)\big) + C_K \, \mathcal{O}(1) \, Q^-,
\end{aligned}$$

which is bounded by $C_K \, \mathcal{O}(1) \, Q^-$ by using the triangle inequality.

Next, if the incoming pattern is made of a shock followed by a rarefaction, then by Lemma 7.4 the outgoing i-wave pattern is made of either a single large shock (and we can rely on the same arguments as above) or else a large shock plus (possibly several) large rarefactions and other (small) waves. To treat the latter case, denote by $\mu_i^\natural(u_l)$ (and $u_l^\natural \in \mathcal{H}_i(u_l)$) the point within the interval $(\mu_i(u_l), \mu_i(u_r))$ at which the shock speed $\overline{\lambda}_i(u_l, u_l^\natural)$ coincides with the characteristic speed $\lambda_i(u_l^\natural)$. We find

$$\begin{aligned}
B(u_-, u_+) &= \Big(\Lambda - \overline{\lambda}_i(u_l, u_l^\natural)\Big)^- \, \big(\mu_i(u_l^\natural) - \mu_i(u_l)\big) + \Gamma_i^-(\Lambda, u_l^\natural, u_r) \\
&\quad - \Big(\Lambda - \overline{\lambda}_i(u_l, u_m)\Big)^- \, \big(\mu_i(u_m) - \mu_i(u_l)\big) \\
&\quad - \int_{\mu_i(u_m)}^{\mu_i(u_r)} \Big(\Lambda - \lambda_i(w_i(\nu; u_m))\Big)^- \, d\nu + C_K \, \mathcal{O}(1) \, Q^-,
\end{aligned}$$

where $\Gamma_i^-(\Lambda, u_l^\natural, u_r)$ is defined in a similar way as $\Gamma_i(u_l^\natural, u_r)$ in Section 7 by taking here into account the negative part. That is, with the same notation as in Section 7,

$$\begin{aligned}\Gamma_i^-(\Lambda, u_l^\natural, u_r) &:= \int_{\mu_i(u^{2p-1})}^{\mu_i(u^{2p})} \Big(\Lambda - \lambda_i(w_i(\nu; u^{2p-1}))\Big)^- \, d\nu \\ &\quad + \sum_p \sigma_i(u^{2p}, u^{2p+1}) \Big(\Lambda - \overline{\lambda}_i(u^{2p}, u^{2p+1})\Big)^- \\ &= \int_{\mu_i(u_l^\natural)}^{\mu_i(u_r)} \Big(\Lambda - \lambda_i(w_i(\nu; u_m))\Big)^- \, d\nu + C_K \, \mathcal{O}(1) \, Q^-.\end{aligned}$$

Consider the following function of the variable (μ, Λ)

$$\begin{aligned}g(\mu, \Lambda) := &\Big(\Lambda - \overline{\lambda}_i(u_l, w_i(\mu; u_m))\Big)^- \, \big(\mu - \mu_i(u_l)\big) \\ &+ \int_{\mu}^{\mu_i(u_r)} \Big(\Lambda - \lambda_i(w_i(\nu; u_m))\Big)^- \, d\nu.\end{aligned}$$

Note that for all μ in $[\mu_i(u_m), \mu_i(u_l^\natural)]$ (which is the interval of interest here) we have

$$\overline{\lambda}_i(u_l, w_i(\nu; u_m)) \leq \lambda_i(u_r). \tag{9.8}$$

We will prove that the function g is non-increasing in μ (modulo a term of order Q^-). We distinguish between three cases and use Lemma 7.4. If $\Lambda < \overline{\lambda}_i(u_l, w_i(\mu; u_m))$ we have

$$\begin{aligned}g(\mu, \Lambda) &= \Big(-\Lambda + \overline{\lambda}_i(u_l, w_i(\mu; u_m))\Big) \, \big(\mu - \mu_i(u_l)\big) \\ &\quad + \int_{\mu}^{\mu_i(u_r)} \Big(-\Lambda + \lambda_i(w_i(\nu; u_m))\Big) \, d\nu \\ &= \Big(-\Lambda + \overline{\lambda}_i(u_l, u_l^\natural)\Big) \, \big(\mu_i(u_l^\natural) - \mu_i(u_l)\big) \\ &\quad + \int_{\mu_i(u_l^\natural)}^{\mu_i(u_r)} \Big(-\Lambda + \lambda_i(w_i(\nu; u_l^\natural))\Big) \, d\nu + C_K \, \mathcal{O}(1) \, Q^-,\end{aligned}$$

thanks to Lemma 7.4. Thus, the function g is constant in this interval.

If $\lambda_i(v(\nu; u_l)) \leq \Lambda < \min\big(\overline{\lambda}_i(u_l, v(\nu; u_l)), \lambda_i(u_r)\big)$, then by calling μ' the point satisfying $\Lambda = \lambda_i(v(\nu; u_l))$ we find

$$\begin{aligned}g(\mu, \Lambda) = &\Big(-\Lambda + \overline{\lambda}_i(u_l, w_i(\mu; u_m))\Big) \, \big(\mu - \mu_i(u_l)\big) \\ &+ \int_{\mu'}^{\mu_i(u_r)} \Big(-\Lambda + \lambda_i(w_i(\nu; u_m))\Big) \, d\nu,\end{aligned}$$

where the latter (integral) term is a constant and the first one is non-increasing.

If now $\overline{\lambda}_i(u_l, v(\nu; u_l)) \leq \Lambda < \lambda_i(u_r)$, then using the notation μ' again, we find

$$g(\mu, \Lambda) = \int_{\mu'}^{\mu_i(u_r)} \Big(-\Lambda + \lambda_i(w_i(\nu; u_m))\Big)\, d\nu,$$

which is constant.

If $\lambda_i(u_r) \leq \Lambda < \overline{\lambda}_i(u_l, v(\nu; u_l))$, then

$$g(\mu, \Lambda) = \Big(-\Lambda + \overline{\lambda}_i(u_l, w_i(\mu; u_m))\Big) \big(\mu - \mu_i(u_l)\big)$$

which is non-increasing. Finally, if $\Lambda > \min\big(\overline{\lambda}_i(u_l, v(\nu; u_l)), \lambda_i(u_r)\big)$, then $g(\mu, \Lambda) \equiv 0$.

This completes the proof that $B(u_-, u_+) \leq 0$ in the monotone case.

In conclusion, for all interactions we found that (9.4) holds, which implies [14] that the generalized Glimm functional is decreasing. This completes the proof of Theorem 9.1. □

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